

Generalized Coprime Graph of Subgroups of a Finite Group**Bushra N. Abdulgaphur*, Sanhan M.S. Khasraw, Rashad R. Haji***Department of Mathematics, College of Education, Salahaddin University-Erbil, Erbil, Kurdistan Region, Iraq***Corresponding author: bushra.abdulgaphur@su.edu.krd*

Abstract. Let $\mathcal{G}$ be a finite group. The generalized coprime graph of $\mathcal{G}$, $\Gamma_{gc}(\mathcal{G})$, is a graph whose vertices are all the proper non-trivial subgroups of $\mathcal{G}$ and two distinct vertices H and K are adjacent if $|H| \nmid |K|$ and $|K| \nmid |H|$. The paper investigates the connectedness, diameter and girth of $\Gamma_{gc}(\mathcal{G})$. Also, we explore the interplay between the algebraic properties of $\mathcal{G}$ and the graph theoretic properties of $\Gamma_{gc}(\mathcal{G})$. Furthermore, the energy of $\Gamma_{gc}(D_{2n})$, where $n \in \{2p, p^k, pq\}$, and $\Gamma_{gc}(\mathbb{Z}_n)$, where $n \in \{pqr, p^kq, p^2q^2\}$, is computed.

1. INTRODUCTION AND PRELIMINARIES

Graph theory provides useful tools for studying the algebraic properties of different algebraic structures. To study the interplay between the algebraic properties of graphs and group theoretic properties, several types of graphs associated to groups are defined, such as the order prime graph [13], the coprime graph [9], a note on coprime graph [6], non-coprime graph [10], and non-commuting graph [1].

For a graph Γ , we denote the vertex set by $V(\Gamma)$ and the edge set by $E(\Gamma)$. A graph whose edge set is empty is called a null graph. A subset E of $V(\Gamma)$, is called independent if no two vertices in E are adjacent. A graph Γ is called r -partite, denoted by $K_{n_1, n_2, \dots, n_r}$, if its vertex set can be partitioned into r independent disjoint sets such that each vertex in one set is adjacent to all vertices in all other independent sets. When $r = 2$, Γ is called bipartite. A star graph is the complete bipartite graph $K_{1, n}$. The graph K_n denotes the complete graph with n vertices. A path graph P_n with n vertices, where all vertices are connected by a single line, and a cycle graph C_n with n vertices consists of a single cycle that includes all of its vertices. A graph Γ is connected if for every pair of vertices (u, v) in $V(\Gamma)$, there exist a path from u to v . For a connected graph Γ , the diameter is the length of the shortest path between any two vertices. A graph is defined as a tree if it is both

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connected and acyclic. The length of the shortest cycle in Γ is called a girth, denoted by $g(\Gamma)$. The fundamental concepts, notations, and definitions of graph theory which are used throughout this work are taken from [3] and [11].

The adjacency matrix $A(\Gamma)$ for a simple graph Γ is a symmetric square matrix where the (i, j) -entry is 1 when there is an edge between v_i and v_j , and is 0 otherwise. The characteristic polynomial $\mathcal{P}(\Gamma, \lambda)$ of a graph Γ is defined as $\mathcal{P}(\Gamma, \lambda) = \det(\lambda I - A(\Gamma))$, where $A(\Gamma)$ is the graph's adjacency matrix and I is the identity matrix. The eigenvalues of Γ are precisely the roots of $\mathcal{P}(\Gamma, \lambda)$, and its spectrum is the nonincreasing list $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ of these eigenvalues [2]. The graph energy is defined as $\mathcal{E}(\Gamma) = \sum_{i=1}^n |\lambda_i|$ [8]. For a positive integer n , $\tau(n)$ counts its positive divisors, $\sigma(n)$ sum of its positive divisors, and $\pi(n)$ is the set of distinct primes dividing n [4].

In [5], Rajkumar and Devi introduced a new type of graph associated to a finite group $\mathcal{G}$. This graph called a coprime graph of subgroups, where the vertex set is the set of all proper nontrivial subgroups of $\mathcal{G}$ and edges exists between two disjoint vertices H and K if and only if the orders of H and K are coprime. In this paper, we introduce the notion of generalized coprime graph of subgroups of a group $\mathcal{G}$, denoted by $\Gamma_{gc}(\mathcal{G})$, as a generalization of coprime graph of subgroups, where the vertex set is the set of all proper nontrivial subgroups of $\mathcal{G}$ and two distinct vertices H and K are adjacent if $|H| \nmid |K|$ and $|K| \nmid |H|$. The paper is organized as follows. In Section 2, the concept of the generalized coprime graph of subgroups of a finite group is introduced and explores some of its properties. Section 3 presents general formulas for the characteristic polynomial and eigenvalue of $\Gamma_{gc}(D_{2n})$. The study also explores the energy of the generalized coprime graph for dihedral groups. Section 4 focuses on finding the energy of $\Gamma_{gc}(\mathbb{Z}_n)$, where $n \in \{pqr, p^2q^2, p^kq\}$ for distinct primes p, q , and r .

The following Lemma, taken from [14], is instrumental in computing the determinants of block matrices that appear in Section 3 and 4.

Lemma 1.1. Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, where A, B, C , and D are all of the same size. If D is invertible, then

$$\det(M) = \det(D) \cdot \det(A - BD^{-1}C).$$

The following Lemma provides the necessary and sufficient conditions for a group of order pq to be cyclic, which is essential for characterizing the subgroup structure used in proof of Theorem 2.3.

Lemma 1.2. [7] For distinct primes p and q with $p < q$, a group of order pq is cyclic if and only if $q \not\equiv 1 \pmod{p}$.

The classification of all proper nontrivial subgroups of the group D_{2n} is given by the following Lemma from [12].

Lemma 1.3. The subgroups of the group D_{2n} which are non-trivial are listed as:

- (i) Cyclic subgroups $H_0^k := \langle a^{\frac{n}{k}} \rangle$, where $|H_0^k| = k$ and k is a divisor of n and $k > 1$;
- (ii) Cyclic subgroups $H_i^1 := \langle a^i b \rangle$, where $|H_i^1| = 2$, for $i = 1, \dots, n$;
- (iii) Dihedral subgroups $H_i^k := \langle a^{\frac{n}{k}}, a^i b \rangle$, where $|H_i^k| = 2k$ and k is a divisor of n , $k \neq 1, n$, and $1 \leq i \leq \frac{n}{k}$.

Dihedral group D_{2n} has exactly $\tau(n) + \sigma(n) - 2$ subgroups in total, because the number of subgroups of D_{2n} are $\tau(n) - 1$ in (i), n in (ii), and $\sigma(n) - n - 1$ in (iii).

2. GENERALIZED COPRIME GRAPH OF SUBGROUPS OF FINITE GROUPS

In this section, the notion of generalized coprime graph of subgroups of a group is introduced as a generalization of coprime graph of subgroups of a group.

Definition 2.1. The generalized coprime graph of subgroups of a group $\mathcal{G}$, denoted as $\Gamma_{gc}(\mathcal{G})$, is a simple graph whose vertex set consists of all proper subgroups of $\mathcal{G}$ which are nontrivial and two distinct vertices H and K are adjacent if $|H| \nmid |K|$ and $|K| \nmid |H|$.

Example 2.1. Consider the cyclic group of order 12, $\mathbb{Z}_{12}$. The proper nontrivial subgroups of $\mathbb{Z}_{12}$ are H_1, H_2, H_3 , and H_4 of orders 2, 3, 4, and 6, respectively. According to the adjacency condition $|H| \nmid |K|$ and $|K| \nmid |H|$, we observe the following:

- H_1 is adjacent to H_2 , since $2 \nmid 3$.
- H_2 is adjacent to H_3 , since $3 \nmid 4$.
- H_3 is adjacent to H_4 , since $4 \nmid 6$.

However, H_1 is not adjacent to H_3 , because $|H_1| \mid |H_3|$. Similarly, H_1 is not adjacent to H_4 , because $|H_1| \mid |H_4|$, and H_2 is not adjacent to H_4 , because $|H_2| \mid |H_4|$. Consequently, the resulting graph, $\Gamma_{gc}(\mathbb{Z}_{12})$ is the path P_4 , with the sequence $H_1 \rightarrow H_2 \rightarrow H_3 \rightarrow H_4$.

Theorem 2.1. For any two groups $\mathcal{G}_1$ and $\mathcal{G}_2$, if $\mathcal{G}_1$ is isomorphic to $\mathcal{G}_2$, then $\Gamma_{gc}(\mathcal{G}_1) \cong \Gamma_{gc}(\mathcal{G}_2)$.

Proof. Let ϕ be a group isomorphism from $\mathcal{G}_1$ to $\mathcal{G}_2$. Construct the mapping $\bar{\cup} : V(\Gamma_{gc}(\mathcal{G}_1)) \rightarrow V(\Gamma_{gc}(\mathcal{G}_2))$ by setting $\bar{\cup}(H) = \phi(H)$, for every H in $V(\Gamma_{gc}(\mathcal{G}_1))$. It follows that $\bar{\cup}$ is a graph isomorphism, since the group isomorphism preserves the order of subgroups. $\square$

Remark 2.1. Note that the converse of Theorem 2.1 does not hold in general, for instance, $\Gamma_{gc}(\mathbb{Z}_{15})$ and $\Gamma_{gc}(\mathbb{Z}_{21})$ are isomorphic, but the groups $\mathbb{Z}_{15}$ and $\mathbb{Z}_{21}$ are not isomorphic.

Theorem 2.2. For a finite group $\mathcal{G}$, the graph $\Gamma_{gc}(\mathcal{G})$ is isomorphic to $\bar{K}_n$ if and only if $\mathcal{G}$ is a p -group.

Proof. Suppose $\Gamma_{gc}(\mathcal{G}) \cong \bar{K}_n$. Then there is no edge in $\Gamma_{gc}(\mathcal{G})$. This means that for any two proper subgroups H and K of $\mathcal{G}$, either $|H| \mid |K|$ or $|K| \mid |H|$. The possible orders of the subgroups must be powers of a single prime p . Therefore, $\mathcal{G}$ is a p -group.

Conversely, let $\mathcal{G}$ be a p -group. The nontrivial subgroups of $\mathcal{G}$ have orders a power of p . Then, for any two subgroups H and K of $\mathcal{G}$, either $|H| \nmid |K|$ or $|K| \nmid |H|$. Then there is no edge between any two distinct vertices in $\Gamma_{gc}(\mathcal{G})$. Hence, $\Gamma_{gc}(\mathcal{G}) \cong \bar{K}_n$. $\square$

Theorem 2.3. Let $\mathcal{G}$ be a finite group of order pq , where p and q are distinct primes.

- (a) if $p \nmid (q-1)$, then $\Gamma_{gc}(\mathcal{G}) = K_2$ if and only if $\mathcal{G} \cong \mathbb{Z}_{pq}$.
- (b) if $p \mid (q-1)$ then
 - (i) $\Gamma_{gc}(\mathcal{G}) = K_2$ if and only if $\mathcal{G}$ is cyclic.
 - (ii) $\Gamma_{gc}(\mathcal{G}) = K_{1,q}$ if and only if $\mathcal{G}$ is nonabelian.

Proof. Case (a): If $p \nmid (q-1)$ with $p < q$, then, by Lemma 1.2, $\mathcal{G}$ is cyclic and isomorphic to $\mathbb{Z}_{pq}$. Since the only proper nontrivial subgroups of $\mathbb{Z}_{pq}$ are of orders p and q and $\gcd(p, q) = 1$, then $\Gamma_{gc}(\mathcal{G}) = K_2$.

Conversely, suppose $\Gamma_{gc}(\mathcal{G}) = K_2$. This means that $\mathcal{G}$ has only two nontrivial subgroups H and K , and $|H| \nmid |K|$ and $|K| \nmid |H|$. Since the intersection of any two subgroups is also a subgroup and $\Gamma(\mathcal{G}) = K_2$, then $H \cap K$ is a trivial subgroup of G . Therefore, $\mathcal{G} \cong \mathbb{Z}_{pq}$.

Case (b): If $p \mid (q-1)$, then there are two possibilities.

- (i) If $\mathcal{G}$ is cyclic, the argument is identical to Case 1.
- (ii) If $\mathcal{G}$ is nonabelian, then $\mathcal{G} \cong \mathbb{Z}_q \rtimes \mathbb{Z}_p$. So $\mathcal{G}$ has one subgroup H of order q and q subgroups M_i each of order p . Since $\gcd(|H|, |M_i|) = 1$, then H is adjacent to each M_i , and M_i is not adjacent to M_j for $i \neq j$ as $\gcd(|M_i|, |M_j|) = |M_i| = p$. Therefore, $\Gamma_{gc}(\mathcal{G}) = K_{1,q}$.

Conversely, suppose $\Gamma_{gc}(\mathcal{G}) = K_{1,q}$, then $\mathcal{G}$ has a proper subgroup, say H of order q , and q subgroups, say M_i , such that H is adjacent to each M_i . This means that $|H| \nmid |M_i|$ and $|M_i| \nmid |H|$. Also, there are q nonadjacent vertices M_i in $\Gamma_{gc}(\mathcal{G})$ so that $|M_j| \nmid |M_i|$ or $|M_i| \nmid |M_j|$ for $i \neq j$. This only happens if $\mathcal{G}$ is a nonabelian group. Thus, $\mathcal{G} \cong \mathbb{Z}_q \rtimes \mathbb{Z}_p$. $\square$

Theorem 2.4. Let $\mathcal{G}$ be a finite non- p -group. Then $\Gamma_{gc}(\mathcal{G})$ is connected.

Proof. Consider a finite group $\mathcal{G}$ whose order is not a power of a single prime. This implies that the order of $\mathcal{G}$ must be divisible by at least two different primes. Our goal is to demonstrate that $\Gamma_{gc}(\mathcal{G})$ is connected, meaning that any two of its vertices can be connected by a path in $\Gamma_{gc}(\mathcal{G})$.

Since the order of $\mathcal{G}$ is not a power of a single prime, let $|\mathcal{G}| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $k \geq 2$. For each prime p_i , there is a Sylow p_i -subgroup P_i of $\mathcal{G}$. The order of P_i is $p_i^{\alpha_i}$ which is coprime to the order of P_j for $i \neq j$.

Consider the Sylow p_1 -subgroup P_1 and Sylow p_2 -subgroup P_2 . The orders of P_1 and P_2 are $p_1^{\alpha_1}$ and $p_2^{\alpha_2}$, respectively. Since $p_1^{\alpha_1} \nmid p_2^{\alpha_2}$ and $p_2^{\alpha_2} \nmid p_1^{\alpha_1}$, P_1 and P_2 are adjacent in $\Gamma_{gc}(\mathcal{G})$.

We claim that any proper subgroup H is connected to at least one Sylow subgroup. This is because the order of H is not divisible by all prime factors of the order of $\mathcal{G}$, ensuring H is not a p -group itself. Thus, H is adjacent to at least one Sylow p_i subgroup P_i because $|H| \nmid |P_i|$ and $|P_i| \nmid |H|$. Consider arbitrary proper subgroups H_1 and H_2 of $\mathcal{G}$. By the above argument, H_1 is

connected to at least one Sylow p_i -subgroup, say P_i . Similarly, H_2 is connected to at least one Sylow p_j -subgroup, say P_j . Since P_i and P_j are adjacent, there exists a path connecting H_1 and H_2 through P_i and P_j , $H_1 \rightarrow P_i \rightarrow P_j \rightarrow H_2$.

We have shown that for any two proper subgroups H_1 and H_2 of $\mathcal{G}$, there exists a path connecting H_1 and H_2 in $\Gamma_{gc}(\mathcal{G})$. Thus, $\Gamma_{gc}(\mathcal{G})$ is connected. $\square$

Theorem 2.5. Let $\mathcal{G}$ be a finite group that is not a p -group. Then, the diameter of the generalized coprime graph, $\text{diam}(\Gamma_{gc}(\mathcal{G}))$ is at most 3.

Proof. We consider the following cases:

Case 1: If $|\mathcal{G}| = p_1 p_2$, where p_1 and p_2 are distinct primes, by Theorem 2.3, we have two possibilities for the structure of $\Gamma_{gc}(\mathcal{G})$, if $\mathcal{G}$ is cyclic, then $\Gamma_{gc}(\mathcal{G}) \cong K_2$. In this case the distance between the two vertices is 1, so $\text{diam}(\Gamma_{gc}(\mathcal{G}))=1$, if $\mathcal{G}$ is non-abelian, the graph $\Gamma_{gc}(\mathcal{G}) \cong K_{1,p_2}$. Since K_{1,p_2} is a star graph, the maximum distance between any two vertices is 2, so $\text{diam}(\Gamma_{gc}(\mathcal{G}))=2$. In both cases, it follows that $\text{diam}(\Gamma_{gc}(\mathcal{G})) \leq 2$.

Case 2: Let $|\mathcal{G}| = p_1 p_2 \cdots p_k$, where $k \geq 3$. Consider the subgroup H of $\mathcal{G}$ with $\pi(H) \neq \pi(\mathcal{G})$, there exist another subgroup K of $\mathcal{G}$ with K not adjacent to H in $\Gamma(\mathcal{G})$. Then by the assumption $\pi(H) \neq \pi(\mathcal{G})$ and $\pi(K) \neq \pi(\mathcal{G})$, so there exist $p_i \in \pi(\mathcal{G})$ such that $p_i \notin \pi(H)$ and $p_i \notin \pi(K)$, then there is a path. $H \rightarrow H_i \rightarrow K$, where H_i is a subgroup of $\mathcal{G}$ of order p_i . Thus, no path in $\Gamma_{gc}(\mathcal{G})$ exceeds length 2, and $\text{diam}(\Gamma_{gc}(\mathcal{G})) = 2$.

Case 3: Let $|\mathcal{G}| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where at least one of the $\alpha_i \geq 2$ and $k \geq 2$. Similar to Case 2, take any two proper subgroups H and K that are not adjacent in $\Gamma_{gc}(\mathcal{G})$ with $\pi(H) \neq \pi(\mathcal{G})$ and $\pi(K) \neq \pi(\mathcal{G})$, and so there exist $p_i \in \pi(\mathcal{G})$ such that $p_i \notin \pi(H)$ and $p_i \notin \pi(K)$, then there is a path. $H \rightarrow H_i \rightarrow K$, where H_i is a subgroup of $\mathcal{G}$ of order p_i . Thus, no path in $\Gamma_{gc}(\mathcal{G})$ exceeds length 2, and $\text{diam}(\Gamma_{gc}(\mathcal{G})) = 2$.

Let $p_1, p_2 \in \pi(\mathcal{G})$ and H be a subgroup of $\mathcal{G}$ with $\pi(H) = \pi(\mathcal{G})$, there exist another subgroup K of $\mathcal{G}$ of order $p_1^{\alpha_1}$. Thus, H and K are adjacent. Then there is a path $H_{p_1} \rightarrow H_{p_2} \rightarrow K \rightarrow H$, where H_{p_1} and H_{p_2} are subgroups of orders p_1 and p_2 , respectively, and H is a subgroup of order $p_1^{\alpha_1-1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. It follows that $\text{diam}(\Gamma_{gc}(\mathcal{G})) \leq 3$. $\square$

Theorem 2.6. Let $\mathcal{G}$ be a finite group of order $p^2 q$, where p and q are distinct primes. If $\mathcal{G} \cong \mathbb{Z}_{pq} \times \mathbb{Z}_p$, then $\Gamma_{gc}(\mathcal{G})$ is a tree graph.

Proof. Since $\mathcal{G} \cong \mathbb{Z}_{pq} \times \mathbb{Z}_p$, the group $\mathcal{G}$ is abelian but not cyclic, its subgroup structure depends on the factors of its order, the divisors of $p^2 q$ are $1, p, q, p^2, pq$, and $p^2 q$. Each divisor corresponds to a potential subgroup. The group $\mathcal{G}$ is non-cyclic, so there are multiple subgroups of the same order. Specifically, there are $p+1$ subgroups of order p , one subgroup of order q , one subgroup of order p^2 , and $p+1$ subgroups of order pq .

The vertices H_{p_i} represent $p+1$ subgroups of order p , and H_{pq_i} represent $p+1$ subgroups of order pq , H_q represents the subgroup of order q , and H_{p^2} represents the subgroup of order p^2 , where $1 \leq i \leq p+1$.

Each of the $p + 1$ vertices H_{p_i} is adjacent to the vertex H_q , as $|H_{p_i}| \nmid |H_q|$ and $|H_q| \nmid |H_{p_i}|$, the vertices H_{p_i} are not adjacent to the vertex H_{p^2} , as $|H_{p_i}| \nmid |H_{p^2}|$, and the vertices H_{p_i} are not adjacent to the vertices H_{pq_i} , as $|H_{p_i}| \nmid |H_{pq_i}|$.

The vertex H_q is adjacent to the vertex H_{p^2} , as $|H_{p^2}| \nmid |H_q|$ and $|H_q| \nmid |H_{p^2}|$, the vertex H_q is adjacent to the $p + 1$ vertices H_{pq_i} , as $|H_q| \nmid |H_{pq_i}|$. The vertex H_{p^2} is adjacent to the $p + 1$ vertices H_{pq_i} , as $|H_{p^2}| \nmid |H_{pq_i}|$ and $|H_{pq_i}| \nmid |H_{p^2}|$. Therefore, $\Gamma_{gc}(\mathcal{G})$ forms a tree graph. $\square$

Theorem 2.7. Let $\mathcal{G}$ be a finite cyclic group. Then $\Gamma_{gc}(\mathcal{G}) \cong P_4$ if and only if $|\mathcal{G}| = p^2q$.

Proof. Assume that $\mathcal{G}$ is a cyclic group of order p^2q . Then $\mathcal{G}$ has proper nontrivial subgroups of orders p, p^2, q , and pq .

Let H_p, H_{p^2}, H_q , and H_{pq} be the proper nontrivial subgroups of order p, p^2, q , and pq , respectively.

Thus H_p is adjacent to H_q , since $|H_p| \nmid |H_q|$ and $|H_q| \nmid |H_p|$ and H_p is not adjacent to H_{p^2} , since $|H_p| \nmid |H_{p^2}|$. The vertex H_q is adjacent to H_{p^2} , since $|H_q| \nmid |H_{p^2}|$ and $|H_{p^2}| \nmid |H_q|$ and H_q is not adjacent to H_{pq} , since $|H_q| \nmid |H_{pq}|$.

The vertex H_{p^2} is adjacent to H_{pq} , since $|H_{p^2}| \nmid |H_{pq}|$ and $|H_{pq}| \nmid |H_{p^2}|$ and H_p is not adjacent to H_{pq} , since $|H_p| \nmid |H_{pq}|$. Therefore, $\Gamma_{gc}(\mathcal{G})$ forms a path graph, P_4 of the form $H_p \rightarrow H_q \rightarrow H_{p^2} \rightarrow H_{pq}$.

Conversely, suppose that $\Gamma_{gc}(\mathcal{G}) \cong P_4$. This means the graph $\Gamma_{gc}(\mathcal{G})$ is a path graph with 4 vertices. The group $\mathcal{G}$ must have exactly 4 proper nontrivial subgroups, and these subgroups must form a path in the graph $\Gamma_{gc}(\mathcal{G})$. Since $\mathcal{G}$ is cyclic, the order of the subgroups are the divisors of $|\mathcal{G}|$. If $|\mathcal{G}|$ has more prime factors and larger powers of primes, $\Gamma_{gc}(\mathcal{G})$ would have more than four vertices or a different adjacency structure. Hence, $|\mathcal{G}| = p^2q$ is the only order in which $\Gamma_{gc}(\mathcal{G}) \cong P_4$. $\square$

Remark 2.2. If $\mathcal{G}$ is not a cyclic group, then Theorem 2.7 is not true in general, for example, the alternating group A_4 is of order 12, that is, $|A_4| = 2^2 \cdot 3$. Then $\Gamma_{gc}(A_4)$ is the complete bipartite graph $K_{4,4}$. The group A_4 has three subgroups H_2 each of order 2, four subgroups H_3 each of order 3, and one subgroup H_4 of order 4.

Theorem 2.8. For a finite group $\mathcal{G}$, the girth $(\Gamma_{gc}(\mathcal{G})) \in \{3, 4, \infty\}$.

Proof. Let $\mathcal{G}$ be a group of order $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where p_i are distinct primes and $\alpha_i \geq 1$, for all $1 \leq i \leq k$.

If $k \geq 3$, then any three subgroups of $\mathcal{G}$ of distinct prime orders are mutually adjacent in $\Gamma_{gc}(\mathcal{G})$, and so $\Gamma_{gc}(\mathcal{G})$ contains C_3 , cycle of length 3, as a subgraph.

It follows that $g(\Gamma_{gc}(\mathcal{G}))$ is 3.

If $k \leq 2$, we have to consider the following cases:

Case 1: If $k = 1$, then $|\mathcal{G}| = p_1^{\alpha_1}$, $\mathcal{G}$ is a p -group, by Theorem 2.2, $\Gamma(\mathcal{G})$ is null graph, then $g(\Gamma_{gc}(\mathcal{G}))$ is ∞ .

Case 2: If $k = 2$, then $|\mathcal{G}| = p_1^{\alpha_1} p_2^{\alpha_2}$

(i) If $\alpha_1, \alpha_2 \geq 2$, then $\mathcal{G}$ has subgroups of order $p_1^{\alpha_1}, p_2^{\alpha_2}$, and $p_1 p_2$. Let be H_1, H_2 , and H_3 , respectively.

Then $\Gamma_{gc}(\mathcal{G})$ contains the cycle $H_1 \rightarrow H_2 \rightarrow H_3 \rightarrow H_1$.

So, the $g(\Gamma_{gc}(\mathcal{G}))$ is 3.

(ii) If $\alpha_1 \geq 2$ and $\alpha_2 = 1$, then $|\mathcal{G}| = p_1^{\alpha_1} p_2$.

Case a: If $\mathcal{G} \cong \mathbb{Z}_{p_1^{\alpha_1} p_2}$, then $\Gamma_{gc}(\mathcal{G})$ is a path graph, $g(\Gamma_{gc}(\mathcal{G}))$ is ∞ .

Case b: If $\mathcal{G} \cong \mathbb{Z}_{p_1 p_2} \times \mathbb{Z}_{p_1}$, then $\Gamma_{gc}(\mathcal{G})$ is a tree graph, $g(\Gamma_{gc}(\mathcal{G}))$ is ∞ .

Case c: If $\mathcal{G} \neq \mathbb{Z}_{p_1^{\alpha_1} p_2}$ and $\mathcal{G} \neq \mathbb{Z}_{p_1 p_2} \times \mathbb{Z}_{p_1}$, then the Sylow p_2 -subgroup is not unique. Let H_1 be a Sylow p_2 -subgroup of order p_2 , H_2 be a subgroup of order p_1 ,

H_3 be a subgroup of order p_1^2 and it is not unique. Then $\Gamma_{gc}(\mathcal{G})$ contains the cycle $H_1 \rightarrow H_3^{(1)} \rightarrow H_2 \rightarrow H_3^{(2)} \rightarrow H_1$, so $g(\Gamma_{gc}(\mathcal{G}))$ is 4.

(iii) If $\alpha_1 = \alpha_2 = 1$, then $|\mathcal{G}| = p_1 p_2$.

a) If $p_1 \nmid (p_2 - 1)$, then by Theorem 2.4, $\Gamma_{gc}(\mathcal{G}) = K_2$, $g(\Gamma_{gc}(\mathcal{G}))$ is ∞ .

b) If $p_1 \mid (p_2 - 1)$, then by Theorem 2.4, $\Gamma_{gc}(\mathcal{G}) = K_{1,p_2}$, $g(\Gamma_{gc}(\mathcal{G}))$ is ∞ .

□

Theorem 2.9. For every finite group $\mathcal{G}$, the graph $\Gamma_{gc}(\mathcal{G})$ is not cycle.

Proof. Let $\Gamma_{gc}(\mathcal{G})$ be the cycle $H_1 \rightarrow H_2 \rightarrow \dots \rightarrow H_n \rightarrow H_1$ of length n . $|H_i| \nmid |H_{i+1}|$ and $|H_{i+1}| \nmid |H_i|$ for all $i, 1 \leq i \leq n-1$.

Assume that $|H_1| = p_1$ and $|H_2| = p_2$. Then $|H_3| = r$ or $|H_3| = p_1^{\alpha_1}$. If $|H_3| = r$, then H_1, H_2 and H_3 are adjacent and so $\Gamma_{gc}(\mathcal{G})$ is a complete graph, contradicts Theorem 2.4 (1). If $|H_3| = p_1^{\alpha_1}$, then $|H_3| \nmid |H_4|$ and $|H_4| \nmid |H_3|$ implies $|H_4| = r$ or $|H_4| = p_2^{\alpha_2}$. If $|H_4| = r$, then H_1, H_2 and H_4 are adjacent, which is impossible. If $|H_4| = p_2^{\alpha_2}$, then $|H_1| \nmid |H_4|$ and $|H_4| \nmid |H_1|$ implies H_1 and H_4 are adjacent.

Here, we have $|\mathcal{G}| = p_1^{\alpha_1} p_2^{\alpha_2}$, $\alpha_1, \alpha_2 \geq 1$.

If $\alpha_1 + \alpha_2 \geq 4$, then $\mathcal{G}$ has more than four nontrivial subgroups, which is impossible.

If $\alpha_1 + \alpha_2 \leq 3$, then either $|\mathcal{G}| = p_1^{\alpha_1} p_2$ or $|\mathcal{G}| = p_1 p_2$. In both cases $\Gamma_{gc}(\mathcal{G})$ is not a cycle, by Theorem 2.3(1), Theorem 2.6, and Theorem 2.7. □

Theorem 2.10. For a finite cyclic group $\mathcal{G}$, the graph $\Gamma_{gc}(\mathcal{G})$, is 3-regular if and only if $|\mathcal{G}| = pqr$, where p, q , and r are distinct primes.

Proof. Suppose that $|\mathcal{G}| = pqr$, where p, q , and r are distinct primes. $\mathcal{G}$ has exactly one Sylow p -subgroup of order p , say H_p , one Sylow q -subgroup of order q , say H_q , and one Sylow r -subgroup of order r , say H_r , and $\mathcal{G}$ has exactly one proper subgroup of order pq , say H_{pq} , one proper subgroup of order pr , say H_{pr} , and one proper subgroup of order qr , say H_{qr} .

The subgroup H_p is adjacent to subgroups H_q, H_r , and H_{qr} , not adjacent to subgroups H_{pq} and H_{pr} . Similarly, the subgroup H_q is adjacent to subgroups H_p, H_r , and H_{pr} , not adjacent to subgroups H_{pq} and H_{qr} . So, we verify that the subgroup H_r is adjacent to the subgroups H_p, H_q and H_{pq} but not to one of the subgroups H_{pr} and H_{qr} . Each subgroup is adjacent to exactly 3 other subgroups because the orders of the subgroups are determined by the unique prime factorization of $|\mathcal{G}| = pqr$.

So, $\Gamma_{gc}(\mathcal{G})$, is 3-regular.

Conversely, assume $\Gamma_{gc}(\mathcal{G})$, is 3-regular. This means every vertex in $\Gamma_{gc}(\mathcal{G})$ has exactly three

neighbors. Let H be a subgroup of $\mathcal{G}$. Since $\Gamma_{gc}(\mathcal{G})$ is 3-regular, H has exactly three neighbors say H_1, H_2 , and H_3 , such that $|H| \nmid |H_i|$ and $|H_i| \nmid |H|$, for $i = 1, 2, 3$.

If $|\mathcal{G}|$ has more than three distinct prime factors, say $|\mathcal{G}| = p_1 p_2 p_3 p_4$, then the number of subgroups and their orders would be more complex, potentially leading to more than three neighbors for some subgroups, and then some vertices have more than three vertices connected to them, contradicting the 3-regularity.

If $|\mathcal{G}|$ has less than 3 prime factors, adjacency can not produce a 3-regular graph, since there will not be at least three distinct subgroup orders. Therefore, $|\mathcal{G}|$ must have exactly three distinct prime factors, say p, q , and r . This ensures that the structure of subgroups and their orders align with the 3-regularity condition. $\square$

Theorem 2.11. For a finite group $\mathcal{G}$ of order $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where p_i 's are distinct primes, $k \geq 2$ and $\alpha_i \geq 1$, for all $1 \leq i \leq k$, the graph $\Gamma_{gc}(\mathcal{G})$ has a complete k -partite subgraph.

Proof. Since $|\mathcal{G}| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, then the subgroups of $\mathcal{G}$ are the Sylow p_i -subgroups. Let S_i be the set of all proper nontrivial subgroups of $\mathcal{G}$ whose orders are powers of p_i . That is S_i consists of all Sylow p_i -subgroups.

Let $H \in S_i, |H|$ is a power of p_i and $K \in S_j, |K|$ is a power of p_j , for $i \neq j$. Since $p_i \neq p_j$, neither $|H|$ divides $|K|$ nor $|K|$ divides $|H|$. Thus, H and K are adjacent in the graph $\Gamma_{gc}(\mathcal{G})$. Therefore, every pair of subgroups in different sets S_i and S_j are adjacent. Hence, the sets $S_1, S_2, \dots, S_k$ are subsets of $V(\Gamma_{gc}(\mathcal{G}))$ which form a k -partite subgraph of the graph $\Gamma_{gc}(\mathcal{G})$. $\square$

Theorem 2.12. Let $\mathbb{Z}_n$ be the cyclic group of order $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where p_i 's are distinct primes and $\alpha_i \geq 1$. If H is a proper nontrivial subgroup of $\mathbb{Z}_n$ of order $p_{i_1}^{\alpha_{i_1}} p_{i_2}^{\alpha_{i_2}} \cdots p_{i_r}^{\alpha_{i_r}}$, then $\deg_{\Gamma_{gc}(\mathbb{Z}_n)}(H) = (\tau(n) - 3) - \left(\tau\left(\frac{n}{|H|}\right) + \tau(|H|) - 4 \right)$.

Proof. It is well-known the fact that for every divisor d of n , the group $\mathbb{Z}_n$ has a unique subgroup of order d . Let H be a proper nontrivial subgroup of $\mathbb{Z}_n$ of order $p_{i_1}^{\alpha_{i_1}} p_{i_2}^{\alpha_{i_2}} \cdots p_{i_r}^{\alpha_{i_r}}$. The number of proper nontrivial subgroups of $\mathbb{Z}_n$ is $\tau(n) - 2 = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_k + 1) - 2$. The degree of H in $\Gamma_{gc}(\mathbb{Z}_n)$ is the number of proper nontrivial subgroups of $\mathbb{Z}_n$ which are adjacent to H .

To compute $\deg_{\Gamma_{gc}(\mathbb{Z}_n)}(H)$, consider the subgroup K of $\mathbb{Z}_n$ which is not adjacent to H in $\Gamma_{gc}(\mathbb{Z}_n)$. Then $|H| \nmid |K|$ or $|K| \nmid |H|$. We need to count the number of subgroups K such that $|H| \nmid |K|$ or $|K| \nmid |H|$.

If $|H| \nmid |K|$, then $|K| = m|H|$ where m is a divisor of $\frac{n}{|H|} - 2$. The number of such divisor is $\tau\left(\frac{n}{|H|}\right) - 2$.

If $|K| \nmid |H|$, then $|K|$ must be a divisor of $|H|$. The number of such divisors is $\tau(|H|)$.

Therefore, $\deg_{\Gamma_{gc}(\mathbb{Z}_n)}(H) = (\tau(n) - 3) - \left(\tau\left(\frac{n}{|H|}\right) + \tau(|H|) - 4 \right)$. $\square$

3. THE ENERGY OF $\Gamma_{gc}(D_{2n})$

In this section, we compute the energy of $\Gamma_{gc}(D_{2n})$, where $n \in \{2p, p^k, pq\}$, and p and q are distinct odd primes with $k \geq 1$. The dihedral group D_{2n} has $2n$ elements, that is $D_{2n} = \{1, a, a^2, \dots, a^{n-1}, b, ab, a^2b, \dots, a^{n-1}b\}$.

Theorem 3.1. Let $\Gamma = \Gamma_{gc}(D_{2n})$. If $n = p^k$, where p is an odd prime and $k \geq 1$, then Γ is a bipartite graph.

Proof. Consider the sets $S_j = \{H \mid |H| = 2p^j\}$ for $0 \leq j \leq k-1$, and $S_k = \{K \mid |K| = p^k; 1 \leq i \leq k\}$. One can see that $|S_j| = p^{k-j}$ for $0 \leq j \leq k-1$. Suppose $S = \bigcup_{j=1}^{k-1} S_j$. Then $|S| = p + p^2 + p^3 + \dots + p^k = \frac{p^{k+1}-p}{p-1}$. Thus, $V(\Gamma) = S_k \cup S$ and $S_k \cap S = \emptyset$, where no two vertices in any of them are adjacent.

Let $H_1 \in S$ and $K_1 \in S_k$. Then $|H_1| = 2p^{j_1}$; for some $j_1, 0 \leq j_1 \leq k-1$ and $|K_1| = p^{i_1}$; for some $i_1, 1 \leq i_1 \leq k$. It is clear that H_1 and K_1 are adjacent if $i_1 > j_1$. Therefore, Γ is a bipartite graph. $\square$

Theorem 3.2. Let $\Gamma = \Gamma_{gc}(D_{2n})$. If $n = p^k$, where p is an odd prime, then the characteristic polynomial of Γ is $\mathcal{P}(\Gamma, \lambda) = \lambda^{\frac{p^{k+1}-(k+1)p+k}{p-1}} \left(\sum_{i=0}^k (-1)^i c_i \lambda^{2k-2i} \right)$, where $c_0 = 1$, $c_1 = \frac{p[1-(k+1)p^k+kp^k]}{(p-1)^2}$, $c_r = p^{\frac{r(r+1)}{2}} + 2 \left(p^{\frac{r(r+1)}{2}} \left(\frac{p^{r(k-r)}-p}{p-1} \right) \right) + 2^{k-r} p^{\frac{r(2k-r+1)}{2}}$, for $2 \leq r \leq k-1$, and $c_k = p^{\frac{k(k+1)}{2}}$.

Proof. From Theorem 3.1, Γ is bipartite. The vertex set $V(\Gamma)$ is partitioned into two parts, S_k with k vertices and $S = \bigcup_{j=1}^{k-1} S_j$ with $\frac{p^{k+1}-p}{p-1}$ vertices. Hence, the adjacency matrix of Γ is the $(k+1) \times (k+1)$ block matrix

$$A(\Gamma) = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1k+1} \\ A_{21} & A_{22} & A_{23} & \cdots & A_{2k+1} \\ A_{31} & A_{32} & A_{33} & \cdots & A_{3k+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{k+11} & A_{k+12} & A_{k+13} & \cdots & A_{k+1k+1} \end{bmatrix},$$

where $A_{1j}, j = 2, 3, \dots, k+1$, is $k \times p^{k-j+2}$ matrix whose entries all ones except the rows $j-2$, which are zeros, $A_{j1} = A_{1j}^T$, for $j = 2, 3, \dots, k+1$, and all other submatrices are zero matrices. This can be simplified as

$$A(\Gamma) = \begin{bmatrix} 0_{k \times k} & B_{k \times \left(\frac{p^{k+1}-p}{p-1} \right)} \\ B^T_{\left(\frac{p^{k+1}-p}{p-1} \right) \times k} & 0_{\left(\frac{p^{k+1}-p}{p-1} \right) \times \left(\frac{p^{k+1}-p}{p-1} \right)} \end{bmatrix}, \text{ where } B = \begin{bmatrix} A_{12} & A_{13} & A_{14} & \cdots & A_{1k+1} \end{bmatrix}.$$

The characteristic polynomial of $A(\Gamma)$ of Γ is

$$\mathcal{P}(\Gamma, \lambda) = |\lambda I - A(\Gamma)| = \lambda^{\frac{p^{k+1}-p}{p-1}-k} \det(\lambda^2 I_k - BB^t), \text{ where}$$

$$BB^T = \begin{bmatrix} p^k & p^k & p^k & \cdots & p^k \\ p^k & p^k + p^{k-1} & p^k + p^{k-1} & \cdots & p^k + p^{k-1} \\ p^k & p^k + p^{k-1} & p^k + p^{k-1} + p^{k-2} & \cdots & p^k + p^{k-1} + p^{k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p^k & p^k + p^{k-1} & p^k + p^{k-1} + p^{k-2} & \cdots & p^k + p^{k-1} + p^{k-2} + \cdots + p \end{bmatrix}$$

$$\text{and } (BB^T)_{ij} = \frac{p^{k+1} - p^{k-\min(i,j)+1}}{p-1}.$$

$$\text{Therefore, } \mathcal{P}(\Gamma, \lambda) = \lambda^{\frac{p^{k+1} - (k+1)p + k}{p-1}} \left(\sum_{i=0}^k (-1)^i c_i \lambda^{2k-2i} \right)$$

$$\text{where } c_0 = 1, c_1 = \text{trace}(BB^T) = \frac{p[1 - (k+1)p^k + kp^k]}{(p-1)^2},$$

$$c_r = p^{\frac{r(r+1)}{2}} + 2 \left(p^{\frac{r(r+1)}{2}} \left(\frac{p^{r(k-r)} - p}{p-1} \right) \right) + 2^{k-r} p^{\frac{r(2k-r+1)}{2}}, \text{ for } 2 \leq r \leq k-1, \text{ and}$$

$$c_k = \det(BB^T) = p^{\frac{k(k+1)}{2}}.$$

□

Theorem 3.3. Let $\Gamma = \Gamma_{gc}(D_{2p})$. If p is an odd prime, then the energy of Γ is $\mathcal{E}(\Gamma) = 2\sqrt{p}$.

Proof. From Theorem 3.2, for $k = 1$, the characteristic polynomial is of the form $\mathcal{P}(\Gamma, \lambda) = \lambda^{p-1}(\lambda^2 - p)$, and the roots of $\mathcal{P}(\Gamma, \lambda)$, are the eigenvalues of Γ , and these eigenvalues are $\lambda = 0$ with multiplicity $(p-1)$, and $\lambda = \pm\sqrt{p}$.

$$\text{Hence, } \mathcal{E}(\Gamma) = \sum_{i=1}^{p+1} |\lambda_i| = \sum_{i=1}^{p-1} |\lambda_i| + |\sqrt{p}| + |-\sqrt{p}| = 2\sqrt{p}.$$

□

Theorem 3.4. Let $\Gamma = \Gamma_{gc}(D_{2p^2})$, where p is an odd prime. Then the energy of Γ is

$$\varepsilon(\Gamma) = 2\sqrt{2p^2 + p + 2\sqrt{p^3}}.$$

Proof. From Theorem 3.2, for $k = 2$, the characteristic polynomial is of the form $\mathcal{P}(\Gamma, \lambda) = \lambda^{p^2+p-2}(\lambda^4 - (2p^2 + p)\lambda^2 + p^3)$, and the roots of $\mathcal{P}(\Gamma, \lambda)$, are the eigenvalues of Γ , and these eigenvalues are $\lambda = 0$ with multiplicity $(p^2 + p - 2)$,

$$\text{and } \lambda_1 = \sqrt{\frac{p((2p+1) + \sqrt{4p^2+1})}{2}}, \lambda_2 = -\sqrt{\frac{p((2p+1) - \sqrt{4p^2+1})}{2}},$$

$$\lambda_3 = \sqrt{\frac{p((2p+1) - \sqrt{4p^2+1})}{2}}, \lambda_4 = -\sqrt{\frac{p((2p+1) + \sqrt{4p^2+1})}{2}}.$$

$$\text{Hence, } \mathcal{E}(\Gamma) = \sum_{i=1}^{p^2+p+2} |\lambda_i| = \sum_{i=1}^{p^2+p-2} |\lambda_i| + \left| \sqrt{\frac{p((2p+1) + \sqrt{4p^2+1})}{2}} \right| + \left| -\sqrt{\frac{p((2p+1) + \sqrt{4p^2+1})}{2}} \right| + \left| \sqrt{\frac{p((2p+1) - \sqrt{4p^2+1})}{2}} \right| + \left| -\sqrt{\frac{p((2p+1) - \sqrt{4p^2+1})}{2}} \right| = 2 \left(\sqrt{\frac{p((2p+1) + \sqrt{4p^2+1})}{2}} + \sqrt{\frac{p((2p+1) - \sqrt{4p^2+1})}{2}} \right).$$

$$\text{Since } (\sqrt{a} + \sqrt{b})^2 = a + b + 2\sqrt{ab}, \text{ then } \left(\sqrt{\frac{p((2p+1) + \sqrt{4p^2+1})}{2}} + \sqrt{\frac{p((2p+1) - \sqrt{4p^2+1})}{2}} \right)^2 =$$

$$\frac{p((2p+1) + \sqrt{4p^2+1})}{2} + \frac{p((2p+1) - \sqrt{4p^2+1})}{2} + 2\sqrt{\left(\frac{p((2p+1) + \sqrt{4p^2+1})}{2} \right) \left(\frac{p((2p+1) - \sqrt{4p^2+1})}{2} \right)}.$$

$$\text{Thus, } \frac{p((2p+1) + \sqrt{4p^2+1})}{2} + \frac{p((2p+1) - \sqrt{4p^2+1})}{2} = 2p^2 + p \text{ and}$$

$$\left(\frac{p((2p+1)+\sqrt{4p^2+1})}{2} \right) \left(\frac{p((2p+1)-\sqrt{4p^2+1})}{2} \right) = p^2.$$

Therefore, $\mathcal{E}(\Gamma) = 2\sqrt{2p^2 + p + 2\sqrt{p^3}}$. $\square$

Theorem 3.5. Let $\Gamma = \Gamma_{gc}(D_{2n})$. If $n = 2p$, where p is an odd prime, then Γ is a bipartite graph.

Proof. Consider the following partition of the vertex set $V(\Gamma)$

$S_1 = \{H_1\} \cup \{H_{1+i}; i = 1, 2, 3\}$, where $|H_1| = p$ and $|H_{1+i}| = 2p$ and

$S_2 = \{H_{4+j}; j = 1, \dots, 2p+1\} \cup \{H_{5+2p+l}; l = 1, \dots, p\}$, where $|H_{4+j}| = 2$, and

$|H_{5+2p+l}| = 4$. So, the total number of vertices is $|S_1| + |S_2| = 4 + (3p+1) = 3p+5$. Clearly, $V(\Gamma) = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$.

The subgroup H_1 in S_1 is adjacent to all subgroups $H_{4+j}, j = 1, \dots, 2p+1$ and $H_{5+2p+l}, l = 1, \dots, p$ in S_2 , since neither order divides the other, the subgroups $H_{1+i}, i = 1, 2, 3$ in S_1 are adjacent to all subgroups $H_{5+2p+l}, l = 1, \dots, p$ in S_2 , since neither order divides the other. Also, any two vertices in $S_i, i = 1, 2$ are nonadjacent, since their orders divide each other. Therefore, Γ is bipartite. $\square$

Theorem 3.6. Let $\Gamma = \Gamma_{gc}(D_{2n})$. If $n = 2p$, where p is an odd prime, then the characteristic polynomial of Γ is $\mathcal{P}(\Gamma, \lambda) = \lambda^{3p+1} (\lambda^4 - (6p+1)\lambda^2 + (6p^2+3p))$.

Proof. From Theorem 3.5, Γ is bipartite with partition S_1, S_2 , where S_1 has 4 vertices H_1 and $H_{1+i}, i = 1, 2, 3$, and S_2 has $3p+1$ vertices $H_{4+j}, j = 1, \dots, 2p+1$ and $H_{5+2p+l}, l = 1, \dots, p$. Hence, the

adjacency matrix of Γ is a $(3p+5)$ by $(3p+5)$ matrix given as $A(\Gamma) = \begin{bmatrix} 0_{4 \times 4} & B_{4 \times (3p+1)} \\ B^T_{(3p+1) \times 4} & 0_{(3p+1) \times (3p+1)} \end{bmatrix}$,

where $B = \begin{bmatrix} A_{12} & A_{13} \end{bmatrix}$, all entries of the first row of the submatrix A_{12} are 1, all other entries are zero, and A_{12} is $4 \times (2p+1)$ and A_{13} is $4 \times p$, all entries are 1. The characteristic polynomial of

$$A(\Gamma) \text{ is } \mathcal{P}(\Gamma, \lambda) = |\lambda I - A(\Gamma)| = \begin{vmatrix} \lambda I_4 & -B \\ -B^T & \lambda I_{3p+1} \end{vmatrix}.$$

Thus,

$$\begin{aligned} \mathcal{P}(\Gamma, \lambda) &= |\lambda I_{3p+1}| \left| \lambda I_4 - (-B) (\lambda I_{3p+1})^{-1} (-B^T) \right| \\ &= \lambda^{3p+1} \left| \lambda I_4 - B (\lambda I_{3p+1})^{-1} (B^T) \right| \\ &= \lambda^{3p+1} \left| \lambda I_4 - B (\lambda I_{3p+1})^{-1} (B^T) \right| \\ &= \lambda^{3p+1} \left| \lambda I_4 - \frac{1}{\lambda} B (\lambda I_{3p+1})^{-1} (B^T) \right| \\ &= \lambda^{3p+1} \left| \lambda I_4 - \frac{1}{\lambda} B (\lambda I_{3p+1})^{-1} (B^T) \right| \\ &= \frac{\lambda^{3p+1}}{\lambda^4} |\lambda^2 I_4 - BB^T| \\ &= \lambda^{3p-3} |\lambda^2 I_4 - BB^T|, \end{aligned}$$

$$\begin{aligned}
\text{where } BB^T &= \begin{bmatrix} 3p+1 & p & p & p \\ p & p & p & p \\ p & p & p & p \\ p & p & p & p \end{bmatrix} \\
\mathcal{P}(\Gamma, \lambda) &= \lambda^{3p-3} \begin{vmatrix} \lambda^2 - (3p+1) & -p & -p & -p \\ -p & \lambda^2 - p & -p & -p \\ -p & -p & \lambda^2 - p & -p \\ -p & -p & -p & \lambda^2 - p \end{vmatrix} \\
&= \lambda^{3p-3} (\lambda^4 - (6p+1)\lambda^2 + (6p^2 + 3p)) \\
&= \lambda^{3p+1} (\lambda^4 - (6p+1)\lambda^2 + (6p^2 + 3p)).
\end{aligned}$$

□

Theorem 3.7. Let $\Gamma = \Gamma_{gc}(D_{2n})$. If $n = 2p$, where p is an odd prime, then the energy of Γ is $\mathcal{E}(\Gamma) = 2 \left(\sqrt{(6p+1) + 2\sqrt{6p^2 + 3p}} \right)$.

Proof. From Theorem 3.6, the roots of the characteristic polynomial

$\mathcal{P}(\Gamma, \lambda) = \lambda^{3p+1} (\lambda^4 - (6p+1)\lambda^2 + (6p^2 + 3p))$, are the eigenvalues of Γ , which are $\lambda = 0$ with multiplicity $(3p+1)$, and

$$\lambda = \pm \sqrt{\frac{1}{2}(6p+1) \pm \frac{1}{2}\sqrt{12p^2 + 1}}.$$

$$\text{Hence, } \mathcal{E}(\Gamma) = \sum_{i=1}^{3p+5} |\lambda_i| = 2 \left(\sqrt{\frac{1}{2}(6p+1) + \frac{1}{2}\sqrt{12p^2 + 1}} + \sqrt{\frac{1}{2}(6p+1) - \frac{1}{2}\sqrt{12p^2 + 1}} \right).$$

$$\begin{aligned}
\text{Since } \left(\sqrt{\frac{1}{2}(6p+1) + \frac{1}{2}\sqrt{12p^2 + 1}} + \sqrt{\frac{1}{2}(6p+1) - \frac{1}{2}\sqrt{12p^2 + 1}} \right)^2 &= \frac{1}{2}(6p+1) + \frac{1}{2}\sqrt{12p^2 + 1} + \\
\frac{1}{2}(6p+1) - \frac{1}{2}\sqrt{12p^2 + 1} + 2\sqrt{\left(\frac{1}{2}(6p+1) + \frac{1}{2}\sqrt{12p^2 + 1}\right)\left(\frac{1}{2}(6p+1) - \frac{1}{2}\sqrt{12p^2 + 1}\right)} &= 12p + 2 + 2\sqrt{6p^2 + 3p}.
\end{aligned}$$

Thus, $\left(\sqrt{\frac{1}{2}(6p+1) + \frac{1}{2}\sqrt{12p^2 + 1}} + \sqrt{\frac{1}{2}(6p+1) - \frac{1}{2}\sqrt{12p^2 + 1}} \right)^2 = 6p + 1 + 2\sqrt{6p^2 + 3p}$, by taking square root of both sides, we obtain

$$\sqrt{\frac{1}{2}(6p+1) + \frac{1}{2}\sqrt{12p^2 + 1}} + \sqrt{\frac{1}{2}(6p+1) - \frac{1}{2}\sqrt{12p^2 + 1}} = \sqrt{(6p+1) + 2\sqrt{6p^2 + 3p}}.$$

$$\text{Therefore, } \mathcal{E}(\Gamma) = 2 \left(\sqrt{(6p+1) + 2\sqrt{6p^2 + 3p}} \right).$$

□

Theorem 3.8. Let $\Gamma = \Gamma_{gc}(D_{2n})$. If $n = pq$, where p and q are distinct odd primes, then Γ is tripartite.

Proof. Consider the following partitions of the vertex set $V(\Gamma)$: $S_1 = \{H_1, H_2\}$, where $|H_1| = p$ and $|H_2| = pq$, $S_2 = \{H_3\} \cup \{H_{3+i}; i = 1, \dots, p\}$, where $|H_3| = q$ and $|H_{3+i}| = 2q$ and $S_3 =$

$\{H_{3+p+j}; j = 1, \dots, pq\} \cup \{H_{3+p+pq+l}; l = 1, \dots, q\}$, where $|H_{3+p+j}| = 2$, and $|H_{3+p+pq+l}| = 2p$. Hence, the total number of vertices is $|S_1| + |S_2| + |S_3| = 2 + (1+p) + (pq+q) = 3+p+q+pq$. Then S_1, S_2 and S_3 form partitions of $V(\Gamma)$, that is $V(\Gamma) = S_1 \cup S_2 \cup S_3$ and $S_i \cap S_j = \emptyset$, if $i \neq j$. The subgroup H_1 in S_1 is adjacent to the subgroups H_3 and $H_{3+i}, i = 1, \dots, p$ in S_2 and the subgroups $H_{3+p+j}, j = 1, \dots, pq$ in S_3 , and the subgroup H_2 in S_1 is adjacent to the subgroups $H_{3+i}, i = 1, \dots, p$ in S_2 and the subgroups $H_{3+p+j}, j = 1, \dots, pq$ and $H_{3+p+pq+l}, l = 1, \dots, q$ in S_3 , the subgroup H_3 in S_2 is adjacent to the subgroups of $H_{3+p+j}, j = 1, \dots, pq$ and $H_{3+p+pq+l}, l = 1, \dots, q$ in S_3 , and the subgroups $H_{3+i}, i = 1, \dots, p$ in S_2 is adjacent to the subgroups $H_{3+p+pq+l}, l = 1, \dots, q$ in S_3 . Also, any two vertices in $S_i, i = 1, 2, 3$ are nonadjacent, since their orders divide each other. Therefore, Γ is 3-partite graph. $\square$

Theorem 3.9. Let $\Gamma = \Gamma_{gc}(D_{2n})$. If $n = pq$, where p and q are distinct odd primes, then the characteristic polynomial of Γ is $\mathcal{P}(\Gamma, \lambda) = \lambda^d (\lambda^4 - A\lambda^2 - B\lambda + C)$, where $d = 2n - \emptyset(n)$, $A = 4pq + 2p + 2q + 1$, $B = 4pq$, and $C = 3pq(pq + 1) + (2pq + 1)(p + q)$.

Proof. From Theorem 3.8, Γ is 3-partite with partitions S_1, S_2 , and S_3 where S_1 has 2 vertices H_1 and H_2 , S_2 has $p + 1$ vertices H_3 and $H_{3+i}, i = 1, \dots, p$, and S_3 has $pq + q$ vertices $H_{3+p+j}, j = 1, \dots, pq$ and $H_{3+p+pq+l}, l = 1, \dots, q$. Hence, the adjacency matrix of Γ is

$$A(\Gamma) = \begin{bmatrix} 0_{2 \times 2} & B_{2 \times (1+p)} & C_{2 \times (pq+q)} \\ B^T & 0_{(1+p) \times (1+p)} & D_{(1+p) \times (pq+q)} \\ C^T & D^T & 0_{(pq+q) \times (pq+q)} \end{bmatrix}, \text{ where } B = \begin{bmatrix} J_{1 \times 1} & J_{1 \times p} \\ 0_{1 \times 1} & J_{1 \times p} \end{bmatrix},$$

$$C = \begin{bmatrix} J_{1 \times pq} & 0_{1 \times p} \\ J_{1 \times pq} & J_{1 \times p} \end{bmatrix}, \text{ and } D = \begin{bmatrix} J_{1 \times pq} & 0_{1 \times p} \\ 0_{p \times pq} & J_{p \times p} \end{bmatrix}. \text{ Here } J \text{ denotes all ones matrix. The characteristic}$$

polynomial of Γ is $\mathcal{P}(\Gamma, \lambda) = |\lambda I - A(\Gamma)| = \lambda^d (\lambda^4 - A\lambda^2 - B\lambda + C)$, where $d = 2n - \emptyset(n)$, $A = 4pq + 2p + 2q + 1$, $B = 4pq$, and $C = 3pq(pq + 1) + (2pq + 1)(p + q)$. $\square$

4. THE ENERGY OF $\Gamma_{gc}(\mathbb{Z}_n)$

In this section we compute the energy of $\Gamma_{gc}(\mathbb{Z}_n)$, where $n \in \{pqr, p^kq, p^2q^2\}$ and p, q , and r are distinct primes.

Theorem 4.1. Let $\Gamma = \Gamma_{gc}(\mathbb{Z}_{pqr})$, where p, q , and r are distinct primes. Then the characteristic polynomial of Γ is $\mathcal{P}(\Gamma, \lambda) = \lambda^2 (\lambda^4 - 9\lambda^2 - 4\lambda + 12)$.

Proof. The cyclic group $\mathbb{Z}_{pqr}$ has the following proper nontrivial subgroups, $H_p, H_q, H_r, H_{pq}, H_{pr}$, and H_{qr} of orders p, q, r, pq, pr , and qr , respectively. Then, the subgroup of order p is adjacent to three subgroups of order q, r , and qr , and not adjacent to the subgroups of orders pq and pr , the subgroup of order q is adjacent to three subgroups of order p, r and pr , and not adjacent to the subgroups of order pq and qr . Similarly, subgroup of order r is adjacent to three subgroups of order p, q and pq , not adjacent to the subgroups of order pr and qr . Hence, the adjacency matrix of Γ is a 6×6 matrix as given

$$A(\Gamma) = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

The characteristic polynomial of Γ is $\mathcal{P}(\Gamma, \lambda) = \lambda^2(\lambda^4 - 9\lambda^2 - 4\lambda + 12)$.

□

Theorem 4.2. Let $\Gamma = \Gamma_{gc}(\mathbb{Z}_{pqr})$, where p, q , and r , are distinct primes. Then the energy of Γ is $\mathcal{E}(\Gamma) = 8$.

Proof. From Theorem 4.1, the roots of the characteristic polynomial $\mathcal{P}(\Gamma, \lambda) = \lambda^2(\lambda^4 - 9\lambda^2 - 4\lambda + 12)$ are the eigenvalues of Γ , which are $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 1, \lambda_4 = 3, \lambda_5 = -2, \lambda_6 = -2$. So $\mathcal{E}(\Gamma) = 8$. □

Theorem 4.3. Let $\Gamma = \Gamma_{gc}(\mathbb{Z}_{p^{k_1}q^{k_2}})$, where p and q are distinct primes and $k_1, k_2 \geq 1$. Then Γ is $(k_2 + 1)$ -partite.

Proof. Consider the sets

$$S_i = \{H \mid |H| = p^j q^i; 0 \leq j \leq k_1 \text{ and } 0 \leq i \leq k_2 - 1\} \text{ and}$$

$$S_{k_2} = \{K \mid |K| = p^s q^{k_2}; 0 \leq s \leq k_1 - 1\}.$$

Then $V(\Gamma) = \cup_{i=1}^{k_2} S_i$ and $\cap_{i=1}^{k_2} S_i = \emptyset$. Within each $S_i, 0 \leq i \leq k_2$ any two subgroups have orders $p^{j_1} q^i$ and $p^{j_2} q^i$ are nonadjacent, since their orders divide each other. Let $H_0 \in S_{i_0}$ and $H_1 \in S_{i_1}$, where $i_0 \neq i_1$ and $|H_0| = p^{j_1} q^{i_0}$ for some j_1 and $|H_1| = p^{j_2} q^{i_1}$ for some j_2 . Then, H_0 is adjacent to H_1 if and only if $i_0 < i_1$ and $j_1 > j_2$. Therefore, Γ is $(k_2 + 1)$ -partite. □

Theorem 4.4. Let $\Gamma = \Gamma_{gc}(\mathbb{Z}_{p^k q})$, where p and q are distinct primes and $k \geq 1$. Then $\lambda = 0$ is not an eigenvalue of Γ .

Proof. The adjacency matrix $A(\Gamma)$, where p and q are distinct primes and $k \in \mathbb{Z}^+$ is of the form

$$A(\Gamma) = \begin{bmatrix} 0_k & B_k \\ B_k^T & 0_k \end{bmatrix}, \text{ where } 0_k \text{ is the } k\text{-by-}k \text{ zero matrix, } B_k \text{ is the } k\text{-by-}k \text{ lower triangular matrix}$$

with all entries ones, and B_k^T is the transpose of B_k . Suppose $\lambda = 0$ is an eigenvalue of $A(\Gamma)$. Then

there exists a nonzero eigenvector $v = \begin{bmatrix} x \\ y \end{bmatrix}$ such that $Av = 0$. Thus $\begin{bmatrix} 0_k & B_k \\ B_k^T & 0_k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$, and so

$B_k y = 0$, which implies $y = 0$ and $B_k^T x = 0$ gives $x = 0$. Thus, $v = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, a contradiction. Therefore,

$\lambda = 0$ is not eigenvalue of Γ . □

Theorem 4.5. Let $\Gamma = \Gamma_{gc}(\mathbb{Z}_{p^kq})$, where p and q are distinct primes and $k \geq 1$. Then the characteristic polynomial of Γ is $\mathcal{P}(\Gamma, \lambda) = \sum_{j=0}^k (-1)^j \binom{k+j}{k-j} \lambda^{2(k-j)}$.

Proof. Consider the sets $S_1 = \{H || H| = p^i; 1 \leq i \leq k\}$ and $S_2 = \{K || K| = p^s q; 0 \leq s < k\}$. Then $V(\Gamma) = \bigcup_{j=1}^2 S_j$. From Theorem 4.3, Γ is bipartite and the adjacency matrix of Γ is $A(\Gamma) = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$, where 0 is the k -by- k zero matrix, B is the k -by- k lower triangular matrix with all entry's ones, and B^T is the transpose of B . The characteristic polynomial of Γ is

$$\mathcal{P}(\Gamma, \lambda) = |A - \lambda I_{2k}| = \begin{vmatrix} -\lambda I_k & B \\ B^T & -\lambda I_k \end{vmatrix}, \quad \text{where } I_k \text{ is } k\text{-by-}k \text{ identity matrix.}$$

$$\begin{aligned} \text{Then } \mathcal{P}(\Gamma, \lambda) &= \begin{vmatrix} -\lambda I_k & B \\ B^T & -\lambda I_k \end{vmatrix} = \det(-\lambda I_k) \det\left((-\lambda I_k) - B^T (-\lambda I_k)^{-1} B\right) \\ &= \det(-\lambda I_k) \det\left((-\lambda I_k) + \frac{1}{\lambda} B^T I_k B\right) \\ &= \det(-\lambda I_k) \det\left(\frac{1}{\lambda} I_k\right) \det\left((-\lambda^2 I_k) + B^T B\right) \\ &= (-\lambda)^k \frac{1}{\lambda^k} \det\left((-\lambda^2 I_k) + B^T B\right) \\ &= (-1)^k \det\left((-\lambda^2 I_k) + B^T B\right) \\ &= (-1)^k \det\left(B^T B - \lambda^2 I_k\right). \end{aligned}$$

Since B is the lower triangular matrix of all ones and B^T is the transpose of B , then the entry (i, j) of the matrix $B^T B$ is of the form $k - \max(i, j) + 1$.

Therefor, $\mathcal{P}(\Gamma, \lambda) = \sum_{j=0}^k (-1)^j \binom{k+j}{k-j} \lambda^{2(k-j)}$.

□

Theorem 4.6. Let $\Gamma = \Gamma_{gc}(\mathbb{Z}_{p^kq})$, where p and q are distinct primes and $k \geq 1$. Then $\mathcal{E}(\Gamma) = 4 \sum_{m=1}^k \sin\left(\frac{(2m-1)\pi}{2(2k+1)}\right)$, for $m = 1, 2, \dots, k$.

Proof. Since Γ is bipartite with the adjacency matrix $A(\Gamma) = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$, by Theorem 4.5, then the eigenvalues of the matrix $B^T B$ are of the form $\lambda_m = 4 \sin^2\left(\frac{(2m-1)\pi}{2(2k+1)}\right)$, for $m = 1, 2, \dots, k$ and the eigenvalues μ_m of the adjacency matrix $A(\Gamma)$ is the square root of the eigenvalues λ_m of $B^T B$,

$$\begin{aligned} \mu_m &= \pm \sqrt{\lambda_m} = \pm \sqrt{4 \sin^2\left(\frac{(2m-1)\pi}{2(2k+1)}\right)} \\ &= \pm 2 \sin\left(\frac{(2m-1)\pi}{2(2k+1)}\right). \end{aligned}$$

The energy of Γ is $\mathcal{E}(\Gamma) = \sum_{m=1}^{2k} |\mu_m| = 4 \sum_{m=1}^k \left(\sin\left(\frac{(2m-1)\pi}{2(2k+1)}\right)\right)$.

□

Theorem 4.7. Let $\Gamma = \Gamma(\mathbb{Z}_{p^2q})$, where p and q are distinct primes. Then $\varepsilon(\Gamma(\mathbb{Z}_{p^2q})) = 2\sqrt{5}$.

Proof. From Theorem 4.5, the characteristic polynomial of Γ is $\mathcal{P}(\Gamma, \lambda) = (\lambda^4 - 3\lambda^2 + 1)$ and from Theorem 4.6 the roots of $\mathcal{P}(\Gamma, \lambda)$ are the eigenvalues of Γ , which are

$$\lambda_1 = \sqrt{\frac{1}{2}(3 + \sqrt{5})}, \lambda_2 = -\sqrt{\frac{1}{2}(3 + \sqrt{5})},$$

$$\lambda_3 = \sqrt{\frac{1}{2}(3 - \sqrt{5})}, \lambda_4 = -\sqrt{\frac{1}{2}(3 - \sqrt{5})}.$$

Thus, $\mathcal{E}(\Gamma) = 2\left(\sqrt{\frac{1}{2}(3 + \sqrt{5})} + \sqrt{\frac{1}{2}(3 - \sqrt{5})}\right)$. Since $(\sqrt{a} + \sqrt{b})^2 = a + b + 2\sqrt{ab}$, then $\left(\sqrt{\frac{1}{2}(3 + \sqrt{5})} + \sqrt{\frac{1}{2}(3 - \sqrt{5})}\right)^2 = \frac{1}{2}(3 + \sqrt{5}) + \frac{1}{2}(3 - \sqrt{5}) + 2\sqrt{\frac{1}{4}(9 - 5)} = 5$, and so $\sqrt{\frac{1}{2}(3 + \sqrt{5})} + \sqrt{\frac{1}{2}(3 - \sqrt{5})} = \sqrt{5}$. Therefore, $\mathcal{E}(\Gamma) = 2\sqrt{5}$. $\square$

Theorem 4.8. Let $\Gamma = \Gamma_{gc}(\mathbb{Z}_{p^2q^2})$, where p and q are distinct primes. Then the characteristic polynomial of Γ is $\mathcal{P}(\Gamma, \lambda) = \lambda^7 - 9\lambda^5 - 2\lambda^4 + 15\lambda^3 - 7\lambda + 2$.

Proof. The cyclic group $\mathbb{Z}_{p^2q^2}$ has the following proper nontrivial subgroups, $H_p, H_q, H_{p^2}, H_{pq}, H_{q^2}, H_{p^2q}$ and H_{pq^2} of orders p, q, p^2, pq, q^2, p^2q and pq^2 , respectively. Then, the subgroup of order p is adjacent to the subgroups of order q and q^2 , the subgroup of order q is adjacent to the subgroups of order p and p^2 and not adjacent to the other subgroups. Similarly, the subgroup of order p^2 is adjacent to subgroups of orders pq, q^2 and pq^2 , the subgroup of order q^2 is adjacent to subgroups of order pq , and qp^2 .

Hence, the adjacency matrix of Γ is a 7×7 matrix, which is

$$A(\Gamma) = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The characteristic polynomial of Γ is $\mathcal{P}(\Gamma, \lambda) = \lambda^7 - 9\lambda^5 - 2\lambda^4 + 15\lambda^3 - 7\lambda + 2$.

$\square$

Theorem 4.9. Let $\Gamma = \Gamma_{gc}(\mathbb{Z}_{p^2q^2})$, where p and q are distinct primes. Then $\varepsilon(\Gamma) \leq 4 + 2\sqrt{2} + 2\sqrt{13}$.

Proof. From Theorem 4.8, the roots of the characteristic polynomial $\mathcal{P}(\Gamma, \lambda) = \lambda^7 - 9\lambda^5 - 2\lambda^4 + 15\lambda^3 - 7\lambda + 2$ are the eigenvalues of Γ , which are $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = -\sqrt{2} - 1, \lambda_4 = -1 + \sqrt{2}$,

$$\lambda_5 = \frac{2}{3} + \frac{13}{9\sqrt[3]{\frac{8}{27} + \frac{\sqrt{237}}{9}i}} + \sqrt[3]{\frac{8}{27} + \frac{\sqrt{237}}{9}i}$$

$$\lambda_6 = \frac{2}{3} + \frac{13}{9\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\sqrt[3]{\frac{8}{27} + \frac{\sqrt{237}}{9}i}} + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\sqrt[3]{\frac{8}{27} + \frac{\sqrt{237}}{9}i}$$

$$\lambda_7 = \frac{2}{3} + \frac{13}{9\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\sqrt[3]{\frac{8}{27} + \frac{\sqrt{237}}{9}i}} + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\sqrt[3]{\frac{8}{27} + \frac{\sqrt{237}}{9}i}$$

$$\begin{aligned}\text{Therefore, } \mathcal{E}(\Gamma) &\leq 1 + 1 + \sqrt{2} + 1 + \sqrt{2} - 1 + 3\left(\frac{2+2\sqrt{13}}{3}\right) \\ &= 4 + 2\sqrt{2} + 2\sqrt{13}.\end{aligned}$$

□

CONCLUSION

In this study, we characterized the structural and spectral properties of the generalized coprime graph $\Gamma_{gc}(\mathcal{G})$. we established sharp bounds its diameter and girth, proving that the typically exhibits a bipartite or star like structure depending on the subgroup lattice. By deriving the characteristic polynomials, we successfully computed the exact energy of generalized coprime graph of dihedral group D_{2n} , for $n \in \{2p, p^k, pq\}$, and the cyclic group $\mathbb{Z}_n$, for $n \in \{pqr, p^kq, p^2q^2\}$. Our finding demonstrates a direct correlation between the prime factorization of the group order and the spectral energy of the graph.

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