

Connections between Extensive Family of Harmonic Mappings and Pascal Distribution Series

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Abstract. Herein, we investigate a comprehensive harmonic family $\Pi_{\eta}(k_3, k_2, k_1, k_0; s_3, s_2, s_1, s_0)$ defined by a coefficient inequality involving cubic polynomial weights. We establish several inclusion relationships between this family and various well-known subfamilies of harmonic mappings through the Pascal distribution series operator. By suitable specializations of the parameters $k_3, k_2, k_1, k_0, s_3, s_2, s_1, s_0$, several previously studied harmonic subfamilies and related inclusion results are recovered as special cases. The results provide a unified framework for studying inclusion relationships associated with the Pascal distribution series operator and may serve as a basis for further investigations of harmonic function families generated by other probability distribution series and related convolution operators.

1. INTRODUCTION AND PRELIMINARIES

A continuous complex function $h = u + iv$ is said to be harmonic in a simply connected complex domain Δ if u and v are real harmonic in Δ . Let b and c be analytic in Δ , then we can write $h = b + \bar{c}$ in any simply connected domain (b the analytic part of h and c the co-analytic part of h). The fact that $|b'(\zeta)| > |c'(\zeta)|$ in Δ is a sufficient and necessary condition for h to be locally univalent and sense preserving.

The family of harmonic functions H of the form $h = b + \bar{c}$ is analytic in $\Theta = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$, where

$$b(\zeta) = \zeta + \sum_{u=2}^{\infty} a_u \zeta^u, \quad c(\zeta) = \sum_{u=1}^{\infty} d_u \zeta^u, \quad |d_1| < 1. \quad (1.1)$$

Also, let UH be the class of harmonic univalent, sense-preserving functions $h = b + \bar{c}$ in Θ .

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If the co-analytic component of a member of the family UH is zero, then the family reduces to the family U of normalized analytic univalent functions.

The family UH and its geometric subfamilies were studied in 1984 by Clunie and Sheil-Small [1], and came up with some bounds on the coefficients. Since then, the family UH and its subfamilies have been the subject of several works, see [2–4].

Define the subfamily UH^0 of UH by (see [1]):

$$UH^0 := \{h = b + \bar{c} \in UH : c'(0) = d_1 = 0\}.$$

If the range $h(\Theta)$ is starlike with regard to the origin, then a sense-preserving harmonic mapping $h \in UH^0$ belongs to the family U^* . A harmonic starlike mapping in Θ is defined as a function $h \in UH^*$. Moreover, if $h(\Theta)$ is a convex domain and $h \in UH^0$, then a function h defined in Θ is a member of the family CH . A function $h \in CH$ is referred to as harmonic convex in Θ (see [5]). Analytically, we have

$$h \in UH^* \text{ iff } \arg\left(\frac{\partial}{\partial\theta} h(re^{i\theta})\right) \geq 0,$$

and

$$h \in CH \text{ iff } \frac{\partial}{\partial\theta} \left\{ \arg\left(\frac{\partial}{\partial\theta} \arg h(re^{i\theta})\right) \right\} \geq 0,$$

$$\varsigma = re^{i\theta} \in \Theta, 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1.$$

Also, assume the family TH of functions in UH that written as $h = b + \bar{c}$, where

$$b(\varsigma) = \varsigma - \sum_{u=2}^{\infty} |a_u| \varsigma^u \quad \text{and} \quad c(\varsigma) = \sum_{u=1}^{\infty} |d_u| \varsigma^u, \quad |d_1| < 1. \quad (1.2)$$

For $0 \leq \varepsilon < 1$, let

$$NH(\varepsilon) := \left\{ h \in H : \operatorname{Re}\left(\frac{h'(\varsigma)}{\varsigma'}\right) \geq \varepsilon, \varsigma = re^{i\theta} \in \Theta \right\},$$

and

$$RH(\varepsilon) := \left\{ h \in H : \operatorname{Re}\left(\frac{h''(\varsigma)}{\varsigma''}\right) \geq \varepsilon, \varsigma = re^{i\theta} \in \Theta \right\},$$

where

$$\varsigma' = \frac{\partial}{\partial\theta} (\varsigma = re^{i\theta}), \varsigma'' = \frac{\partial}{\partial\theta} (\varsigma'), h'(\varsigma) = \frac{\partial}{\partial\theta} h(re^{i\theta}), \text{ and } h'' = \frac{\partial}{\partial\theta} (h'(\varsigma)).$$

Define

$$TNH(\varepsilon) := NH(\varepsilon) \cap TH \quad \text{and} \quad TRH(\varepsilon) := RH(\varepsilon) \cap TH.$$

The families TH , $NH(\varepsilon)$, $TNH(\varepsilon)$, $RH(\varepsilon)$ and $TRH(\varepsilon)$ were introduced and studied in [6] and [7].

The importance of special functions in geometric function theory is widely acknowledged [8], especially in the de Branges solution of the famous Bieberbach conjecture [9, 10]. The geometric

characteristics of analytic functions that connect to hypergeometric functions (see [11–15]), Struve functions (see [16, 17]), Bessel functions (see [18, 19]), Error Function (see [20, 21]), Hermite and Horadam Polynomials (see [22, 23]), Poisson distribution series (see [24–26]), Pascal distribution series (see [27, 28]), and Mittag-Leffler function [29, 30]) have recently been the focus of studies.

If the variable Y takes the values $0, 1, 2, 3, \dots$, then it is said to be a Pascal distribution with probabilities

$(1 - \varrho)^\beta, \frac{\varrho\beta(1 - \varrho)^\beta}{1!}, \frac{\varrho^2\beta(\beta + 1)(1 - \varrho)^\beta}{2!}, \frac{\varrho^3\beta(\beta + 1)(\beta + 2)(1 - \varrho)^\beta}{3!}, \dots$, respectively, where ϱ and β are called the parameters, and thus

$$PA(Y = r) = \binom{r + \beta - 1}{\beta - 1} \varrho^r (1 - \varrho)^\beta, \quad r = 0, 1, 2, 3, \dots$$

Recently, El-Deeb et al. [31] introduced a power series whose coefficients are probabilities of the Pascal distribution, that is

$$P_\varrho^\beta(\varsigma) := \varsigma + \sum_{u=2}^{\infty} \binom{u + \beta - 2}{\beta - 1} \varrho^{u-1} (1 - \varrho)^\beta \varsigma^u, \quad \varsigma \in \Theta,$$

where $\beta \geq 1$, $0 \leq \varrho \leq 1$, and observe that, by the ratio test, the radius of convergence of the aforementioned series is infinite.

For $\beta, \tau \geq 1$ and $0 \leq \varrho, \phi \leq 1$; the following operator is introduced [32]

$$P_{\varrho, \phi}^{\beta, \tau} b(\varsigma) = P_\varrho^\beta(\varsigma) * b(\varsigma) + \overline{P_\phi^\tau(\varsigma) * c(\varsigma)} = A(\varsigma) + \overline{B(\varsigma)}, \quad (1.3)$$

where

$$A(\varsigma) = \varsigma + \sum_{u=2}^{\infty} \binom{u + \beta - 2}{\beta - 1} \varrho^{u-1} (1 - \varrho)^\beta a_u \varsigma^u, \quad (1.4)$$

and

$$B(\varsigma) = d_1 \varsigma + \sum_{u=2}^{\infty} \binom{u + \delta - 2}{\delta - 1} \varphi^{u-1} (1 - \varphi)^\delta d_u \varsigma^u. \quad (1.5)$$

Definition 1.1. Let $h = b + \bar{c}$ where b and c are given by (1.2), then $h \in \Pi_\eta(k_3, k_2, k_1, k_0; s_3, s_2, s_1, s_0)$ if

$$\sum_{u=2}^{\infty} (k_3 u^3 + k_2 u^2 + k_1 u + k_0) |a_u| + \sum_{u=1}^{\infty} (s_3 u^3 + s_2 u^2 + s_1 u + s_0) |d_u| \leq \eta,$$

where $k_3, k_2, k_1, k_0, s_3, s_2, s_1, s_0 \in \mathbb{R}$ and $\eta > 0$.

Remark 1.1. When $h \in \Pi_\eta(k_3, k_2, k_1, k_0; s_3, s_2, s_1, s_0)$, then

$$|a_u| \leq \frac{\eta}{k_3 u^3 + k_2 u^2 + k_1 u + k_0}, \quad u \geq 2,$$

and

$$|d_u| \leq \frac{\eta}{s_3 u^3 + s_2 u^2 + s_1 u + s_0}, \quad u \geq 1.$$

Remark 1.2. Through specialization for the coefficients $k_3, k_1, k_2, k_0, s_3, s_2, s_1, s_0$ and η in Definition 1.1, we get the following:

(1) A function $h \in H(\alpha, \mu) \equiv \Pi_{1-\mu}(\alpha, 0, 1 - \alpha, 0; \alpha, 0, 1 - \alpha, 0)$ (which is due to Sokół et al. [33]) if

$$\sum_{u=2}^{\infty} u(1 + \alpha(u^2 - 1)) |a_u| + \sum_{u=1}^{\infty} u(1 + \alpha(u^2 - 1)) |d_u| \leq 1 - \mu, \quad (\alpha \geq 0, 0 \leq \mu < 1).$$

(2) A function $h \in HUC(\alpha, \mu) \equiv \Pi_{1-\mu}(0, \alpha + 1, -(\alpha + \mu), 0; 0, \alpha + 1, \alpha + \mu, 0)$ (which is due to Kim et al. [34]) if

$$\sum_{u=2}^{\infty} u(u(\alpha + 1) - (\alpha + \mu)) |a_u| + \sum_{u=1}^{\infty} u(u(\alpha + 1) + (\alpha + \mu)) |d_u| \leq 1 - \mu, \quad (\alpha \geq 0, 0 \leq \mu < 1).$$

(3) A function $h \in SH(J, L) \equiv \Pi_{2(J-L)}(0, L + 1, -J - 1, 0; 0, L + 1, J + 1, 0)$ (which is due to Dziok [35]) if

$$\sum_{u=1}^{\infty} u[u(L + 1) - J - 1] |a_u| + \sum_{u=1}^{\infty} u[u(L + 1) + J + 1] |d_u| \leq 2(J - L), \quad (-L \leq J < L \leq 1).$$

(4) A function $h \in GH(\delta, \mu) \equiv \Pi_{1-\mu}(0, 0, 2, -\delta(\mu + 1); 0, 0, 2, \delta(\mu + 1))$ (which is due to Porwal et al. [36]) if

$$\sum_{u=2}^{\infty} (2u - \delta(\mu + 1)) |a_u| + \sum_{u=1}^{\infty} (2u + \delta(\mu + 1)) |d_u| \leq 1 - \mu, \quad (0 \leq \delta < 1, 0 \leq \mu < 1).$$

(5) A function $h \in S_H^*(\delta, \mu) \equiv \Pi_{1-\mu}(0, 0, 1 - \delta\mu, -\mu(1 - \delta); 0, 0, 1 - \delta\mu, \mu(1 - \delta))$ (which is due to Öztürk et al. [37]) if

$$\sum_{u=2}^{\infty} (u(1 - \delta\mu) - \mu(1 - \delta)) |a_u| + \sum_{u=1}^{\infty} (u(1 - \delta\mu) + \mu(1 - \delta)) |d_u| \leq 1 - \mu, \quad (0 \leq \delta < 1, 0 \leq \mu < 1).$$

(6) A function $h \in G_H(\delta, \mu) \equiv \Pi_{1-\mu}(0, 0, 2, -\alpha - 1; 0, 0, 2, \alpha + 1)$ (which is due to Rosy et al. [38]) if

$$\sum_{u=2}^{\infty} (2u - \alpha - 1) |a_u| + \sum_{u=1}^{\infty} (2u + \alpha + 1) |d_u| \leq 1 - \mu, \quad (0 \leq \delta < 1, 0 \leq \mu < 1).$$

(7) A function $h \in G_H(\alpha, \gamma, \mu) \equiv \Pi_{1-\mu}(0, 0, \gamma + 1, -\alpha(\gamma + \mu); 0, 0, \gamma + 1, \alpha(\gamma + \mu))$ (which is due to Ahuja et al. [39]) if

$$\sum_{u=2}^{\infty} (u(\gamma + 1) - \alpha(\gamma + \mu)) |a_u| + \sum_{u=1}^{\infty} (u(\gamma + 1) + \alpha(\gamma + \mu)) |d_u| \leq 1 - \mu,$$

$$(0 \leq \gamma < \infty, 0 \leq \alpha < 1, 0 \leq \mu < 1).$$

2. PRELIMINARY LEMMAS

In order to establish our main results, we require the following Lemmas.

Lemma 2.1. [5] Let $h = b + \bar{c}$ where b and c are given by (1.2) and suppose that $0 \leq \varepsilon < 1$. Then $h \in TNH(\varepsilon)$ if and only if

$$\sum_{u=2}^{\infty} u|a_u| + \sum_{u=1}^{\infty} u|d_u| \leq 1 - \varepsilon.$$

Moreover, if $h \in TNH(\varepsilon)$, then

$$|a_u| \leq \frac{1-\varepsilon}{u}, u \geq 2, \quad (2.1)$$

and

$$|d_u| \leq \frac{1-\varepsilon}{u}, u \geq 1. \quad (2.2)$$

Lemma 2.2. [6] Let $h = b + \bar{c}$ where b and c are given by (1.2), and suppose that $0 \leq \varepsilon < 1$. Then $h \in TRH(\varepsilon)$ if

$$\sum_{u=2}^{\infty} u^2 |a_u| + \sum_{u=1}^{\infty} u^2 |d_u| \leq 1 - \varepsilon.$$

Moreover, if $h \in TRH(\varepsilon)$, then

$$|a_u| \leq \frac{1-\varepsilon}{u^2}, u \geq 2 \quad (2.3)$$

and

$$|d_u| \leq \frac{1-\varepsilon}{u^2}, u \geq 1. \quad (2.4)$$

Lemma 2.3. [1] If $h = b + \bar{c} \in NH(\varepsilon)$ where b and c are given by (1.1) with $d_1 = 0$, then

$$|a_u| \leq \frac{(2u+1)(u+1)}{6} \text{ and } |d_u| \leq \frac{(2u-1)(u-1)}{6}. \quad (2.5)$$

Lemma 2.4. [1] If $h = b + \bar{c} \in RH(\varepsilon)$ where b and c are given by (1.1) with $d_1 = 0$, then

$$|a_u| \leq \frac{u+1}{2} \text{ and } |d_u| \leq \frac{u-1}{2}. \quad (2.6)$$

In order to prove our main results, we will use the following notations, for $\beta \geq 1$ and $0 \leq \varrho < 1$:

$$\begin{aligned} \sum_{u=0}^{\infty} \binom{u+\beta-1}{\beta-1} \varrho^u &= \frac{1}{(1-\varrho)^\beta}, \quad \sum_{u=0}^{\infty} \binom{u+\beta-2}{\beta-2} \varrho^u = \frac{1}{(1-\varrho)^{\beta-1}}, \\ \sum_{u=0}^{\infty} \binom{u+\beta}{\beta} \varrho^u &= \frac{1}{(1-\varrho)^{\beta+1}}, \quad \sum_{u=0}^{\infty} \binom{u+\beta+1}{\beta+1} \varrho^u = \frac{1}{(1-\varrho)^{\beta+2}}. \end{aligned}$$

By simple calculations we derive the following relations:

$$\begin{aligned} \sum_{u=2}^{\infty} \binom{u+\beta-2}{\beta-1} \varrho^{u-1} &= \sum_{u=0}^{\infty} \binom{u+\beta-1}{\beta-1} \varrho^u - 1 = \frac{1}{(1-\varrho)^\beta} - 1, \\ \sum_{u=2}^{\infty} (u-1) \binom{u+\beta-2}{\beta-1} \varrho^{u-1} &= \varrho \beta \sum_{u=0}^{\infty} \binom{u+\beta}{\beta} \varrho^u = \frac{\varrho \beta}{(1-\varrho)^{\beta+1}}, \\ \sum_{u=3}^{\infty} (u-1)(u-2) \binom{u+\beta-2}{\beta-1} \varrho^{u-1} \\ &= \varrho^2 \beta(\beta+1) \sum_{u=0}^{\infty} \binom{u+\beta+1}{\beta+1} \varrho^u = \frac{\varrho^2 \beta(\beta+1)}{(1-\varrho)^{\beta+2}} = 2\varrho^2 \frac{\binom{\beta+1}{\beta-1}}{(1-\varrho)^{\beta+2}}, \end{aligned}$$

and in general, we have

$$\sum_{u=i+1}^{\infty} (u-1)(u-2) \cdots (u-i) \binom{u+\beta-2}{\beta-1} \varrho^{u-1}$$

$$\begin{aligned}
&= \varrho^j \beta(\beta+1) \cdots (\beta+i-1) \sum_{u=0}^{\infty} \binom{u+\beta+1}{\beta+1} \varrho^u \\
&= \frac{\varrho^j \beta(\beta+1) \cdots (\beta+i-1)}{(1-\varrho)^{\beta+i}} = i! \varrho^j \frac{\binom{\beta+i-1}{\beta-1}}{(1-\varrho)^{\beta+i}}.
\end{aligned}$$

Inspired by several earlier results and connections regarding the relationships among different subfamilies of harmonic univalent functions, utilizing Pascal distribution series (see, [40–42]), we obtain some properties between some families of analytic functions with inclusive family $\Pi_{\eta}(k_3, k_2, k_1, k_0; s_3, s_2, s_1, s_0)$.

3. INCLUSION RELATIONS OF THE FAMILY $\Pi_{\eta}(k_3, k_2, k_1, k_0; s_3, s_2, s_1, s_0)$

Let

$$NH^0(\varepsilon) := \{h \in NH(\varepsilon) : d_1 = 0\},$$

and

$$RH^0(\varepsilon) := \{h \in \mathcal{RH}(\varepsilon) : d_1 = 0\}.$$

In this section, we prove some inclusion relations of the harmonic inclusive family $\Pi_{\eta}(k_3, k_2, k_1, k_0; s_3, s_2, s_1, s_0)$ with the families $NH^0(\varepsilon)$, $RH^0(\varepsilon)$, $TNH(\varepsilon)$, $TRH(\varepsilon)$.

Theorem 3.1. *Let $\beta, \delta > 0$, $0 \leq \varrho, \varphi < 1$, $k_3, k_2, k_1, k_0, s_3, s_2, s_1, s_0 \in \mathbb{R}$ and $\eta > 0$. If*

$$\begin{aligned}
&\left[240k_3\varrho^5 \frac{\binom{\beta+4}{\beta-1}}{(1-\varrho)^5} + (792k_3 + 48k_2)\varrho^4 \frac{\binom{\beta+3}{\beta-1}}{(1-\varrho)^4} \right. \\
&+ (966k_3 + 138k_2 + 12k_1)\varrho^3 \frac{\binom{\beta+2}{\beta-1}}{(1-\varrho)^3} + (522k_3 + 138k_2 + 30k_1 + 4k_0)\varrho^2 \frac{\binom{\beta+1}{\beta-1}}{(1-\varrho)^2} \\
&+ (114k_3 + 54k_2 + 24k_1 + 9k_0) \frac{\varrho \binom{\beta}{\beta-1}}{1-\varrho} + 6(k_3 + k_2 + k_1 + k_0)(1 - (1-\varrho)^{\beta}) \\
&+ 240s_3\varphi^5 \frac{\binom{\delta+4}{\delta-1}}{(1-\varphi)^5} + (648s_3 + 48s_2)\varphi^4 \frac{\binom{\delta+3}{\delta-1}}{(1-\varphi)^4} + (606s_3 + 102s_2 + 12s_1)\varphi^3 \frac{\binom{\delta+2}{\delta-1}}{(1-\varphi)^3} \\
&\left. + (222s_3 + 66s_2 + 18s_1 + 4s_0)\varphi^2 \frac{\binom{\delta+1}{\delta-1}}{(1-\varphi)^2} + (24s_3 + 12s_2 + 6s_1 + 3s_0) \frac{\varphi \binom{\delta}{\delta-1}}{1-\varphi} \right] \leq 6\eta \quad (3.1)
\end{aligned}$$

then

$$P_{\varrho, \varphi}^{\beta, \tau}(NH^0(\varepsilon)) \subset \Pi_{\eta}(k_3, k_2, k_1, k_0; s_3, s_2, s_1, s_0).$$

Proof. Let $h = b + \bar{c} \in NH^0(\varepsilon)$ where b and c are of the form (1.2) with $d_1 = 0$. We need to show that $P_{\varrho, \varphi}^{\beta, \tau}(h) = A + B \in \Pi_{\eta}(k_3, k_2, k_1, k_0; s_3, s_2, s_1, s_0)$, where A and B defined by (1.4) and (1.5) with $d_1 = 0$ are analytic functions in Θ . From Definition 1.1, we need to prove that

$$S(\beta, \delta, \varrho, \varphi) \leq \eta,$$

where

$$S(\beta, \delta, \varrho, \varphi) = \sum_{u=2}^{\infty} (k_3 u^3 + k_2 u^2 + k_1 u + k_0) \left| \binom{u+\beta-2}{\beta-1} \varrho^{u-1} (1-\varrho)^{\beta} a_u \right| \\ + \sum_{u=2}^{\infty} (s_3 u^3 + s_2 u^2 + s_1 u + s_0) \left| \binom{u+\delta-2}{\delta-1} \varphi^{u-1} (1-\varphi)^{\delta} d_u \right|. \quad (3.2)$$

Using Lemma 2.3, we get

$$S(\beta, \delta, \varrho, \varphi) \\ \leq \frac{1}{6} \left[\sum_{u=2}^{\infty} (2u+1)(u+1) (k_3 u^3 + k_2 u^2 + k_1 u + k_0) \binom{u+\beta-2}{\beta-1} \varrho^{u-1} (1-\varrho)^{\beta} \right. \\ \left. + \sum_{u=2}^{\infty} (2u-1)(u-1) (s_3 u^3 + s_2 u^2 + s_1 u + s_0) \binom{u+\delta-2}{\delta-1} \varphi^{u-1} (1-\varphi)^{\delta} \right] \\ = \frac{1}{6} \left[\sum_{u=2}^{\infty} [2k_3 u^5 + (3k_3 + 2k_2)u^4 + (k_3 + 3k_2 + 2k_1)u^3 \right. \\ \left. + (k_2 + 3k_1 + 2k_0)u^2 + (k_1 + 3k_0)u + k_0] \binom{u+\beta-2}{\beta-1} \varrho^{u-1} (1-\varrho)^{\beta} \right] \\ \frac{1}{6} \left[\sum_{u=2}^{\infty} [2s_3 u^5 + (2s_2 - 3s_3)u^4 + (s_3 - 3s_2 + 2s_1)u^3 \right. \\ \left. + (s_2 - 3s_1 + 2s_0)u^2 + (s_1 - 3s_0)u + s_0] \binom{u+\delta-2}{\delta-1} \varphi^{u-1} (1-\varphi)^{\delta} \right]. \quad (3.3)$$

Writing

$$u = (u-1) + 1, \quad (3.4)$$

$$u^2 = (u-1)(u-2) + 3(u-1) + 1, \quad (3.5)$$

$$u^3 = (u-1)(u-2)(u-3) + 6(u-1)(u-2) + 7(u-1) + 1, \quad (3.6)$$

$$u^4 = (u-1)(u-2)(u-3)(u-4) + 10(u-1)(u-2)(u-3) + 25(u-1)(u-2) + 15(u-1) + 1, \quad (3.7)$$

and

$$u^5 = (u-1)(u-2)(u-3)(u-4)(u-5) + 15(u-1)(u-2)(u-3)(u-4) \\ + 65(u-1)(u-2)(u-3) + 90(u-1)(u-2) + 31(u-1) + 1,$$

in (3.3), we have

$$S(\beta, \delta, \varrho, \varphi) \leq \frac{1}{6} \left[\sum_{u=2}^{\infty} [2k_3 (u-1)(u-2)(u-3)(u-4)(u-5) \right. \\ + (33k_3 + 2k_2)(u-1)(u-2)(u-3)(u-4) \\ + (161k_3 + 23k_2 + 2k_1)(u-1)(u-2)(u-3) \\ \left. + (261k_3 + 69k_2 + 15k_1 + 2k_0)(u-1)(u-2) \right]$$

$$\begin{aligned}
& + (114k_3 + 54k_2 + 24k_1 + 9k_0) (u-1) + 6 (k_3 + k_2 + k_1 + k_0) \left[\binom{u+\beta-2}{\beta-1} \varrho^{u-1} (1-\varrho)^\beta \right. \\
& + \sum_{u=2}^{\infty} [2s_3(u-1)(u-2)(u-3)(u-4)(u-5) \\
& + (27s_3 + 2s_2)(u-1)(u-2)(u-3)(u-4) \\
& + (101s_3 + 17s_2 + 2s_1) (u-1) (u-2) (u-3) \\
& + (111s_3 + 33s_2 + 9s_1 + 2s_0) (u-1) (u-2) \\
& \left. + (24s_3 + 12s_2 + 6s_1 + 3s_0) (u-1) \right] \left[\binom{u+\delta-2}{\delta-1} \varphi^{u-1} (1-\varphi)^\delta \right] \\
& = \frac{1}{6} \left[240k_3 \varrho^5 \frac{\binom{\beta+4}{\beta-1}}{(1-\varrho)^5} + (792k_3 + 48k_2) \varrho^4 \frac{\binom{\beta+3}{\beta-1}}{(1-\varrho)^4} \right. \\
& + (966k_3 + 138k_2 + 12k_1) \varrho^3 \frac{\binom{\beta+2}{\beta-1}}{(1-\varrho)^3} + (522k_3 + 138k_2 + 30k_1 + 4k_0) \varrho^2 \frac{\binom{\beta+1}{\beta-1}}{(1-\varrho)^2} \\
& + (114k_3 + 54k_2 + 24k_1 + 9k_0) \frac{\varrho \binom{\beta}{\beta-1}}{1-\varrho} + 6 (k_3 + k_2 + k_1 + k_0) (1 - (1-\varrho)^\beta) \\
& + 240s_3 \varphi^5 \frac{\binom{\delta+4}{\delta-1}}{(1-\varphi)^5} + (648s_3 + 48s_2) \varphi^4 \frac{\binom{\delta+3}{\delta-1}}{(1-\varphi)^4} + (606s_3 + 102s_2 + 12s_1) \varphi^3 \frac{\binom{\delta+2}{\delta-1}}{(1-\varphi)^3} \\
& \left. + (222s_3 + 66s_2 + 18s_1 + 4s_0) \varphi^2 \frac{\binom{\delta+1}{\delta-1}}{(1-\varphi)^2} + (24s_3 + 12s_2 + 6s_1 + 3s_0) \frac{\varphi \binom{\delta}{\delta-1}}{1-\varphi} \right].
\end{aligned}$$

But the last expression is bounded above by η if (3.1) holds. $\square$

Theorem 3.2. Let $\beta, \delta > 0, 0 \leq \varrho, \varphi < 1, k_3, k_2, k_1, k_0, s_3, s_2, s_1, s_0 \in \mathbb{R}$ and $\eta > 0$. If

$$\begin{aligned}
& 24k_3 \varrho^4 \frac{\binom{\beta+3}{\beta-1}}{(1-\varrho)^4} + 6(11k_3 + k_2) \varrho^3 \frac{\binom{\beta+2}{\beta-1}}{(1-\varrho)^3} + 2(31k_3 + 7k_2 + k_1) \varrho^2 \frac{\binom{\beta+1}{\beta-1}}{(1-\varrho)^2} + \\
& (22k_3 + 10k_2 + 4k_1 + k_0) \varrho \frac{\binom{\beta}{\beta-1}}{1-\varrho} + 2(k_3 + k_2 + k_1 + k_0) (1 - (1-\varrho)^\beta) \\
& + 24s_3 \varphi^4 \frac{\binom{\delta+3}{\delta-1}}{(1-\varphi)^4} + 6(9s_3 + s_2) \varphi^3 \frac{\binom{\delta+2}{\delta-1}}{(1-\varphi)^3} \\
& + 2(19s_3 + 5s_2 + s_1) \varphi^2 \frac{\binom{\delta+1}{\delta-1}}{(1-\varphi)^2} + (8s_3 + 4s_2 + 2s_1 + s_0) \varphi \frac{\binom{\delta}{\delta-1}}{1-\varphi} \Big] \\
& \leq 2\eta,
\end{aligned} \tag{3.8}$$

then

$$P_{\varrho, \phi}^{\beta, \tau}(RH^0(\varepsilon)) \subset \Pi_\eta(k_3, k_2, k_1, k_0; s_3, s_2, s_1, s_0).$$

Proof. Let $h = b + \bar{c} \in RH(\varepsilon)$ where b and c are of the form (1.2) with $d_1 = 0$. We need to show that $P_{\varrho, \phi}^{\beta, \tau}(h) = A + B \in \Pi_{\eta}(k_3, k_2, k_1, k_0; s_3, s_2, s_1, s_0)$, where A and B defined by (1.4) and (1.5) with $d_1 = 0$ are analytic functions in Θ . From Definition 1.1, we need to prove that

$$S(\beta, \delta, \varrho, \varphi) \leq \eta,$$

where $S(\beta, \delta, \varrho, \varphi)$ as given in (3.2). by Lemma 2.4, we get

$$\begin{aligned} S(\beta, \delta, \varrho, \varphi) &\leq \frac{1}{2} \left[\sum_{u=2}^{\infty} (u+1) (k_3 u^3 + k_2 u^2 + k_1 u + k_0) \binom{u+\beta-2}{\beta-1} \varrho^{u-1} (1-\varrho)^{\beta} \right. \\ &\quad \left. + \sum_{u=2}^{\infty} (u-1) (s_3 u^3 + s_2 u^2 + s_1 u + s_0) \binom{u+\delta-2}{\delta-1} \varphi^{u-1} (1-\varphi)^{\delta} \right] \\ &= \frac{1}{2} \left[\sum_{u=2}^{\infty} \left[\begin{array}{c} k_3 u^4 + (k_3 + k_2) u^3 + (k_2 + k_1) u^2 \\ + (k_1 + k_0) u + k_0 \end{array} \right] \binom{u+\beta-2}{\beta-1} \varrho^{u-1} (1-\varrho)^{\beta} \right. \\ &\quad \left. + \sum_{u=2}^{\infty} \left[\begin{array}{c} s_3 u^4 + (s_2 - s_3) u^3 + (s_1 - s_2) u^2 + (s_0 - s_1) u - s_0 \end{array} \right] \binom{u+\delta-2}{\delta-1} \varphi^{u-1} (1-\varphi)^{\delta} \right]. \quad (3.9) \end{aligned}$$

Using the equations (3.4)-(3.7) in (3.9), we have

$$\begin{aligned} S(\beta, \delta, \varrho, \varphi) &\leq \frac{1}{2} \left[\sum_{u=2}^{\infty} [k_3 (u-1)(u-2)(u-3)(u-4) \right. \\ &\quad + (11k_3 + k_2)(u-1)(u-2)(u-3) + (31k_3 + 7k_2 + k_1)(u-1)(u-2) \\ &\quad + (22k_3 + 10k_2 + 4k_1 + k_0)(u-1) + 2(k_3 + k_2 + k_1 + k_0)] \binom{u+\beta-2}{\beta-1} \varrho^{u-1} (1-\varrho)^{\beta} \\ &\quad + \sum_{u=2}^{\infty} [s_3 (u-1)(u-2)(u-3)(u-4) + (9s_3 + s_2)(u-1)(u-2)(u-3) \\ &\quad + (19s_3 + 5s_2 + s_1)(u-1)(u-2) \\ &\quad + (8s_3 + 4s_2 + 2s_1 + s_0)(u-1)] \binom{u+\delta-2}{\delta-1} \varphi^{u-1} (1-\varphi)^{\delta} \Big] \\ &= \frac{1}{2} \left[24k_3 \varrho^4 \frac{\binom{\beta+3}{\beta-1}}{(1-\varrho)^4} + 6(11k_3 + k_2) \varrho^3 \frac{\binom{\beta+2}{\beta-1}}{(1-\varrho)^3} + 2(31k_3 + 7k_2 + k_1) \varrho^2 \frac{\binom{\beta+1}{\beta-1}}{(1-\varrho)^2} + \right. \\ &\quad (22k_3 + 10k_2 + 4k_1 + k_0) \varrho \frac{\binom{\beta}{\beta-1}}{1-\varrho} + 2(k_3 + k_2 + k_1 + k_0)(1 - (1-\varrho)^{\beta}) \\ &\quad + 24s_3 \varphi^4 \frac{\binom{\delta+3}{\delta-1}}{(1-\varphi)^4} + 6(9s_3 + s_2) \varphi^3 \frac{\binom{\delta+2}{\delta-1}}{(1-\varphi)^3} \\ &\quad \left. + 2(19s_3 + 5s_2 + s_1) \varphi^2 \frac{\binom{\delta+1}{\delta-1}}{(1-\varphi)^2} + (8s_3 + 4s_2 + 2s_1 + s_0) \varphi \frac{\binom{\delta}{\delta-1}}{1-\varphi} \right]. \end{aligned}$$

But the last expression is bounded above by η if (3.8) holds. $\square$

Next we determine connection between $TNH(\varepsilon)$ and $\Pi_\eta(k_3, k_2, k_1, 0; s_3, s_2, s_1, 0)$.

Theorem 3.3. Let $\beta, \delta > 0, 0 \leq \varrho, \varphi < 1, 0 \leq \varepsilon < 1, k_3, k_2, k_1, s_3, s_2, s_1 \in \mathbb{R}$ and $\eta > 0$. If

$$\begin{aligned} & (1 - \varepsilon) \left[2k_3 \varrho^2 \frac{\binom{\beta+1}{\beta-1}}{(1 - \varrho)^2} + (3k_3 + k_2) \varrho \frac{\binom{\beta}{\beta-1}}{1 - \varrho} + (k_3 + k_2 + k_1)(1 - (1 - \varrho^\beta)) \right. \\ & \left. 2s_3 \varphi^2 \frac{\binom{\delta+1}{\delta-1}}{(1 - \varphi)^2} + (3s_3 + s_2) \varphi \frac{\binom{\delta}{\delta-1}}{1 - \varphi} + (s_3 + s_2 + s_1)(1 - (1 - \varphi^\delta)) \right] \\ & \leq \eta - (s_3 + s_2 + s_1) |d_1| \end{aligned}$$

then

$$P_{\varrho, \phi}^{\beta, \tau}(TNH(\varepsilon)) \subset \Pi_\eta(k_3, k_2, k_1, 0; s_3, s_2, s_1, 0).$$

Proof. Let $h = b + \bar{c} \in TNH(\varepsilon)$ where b and c are given by (1.2). From Definition 1.1, it is enough to show that $Q(\beta, \delta, \varrho, \varphi) \leq \eta$, where

$$\begin{aligned} Q(\beta, \delta, \varrho, \varphi) &= \sum_{u=2}^{\infty} (k_3 u^3 + k_2 u^2 + k_1 u) \left| \binom{u + \beta - 2}{\beta - 1} \varrho^{u-1} (1 - \varrho)^\beta a_u \right| \\ &+ (s_3 + s_2 + s_1) |d_1| + \sum_{u=2}^{\infty} (s_3 u^3 + s_2 u^2 + s_1 u) \left| \binom{u + \delta - 2}{\delta - 1} \varphi^{u-1} (1 - \varphi)^\delta d_u \right|. \quad (3.10) \end{aligned}$$

Using the inequalities (2.1) and (2.2), it follows that

$$\begin{aligned} & Q(\beta, \delta, \varrho, \varphi) \\ & \leq (1 - \varepsilon) \left[\sum_{u=2}^{\infty} (k_3 u^2 + k_2 u + k_1) \binom{u + \beta - 2}{\beta - 1} \varrho^{u-1} (1 - \varrho)^\beta \right. \\ & \left. + \sum_{u=2}^{\infty} (s_3 u^2 + s_2 u + s_1) \binom{u + \delta - 2}{\delta - 1} \varphi^{u-1} (1 - \varphi)^\delta \right] + (s_3 + s_2 + s_1) |d_1| \\ & = (1 - \varepsilon) \left[\sum_{u=2}^{\infty} [k_3 (u - 1)(u - 2) + (3k_3 + k_2)(u - 1) + k_3 + k_2 + k_1] \binom{u + \beta - 2}{\beta - 1} \varrho^{u-1} (1 - \varrho)^\beta \right. \\ & \left. + \sum_{u=2}^{\infty} [s_3 (u - 1)(u - 2) + (3s_3 + s_2)(u - 1) + s_3 + s_2 + s_1] \binom{u + \delta - 2}{\delta - 1} \varphi^{u-1} (1 - \varphi)^\delta \right] \\ & + (s_3 + s_2 + s_1) |d_1| \\ & = (1 - \varepsilon) \left[2k_3 \varrho^2 \frac{\binom{\beta+1}{\beta-1}}{(1 - \varrho)^2} + (3k_3 + k_2) \varrho \frac{\binom{\beta}{\beta-1}}{1 - \varrho} + (k_3 + k_2 + k_1)(1 - (1 - \varrho^\beta)) \right. \\ & \left. 2s_3 \varphi^2 \frac{\binom{\delta+1}{\delta-1}}{(1 - \varphi)^2} + (3s_3 + s_2) \varphi \frac{\binom{\delta}{\delta-1}}{1 - \varphi} + (s_3 + s_2 + s_1)(1 - (1 - \varphi^\delta)) \right] + (s_3 + s_2 + s_1) |d_1| \\ & \leq \eta. \end{aligned}$$

Which is completes the proof of Theorem 3.3. $\square$

Next we find the relationship between the families $TRH(\varepsilon)$ and $\Pi_\eta(k_3, k_2, k_1, 0; s_3, s_2, s_1, 0)$.

Theorem 3.4. Let $\beta, \delta > 0, 0 \leq \varrho, \varphi < 1, 0 \leq \varepsilon < 1, k_3, k_2, k_1, s_3, s_2, s_1 \in \mathbb{R}$ and $\eta > 0$. If

$$\begin{aligned} & (1 - \varepsilon) \left[k_3 \varrho \frac{\binom{\beta}{\beta-1}}{1 - \varrho} + (k_3 + k_2)(1 - (1 - \varrho^\beta)) \right. \\ & + \frac{k_1}{\varrho(\beta - 1)} \left((1 - \varrho) - (1 - \varrho)^\beta - \varrho(\beta - 1)(1 - \varrho)^\beta \right) \\ & + s_3 \varphi \frac{\binom{\delta}{\delta-1}}{1 - \varphi} + (s_3 + s_2)(1 - (1 - \varphi^\delta)) \\ & \left. + \frac{s_1}{\varphi(\delta - 1)} \left((1 - \varphi) - (1 - \varphi)^\delta - \varphi(\delta - 1)(1 - \varphi)^\delta \right) \right] \\ & \leq \eta - (s_3 + s_2 + s_1)|d_1|, \end{aligned} \quad (3.11)$$

then

$$P_{\varrho, \phi}^{\beta, \tau}(TRH(\varepsilon)) \subset \Pi_\eta(k_3, k_2, k_1, 0; s_3, s_2, s_1, 0).$$

Proof. From Definition 1.1, we need only to prove that $Q(\beta, \delta, \varrho, \varphi) \leq \eta$, where $Q(\beta, \delta, \varrho, \varphi)$ as given in (3.10). Using the inequalities (2.3) and (2.4), it follows that

$$\begin{aligned} Q(\beta, \delta, \varrho, \varphi) &= \sum_{u=2}^{\infty} (k_3 u^3 + k_2 u^2 + k_1 u) \left| \binom{u + \beta - 2}{\beta - 1} \varrho^{u-1} (1 - \varrho)^\beta a_u \right| \\ &+ (s_3 + s_2 + s_1) |d_1| + \sum_{u=2}^{\infty} (s_3 u^3 + s_2 u^2 + s_1 u) \left| \binom{u + \delta - 2}{\delta - 1} \varphi^{u-1} (1 - \varphi)^\delta d_u \right| \\ &\leq (1 - \varepsilon) \left[\sum_{u=2}^{\infty} \left(k_3 u + k_2 \frac{k_1}{u} \right) \binom{u + \beta - 2}{\beta - 1} \varrho^{u-1} (1 - \varrho)^\beta \right. \\ &+ \sum_{u=2}^{\infty} \left(s_3 u + s_2 \frac{k_1}{u} \right) \binom{u + \delta - 2}{\delta - 1} \varphi^{u-1} (1 - \varphi)^\delta \left. \right] + (s_3 + s_2 + s_1) |d_1| \\ &\leq (1 - \varepsilon) \left[\sum_{u=2}^{\infty} \left(k_3(u - 1) + k_3 + k_2 + \frac{k_1}{u} \right) \binom{u + \beta - 2}{\beta - 1} \varrho^{u-1} (1 - \varrho)^\beta \right. \\ &+ \sum_{u=2}^{\infty} \left(s_3(u - 1) + s_3 + s_2 + \frac{s_1}{u} \right) \binom{u + \delta - 2}{\delta - 1} \varphi^{u-1} (1 - \varphi)^\delta \left. \right] + (s_3 + s_2 + s_1) |d_1| \\ &= (1 - \varepsilon) \left[k_3 \varrho \frac{\binom{\beta}{\beta-1}}{1 - \varrho} + (k_3 + k_2)(1 - (1 - \varrho^\beta)) \right. \\ &+ \frac{k_1}{\varrho(\beta - 1)} \left((1 - \varrho) - (1 - \varrho)^\beta - \varrho(\beta - 1)(1 - \varrho)^\beta \right) + s_3 \varphi \frac{\binom{\delta}{\delta-1}}{1 - \varphi} + (s_3 + s_2)(1 - (1 - \varphi^\delta)) \end{aligned}$$

$$+ \frac{s_1}{\varphi(\delta-1)} \left((1-\varphi) - (1-\varphi)^\delta - \varphi(\delta-1)(1-\varphi)^\delta \right) \Big] + (s_3 + s_2 + s_1)|d_1| \\ \leq \eta.$$

Then inequality (3.11) completes the proof. $\square$

Theorem 3.5. Let $\beta, \delta > 0, 0 \leq \varrho, \varphi < 1, k_3, k_2, k_1, k_0, s_3, s_2, s_1, s_0 \in \mathbb{R}$ and $\eta > 0$. If

$$(1 - \varrho^\beta) - (1 - \varphi^\delta) \geq 1 + \frac{(s_3 + s_2 + s_1 + s_0)|d_1|}{\eta} \quad (3.12)$$

then

$$P_{\varrho, \phi}^{\beta, \tau}(\Pi_\eta(k_3, k_2, k_1, 0; s_3, s_2, s_1, s_0)) \subset \Pi_\eta(k_3, k_2, k_1, 0; s_3, s_2, s_1, s_0).$$

Proof. Where $Q(\beta, \delta, \varrho, \varphi)$ as given in (3.10) and using Remark 1.1, we obtain

$$\begin{aligned} Q(\beta, \delta, \varrho, \varphi) &\leq \eta \left[\sum_{u=2}^{\infty} \binom{u+\beta-2}{\beta-1} \varrho^{u-1} (1-\varrho)^\beta + \sum_{u=2}^{\infty} \binom{u+\delta-2}{\delta-1} \varphi^{u-1} (1-\varphi)^\delta \right] \\ &\quad + (s_3 + s_2 + s_1 + s_0)|d_1| \\ &= \eta \left[(1 - (1 - \varrho^\beta)) + (1 - (1 - \varphi^\delta)) \right] + (s_3 + s_2 + s_1 + s_0)|d_1| \\ &= \eta \left[(2 - (1 - \varrho^\beta) - (1 - \varphi^\delta)) \right] + (s_3 + s_2 + s_1 + s_0)|d_1| \leq \eta. \end{aligned}$$

Then inequality (3.12) completes the proof. $\square$

Remark 3.1. Particularization of the parameters $k_3, k_2, k_1, k_0, s_3, s_2, s_1, s_0$ and η in our theorems obtain many connections between various families of harmonic univalent functions, some of which have been examined by numerous authors. For example:

- (1) Take $k_3 = s_3 = \alpha, k_2 = s_2 = 0, k_1 = s_1 = 1 - \alpha, k_0 = s_0 = 0, \eta = 1 - \mu$, we obtain the connections between various families of harmonic functions and the family $H(\alpha, \mu)$, obtained by Frasin et al. [43];
- (2) Take $k_3 = 0, k_2 = \alpha + 1, k_1 = -(\alpha + \mu), k_0 = 0; s_3 = 0, s_2 = \alpha + 1, s_1 = \alpha + \mu, s_0 = 0$ and $\eta = 1 - \mu$, we obtain the connections between various families of harmonic functions and the family $HUC(\alpha, \mu)$, obtained by Yaşar [40];
- (3) Take $k_3 = 0, k_2 = L + 1, k_1 = -J - 1, k_0 = 0; s_3 = 0, s_2 = L + 1, s_1 = J + 1, s_0 = 0$ and $\eta = 2(J - L)$, we obtain the connections between various families of harmonic functions and the family $SH(J, L)$, obtained by Tokgöza et al. [41];
- (4) Take $k_3 = 0, k_2 = 0, k_1 = 2, k_0 = -\delta(\mu + 1); s_3 = 0, s_2 = 0, s_1 = 2, s_0 = \delta(\mu + 1)$ and $\eta = 1 - \mu$, we obtain the connections between various families of harmonic functions and the family $GH(\delta, \mu)$, obtained by Yalçın et al. [42].

Remark 3.2. Particularization of the parameters $k_3, k_2, k_1, k_0, s_3, s_2, s_1, s_0$ and η in our theorems obtain many connections between various families of harmonic univalent functions, which has not yet been studied. For example:

- (1) Take $k_3 = 0, k_2 = 0, k_1 = 1 - \delta\mu, k_0 = -\mu(1 - \delta); s_3 = 0, s_2 = 0, s_1 = 1 - \delta\mu, s_0 = \mu(1 - \delta)$ and $\eta = 1 - \mu$, we get the connections between various families of harmonic functions and the family $S_H^*(\delta, \mu)$.
- (2) Take $k_3 = 0, k_2 = 0, k_1 = 2, k_0 = -\alpha - 1; s_3 = 0, s_2 = 0, s_1 = 2, s_0 = \alpha + 1$ and $\eta = 1 - \mu$, we get the connections between various families of harmonic functions and the family $G_H(\delta, \mu)$.
- (3) Take $k_3 = 0, k_2 = 0, k_1 = \gamma + 1, k_0 = -\alpha(\gamma + \mu); s_3 = 0, s_2 = 0, s_1 = \gamma + 1, s_0 = \alpha(\gamma + \mu)$ and $\eta = 1 - \mu$, we get the connections between various families of harmonic functions and the family $G_H(\alpha, \gamma, \mu)$.

4. CONCLUSIONS

The aim of this study is to establish several inclusion relations between the harmonic comprehensive family

$$\Pi_\eta(k_3, k_2, k_1, k_0; s_3, s_2, s_1, s_0)$$

and the classes $NH^0(\varepsilon)$, $RH^0(\varepsilon)$, $TNH(\varepsilon)$, and $TRH(\varepsilon)$ of harmonic functions associated with the operator $P_{\rho, \phi}^{\beta, \tau}$ defined by the Pascal distribution series. The obtained results provide a generalized framework that encompasses several previously established results as special cases. In particular, appropriate choices of the parameters in the comprehensive family yield connections with various known subclasses associated with the Pascal distribution series. Furthermore, the results provide a useful framework for investigating new properties of harmonic function classes and related operators generated by probability distribution series. Thus, the present study not only unifies a number of existing results but also provides a basis for further investigations of harmonic function classes associated with the Pascal distribution.

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