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STRONG METRIZABILITY FOR CLOSED OPERATORS AND THE SEMI-FREDHOLM OPERATORS BETWEEN TWO HILBERT SPACES

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Abstract. To be able to refine the completion of $\mathcal{C}(H_1, H_2)$, the set of all closed densely defined linear operators between two Hilbert spaces H_1 and H_2, we define in this paper some new strictly stronger metrics than the gap metric g and we characterize the closure with respect to theses metrics of the subset $\mathcal{L}(H_1, H_2)$ of bounded elements of $\mathcal{C}(H_1, H_2)$. In addition, several operator norm inequalities concerning the equivalence of some metrics on $\mathcal{L}(H_1, H_2)$ are presented. We also establish the semi-Fredholmness and Fredholmness of unbounded operators in terms of bounded pure contractions.

1. Introduction

Let H, H_1, H_2 be a complex Hilbert spaces endowed with the appropriate scalar product and the associated norm. The inner product in $H_1 \times H_2$ is defined by $<(x, y); (x', y')> = < x; x' > + < y; y' >$. For T linear operator from H_1 to H_2, the symbols $\mathcal{D}(T) \subset H_1$, $N(T) \subset H_1$ and $R(T) \subset H_2$ will denote the domain, null space and the range space of T, respectively. The set $\mathcal{G}(T) = \{(x, Tx) : x \in \mathcal{D}(T)\} \subset H_1 \times H_2$ is called the graph of T. The operator T is closed if and only if $\mathcal{G}(T)$ is a closed subset of $H_1 \times H_2$, and is densely defined if $\overline{\mathcal{D}(T)} = H_1$, where $\overline{\mathcal{D}(T)}$ denote the closure of $\mathcal{D}(T)$ in H_1. The set of all closed and densely defined linear operators from H_1 to H_2 will be denoted by $\mathcal{C}(H_1, H_2)$. Denote by $\mathcal{L}(H_1, H_2)$ the Banach space of all bounded linear operators from H_1 to H_2. If $H_1 = H_2$, write $\mathcal{C}(H_1, H_2) = \mathcal{C}(H_1)$ and $\mathcal{L}(H_1, H_2) = \mathcal{L}(H_1)$. If $T \in \mathcal{C}(H_1, H_2)$, the adjoint T^* of T exists, is unique and $T^* \in \mathcal{C}(H_2, H_1)$. An operator $A \in \mathcal{L}(H_1, H_2)$ is a pure contraction if $\|Ax\|_2 < \|x\|_1$ for all nonzero x in H_1. We denote by $\mathcal{L}_0(H_1, H_2)$ the set of all pure contractions.

In [9] W. E. Kaufman showed that if $T \in \mathcal{C}(H)$ then T is represented as $T = \Gamma(A) = A(I - A^*A)^{-1/2}$ using a unique pure contraction A defined in H, where I denote the identity in H.

Since the publication of Kaufman [9] in 1978 and its follows papers, this Kaufman's representation is used to reformulate questions about unbounded operators in terms of bounded ones:

- In [9] [11], Kaufman proved that the map Γ preserves many properties of operators: self-adjontness, nonnegative conditions, normality and quasinormality. In [12] he also defined by the use of Γ^{-1} a metric in the space of closed densely defined Hilbert space which is stronger than the gap metric and sharing many of its properties. On the the bounded operators it is equivalent to the metric generated by the usual operator-norm.
- In [5] Hirasawa showed that a pure contraction A is hyponormal if and only if $\Gamma(A)$ is formally hyponormal, and if A is quasinormal then $T^n = A^n(I - A^*A)^{-n/2}$ is quasinormal for all integers $n \geq 2$.
- In [2], Cordes and Labrousse proved that if a closed and densely defined operator T is semi Fredholm then so is the bounded operator $\Gamma^{-1}(T) = T(I + T^*T)^{-1/2}$.

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- In [1], Benharrat and Messirdi prove that if a pure contraction A on a Hlibert space H is semi Fredholm, then the closed densely defined linear operator $\lambda I - T = \lambda I - A(I - A^*A)^{-1/2}$ is semi Fredholm operator for all $\lambda \in \mathbb{C}$ such that $|\lambda| < \frac{\gamma(A)}{1+\gamma(A)}$.

Recently, J. J. Koliha in [14] extend Kaufman's results to operators between two Hilbert spaces and showed that the mapping Γ maps the set $\mathcal{L}_0(H_1, H_2)$ one-to-one onto the set $\mathcal{C}(H_1, H_2)$. More precisely, we have the following result.

Theorem 1.1. [14, Theorem 5.] *Let $\mathcal{L}_0(H_1, H_2)$ be the set of all pure contractions from H_1 to H_2, $\mathcal{C}(H_1, H_2)$ the set of all closed and densely defined linear operators from H_1 to H_2, and $G \in \mathcal{L}^+(H_1)$ a positive bijection. The mapping Γ_G defined by*

$$\Gamma_G(A) = AG^{1/2}(G^{1/2}(I - A^*A)G^{1/2})^{-1/2}G^{1/2}, \quad A \in \mathcal{L}_0(H_1, H_2),$$

is a bijection of $\mathcal{L}_0(H_1, H_2)$ onto $\mathcal{C}(H_1, H_2)$ with the inverse

$$\Gamma_G^{-1}(T) = T(G + T^*T)^{-1/2}, \quad T \in \mathcal{C}(H_1, H_2).$$

In this paper, by the use of the generalized Kaufman's representation, we discuss some metrics in the space $\mathcal{C}(H_1, H_2)$ endowed with the gap metric. More precisely, in Section 2, we define some metrics on $\mathcal{C}(H_1, H_2)$ equivalent to the gap metric. In Section 3, we define on $\mathcal{C}(H_1, H_2)$ a metric in term of Γ^{-1}, strictly stronger than the gap metric and is equivalent to the metric associated to the operator-norm in $\mathcal{L}(H_1, H_2)$. We characterize essentially the closure of $\mathcal{L}(H_1, H_2)$ in $\mathcal{C}(H_1, H_2)$ for this metric. In section 4, we prove some operator norm inequalities for bounded operators between two Hilbert spaces. In the last section, we establish some characterizations of Fredholm unbounded operators in terms of bounded pure contractions by treating unbounded operators between two Hilbert spaces rather than restricting the investigation to operators on a single space.

2. A strong metric for closed operators between two Hilbert spaces

Recall that, if $T \in \mathcal{C}(H_1, H_2)$, then the operator $R_T = (I + T^*T)^{-1}$ is self-adjoint positive operator defined on all H_1, and has a unique positive definite self-adjoint square root, which we denoted by S_T. The fundamental properties of R_T and S_T are, see [15]:

$$R_T, S_T \in \mathcal{L}(H_1, H_2), \quad \|R_T\| \leq 1, \quad \|S_T\| \leq 1,$$

$$\|TR_T\| \leq 1, \quad \|TS_T\| \leq 1,$$

and

$$(TR_T)^* = T^*R_{T^*}, \quad (TS_T)^* = T^*S_{T^*}, \quad T^*S_{T^*}TS_T = I - R_T. \tag{2.1}$$

In the sequel $\mathcal{L}^+(H_1)$ denotes the set of all positive bijective operators in $\mathcal{L}(H_1)$. Let G be an element of $\mathcal{L}^+(H_1)$. By Theorem 1.1, for a given pure contraction $A \in \mathcal{L}_0(H_1, H_2)$, there exists a unique operator $T \in \mathcal{C}(H_1, H_2)$ such that the equation

$$XGX = I - A^*A, \tag{2.2}$$

admits a unique solution given by

$$X = (G + T^*T)^{-1/2} = G^{-1/2}(G^{1/2}(I - A^*A)G^{1/2})^{1/2}G^{-1/2}, \tag{2.3}$$

with $T = \Gamma_G(A)$. Further, the operator $TG^{-1/2}$ has domain $G^{1/2}\mathcal{D}(T)$, which is clearly dense in H_1. Therefore $R_{TG^{-1/2}}$ is defined on all H_1 and

$$(G + T^*T)^{-1} = G^{-1/2}R_{TG^{-1/2}}G^{-1/2}. \tag{2.4}$$

Hence $(G + T^*T)^{-1} \in \mathcal{L}(H_1)$.

Let $P_{\mathcal{G}(T)}$ denote the correspondence which assigns to each $T \in \mathcal{C}(H_1, H_2)$ the orthogonal projection from $H_1 \times H_2$ onto the graph $\mathcal{G}(T)$ of T.

Lemma 2.1. *Let $T \in \mathcal{C}(H_1, H_2)$ and $G \in \mathcal{L}^+(H_1)$. Then*

$$P_{\mathcal{G}(TG^{-1/2})} = \begin{pmatrix} G^{1/2}(G+T^*T)^{-1}G^{1/2} & G^{1/2}(G+T^*T)^{-1}T^* \\ T(G+T^*T)^{-1}G^{1/2} & I-T(G+T^*T)^{-1}T^* \end{pmatrix}. \tag{2.5}$$

Proof. Let $v \in \mathcal{D}(G^{1/2}T^*)$ and $X = (u,v) \in H_1 \times H_2$ such that $P_{\mathcal{G}(TG^{-1/2})}X = (x, TG^{-1/2}x)$, for $x \in G^{1/2}\mathcal{D}(T)$. Since $\mathcal{G}(TG^{-1/2})^{\perp} = V(\mathcal{G}(G^{-1/2}T^*))$ with the isomorphism V from $H_1 \times H_2$ to $H_2 \times H_1$ defined by $V(x_1, x_2) = (-x_2, x_1)$, we get $P_{\mathcal{G}(TG^{-1/2})}X$ from the decomposition

$$X = (u,v) = (x, TG^{-1/2}x) + (-G^{1/2}T^*y, y), \quad y \in \mathcal{D}(G^{1/2}T^*), \quad x \in G^{1/2}\mathcal{D}(T),$$

where x and y are the solutions of the system:

$$\begin{cases} u = x - G^{1/2}T^*y \\ v = TG^{-1/2}x + y. \end{cases}$$

By solving this system we get

$$\begin{cases} x = G^{1/2}(G+T^*T)^{-1}G^{1/2}u + G^{1/2}(G+T^*T)^{-1}T^*v \\ y = T(G+T^*T)^{-1}G^{1/2}u + (I - T(G+T^*T)^{-1}T^*)v. \end{cases}$$

□

By (2.4), we can rewrite (2.5) as follows

$$P_{\mathcal{G}(TG^{-1/2})} = \begin{pmatrix} R_{TG^{-1/2}} & R_{TG^{-1/2}}G^{-1/2}T^* \\ TG^{-1/2}R_{TG^{-1/2}} & I - TG^{-1/2}R_{TG^{-1/2}}G^{-1/2}T^* \end{pmatrix}. \tag{2.6}$$

If $H_1 = H_2$, we also have

$$P_{\mathcal{G}(TG^{-1/2})} = \begin{pmatrix} R_{TG^{-1/2}} & G^{-1/2}T^*R_{G^{-1/2}T^*} \\ TG^{-1/2}R_{TG^{-1/2}} & I - R_{G^{-1/2}T^*} \end{pmatrix}. \tag{2.7}$$

In the case of $G = I_{H_1}$ Lemma 2.1 reduces to the following well-known statement, see [15].

Corollary 2.2. *Let $T \in \mathcal{C}(H_1, H_2)$. Then the orthogonal projection $P_{\mathcal{G}(T)}$ in $H_1 \oplus H_2$ onto the graph $\mathcal{G}(T)$ of T, is given by*

$$P_{\mathcal{G}(T)} = \begin{pmatrix} R_T & T^*R_{T^*} \\ TR_T & I - R_{T^*} \end{pmatrix}. \tag{2.8}$$

Definition 2.3. Let $G \in \mathcal{L}^+(H_1)$ and $T, S \in \mathcal{C}(H_1, H_2)$. The gap metric between T and S associated to G is defined by

$$\mathbf{g}_G(T,S) = \left\| P_{\mathcal{G}(TG^{-1/2})} - P_{\mathcal{G}(SG^{-1/2})} \right\|. \tag{2.9}$$

Note that if $G = I_{H_1}$, we have the usual gap metric (cf. [8, p. 201]) for $T, S \in \mathcal{C}(H_1, H_2)$,

$$g(T,S) = \left\| P_{\mathcal{G}(T)} - P_{\mathcal{G}(S)} \right\| \text{ for all } T, S \in \mathcal{C}(H_1, H_2). \tag{2.10}$$

Thus, for an infinite sequence (T_n) of $\mathcal{C}(H_1, H_2)$, $g(T_n, T) \to 0$ if and only if each the following conditions hold

(i) $\|R_{T_n} - R_T\| \to 0$,
(ii) $\|T_nR_{T_n} - TR_T\| \to 0$,
(iii) $\left\|R_{T_n^*} - R_{T^*}\right\| \to 0$,
(iv) $\left\|T_n^*R_{T_n^*} - T^*R_{T^*}\right\| \to 0$.

Similarly, we can express the convergence with respect to the metric $\mathbf{g}_G$ as follows :

Proposition 2.4. *For an infinite sequence (T_n) of $\mathcal{C}(H_1, H_2)$, $g(T_n, T) \to 0$ in the sens of Definition 2.3 if and only if each the following conditions hold*

(i) $\left\|R_{T_nG^{-1/2}} - R_{TG^{-1/2}}\right\| \to 0$,
(ii) $\left\|T_nG^{-1/2}R_{T_nG^{-1/2}} - TG^{-1/2}R_{TG^{-1/2}}\right\| \to 0$,
(iii) $\left\|T_nG^{-1/2}R_{T_nG^{-1/2}}G^{-1/2}T_n^* - TG^{-1/2}R_{TG^{-1/2}}G^{-1/2}T^*\right\| \to 0$,
(iv) $\left\|R_{T_nG^{-1/2}}G^{-1/2}T_n^* - R_{TG^{-1/2}}G^{-1/2}T^*\right\| \to 0$.

with $G \in \mathcal{L}^+(H_1)$.

By (2.9) and (2.10) we can deduce that

$$\mathbf{g}_G(T,S) = g(TG^{-1/2}, SG^{-1/2}).$$

Let M, N be two closed linear subspaces of the Hilbert space H. Denote by P_M and P_N the orthogonal projection onto M and N respectively. Set

$$\delta(M,N) = \|(I - P_N)P_M\|,$$

δ is a pseudo-distance, for its properties we can see also [2]. We define another metric on $\mathcal{C}(H_1, H_2)$ as follows,

$$\mathbf{d}_G(T,S) = \left\|(I - P_{\mathcal{G}(TG^{-1/2})})P_{\mathcal{G}(SG^{-1/2})}\right\| + \left\|(I - P_{\mathcal{G}(S)}G^{-1/2})P_{\mathcal{G}(TG^{-1/2})}\right\|,$$

for all $T, S \in \mathcal{C}(H_1, H_2)$ with $G \in \mathcal{L}^+(H_1)$.

We notice that

$$\mathbf{g}_G(T,S) = \max\{\delta(\mathcal{G}(TG^{-1/2}), \mathcal{G}(SG^{-1/2})), \delta(\mathcal{G}(SG^{-1/2}), \mathcal{G}(TG^{-1/2}))\}$$

and

$$\mathbf{d}_G(T,S) = \delta(\mathcal{G}(TG^{-1/2}), \mathcal{G}(SG^{-1/2})) + \delta(\mathcal{G}(SG^{-1/2}), \mathcal{G}(TG^{-1/2})).$$

The following result is immediately obtained.

Corollary 2.5. *If* $G \in \mathcal{L}^+(H_1)$*; then* $\boldsymbol{d}_G$ *and* $\boldsymbol{g}_G$ *are equivalent metrics on* $\mathcal{C}(H_1, H_2)$*, in particular we have*

$$\boldsymbol{g}_G(T,S) \le \boldsymbol{d}_G(T,S) \le 2\boldsymbol{g}_G(T,S).$$

Put $\mathcal{P} = (I - P_{\mathcal{G}(TG^{-1/2})})P_{\mathcal{G}(SG^{-1/2})}$, let us remark that

$$\mathcal{P} = \mathcal{U}_T \begin{bmatrix} 0 & 0 \\ TG^{-1/2}S_{TG^{-1/2}}S_{SG^{-1/2}} - S^*_{TG^{-1/2}}SG^{-1/2}S_{SG^{-1/2}} & 0 \end{bmatrix} \mathcal{U}_S. \tag{2.11}$$

with

$$\mathcal{U}_T = \begin{bmatrix} S_{TG^{-1/2}} & T_{TG^{-1/2}}G^{-1/2}T^* \\ TG^{-1/2}S_{TG^{-1/2}} & S^*_{TG^{-1/2}} \end{bmatrix}.$$

Then we can deduce that

Corollary 2.6. *If* $G \in \mathcal{L}^+(H_1)$*, then for* $T, S \in \mathcal{C}(H_1, H_2)$ *we have*

$$\begin{aligned} \boldsymbol{d}_G(T,S) &= \left\|TG^{-1/2}S_{TG^{-1/2}}S_{SG^{-1/2}} - S^*_{TG^{-1/2}}SG^{-1/2}S_{SG^{-1/2}}\right\| \\ &+ \left\|SG^{-1/2}S_{SG^{-1/2}}S_{TG^{-1/2}} - S^*_{SG^{-1/2}}TG^{-1/2}S_{TG^{-1/2}}\right\|. \end{aligned}$$

Furthermore, if T, S *are bounded, then*

$$\boldsymbol{d}_G(T,S) = \left\|G^{-1/2}S^*_{TG^{-1/2}}(T-S)S_{SG^{-1/2}}\right\| + \left\|G^{-1/2}S^*_{SG^{-1/2}}(S-T)S_{TG^{-1/2}}\right\|.$$

For an operator $G \in \mathcal{L}^+(H_1)$, we define a third metric on $\mathcal{C}(H_1, H_2)$ by

$$\mathbf{p}_G(T,S) = \left[\|R_{TG^{-1/2}} - R_{SG^{-1/2}}\|^2 + \left\|TG^{-1/2}R_{TG^{-1/2}} - SG^{-1/2}R_{SG^{-1/2}}\right\|^2\right]^{1/2}.$$

It easy to see that

$$\mathbf{p}_G(T,S) \le \mathbf{g}_G(T,S).$$

Hence

Theorem 2.7. *The topology induced from the gap metric* $\boldsymbol{g}_G$ *on* $\mathcal{C}(H_1, H_2)$ *is strictly stronger than that induced from* $\boldsymbol{p}_G$*.*

The following example exclude the possibility that the metrics $\mathbf{p}_G$ and $\mathbf{g}_G$ generate the same topology even in the case of $G = I_{H_1}$.

Example 2.8. Let H_1 and H_2 two separable Hilbert spaces and $\{\phi_n\}$, $\{\psi_n\}$ an orthonormal basis in H_1, H_2 respectively. Put for $n \in \mathbb{N}^*$,

$$T_n\phi_k = \begin{cases} k\psi_k, & \text{if } k < n \\ -k\psi_{k+1} & \text{if } k \geq n. \end{cases}$$

Then,

$$T_n^*\psi_k = \begin{cases} k\phi_k, & \text{if } k < n \\ 0 & \text{if } k = n \\ -k\phi_k & \text{if } k > n \end{cases}$$

and thus,

$$R_{T_n}\phi_k = \begin{cases} \frac{1}{1+k^2}\phi_k, & \text{if } k < n \\ \phi_n & \text{if } k = n \\ \frac{1}{1+(k-1)^2}\phi_k & \text{if } k > n \end{cases}$$

Define the operator T by $T\phi_k = k\psi_k$, $k \in \mathbb{N}^*$. Then, $R_T = R_{T^*} = R_{T_n^*}$, $\left\|R_{T^*} - R_{T_n^*}\right\| = 0$, and

$$T^*R_{T^*} - T_n^*R_{T_n^*}\psi_k = \begin{cases} 0 & \text{if } k < n \\ \frac{n}{1+n^2}\phi_n & \text{if } k = n \\ \frac{k}{1+k^2}\phi_k & \text{if } k > n \end{cases}$$

Thus

$$\|TR_T - T_nR_{T_n}\| = \left\|T^*R_{T^*} - T_n^*R_{T_n^*}\right\| \leq \frac{2}{\sqrt{1+n^2}} \to 0.$$

On the other hand,

$$(R_T - R_{T_n})\phi_k = \begin{cases} 0 & \text{if } k < n \\ (\frac{n}{1+n^2} - 1)\phi_n & \text{if } k = n \\ \frac{1-2k}{(1+k^2)(1+(k-1)^2}\phi_k & \text{if } k > n \end{cases}$$

Then

$$\|R_T - R_{T_n}\| \geq \frac{n^2}{\sqrt{1+n^2}} \to 1.$$

Finally, if we put $S_n = T_n^*$ and $S = T^*$, we get $\mathbf{p}_G(S, S_n) \to 0$ and $\mathbf{g}_G(S, S_n) \to 1$.

3. A new strong metric than the gap metric

Let $G \in \mathcal{L}^+(H_1)$ and $T, S \in \mathcal{C}(H_1, H_2)$. We define another metric in terms of Γ_G^{-1}, given in Theorem 1.1, as follows,

$$\mathbf{q}_G(T, S) = \left\|\Gamma_G^{-1}(T) - \Gamma_G^{-1}(S)\right\|.$$

Clearly $\mathcal{C}(H_1, H_2)$ is isometric to the subset $\mathcal{L}_0(H_1, H_2)$ of the unit ball in $\mathcal{L}(H_1, H_2)$ under the operator-norm, so that $\mathbf{q}_G(T, S) \leq 2$ for all $T, S \in \mathcal{C}(H_1, H_2)$. The related convergence in the space $\mathcal{C}(H_1, H_2)$, called quotient-convergence associated to G. The purpose of the following theorem is to prove that the metric $\mathbf{q}_G$ is stronger than $\mathbf{d}_G$.

Theorem 3.1. *Let $G \in \mathcal{L}^+(H_1)$. The metric topology induced by $\mathbf{q}_G$ is stronger than that induced by the gap metric $\mathbf{g}_G$ in $\mathcal{C}(H_1, H_2)$.*

Proof. Let $T \in \mathcal{C}(H_1, H_2)$ and (T_n) an infinite sequence of $\mathcal{C}(H_1, H_2)$, such that $\mathbf{q}_G(T_n, T) \to 0$. By Theorem 1.1 then we can write $T = \Gamma_G(A)$ (resp $T_n = \Gamma_G(A_n)$) with a unique positive contraction $A \in \mathcal{L}_0(H_1, H_2)$ (resp. $A_n \in \mathcal{L}_0(H_1, H_2)$ for all n). Thus,

$$(G + T^*T)^{-1/2}G(G + T^*T)^{-1/2} = I - A^*A,$$

and

$$(G + T_n^*T_n)^{-1/2}G(G + T_n^*T_n)^{-1/2} = I - A_n^*A_n.$$

Therefore, by (2.3) the orthogonal projections $P_{\mathcal{G}(TG^{-1/2})}$ and $P_{\mathcal{G}(T_nG_n^{-1/2})}$ are easily computed from (2.3), and we obtain respectively,

$$P_{\mathcal{G}(TG^{-1/2})} = \begin{pmatrix} UG^{-1}U & UG^{-1}UG^{-1/2}(\Gamma_G(A))^* \\ AG^{-1/2}U & I - AG^{-1/2}UG^{-1/2}(\Gamma_G(A))^* \end{pmatrix}, \tag{3.1}$$

and

$$P_{\mathcal{G}(T_nG^{-1/2})} = \begin{pmatrix} U_nG^{-1}U_n & U_nG^{-1}U_nG^{-1/2}(\Gamma_G(A_n))^* \\ A_nG^{-1/2}U_n & I - A_nG^{-1/2}U_nG^{-1/2}(\Gamma_G(A_n))^* \end{pmatrix}, \tag{3.2}$$

where $U = (G^{1/2}(I - A^*A)G^{1/2})^{1/2}$ and $U_n = (G^{1/2}(I - A_n^*A_n)G^{1/2})^{1/2}$ for all $n \in \mathbb{N}$.

Consequently, if A_n converges to A in $\mathcal{L}_0(H_1, H_2)$, then U_n converges to U and this assures the convergence $P_{\mathcal{G}(T_nG^{-1/2})} \longrightarrow P_{\mathcal{G}(TG^{-1/2})}$ as $n \longrightarrow \infty$, hence $\mathbf{g}_G(T_n, T) \to 0$. □

In the following example, we show that is not possible that the metrics $\mathbf{q}_G$ and $\mathbf{g}_G$ generate the same topology even for $G = I_{H_1}$.

Example 3.2. Let H_1 and H_2 be two separable Hilbert spaces and $\{\phi_n\}$, $\{\psi_n\}$ an orthonormal basis in H_1, H_2 respectively. Put for $n \in \mathbb{N}^*$,

$$T_n\phi_k = \begin{cases} k\psi_k, & \text{if } k < n \\ -k\psi_k & \text{if } k \geq n. \end{cases}$$

Then,

$$T_n^*\psi_k = \begin{cases} k\phi_k, & \text{if } k < n \\ -k\phi_k & \text{if } k \geq n \end{cases}$$

and thus,

$$R_{T_n}\phi_k = \frac{1}{1+k^2}\phi_k, \qquad R_{T_n^*}\psi_k = \frac{1}{1+k^2}\psi_k.$$

If we define the operator T by $T\phi_k = k\psi_k$, $k \in \mathbb{N}^*$. Then, $T = \Gamma(A)$ where $A\phi_k = \frac{k}{\sqrt{1+k^2}}\psi_k$, $A \in \mathcal{L}_0(H_1, H_2)$, we see that the conditions (i)-(iv) of Proposition 2.4 holds. Thus $\mathbf{g}_G(T_n, T) \to 0$. On the other hand, as $n \longrightarrow \infty$,

$$\mathbf{q}_G(T_n, T) = \|A_n - A\| = \frac{2n}{\sqrt{1+n^2}} \to 2.$$

We have also the following result:

Corollary 3.3. *The topology induced on $\mathcal{C}(H_1, H_2)$ by the metric $\mathbf{q}_G$ is strictly stronger than the topology induced by the metric $\mathbf{d}_G$.*

Lemma 3.4. *Let $G \in \mathcal{L}^+(H_1)$. An operator $T \in \mathcal{C}(H_1, H_2)$ is bounded if and only if $\left\|\Gamma_G^{-1}(T)\right\| < 1$. In this case,*

$$\|T\| = \frac{\left\|\Gamma_G^{-1}(T)\right\| \left\|G^{1/2}\right\|^2}{\sqrt{1 - \left\|\Gamma_G^{-1}(T)\right\|^2}}. \tag{3.3}$$

Proof. Let $T \in \mathcal{C}(H_1, H_2)$ a bounded operator, then for all $x \in H_1$, we have

$$\begin{aligned} \|x\|^2 &= \left\|G^{1/2}(G + T^*T)^{-1/2}x\right\|^2 + \left\|T(G + T^*T)^{-1/2}x\right\|^2 \\ &\leq \left[1 + \left\|G^{-1/2}\right\|^2 \|T\|^2\right] \left\|G^{1/2}(G + T^*T)^{-1/2}x\right\|^2. \end{aligned}$$

Thus,

$$\left\|G^{1/2}(G + T^*T)^{-1/2}x\right\|^2 \geq \frac{1}{1 + \left\|G^{-1/2}\right\|^2 \|T\|^2} \|x\|^2.$$

Consequently

$$\left\|T(G + T^*T)^{-1/2}x\right\|^2 = \|x\|^2 - \left\|G^{1/2}(G + T^*T)^{-1/2}x\right\|^2 \leq \frac{\|T\|^2}{\left\|G^{1/2}\right\|^2 + \|T\|^2} \|x\|^2.$$

Hence

$$\left\|T(G + T^*T)^{-1/2}\right\| \leq \frac{\|T\|}{\sqrt{\left\|G^{1/2}\right\|^2 + \|T\|^2}} < 1. \tag{3.4}$$

Conversely, assume that $\left\|\Gamma_G^{-1}(T)\right\| < 1$. Then $I - (\Gamma_G^{-1}(T))^*\Gamma_G^{-1}(T)$ is invertible and for all $x \in H_1$, we have

$$\begin{aligned}\left\langle G^{1/2}(I - (\Gamma_G^{-1}(T))^*\Gamma_G^{-1}(T))G^{1/2}x, x\right\rangle &= \left\|G^{1/2}x\right\|^2 - \left\|\Gamma_G^{-1}(T)G^{1/2}x\right\|^2 \\ &\geq \left[1 - \left\|\Gamma_G^{-1}(T)\right\|^2\right]\left\|G^{1/2}x\right\|^2.\end{aligned}$$

Hence

$$\left\|\left[G^{1/2}(I - (\Gamma_G^{-1}(T))^*\Gamma_G^{-1}(T))G^{1/2}\right]^{-1/2}\right\| \leq \frac{1}{\sqrt{1 - \left\|\Gamma_G^{-1}(T)\right\|^2}}.$$

Since $T = \Gamma_G^{-1}(T)G^{1/2}\left[G^{1/2}(I - (\Gamma_G^{-1}(T))^*\Gamma_G^{-1}(T))G^{1/2}\right]^{-1/2}G^{1/2}$, we obtain

$$\|T\| \leq \frac{\left\|\Gamma_G^{-1}(T)\right\|\left\|G^{1/2}\right\|^2}{\sqrt{1 - \left\|\Gamma_G^{-1}(T)\right\|^2}}. \tag{3.5}$$

This implies the boundedness of T. Furthermore, by (3.4) and (3.5) we obtain (3.3). $\square$

Theorem 3.5. *$\mathcal{L}(H_1, H_2)$ is dense open subset of $\mathcal{C}(H_1, H_2)$ endowed with the metric $\mathbf{q}_G$.*

Proof. $\mathcal{L}(H_1, H_2)$ is an open subset of $\mathcal{C}(H_1, H_2)$ with respect to the metric $\mathbf{q}_G$ follows immediately from Lemma 3.4. Now suppose that $T \in \mathcal{C}(H_1, H_2)$, then $\Gamma_G^{-1}(T)$ is in the unit closed ball of $\mathcal{L}(H_1, H_2)$, relative to the operator-norm. Hence $\{\frac{n}{n+1}\Gamma_G^{-1}(T)\}$ is a sequence $\{A_n\}$ of operators such that for each n, $\|A_n\| < 1$ and $\left\|A_n - \Gamma_G^{-1}(T)\right\| \longrightarrow 0$. For each n, we put $T_n = \Gamma_G(A_n)$, by Theorem 1.1 each T_n is in $\mathcal{L}(H_1, H_2)$ and, clearly, $\mathbf{q}_G(T_n, T) = \left\|\Gamma_G^{-1}(T_n) - \Gamma_G^{-1}(T)\right\| \longrightarrow 0$. This complete the proof. $\square$

Definition 3.6. Let $T_1, T_2 \in \mathcal{C}(H_1, H_2)$. We put

$$\Sigma_G(T_1, T_2) = \left[2\mathbf{q}_G(T_1, T_2)^2 + \left\|S_{T_1G^{-1/2}} - S_{T_2G^{-1/2}}\right\|^2 + \left\|S_{G^{-1/2}T_1^*} - S_{G^{-1/2}T_2^*}\right\|^2\right]^{1/2}.$$

Σ_G is a metric on $\mathcal{C}(H_1, H_2)$ and note that the sequence defined in the Example 2.8 converges on $\mathcal{C}(H_1, H_2)$ for the metric $\mathbf{q}_G$ but is not convergent for the metric Σ_G.

Theorem 3.7. *The topology induced on $\mathcal{C}(H_1, H_2)$ by the metric Σ_G is strictly stronger than the topology induced from the metric $\mathbf{q}_G$.*

By Theorem 3.7 and Theorem 3.3 we obtain the following results.

Corollary 3.8. *The topology induced on $\mathcal{C}(H_1, H_2)$ by the metric Σ_G is strictly stronger than the topology induced from the gap metric $\mathbf{g}_G$.*

Theorem 3.9. *$\mathcal{L}(H_1, H_2)$ is dense open subset of $\mathcal{C}(H_1, H_2)$ endowed with the metric Σ_G.*

For the proof of this theorem we need the following lemma

Lemma 3.10. *If $T \in \mathcal{C}(H_1, H_2)$ and $B \in \mathcal{L}(H_1, H_2)$ such that*

$$\Sigma_G(T, B) < \frac{1}{\sqrt{1 + \left\|BG^{-1/2}\right\|^2}},$$

then $T \in \mathcal{L}(H_1, H_2)$.

Proof. Let $x \in G^{1/2}\mathcal{D}(T)$, then for all $y \in H_2$,

$$\begin{aligned}\left\langle TG^{-1/2}x, y\right\rangle - \left\langle x, G^{-1/2}B^*y\right\rangle &= \left\langle (x, TG^{-1/2}x), (-G^{-1/2}B^*y, y)\right\rangle \\ &= \left\langle P_{\mathcal{G}(TG^{-1/2})}(x, TG^{-1/2}x), (I - P_{\mathcal{G}(BG^{-1/2})})(-G^{-1/2}B^*y, y)\right\rangle,\end{aligned}$$

by using Schwarz inequality,

$$\left|\left\langle (T-B)G^{-1/2}x, y\right\rangle\right| \leq \mathbf{g}_G(T,B)\left\|(x, TG^{-1/2}x)\right\|\left\|(-G^{-1/2}B^*y, y)\right\|. \tag{3.6}$$

Setting $y=(T-B)G^{-1/2}x$ in (3.6) it follows that

$$\left\|(T-B)G^{-1/2}x\right\| \leq \Sigma_G(T,B)\sqrt{\|x\|^2+\left\|TG^{-1/2}x\right\|^2}\sqrt{1+\left\|BG^{-1/2}\right\|^2}.$$

Let us put $\Sigma_G(T,B)\sqrt{1+\left\|BG^{-1/2}\right\|^2}=1-\epsilon$, $\epsilon>0$. Thus,

$$\left\|TG^{-1/2}x\right\| \leq \left\|BG^{-1/2}x\right\| + (1-\epsilon)[\|x\| + \left\|TG^{-1/2}x\right\|],$$

finally

$$\left\|TG^{-1/2}x\right\| \leq \frac{1}{\epsilon}[\left\|G^{1/2}\right\| + \left\|BG^{-1/2}\right\|]\left\|G^{-1/2}x\right\|,$$

what shows that T is bounded from H_1 to H_2. $\square$

Proof of Theorem 3.9. From Lemma 3.10 $\mathcal{L}(H_1,H_2)$ is an open subset of $\mathcal{C}(H_1,H_2)$. Now, we show the density. Let $T\in\mathcal{C}(H_1,H_2)$, then there exists an unique pure contraction A such that $A=\Gamma_G^{-1}(T)$. We put $T_n=\frac{n}{n+1}A$ as in the proof of Theorem 3.5. Then $T_n\in\mathcal{L}(H_1,H_2)$ and $\mathbf{q}_G(T_n,T)\longrightarrow 0$. On the other hand, let $R_T^G=(G+T^*T)^{-1}$ and $S_T^G=(G+T^*T)^{-1/2}$, then

$$\begin{aligned}
\left\|R_{T_n}^G - R_T^G\right\| &= \left\|S_{T_n}^G T_n^* T_n S_{T_n}^G - S_T^G T^* T S_T^G\right\| \\
&= \left\|S_{T_n}^G T_n^* T_n S_{T_n}^G + S_{T_n}^G T_n^* T S_T^G - S_{T_n}^G T_n^* T S_T^G - S_T^G T^* T S_T^G\right\| \\
&\leq \left\|S_{T_n}^G T_n^*\right\|\left\|T_n S_{T_n}^G - T S_T^G\right\| + \left\|S_{T_n}^G T_n^* - S_T^G T^*\right\|\left\|T S_T^G\right\| \\
&\leq [\left\|S_{T_n}^G T_n^*\right\| + \left\|T S_T^G\right\|]\,\mathbf{q}_G(T_n,T).
\end{aligned}$$

Thus, $\lim_{n\to+\infty}\left\|R_{T_n}^G - R_T^G\right\|=0$, hence $\lim_{n\to+\infty}\left\|S_{T_n}^G - S_T^G\right\|=0$. By (2.4), we observe that $S_{TG^{-1/2}}=(G^{1/2}R_T^G G^{1/2})^{1/2}$, then we conclude that

$$\lim_{n\to+\infty}\left\|S_{T_nG^{-1/2}} - S_{TG^{-1/2}}\right\| = \left\|S_{G^{-1/2}T_n^*} - S_{G^{-1/2}T^*}\right\| = 0.$$

Thus $\Sigma_G(T_n,T)\longrightarrow 0$, this shows the density of $\mathcal{L}(H_1,H_2)$ in $\mathcal{C}(H_1,H_2)$. $\square$

4. Some equivalent metrics for bounded operators between two Hilbert spaces

In this section we present several operator norm inequalities to compare the metric $\mathbf{q}_G$, the gap metric $\mathbf{g}_G$, and the usual operator norm metric. More presicily, we show that these three metrics are equivalent in $\mathcal{L}(H_1,H_2)$. Our results extend those obtained in [12] and [13] to the bounded operators between two Hilbert spaces.

Lemma 4.1. *If $T_1,T_2\in\mathcal{L}(H_1,H_2)$, then*

$$\begin{aligned}
\|T_1-T_2\| \leq{}& \frac{1}{2}\left\|(S_{T_1}^G)^{-1} + (S_{T_2}^G)^{-1}\right\|\boldsymbol{q}_G(T_1,T_2) \\
&+ \frac{1}{2}\left\|(R_{T_1}^G)^{-1}\right\|\left\|(R_{T_2}^G)^{-1}\right\|\left\|\Gamma_G^{-1}(T_1)+\Gamma_G^{-1}(T_2)\right\|^2\boldsymbol{q}_G(T_1,T_2).
\end{aligned}$$

Proof. Let $T_1, T_2 \in \mathcal{L}(H_1, H_2)$, we have

$$\begin{aligned}
\|T_1 - T_2\| &= \left\|\Gamma_G^{-1}(T_1)(S_{T_1}^G)^{-1} - \Gamma_G^{-1}(T_2)(S_{T_2}^G)^{-1}\right\| \\
&\leq \frac{1}{2}\mathbf{q}_G(T_1, T_2)\left\|(S_{T_1}^G)^{-1} + (S_{T_2}^G)^{-1}\right\| + \frac{1}{2}\left\|T_1 S_{T_1}^G + T_2 S_{T_2}^G\right\| \left\|(S_{T_1}^G)^{-1} - (S_{T_2}^G)^{-1}\right\| \\
&\leq \frac{1}{2}\mathbf{q}_G(T_1, T_2)\left\|(S_{T_1}^G)^{-1} + (S_{T_2}^G)^{-1}\right\| + \frac{1}{4}\left\|T_1 S_{T_1}^G + T_2 S_{T_2}^G\right\| \left\|(R_{T_1}^G)^{-1} - (R_{T_2}^G)^{-1}\right\| \\
&\leq \frac{1}{2}\mathbf{q}_G(T_1, T_2)\left\|(S_{T_1}^G)^{-1} + (S_{T_2}^G)^{-1}\right\| \\
&\quad + \frac{1}{4}\left\|(R_{T_1}^G)^{-1}\right\| \left\|(R_{T_2}^G)^{-1}\right\| \left\|T_1 S_{T_1}^G + T_2 S_{T_2}^G\right\| \left\|R_{T_1}^G - R_{T_2}^G\right\|.
\end{aligned}$$

Since

$$\begin{aligned}
\left\|R_{T_1}^G - R_{T_2}^G\right\| &= \left\|(\Gamma_G^{-1}(T_1))^*\Gamma_G^{-1}(T_1) - (\Gamma_G^{-1}(T_2))^*\Gamma_G^{-1}(T_2)\right\| \\
&\leq \mathbf{q}_G(T_1, T_2)\left\|\Gamma_G^{-1}(T_1) + \Gamma_G^{-1}(T_2)\right\|,
\end{aligned} \tag{4.1}$$

it follows the desired inequality. □

Lemma 4.2. *If $T_1, T_2 \in \mathcal{L}(H_1, H_2)$, then*

$$\boldsymbol{q}_G(T_1, T_2) \leq (1 + \frac{1}{4}\|T_1 + T_2\|^2)\|T_1 - T_2\|.$$

Proof. Let $T_1, T_2 \in \mathcal{L}(H_1, H_2)$, we have

$$\begin{aligned}
\mathbf{q}_G(T_1, T_2) &= \left\|\Gamma_G^{-1}(T_1) - \Gamma_G^{-1}(T_2)\right\| \\
&= \left\|\frac{1}{2}(T_1 - T_2)(S_{T_1}^G + S_{T_2}^G) + \frac{1}{2}(T_1 + T_2)(S_{T_1}^G - S_{T_2}^G)\right\| \\
&\leq \|T_1 - T_2\| + \frac{1}{2}\|T_1 + T_2\| \left\|S_{T_1}^G - S_{T_2}^G\right\| \\
&\leq \|T_1 - T_2\| + \frac{1}{2}\|T_1 + T_2\| \left\|(S_{T_1}^G)^{-1} - (S_{T_2}^G)^{-1}\right\| \\
&\leq \|T_1 - T_2\| + \frac{1}{4}\|T_1 + T_2\| \left\|(R_{T_1}^G)^{-1} - (R_{T_2}^G)^{-1}\right\| \\
&= \|T_1 - T_2\| + \frac{1}{4}\|T_1 + T_2\| \|T_1^*T_1 - T_2^*T_2\| \\
&\leq \|T_1 - T_2\|(1 + \frac{1}{4}\|T_1 + T_2\|^2).
\end{aligned}$$

□

Combining Lemma 4.1 and Lemma 4.2, we obtain the following result:

Theorem 4.3. *Let $G \in \mathcal{L}^+(H_1)$. The restriction of the metric $\boldsymbol{q}_G$ to $\mathcal{L}(H_1, H_2)$ is equivalent to the operator-norm.*

Note that this theorem is extended to the unbounded operators between two Hilbert spaces, the result was shown by W. E. Kaufman in [12, Theorem 2] in the case of unbounded operators defined on a single Hilbert space and when $G = I_{H_1}$.

Lemma 4.4. *If $T_1, T_2 \in \mathcal{L}(H_1, H_2)$, then*

$$\boldsymbol{q}_G(T_1, T_2) \leq \frac{1}{2}\left[\left\|(S_{T_1}^G)^{-1} + (S_{T_2}^G)^{-1}\right\| + \frac{1}{2}\|G + T_1^*T_1\| \|G + T_2^*T_2\|\right]\boldsymbol{g}_G(T_1, T_2).$$

Proof. Let $T_1, T_2 \in \mathcal{L}(H_1, H_2)$, we have

$$\begin{aligned}
&\mathbf{q}_G(T_1,T_2) = \left\|T_1 R^G_{T_1}(S^G_{T_1})^{-1} - T_2 R^G_{T_2}(S^G_{T_2})^{-1}\right\| \\
&\le \frac{1}{2}\left\|T_1 R^G_{T_1} - T_2 R^G_{T_2}\right\| \left\|(S^G_{T_1})^{-1} + (S^G_{T_2})^{-1}\right\| \\
&\quad + \frac{1}{2}\left\|T_1 R^G_{T_1} + T_2 R^G_{T_2}\right\| \left\|(S^G_{T_1})^{-1} - (S^G_{T_2})^{-1}\right\| \\
&\le \frac{1}{2}\mathbf{g}_G(T_1,T_2)\left\|(S^G_{T_1})^{-1} + (S^G_{T_2})^{-1}\right\| + \frac{1}{2}\left\|(S^G_{T_1})^{-1} - (S^G_{T_2})^{-1}\right\| \\
&\le \frac{1}{2}\mathbf{g}_G(T_1,T_2)\left\|(S^G_{T_1})^{-1} + (S^G_{T_2})^{-1}\right\| + \frac{1}{4}\left\|(R^G_{T_1})^{-1} - (R^G_{T_2})^{-1}\right\| \\
&\le \frac{1}{2}\mathbf{g}_G(T_1,T_2)\left\|(S^G_{T_1})^{-1} + (S^G_{T_2})^{-1}\right\| + \frac{1}{4}\left\|(R^G_{T_1})^{-1}\right\|\left\|(R^G_{T_2})^{-1}\right\|\left\|(R^G_{T_1}) - (R^G_{T_2}\right\| \\
&\le \frac{1}{2}\mathbf{g}_G(T_1,T_2)\left\|(S^G_{T_1})^{-1} + (S^G_{T_2})^{-1}\right\| + \frac{1}{4}\left\|(R^G_{T_1})^{-1}\right\|\left\|(R^G_{T_2})^{-1}\right\|\mathbf{g}_G(T_1,T_2).
\end{aligned}$$

□

Lemma 4.5. *If $T_1, T_2 \in \mathcal{L}(H_1, H_2)$, then*

$$\begin{aligned}
\boldsymbol{g}^2_G(T_1,T_2) \le& \left[(\left\|G^{1/2}\right\|^4 + 1)\left\|\Gamma^{-1}_G(T_1) + \Gamma^{-1}_G(T_2)\right\|^2 + \left\|G^{1/2}\right\|^2\right]\boldsymbol{q}^2_G(T_1,T_2) \\
&+ 2\left\|G^{1/2}\right\|^2\left\|\Gamma^{-1}_G(T_1) + \Gamma^{-1}_G(T_2)\right\|^{3/2}\boldsymbol{q}^{3/2}_G(T_1,T_2) \\
&+ \frac{1}{2}\left\|G^{1/2}\right\|^2\left\|\Gamma^{-1}_G(T_1) + \Gamma^{-1}_G(T_2)\right\|^3\boldsymbol{q}_G(T_1,T_2).
\end{aligned}$$

Proof. By using the representation (2.5), we get

$$\begin{aligned}
\mathbf{g}^2_G(T_1,T_2) \le& \left\|G^{1/2}\right\|^4\left\|R^G_{T_1} - R^G_{T_2}\right\|^2 + 2\left\|G^{1/2}\right\|^2\left\|T^G_1 R^G_{T_1} - T_2 R^G_{T_2}\right\|^2 \\
&+ \left\|T_1 R^G_{T_1} T^*_1 - T_2 R^G_{T_2} T^*_2\right\|^2.
\end{aligned}$$

We have

$$\begin{aligned}
\left\|T_1 R^G_{T_1} - T_2 R^G_{T_2}\right\| &\le \frac{1}{2}\mathbf{q}_G(T_1,T_2)\left\|S^G_{T_1} + S^G_{T_2}\right\| + \frac{1}{2}\left\|\Gamma^{-1}_G(T_1) + \Gamma^{-1}_G(T_2)\right\|\left\|S^G_{T_1} - S^G_{T_2}\right\| \\
&\le \mathbf{q}_G(T_1,T_2) + \frac{1}{2}\left\|\Gamma^{-1}_G(T_1) + \Gamma^{-1}_G(T_2)\right\|\left\|S^G_{T_1} - S^G_{T_2}\right\| \\
&\le \mathbf{q}_G(T_1,T_2) + \frac{1}{2}\left\|\Gamma^{-1}_G(T_1) + \Gamma^{-1}_G(T_2)\right\|\left\|R^G_{T_1} - R^G_{T_2}\right\|^{1/2} \\
&\le \mathbf{q}_G(T_1,T_2) + \frac{1}{2}\left\|\Gamma^{-1}_G(T_1) + \Gamma^{-1}_G(T_2)\right\|^{3/2}\mathbf{q}^{1/2}_G(T_1,T_2).
\end{aligned}$$

In view of these estimations and the fact that, by (4.1), both $\left\|T_1 R^G_{T_1} T^*_1 - T_2 R^G_{T_2} T^*_2\right\|$ and $\left\|R^G_{T_1} - R^G_{T_2}\right\|$ are majorized by $\mathbf{q}_G(T_1,T_2)\left\|\Gamma^{-1}_G(T_1) + \Gamma^{-1}_G(T_2)\right\|$ we get the required inequality. □

Combining Lemma 4.4 and Lemma 4.5 we obtain:

Theorem 4.6. *In $\mathcal{L}(H_1, H_2)$ the metric $\boldsymbol{q}_G$ is equivalent to the gap metric $\boldsymbol{g}_G$.*

Combining Theorem 4.3, Theorem 4.6 and Corollary 2.5 we deduce:

Corollary 4.7. *The metrics $\boldsymbol{q}_G$, $\boldsymbol{g}_G$, $\boldsymbol{p}_G$ and the operator-norm metric are equivalent on $\mathcal{L}(H_1, H_2)$.*

5. Pure contractions and semi-Fredholm operators

In this section, by the use of the generalized Kaufman's representation, we present some results concerning the characterization of unbounded semi Fredholm operators in terms of bounded ones. We begin by introduce now some important classes of operators in Fredholm theory. In the sequel, for every $T \in \mathcal{C}(H_1, H_2)$, let $\alpha(T)$ and $\beta(T)$ be the nullity and the deficiency of T defined as $\alpha(T) := \dim N(T)$, and $\beta(T) := \operatorname{codim} R(T)$. If the range $R(T)$ of T is closed and $\alpha(T) < \infty$ (resp. $\beta(T) < \infty$), then T is called an upper (resp. a lower) semi-Fredholm operator. If T is either upper or lower semi-Fredholm, then T is called a semi-Fredholm operator, and the index of T is defined by $\operatorname{ind}(T) := \alpha(T) - \beta(T)$. If both $\alpha(T)$ and $\beta(T)$ are finite, then T is a called a Fredholm operator. In the following, A denotes a pure contraction from H_1 to H_2, and T the closed and densely-defined operator $\Gamma_G(A) = AG^{1/2}B^{-1}G^{1/2}$ from H_1 to H_2, with $B = (G^{1/2}(I - A^*A)G^{1/2})^{1/2}$ such that $G^{-1/2}BG^{-1/2}$ is the unique solution of the equation (2.2) with $G \in \mathcal{GL}^+(H_1)$. Note that since A is a pure contraction, B is a positive and injective element of $\mathcal{L}(H_1)$.

Recall that the reduced minimum modulus of a non-zero operator T is defined by

$$\gamma(T) = \inf_{x \in N(T)^\perp} \frac{\|Tx\|}{\|x\|}$$

If $T = 0$ then we take $\gamma(T) = \infty$. Note that (see [8]):

$$\gamma(T) > 0 \Leftrightarrow R(T) \text{ is closed.}$$

Lemma 5.1 ([8])**.** (1) *If* $\delta(M, N) < 1$ *then* $\dim M \leq \dim N$.
(2) $\delta(M, N) = \delta(N^\perp, M^\perp)$.

The main results of this section is:

Theorem 5.2. *Let* $A \in \mathcal{L}_0(H_1, H_2)$. *If* A *is upper semi-Fredholm operator then* $\lambda C - \Gamma_G(A)$ *is upper semi-Fredholm operator for all* $C \in \mathcal{L}(H_1, H_2)$ *and* $|\lambda| < \frac{\gamma(A)}{1+\gamma(A)} \frac{\|G^{1/2}\|}{\|C\|}$.

Proof. Let $A \in \mathcal{L}_0(H_1, H_2)$, $C \in \mathcal{L}(H_1, H_2)$ and B denote the positive member $(G^{1/2}(I - A^*A)G^{1/2})^{1/2}$ of $\mathcal{L}_0(H)$. Since A is a pure contraction, B is one-to-one with dense range in H_1, and the fact that $\lambda C - \Gamma_G(A) = (\lambda CG^{-1/2}BG^{-1/2} - A)G^{1/2}B^{-1}G^{1/2}$, it follows that to prove $\lambda C - \Gamma_G(A)$ is upper semi-Fredholm operator it suffices to prove that $(\lambda CG^{-1/2}BG^{-1/2} - A)$ is upper semi-Fredholm one.

For each nonzero x in H_1, $\|x\|^2 - \|Ax\|^2 = \left\|BG^{-1/2}x\right\|^2$; thus

$$\left\|BG^{-1/2}x\right\| \leq \|x\| + \|Ax\|.$$

Hence

$$\begin{aligned} \left\|CG^{-1/2}BG^{-1/2}x\right\| &\leq \left\|G^{-1/2}\right\| \left\|BG^{-1/2}x\right\| \\ &\leq \|C\| \left\|G^{-1/2}\right\| (\|x\| + \|Ax\|). \end{aligned} \tag{5.1}$$

Let λ in $\mathbb{C}$. We prove that if $|\lambda| < \frac{\|G^{1/2}\|\gamma(A)}{\|C\|(1+\gamma(A))}$ then $0 < \gamma(\lambda CG^{-1/2}BG^{-1/2} - A) < \infty$ and hence $R(\lambda CG^{-1/2}BG^{-1/2} - A)$ is closed. First if we use (5.1) with λx instead of x and by [7, Theorem 1a], we obtain that $\gamma(\lambda CG^{-1/2}BG^{-1/2} - A) > 0$ for $|\lambda| < \frac{\|G^{1/2}\|\gamma(A)}{\|C\|(1+\gamma(A))}$. Now to prove that $\gamma(\lambda CG^{-1/2}BG^{-1/2} - A) < \infty$, we proceed by contraposition. In fact $\gamma(\lambda CG^{-1/2}BG^{-1/2} - A) = \infty$ implies that $(\lambda CG^{-1/2}BG^{-1/2} - A)x = 0$ for all $x \in H_1$. Hence

$$\|Ax\| = |\lambda| \left\|CG^{-1/2}BG^{-1/2}x\right\| \leq \|C\| \, |\lambda| \left\|G^{-1/2}\right\| (\|x\| + \|Ax\|),$$

and so

$$\gamma(A) \|x\| \leq \|Ax\| \leq \frac{|\lambda| \, \|C\| \left\|G^{-1/2}\right\|}{1 - |\lambda| \, \|C\| \left\|G^{-1/2}\right\|} \|x\| \tag{5.2}$$

for $x \in N(A)^{\perp}$ with $x \neq 0$. It follows that $|\lambda| \geq \frac{\|G^{1/2}\|\gamma(A)}{\|C\|(1+\gamma(A))}$. We next prove that

$$\delta(N(\lambda CG^{-1/2}BG^{-1/2} - A), N(A)) \leq \frac{|\lambda| \, \|C\| \, \|G^{-1/2}\|}{(1 - |\lambda| \, \|C\| \, \|G^{-1/2}\|)\gamma(A)}. \tag{5.3}$$

Let $x \in H_1$,

$$\gamma(A) \left\|(I - P_{N(A)})P_{N(\lambda CG^{-1/2}BG^{-1/2}-A)}x\right\| \leq \left\|AP_{N(\lambda CG^{-1/2}BG^{-1/2}-A)}x\right\|.$$

Since $P_{N(\lambda CG^{-1/2}BG^{-1/2}-A)}x \in N(\lambda CG^{-1/2}BG^{-1/2} - A)$ by the same calculation given before we have

$$\gamma(A) \left\|(I - P_{N(A)})P_{N(\lambda CG^{-1/2}BG^{-1/2}-A)}x\right\| \leq \frac{|\lambda| \, \|C\| \, \|G^{-1/2}\|}{(1 - |\lambda| \, \|C\| \, \|G^{-1/2}\|)} \|x\|.$$

Recalling the definition of $\delta(N, M)$, this proves (5.3). The right side of (5.3) is smaller than one if $|\lambda| < \frac{\|G^{1/2}\|\gamma(A)}{\|C\|(1+\gamma(A))}$, thus Lemma 5.1 shows that

$$\alpha(\lambda CG^{-1/2}BG^{-1/2} - A) \leq \alpha(A) \qquad \text{for } |\lambda| < \frac{\|G^{1/2}\| \gamma(A)}{\|C\| (1 + \gamma(A))}. \tag{5.4}$$

We then conclude that $\lambda CG^{-1/2}BG^{-1/2} - A$ is upper semi-Fredholm operator for $|\lambda| < \frac{\|G^{1/2}\|\gamma(A)}{\|C\|(1+\gamma(A))}$. This complete the proof of the theorem. □

Theorem 5.3. *Let $A \in \mathcal{L}_0(H_1, H_2)$ is a lower semi-Fredholm operator. Then $\lambda C - \Gamma_G(A)$ is a lower semi-Fredholm operator for all $C \in \mathcal{L}(H_1, H_2)$ and λ such that $|\lambda| < \frac{\|G^{1/2}\|\gamma(A)}{\|C\|(1+\gamma(A))}$.*

Proof. Since $R(A)$ is closed, by the first part of the proof of Theorem 5.2, $R(\lambda CG^{-1/2}BG^{-1/2} - A)$ is closed and $R(\lambda CG^{-1/2}BG^{-1/2} - A) = N(\lambda G^{-1/2}B^*G^{-1/2}C^* - A^*)^{\perp}$ for all $|\lambda| < \frac{\|G^{1/2}\|\gamma(A)}{\|C\|(1+\gamma(A))}$. From (5.3) we deduce that

$$\begin{aligned} \delta(R(\lambda CG^{-1/2}BG^{-1/2} - A)^{\perp}, R(A)^{\perp}) &= \delta(N(\overline{\lambda}G^{-1/2}B^*G^{-1/2}C^* - A^*), N(A^*)) \\ &\leq \frac{|\lambda| \, \|C\| \, \|G^{-1/2}\|}{(1 - |\lambda| \, \|C\| \, \|G^{-1/2}\|)\gamma(A)}, \end{aligned}$$

because $\gamma(A) = \gamma(A^*)$. Now by Lemma 5.1 we have

$$\beta(\lambda CG^{-1/2}BG^{-1/2} - A) \leq \beta(A) \quad \text{for } |\lambda| < \frac{\|G^{1/2}\| \gamma(A)}{\|C\| (1 + \gamma(A))}.$$

Consequently, $\lambda CG^{-1/2}BG^{-1/2} - A$ is lower semi-Fredholm one for all $|\lambda| < \frac{\|G^{1/2}\|\gamma(A)}{\|C\|(1+\gamma(A))}$ and hence $\lambda C - \Gamma_G(A)$ is lower semi-Fredholm operator for all λ such that $|\lambda| < \frac{\|G^{1/2}\|\gamma(A)}{\|C\|(1+\gamma(A))}$. □

Corollary 5.4. *If $A \in \mathcal{L}_0(H_1, H_2)$ is a semi-Fredholm operator (resp. Fredholm operator), then $\lambda C - \Gamma_G(A)$ is a semi-Fredholm operator (resp. Fredholm operator) for all $C \in \mathcal{L}(H_1, H_2)$ and λ such that $|\lambda| < \frac{\|G^{1/2}\|\gamma(A)}{\|C\|(1+\gamma(A))}$.*

We proceed as in the proof of [2, Lemma 5.2, p. 708], by taking in count that the operator T is defined between two Hilbert spaces, we can easily check the following result.

Proposition 5.5. *If $T \in \mathcal{C}(H_1, H_2)$ is a semi-Fredholm operator (resp. Fredholm operator), then $A = T(G + T^*T)^{-1/2}$ is a semi-Fredholm operator (resp. Fredholm operator) from H_1 to H_2, and $N(A) = N(T), N(A^*) = N(T^*)$.*

Proof. It is easy to see that $N(T) = \{x \in H_1 : (G + T^*T)^{-1}Gx = x\}$. Since $G \in \mathcal{GL}^+(H_1)$, $(G+T^*T)^{-1}$ is bounded self-adjoint and $(G+T^*T)^{-1}G$ leaves $N(T)$ as well as $N(T)^{\perp}$ invariant, so this two subspaces are invariant by $(G + T^*T)^{-1}$ and its square root. Accordingly $N(T) = N(A)$. It is also clear that $y \in N(A^*)$ if and only if $\langle T(G + T^*T)^{-1/2}x, y\rangle = 0$ for all $x \in H_1$ i.e. $\langle T(G + T^*T)^{-1}z, y\rangle = 0$ for all $z \in H_1$ i.e if $y \in N((T(G + T^*T)^{-1})^*) = N(T^*)$. Thus $N((T(G + T^*T)^{-1/2})^*) = N(T^*)$. □

By Proposition 5.5 and Corollary 5.4 we obtain the following results

Theorem 5.6. *Let $A \in \mathcal{L}_0(H_1, H_2)$. Then A is a semi-Fredholm operator (resp. Fredholm operator) if and only if $\Gamma_G(A)$ is a semi-Fredholm operator (resp. Fredholm operator). In this case $ind(A) = ind(\Gamma_G(A))$.*

Remark 5.7. Theorems 5.2, 5.3 and 5.6 generalize [1, Theorem 1], [1, Theorem 2] and [1, Theorem 3] respectively, by taking $H_1 = H_2 = H$ and $G = C = I$.

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