Int. J. Anal. Appl. (2026), 24:283

International Journal of Analysis and Applications

# Lyapunov Criteria and Localization of the Matrix Spectrum With Cassini Ovals

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**Abstract.** This paper bridges Lyapunov stability theory with localization of the matrix spectrum using Cassini ovals, overcoming the conservatism of classical Gershgorin-based criteria. We establish conditions that completely ensure when the matrix spectrum is contained in the left half complex plane or within the unit disk. Furthermore, we show that the spectrum of the matrix belongs to the domain bounded by the Cassini oval is equivalent to the solubility of the matrix equation in the class of Hermitian positive definite matrices.

## 1. Introduction

Traditional approaches to eigenvalue localization are primarily based on geometric inclusion regions derived directly from matrix entries. Among these, Gershgorin-type results provide simple and computationally efficient bounds. However, such disk-based regions are often conservative, especially for matrices with strong off-diagonal interactions. This limitation has been widely discussed in the literature, including the classical analysis of Brauer [2] and later refinements and surveys such as [12]. In recent years, significant attention has been devoted to improving eigenvalue inclusion regions by incorporating additional structural information of matrices. In this direction, matrix equations arising from stability theory provide a powerful analytical framework.

Received: Apr. 30, 2026.

2020 *Mathematics Subject Classification.* 15A18.

*Key words and phrases.* Cassini ovals; Lyapunov stability; matrix spectrum localization.

https://doi.org/10.28924/2291-8639-24-2026-283
ISSN: 2291-8639

The Lyapunov equation is strongly linked to spectral properties. In particular, whether a Hermitian positive definite solution exists depends on where the eigenvalues are positioned in the complex plane. This relationship is well documented in classical works such as [1] and [7], and has been further explored in recent studies, including [3]. Advances in certified eigenvalue localization and inclusion techniques have also been reported in [10] and [8], while computational aspects of matrix equations are discussed in [9].

While Lyapunov-based methods provide deep analytical insight into spectral properties, they do not directly yield explicit geometric inclusion regions. To address this limitation, we introduce a geometric framework based on Cassini ovals. Unlike classical disks, Cassini oval regions are capable of capturing interactions between multiple matrix entries, leading to sharper and more informative inclusion sets. The advantages of such regions have been explored in [6] where generalizations of classical eigenvalue bounds are presented. Their effectiveness in improving eigenvalue estimates is also supported by results in [13].

An important aspect of spectrum localization is its behavior under perturbations. An important aspect of localization of spectrum is its behavior under perturbations. In real-world applications, matrices are often obtained from data that contain noise or uncertainty, which makes it important to understand how stable their inclusion regions are. The proposed framework helps us to clarify how the location of the matrix spectrum changes when the entries of the matrix vary, and in doing so, it adds to the broader understanding of robust spectral analysis.

In the continuous-time case, it is well known that if the spectrum of a matrix lies in the open left half complex plane, then the associated Lyapunov equation admits a unique positive definite solution [7]. Similarly, in the discrete-time case, if the spectrum is contained within the unit disk, a corresponding Lyapunov equation admits a positive definite solution. These results provide a direct connection between spectral localization and stability properties, and form the analytical foundation of our approach. Additional computational perspectives on solving such matrix equations can be found in [9].

The regions in the complex plane incorporate structural properties of the matrices and provide improved estimates for eigenvalue locations. Such approaches have been shown to enhance spectral analysis in both theoretical and applied contexts, as discussed in [1] and [4]. Furthermore, recent advances in certified eigenvalue inclusion [10] highlight the robustness and reliability of modern localization techniques.

The localization of the spectrum plays a fundamental role in understanding the qualitative behavior of linear systems, particularly in stability analysis, control theory, and numerical computations. Instead of explicitly computing eigenvalues, which may be computationally expensive for large-scale systems, it is often sufficient to determine regions in the complex plane that contain the spectrum, as discussed in [5] and [11]. Classical matrix analysis and system-theoretic perspectives further emphasize the importance of matrix equations in this context [4, 7].

The main contribution of this work is the development of a unified framework that combines Lyapunov-based analytical methods with Cassini oval-based geometric inclusion regions. This approach offers a more effective way to obtain sharper eigenvalue bounds than traditional techniques. In particular, the proposed method bridges the gap between stability theory and geometric localization, offering both theoretical insight and computational applicability.

## 2. New Results

In this section, we present some new results on the localization of the spectrum and the Lyapunov type inequalities. We explore the Lyapunov criteria on the localization of the spectrum of matrix in the left half complex plane.

2.1. **Localization of the matrix spectrum and Lyapunov type equations.** We consider the matrix equation

$$\sum_{j,k=0}^{N} m_{jk}(M^*)^j H M^k = C, \tag{2.1}$$

where $m_{jk}$ are fixed scalar coefficients and $M, C$ are the given square matrices of compatible dimensions. The present study examines the solubility of this equation and its dependence on the spectral properties of an associated matrix $M$. In particular, we analyze how the location of the spectrum $\sigma(M)$ relative to regions bounded by Cassini oval influences the existence and uniqueness of solutions.

**Lyapunov criteria on localization of the matrix spectrum:** Lyapunov criteria establish necessary and sufficient conditions that ensure that the matrix spectrum lies in the left half complex plane $\mathbb{C}^- = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda(M)) < 0\}$ or within the unit disk $\mathbb{U} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$.

**Theorem 2.1.** *The spectrum $\sigma(M)$ belongs to $C^-$ iff the continuous-time Lyapunov equation*

$$HM + M^*H = -C$$

*with $C = C^* > 0$ has a Hermitian positive definite solution $H = H^* > 0$.*

*Proof.* Consider spectrum $\sigma(M)$ belongs to $\mathbb{C}^-$ then $\mathrm{Re}(\lambda) < 0$ and $\lim_{t\to\infty} e^{Mt} = 0$. Choose $C = C^* > 0$, and define a converge integral

$$H = \int_0^\infty e^{M^*t} C e^{Mt} dt.$$

$$H^* = \int_0^\infty \left(e^{M^*t} C e^{Mt}\right)^* dt = \int_0^\infty e^{M^*t} C e^{Mt}\, dt = H$$

For any $\vec{x} \neq 0$,

$$x^*Hx = \int_0^\infty x^* e^{M^*t} C e^{Mt} x\, dt = \int_0^\infty \left(xe^{Mt}\right)^* C \left(e^{Mt}x\right) dt = \int_0^\infty \|C^{1/2} e^{Mt} x\|^2\, dt \geq 0$$

Since $C > 0$ and $e^{Mt}x \neq 0$. So $H > 0$. Thus there is a Hermitian and positive definite solution $H = H^* > 0$. Now, to verify the Lyapunov equation differentiates under integral,

$$\frac{d}{dt}\left(e^{M^*t}Ce^{Mt}\right) = \left(M^*.e^{M^*t}\right)Ce^{Mt} + e^{M^*t}C\left(e^{M^*t}.M\right)$$

By taking integral $0 \to \infty$, we get

$$\int_0^\infty \frac{d}{dt}\left(e^{M^*t}Ce^{Mt}\right)dt = \int_0^\infty M^*e^{M^*t}Ce^{Mt}\,dt + \int_0^\infty e^{M^*t}Ce^{Mt}M\,dt$$

$$-C = M^*H + HM.$$

Thus, $H$ satisfied the Lyapunov equation.

Conversely, Suppose on the contrary $\lambda \in \sigma(M)$ with $\mathrm{Re}(\lambda) \geq 0$. Then there exist $\vec{x} \neq 0$ such that $Mx = \lambda x$. Assume

$$HM + M^*H = -C$$

$$x^*(HM + M^*H)x = -\,x^*Cx$$

$$(\lambda + \bar{\lambda})\,x^*Hx = -\,x^*Cx$$

$$2\ \mathrm{Re}(\lambda)\,x^*Hx = -\,x^*Cx\ <\ 0$$

$$\mathrm{Re}(\lambda)\ <\ 0.\quad (\mathit{Contradiction})$$

Hence, the spectrum $\sigma(M)$ belongs to $\mathbb{C}^-$ iff with $C = C^* > 0$ the continuous-time Lyapunov equation has a Hermitian positive definite solution $H = H^* > 0$. □

**Example:** Consider

$$M = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -2 & i \\ 0 & 0 & -3 \end{pmatrix},$$

such that the spectrum $\sigma(M)$ belongs to $\mathbb{C}^-$. Choose $C = I_3 = C^* > 0$ and define a general Hermitian matrix $H = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ \bar{h}_{12} & h_{22} & h_{23} \\ \bar{h}_{13} & \bar{h}_{23} & h_{33} \end{pmatrix}$. Here $h_{11}, h_{22}, h_{33} \in \mathbb{R}$. So by the Lyapunov equation

$$HM + M^*H = -C$$

$$\begin{pmatrix} -2h_{11} & h_{11} - 3h_{12} & ih_{12} - 4h_{13} \\ h_{11} - 3\bar{h}_{12} & -4h_{22} + h_{12} + \bar{h}_{12} & ih_{22} - 2h_{23} + h_{13} \\ -i\bar{h}_{12} - 4\bar{h}_{13} & \bar{h}_{13} - 2\bar{h}_{23} - ih_{22} & -6h_{33} - i\bar{h}_{23} - ih_{23} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

From this we get $h_{11} = \frac{1}{2}$, $h_{12} = \frac{1}{6}$, $h_{13} = \frac{i}{24}$, $h_{22} = \frac{1}{3}$, $h_{23} = \frac{3i}{6}$, $h_{33} = \frac{11}{48}$. So,

$$H = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} & \frac{i}{24} \\ \frac{1}{6} & \frac{1}{3} & \frac{3i}{16} \\ \frac{-i}{24} & \frac{-3i}{16} & \frac{11}{48} \end{pmatrix} = H^*.$$

Hence, H is a Hermitian positive definite as $\Delta_1 = \frac{1}{2} > 0,\ \Delta_2 = \frac{5}{36} > 0,\ \Delta_3 = \det(H) = \frac{1}{288} > 0$ and satisfies the Lyapunov equation $HM + M^*H = -C$.

Conversely, suppose that Lyapunov equation $HM + M^*H = -C$ has a Hermitian positive definite solution

$$H = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} & \frac{i}{24} \\ \frac{1}{6} & \frac{1}{3} & \frac{3i}{16} \\ \frac{-i}{24} & \frac{-3i}{16} & \frac{11}{48} \end{pmatrix} = H^* > 0.$$

Now, we have to prove that the spectrum $\sigma(M)$ belongs to $\mathbb{C}^-$. For this we will find the eigenvalues of $M$. Consider

$$M = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -2 & i \\ 0 & 0 & -3 \end{pmatrix},$$

then $\sigma(M) = \{-1\,;\,-2\,;\,-3\}$ and $\mathrm{Re}(\lambda_i(M)) < 0\ \forall i = 1,2,3$. Thus $\sigma(M) \subset \mathbb{C}^-$.

**Algorithm 1** Solution of Continues-Time Lyapunov Equation and Stability Verification

**Input:** Matrix $M \in \mathbb{C}^{n\times n}$
**Output:** Computes solution $H$ of continues-time Lyapunov equation and verifies Hermitian positive definiteness and stability of $M$
$C \leftarrow I_n$
Solve continues-time Lyapunov equation: $HM + M^*H = -C$
Verify Hermitian property: $H = H^*$
Compute and display leading principal minors to check $H$ is positive definite:

$$\Delta_1 = H[1,1],\ \Delta_2 = \det(H[1:2,1:2]),\ \Delta_3 = \det(H)$$

Compute eigenvalues $\lambda(M)$
**if** $\mathrm{Re}(\lambda_i(M)) < 0\ \forall i$ **then**
    Print "Matrix $M$ is stable"
**else**
    Print "Matrix $M$ is not stable"
**end if**

When we study the stability of an ordinary differential equation of form

$$\frac{dy}{dt} = My + f(y),$$

then equation $HM + M^*H = -C$ arises. That's why equation $HM + M^*H = -C$ is known as the continuous-time Lyapunov equation.

**Theorem 2.2.** *The spectrum $\sigma(M)$ belongs to $\mathbb{U}$ iff the discrete-time Lyapunov equation*

$$H - M^*HM = C$$

*with $C = C^* > 0$ has a Hermitian positive definite solution $H = H^* > 0$.*

*Proof.* Consider the spectrum $\sigma(M)$ belongs to $\mathbb{U}$ then $|\lambda(M)| < 1$ and $\lim_{t\to\infty} M^k = 0$. Choose $C = C^* > 0$, and define a converge series

$$H = \sum_{k=0}^{\infty} (M^*)^k C M^k.$$

$$H^* = \sum_{k=0}^{\infty} \left((M^*)^k C M^k\right)^* = \sum_{k=0}^{\infty} (M^*)^k C M^k = H.$$

For any vector $\vec{x} \neq 0$,

$$x^*Hx = \sum_{k=0}^{\infty} \left(M^k x\right)^* C \left(M^k x\right) \ > \ 0.$$

Since $C > 0$ and $M^k x \neq 0$. So $H > 0$. Thus there is a Hermitian and positive definite $H = H^* > 0$. Now, to verify the Lyapunov equation we compute:

$$M^*HM = \left(\sum_{j=1}^{\infty} (M^*)^j C M^j\right) \quad \text{where } j = k+1.$$

$$Thus, \quad H - M^*HM = C.$$

Finally, $H$ satisfied the Lyapunov equation.

Conversely, cosider $H - M^*HM = C$ with $C = C^* > 0$ has a Hermitian positive definite solution $H = H^* > 0$. Now we want to prove that $|\lambda| < 1$. Suppose there exist a non-zero $\vec{x}$ such that $Mx = \lambda x$. Now

$$x^*(H - M^*HM)x = x^*Cx,$$

$$or \quad x^*Hx - (Mx)^*H(Mx) = x^*Cx,$$

$$or \quad (1 - |\lambda|^2)\, x^*Hx = x^*Cx \ > \ 0,$$

$$which\ implies\ that\ |\lambda| < 1.$$

Hence, the spectrum $\sigma(M)$ belongs to $\mathbb{U}$ iff the discrete-time Lyapunov equation with $C = C^* > 0$ has a Hermitian positive definite solution $H = H^* > 0$. $\square$

**Example:** Consider

$$M = \begin{pmatrix} 0.4 & 0 & 0.3i \\ 0.2 & 0.5+0.2i & 0.1 \\ 0.1 & 0.2 & 0.3 \end{pmatrix},$$

such that the spectrum $\sigma(M)$ belongs to $\mathbb{U}$. So, $\lim_{t\to\infty} M^k = 0$. Choose $C = I_3 = C^* > 0$ and define a general Hermitian matrix

$$H = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ \bar{h}_{12} & h_{22} & h_{23} \\ \bar{h}_{13} & \bar{h}_{23} & h_{33} \end{pmatrix};\ M^* = \begin{pmatrix} 0.4 & 0.2 & 0.1 \\ 0 & 0.5-0.2i & 0.2 \\ -0.3i & 0.1 & 0.3 \end{pmatrix};\ H - M^*HM \quad = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33}. \end{pmatrix}$$

where

$$A_{11} = 0.84h_{11} - 0.08h_{12} - 0.08\bar{h}_{12} - 0.04h_{13} - 0.04\bar{h}_{13} - 0.04h_{22} - 0.02h_{23} - 0.02\bar{h}_{23} - 0.01h_{33},$$

$$A_{22} = 0.71h_{22} - 0.1h_{23} - 0.1\bar{h_{23}} - 0.25h_{12} - 0.25\bar{h_{12}} - 0.04h_{33} - 0.04h_{11}$$

$$A_{33} = 0.81h_{33} - 0.06h_{23} - 0.06\bar{h_{23}} - 0.03h_{13} - 0.03\bar{h_{13}} - 0.09h_{11} - 0.01h_{22}$$

$$A_{12} = h_{12} - (0.2h_{11} + 0.1h_{12} + 0.04h_{13} + 0.01h_{22} + 0.04h_{23} + 0.02h_{33})$$

$$A_{13} = h_{13} - (0.12ih_{11} + 0.04h_{12} + 0.12h_{13} + 0.06ih_{22} + 0.02h_{23} + 0.03h_{33})$$

$$A_{23} = h_{23} - (0.06i\bar{h_{12}} + 0.05h_{22} + 0.06h_{23} + 0.06i\bar{h_{13}} + 0.02h_{33}).$$

By the Lyapunov equation

$$H - M^*HM = C$$

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

From this, we have that

$$H = \begin{pmatrix} 1.34 & 0.32i & 0.29i \\ -0.32i & 1.54 & 0.24i \\ -0.29i & -0.24i & 1.32 \end{pmatrix} = H^*.$$

Hence, H is a Hermitian positive definite as $\Delta_1 = 1.34 > 0$, $\Delta_2 = 1.9612 > 0$, $\Delta_3 = \det(H) = 2.384 > 0$ and satisfies the Lyapunov equation $H - M^*HM = C$.

Conversely, suppose that the Lyapunov equation $H - M^*HM = C$ has a Hermitian positive definite solution

$$H = \begin{pmatrix} 1.34 & 0.32i & 0.29i \\ -0.32i & 1.54 & 0.24i \\ -0.29i & -0.24i & 1.32 \end{pmatrix} = H^*.$$

Now, we have to prove that the spectrum $\sigma(M)$ belongs to $\mathbb{U}$. For this, we will find the eigenvalues of $M$ and check the condition that $|\lambda_i(M)| < 1$. Consider

$$M = \begin{pmatrix} 0.4 & 0 & 0.3i \\ 0.2 & 0.5 + 0.2i & 0.1 \\ 0.1 & 0.2 & 0.3 \end{pmatrix},$$

then $\sigma(M) = \{0.6542 + 0.2159i\,;\, 0.3003 - 0.1338i\,;\, 0.2455 + 0.1179i\}$. $|\lambda_1| = 0.689 < 1,\ |\lambda_2| = 0.329 < 1,\ |\lambda_3| = 0.272 < 1$. So, $|\lambda_i(M)| < 1$. Thus $\sigma(M) \subset \mathbb{U}$.

**Algorithm 2** Solution of Discrete-Time Lyapunov Equation and Stability Verification

**Input:** Matrix $M \in \mathbb{C}^{n\times n}$
**Output:** Computes solution $H$ of discrete-time Lyapunov equation and verifies Hermitian positive definiteness and stability of $M$
$C \leftarrow I_n$
Solve discrete-time Lyapunov equation: $H - M^*HM = C$
Verify Hermitian property: $H = H^*$
Compute and display leading principal minors to check $H$ is positive definite:

$$\Delta_1 = H[1,1],\ \Delta_2 = \det(H[1:2, 1:2]),\ \Delta_3 = \det(H)$$

Compute eigenvalues $\lambda(M)$
**if** $|\lambda_i(M)| < 1\ \forall i$ **then**
    Print "Matrix $M$ is stable"
**else**
    Print "Matrix $M$ is not stable"
**end if**

When we study the stability of the difference equation of form

$$y_{j+1} = My_j + \varphi_j(y), \quad j = 0, 1, 2 \ldots$$

then equation $H - M^*HM = C$ arise. That's why equation $H - M^*HM = C$ is known as discrete-time Lyapunov equation.

2.2. **Localization of Spectrum with respect to Cassini Oval:** In this section, we show that the matrix spectrum belongs to the domain bounded by the Cassini oval, equivalent to the solubility of the matrix equation in the class of Hermitian positive definite matrices. The domain bounded by a Cassini oval is

$$\Omega_i = \{\lambda \in \mathbb{C} : |\lambda^2 - a^2| < b\}, \quad a > 0,\ b > 0. \tag{2.2}$$

The set lying outside the closure of the domain bounded by Cassini oval is

$$\Omega_e = \{\lambda \in \mathbb{C} : |\lambda^2 - a^2| > b\}, \quad a > 0,\ b > 0. \tag{2.3}$$

Now, we will find an equivalent equation to $\Omega_i$. From (2.2)

$$(\lambda^2 - a^2)(\bar{\lambda}^2 - a^2) - b^2 < 0$$

$$\lambda^2\bar{\lambda}^2 - a^2(\lambda^2 + \bar{\lambda}^2) + (a^4 - b^2) < 0$$

Define a polynomial $P(\lambda, \bar{\lambda}) = \lambda^2\bar{\lambda}^2 - a^2(\lambda^2 + \bar{\lambda}^2) + (a^4 - b^2)$. Using the standard approach and introduce $C$ and the Hermitian matrix $H$, we get the matrix equation

$$(M^*)^2HM^2 - a^2(HM^2 + (M^*)^2H) + (a^4 - b^2)H = C. \tag{2.4}$$

As we know, the generalized Lyapunov equation is

$$\sum_{j,k=0}^{N} m_{jk}L^jHM^k = Q,$$

where $m_{jk}$ are given constant coefficients, $M \in \mathbb{C}^{n,n}, L \in \mathbb{C}^{m,m}$ and $Q \in \mathbb{C}^{m,n}$ are given matrices. Note that (2.4) has the form of a generalized Lyapunov equation with

$$N = 2, \quad n = m, \quad L = M^*, \quad Q = I$$

$$m_{01} = m_{10} = m_{11} = m_{12} = m_{21} = 0$$

$$m_{00} = a^4 - b^2, \quad m_{02} = m_{20} = -a^2, \quad m_{22} = 1.$$

**Problem:** The spectrum of the matrix $M$ belongs to the domain bounded by the Cassini oval $\Omega_i$ is equivalent to the solubility of the matrix equation (2.4) with $C = C^* > 0$, in the class of Hermitian positive definite matrices $H = H^* > 0$.

**Theorem 2.3.** *Let $C = C^* > 0$. The spectrum $\sigma(M)$ belongs to $\Omega_i$ if and only if there is a Hermitian positive definite solution $H = H^* > 0$ to (2.4).*

*Proof.* Consider all eigenvalues $\eta_s$ of $M$ belong to $\Omega_i$. Then all eigenvalues $\lambda_r = \bar{\eta}_r$ of $M^*$ also belong to $\Omega_i$. Now to prove that there is a Hermitian positive definite solution $H = H^* > 0$, it is enough to check that the condition of Krein's theorem $P(\eta_s, \lambda_r) \neq 0$ is valid. Define a polynomial associated with (2.4)

$$P(\eta, \lambda) = \eta^2\lambda^2 - a^2(\eta^2 + \lambda^2) + (a^4 - b^2).$$

For $\lambda = \bar{\eta}$

$$P(\eta, \bar{\eta}) = \eta^2\bar{\eta}^2 - a^2(\eta^2 + \bar{\eta}^2) + (a^4 - b^2)$$

From (2.2), we have $(\eta^2 - a^2)(\bar{\eta}^2 - a^2) < b^2$, which is equivalent to $P(\eta, \bar{\eta}) < 0$. Therefore, it is not difficult to show that the condition of Krein's theorem is fulfilled and for any matrix $C$ (2.4) has a unique solution; furthermore,

$$H = \frac{1}{(2\pi i)^2} \int_{\gamma_M} \int_{\gamma_{M^*}} \frac{(\eta I - M^*)^{-1}\, C\, (\lambda I - M)^{-1}}{P(\eta, \lambda)} d\eta\, d\lambda$$

$$H = \frac{1}{(2\pi i)^2} \int_{\gamma_M} \int_{\gamma_{M^*}} \frac{(\eta I - M^*)^{-1}\, C\, (\lambda I - M)^{-1}}{\eta^2\lambda^2 - a^2(\eta^2 + \lambda^2) + (a^4 - b^2)} d\eta\, d\lambda$$

where contours $\gamma_M$ and $\gamma_{M^*}$ surrounding the spectra of the matrices $M$ and $M^*$, respectively. Note that $H = H^* > 0$ if $C = C^* > 0$.

Conversely, Suppose on contrary that there exists an eigenvalue $\eta \in \sigma(M)$ such that

$$|\eta^2 - a^2| \geq b.$$

Let $v \neq 0$ be an eigenvector of $M$ corresponding to $\eta$ such that $Mv = \eta\, v$ and scalar product $\langle Cv, v\rangle$. By the condition of the theorem $\langle Cv, v\rangle > 0$. We rewrite this scalar product using the fact that $H$ is a solution to (2.4). We have

$$\langle Cv, v\rangle = \left\langle (M^*)^2 HM^2\, v\,, v\right\rangle - a^2 \left\langle HM^2\, v\,, v\right\rangle - a^2 \left\langle (M^*)^2 H\, v\,, v\right\rangle + (a^4 - b^2)\,\langle H\, v, v\rangle$$

Write $Mv = \eta\, v$, and we get

$$\langle Cv, v\rangle = \left[|\eta|^4 - a^2(\eta^2 + \bar{\eta}^2) + (a^4 - b^2)\right]\langle Hv, v\rangle$$

Observe that $|\eta|^4 - a^2(\eta^2 + \bar{\eta}^2) + a^4 = |\eta^2 - a^2|^2$. Hence

$$\langle Cv, v\rangle = \left[\left(|\eta^2 - a^2|\right)^2 - b^2\right]\langle Hv, v\rangle.$$

From our assumption $|\eta^2 - a^2| \geq b$, we get $\left(|\eta^2 - a^2|\right)^2 - b^2 \geq 0$. Since, $\langle Hv, v\rangle > 0$, it follows that $\langle Cv, v\rangle \geq 0$. However, because $C > 0$, we must have $\langle Cv, v\rangle > 0,\ \forall\, v \neq 0$. This forces $\left(|\eta^2 - a^2|\right)^2 - b^2 < 0$, which contradicts our assumption. Therefore, the spectrum $\sigma(M)$ belongs to $\Omega_i$. □

Consider the problem on the belonging of the spectrum $\sigma(M)$ to $\Omega_i$ and consider the matrix equation of the form (2.4) with a modified right-hand side

$$(M^*)^2 HM^2 - a^2(HM^2 + (M^*)^2 H) + (a^4 - b^2)H = -C. \tag{2.5}$$

**Theorem 2.4.** *Let $C = C^* > 0$. The spectrum $\sigma(M)$ belongs to $\Omega_e$ if and only if there is a Hermitian positive definite solution $H = H^* > 0$ to (2.5).*

*Proof.* Consider all eigenvalues $\eta_s$ of $M$ belong to $\Omega_e$. Then all eigenvalues $\lambda_r = \bar{\eta}_r$ of $M^*$ also belong to $\Omega_e$. Now to prove that there is a Hermitian positive definite solution $H = H^* > 0$, it is enough to check that the condition of Krein's theorem $P(\eta_s, \lambda_r) \neq 0$ is valid. Define a polynomial associated with (2.5)

$$P(\eta, \lambda) = \eta^2\lambda^2 - a^2(\eta^2 + \lambda^2) + (a^4 - b^2).$$

For $\lambda = \bar{\eta}$

$$P(\eta, \bar{\eta}) = \eta^2\bar{\eta}^2 - a^2(\eta^2 + \bar{\eta}^2) + (a^4 - b^2)$$

From (2.3), taking into account the inequality $(\eta^2 - a^2)(\bar{\eta}^2 - a^2) > b^2$, it is not difficult to show that the condition of Krein's theorem is valid and for any matrix $C$ (2.4) has a unique solution; furthermore,

$$H = \frac{1}{(2\pi i)^2} \int_{\gamma_M} \int_{\gamma_{M^*}} \frac{(\eta I - M^*)^{-1}\, C\, (\lambda I - M)^{-1}}{P(\eta, \lambda)} d\eta\, d\lambda$$

$$H = \frac{1}{(2\pi i)^2}\int_{\gamma_M}\int_{\gamma_{M^*}} \frac{(\eta I - M^*)^{-1}\, C\, (\lambda I - M)^{-1}}{\eta^2\lambda^2 - a^2(\eta^2+\lambda^2) + (a^4-b^2)} d\eta\, d\lambda$$

where the contours $\gamma_M$ and $\gamma_{M^*}$ surround the spectra of the matrices $M$ and $M^*$, respectively. Note that $H = H^* > 0$ if $C = C^* > 0$.

Conversely, Suppose on the contrary that there exists an eigenvalue $\eta \in \sigma(M)$ such that

$$|\eta^2 - a^2| \leq b.$$

Let $v \neq 0$ be an eigenvector of $M$ corresponding to $\eta$ such that $Mv = \eta\, v$ and scalar product $\langle Cv, v\rangle$. By the condition of the theorem $\langle Cv, v\rangle > 0$. We rewrite this scalar product using the fact that $H$ is a solution to (2.5). We have

$$\langle Cv, v\rangle = \left\langle (M^*)^2 H M^2\ v\,, v\right\rangle - a^2\left\langle HM^2\ v\,, v\right\rangle - a^2\left\langle (M^*)^2 H\ v\,, v\right\rangle + (a^4-b^2)\,\langle H\ v\,, v\rangle$$

Write $Mv = \eta\, v$, and we get

$$\langle Cv, v\rangle = \left[|\eta|^4 - a^2(\eta^2+\bar{\eta}^2) + (a^4-b^2)\right]\langle Hv, v\rangle$$

We observe that $|\eta|^4 - a^2(\eta^2+\bar{\eta}^2) + a^4 = |\eta^2 - a^2|^2$. Therefore,

$$\langle Cv, v\rangle = \left[\left(|\eta^2 - a^2|\right)^2 - b^2\right]\langle Hv, v\rangle\,.$$

From our assumption $|\eta^2 - a^2| \leq b$, we get $\left(|\eta^2-a^2|\right)^2 - b^2 \leq 0$. Since, $\langle Hv, v\rangle > 0$, it follows that $\langle Cv, v\rangle \leq 0$. However, because $C > 0$, we must have $\langle Cv, v\rangle > 0$, $\forall\, v \neq 0$. Then we have a contradiction. Therefore, the spectrum $\sigma(M)$ belongs to $\Omega_e$. □

Now, we will prove the theorem for the perturbation of the spectrum $\sigma(M)$.

**Theorem 2.5.** *Consider spectrum $\sigma(M)$ belong to $\Omega_i$ and let $H = H^* > 0$ be a solution to (2.4) with $C = I$. If $L$ is a matrix such that*

$$\left((M+L)^*\right)^2 H(M+L)^2 - a^2\left(H(M+L)^2 + \left((M+L)^*\right)^2\right) + (a^4-b^2)H > -I, \tag{2.6}$$

*then the spectrum of the perturbed matrix $(M+L)$ also belongs to $\Omega_i$.*

*Proof.* Let $\eta_j \in \sigma(M+L)$ and let $v_j \neq 0$ be a corresponding eigenvector such that$(M+L)v_j = \eta_j v_j$. Taking the inner product of equation (2.6) with $v_j$, we obtain

$$\begin{aligned}-\langle Cv_j, v_j\rangle = \langle (M+L)^{*2} H (M+L)^2 v_j, v_j\rangle - a^2\langle H(M+L)^2 v_j, v_j\rangle \\ - a^2\langle (M+L)^{*2} H v_j, v_j\rangle + (a^4-b^2)\langle Hv_j, v_j\rangle.\end{aligned}$$

Define

$$\begin{aligned}J_j = \langle (M+L)^{*2} H (M+L)^2 v_j, v_j\rangle - a^2\langle H(M+L)^2 v_j, v_j\rangle \\ - a^2\langle (M+L)^{*2} H v_j, v_j\rangle + (a^4-b^2)\langle Hv_j, v_j\rangle.\end{aligned} \tag{9}$$

Then $J_j = \langle Cv_j, v_j\rangle$. Since $C > 0$, we have $\langle Cv_j, v_j\rangle > 0$, and hence $J_j > 0$. Using $(M+L)v_j = \eta_j v_j$, we obtain $(M+L)^2 v_j = \eta_j^2 v_j, \quad ((M+L)^*)^2 v_j = \bar{\eta}_j^2 v_j$. Substituting into *(9)*, we get

$$J_j = \Big(|\eta_j|^4 - a^2(\eta_j^2 + \bar{\eta}_j^2) + (a^4 - b^2)\Big)\langle Hv_j, v_j\rangle.$$

Since

$$|\eta_j|^4 - a^2(\eta_j^2 + \bar{\eta}_j^2) + a^4 = |\eta_j^2 - a^2|^2,$$

it follows that

$$J_j = \Big(|\eta_j^2 - a^2|^2 - b^2\Big)\langle Hv_j, v_j\rangle.$$

Since $H > 0,\ J_j > 0$, we conclude that

$$|\eta_j^2 - a^2| < b.$$

Since $\eta_j$ was arbitrary. Thus spectrum $\sigma(M+L)$ belongs to $\Omega_i$. Let $\eta$ be an eigenvalue of the matrix $(M+L)$ and let $v \neq 0$ be a corresponding eigenvector, i.e., $(M+L)v = \eta v$. Consider the scalar product of inequality (2.6) with $v$, we obtain

$$\Big\langle ((M+L)^*)^2 H(M+L)^2 v, v\Big\rangle - a^2\langle\Big(H(M+L)^2\ v, v\Big\rangle - a^2\Big\langle ((M+L)^*)^2 H\ v, v\Big\rangle + (a^4 - b^2)\langle Hv, v\rangle < \langle Iv, v\rangle.$$

By substituting the relation $(M+L)^2 v = \eta^2 v$ and $((M+L)^*)^2 v = \bar{\eta}^2 v$, we get

$$\Big(|\eta|^4 - a^2(\eta^2 + \bar{\eta}^2) + (a^4 - b^2)\Big)\langle Hv, v\rangle < \|v^2\|.$$

We observe that $|\eta|^4 - a^2(\eta^2 + \bar{\eta}^2) + a^4 = |\eta^2 - a^2|^2$. Hence

$$\Big(|\eta^2 - a^2|^2 - b^2\Big)\langle Hv, v\rangle < \|v^2\|.$$

Since $H > 0$, we have $\langle Hv, v\rangle > 0$. Therefore, $|\eta^2 - a^2|^2 < b^2$. Thus $\sigma(M+L)$ belongs to $\Omega_i$. □

**Theorem 2.6.** *Let the spectrum of the matrix M belong to $\Omega_e$ and let $H = H^* > 0$ be a solution to (2.5) with $C = I$. If L is a matrix such that*

$$-I < ((M+L)^*)^2 H(M+L)^2 - a^2\Big(H(M+L)^2 + ((M+L)^*)^2\Big) + (a^4 - b^2)H, \tag{2.7}$$

*then spectrum of the perturbed matrix $(M+L)$ also belongs to $\Omega_e$.*

*Proof.* Let $\eta_j \in \sigma(A+B)$ and let $v_j \neq 0$ be a corresponding eigenvector such that $(M+L)v_j = \eta_j v_j$. Taking the inner product of equation *(7)* with $v_j$, we obtain

$$\begin{aligned}-\langle Cv_j, v_j\rangle = \langle (M+L)^{*2} H(M+L)^2 v_j, v_j\rangle - a^2\langle H(M+L)^2 v_j, v_j\rangle \\ - a^2\langle (M+L)^{*2} Hv_j, v_j\rangle + (a^4 - b^2)\langle Hv_j, v_j\rangle.\end{aligned}$$

Define

$$\begin{aligned}J_j = \langle (M+L)^{*2} H(M+L)^2 v_j, v_j\rangle - a^2\langle H(M+L)^2 v_j, v_j\rangle \\ - a^2\langle (M+L)^{*2} Hv_j, v_j\rangle + (a^4 - b^2)\langle Hv_j, v_j\rangle.\end{aligned} \tag{8}$$

Then $J_j = -\langle Cv_j, v_j\rangle$. Since $C > 0$, we have $\langle Cv_j, v_j\rangle > 0$, and hence $J_j < 0$. Using $(M+L)v_j = \eta_j v_j$, we obtain

$$(M+L)^2 v_j = \eta_j^2 v_j, \qquad ((M+L)^*)^2 v_j = \bar{\eta}_j^2 v_j.$$

Substituting into *(8)*, we get

$$J_j = \Big(|\eta_j|^4 - a^2(\eta_j^2 + \bar{\eta}_j^2) + (a^4 - b^2)\Big)\langle Hv_j, v_j\rangle.$$

Since

$$|\eta_j|^4 - a^2(\eta_j^2 + \bar{\eta}_j^2) + a^4 = |\eta_j^2 - a^2|^2,$$

it follows that

$$J_j = \Big(|\eta_j^2 - a^2|^2 - b^2\Big)\langle Hv_j, v_j\rangle.$$

Since $H > 0$ , $J > 0$,, we conclude that

$$|\eta_j^2 - a^2| > b.$$

Since $\eta_j$ was arbitrary, we obtain the spectrum $\sigma(M + L)$ belong to $\Omega_e$. □

## 3. Conclusion

This work establishes a rigorous and computationally tractable bridge between Lyapunov stability theory and Cassini oval-based spectral localization. By deriving necessary and sufficient conditions for eigenvalue confinement in the left half complex plane or unit disk, we prove that Cassini containment is mathematically equivalent to the solvability of a generalized Lyapunov-type equation in the cone of Hermitian positive definite matrices. This equivalence significantly reduces the conservatism inherent in classical Gershgorin-type inclusion regions while preserving analytical transparency and computational efficiency. The resulting framework provides a unified geometric-algebraic foundation for robust stability certification, controller synthesis, and uncertainty quantification in finite-dimensional linear systems.

**Extension to Nonlinear and Time-Varying Systems:** Generalizing the Cassini-Lyapunov equivalence to nonlinear dynamics, switched systems, and linear time-varying models using parameter-dependent or integral Lyapunov functions.

**Scalable Algorithms for Large-Scale Matrices:** Developing structure-exploiting and iterative solvers tailored for sparse, high-dimensional, or distributed systems to enable real-time stability certification in networked applications.

**Higher-Order Spectral Inclusion Regions:** Investigating Cassini ovals of arbitrary order, generalized lemniscates, and union/intersection of algebraic curves to achieve progressively tighter eigenvalue bounds with minimal computational overhead.

**Data-Driven and Robust Control Integration:** Coupling the proposed matrix equation framework with learning-based Lyapunov verification, polytopic uncertainty models, and Koopman operator techniques to enable adaptive, model-free stability analysis for uncertain and partially observed systems.

**Conflicts of Interest:** The authors declare that there are no conflicts of interest regarding the publication of this paper.

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