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On σ-Twisted Functional Identities and Elementary Operators Over Division Rings

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Abstract. Let $\mathcal{D}$ be a division ring with centre $\mathcal{Z}(\mathcal{D})$ and σ be an automorphism of $\mathcal{D}$. For additive maps $f : \mathcal{D} \to \mathcal{D}$, we study the functional identity $\sigma(x)G(x)f(x) = H(x)$, where $G(X)$ and $H(X)$ are non-zero generalized polynomials in the variable X over $\mathcal{D}$. By applying the inverse automorphism, we reduce the problem to the classical case and prove that either $[\mathcal{D} : \mathcal{Z}(\mathcal{D})] < \infty$ or f is a σ-elementary operator, that is, a finite sum of terms $a\,\sigma(x)\,b$ with $a, b \in \mathcal{D}$. We also consider additive solutions of $f(x) = \sigma(x)^n g(x^{-1})$ for $n > 2$ and obtain the corresponding forms under the stated hypotheses.

1. Introduction

Throughout the paper $\mathcal{D}$ is a division ring with centre $\mathcal{Z} = \mathcal{Z}(\mathcal{D})$ and σ is a fixed automorphism of $\mathcal{D}$. A *generalized polynomial* in one indeterminate X over $\mathcal{D}$ is an element of the free product $\mathcal{D} *_{\mathcal{Z}} \mathcal{Z}[X]$, denoted by $\mathcal{D}\{X\}$; more generally $\mathcal{D}\{X_1, \ldots, X_m\}$ is the free product of $\mathcal{D}$ and the free $\mathcal{Z}$-algebra $\mathcal{Z}\langle X_1, \ldots, X_m\rangle$. A non-zero $P \in \mathcal{D}\{X_1, \ldots, X_m\}$ is a *generalized polynomial identity* (GPI) for $\mathcal{D}$ if $P(x_1, \ldots, x_m) = 0$ for all $x_i \in \mathcal{D}$. Martindale's theorem [2] states that a division ring admitting a non-trivial GPI is necessarily finite-dimensional over its centre. Standard background on noncommutative rings, division rings, generalized polynomial identities, and functional identities can be found in [3,4,6,7]. Related inverse-type rational functional identities on division rings were studied in [5].

An additive map $f : \mathcal{D} \to \mathcal{D}$ is called *elementary* if there exist finitely many $a_i, b_i \in \mathcal{D}$ such that $f(x) = \sum_i a_i x b_i$ for all $x \in \mathcal{D}$. This notion plays a central role in the theory of functional identities. In a recent landmark paper, Lee and Lin [1] proved the following: if f is additive and satisfies $G(x)f(x) = H(x)$ for non-zero $G, H \in \mathcal{D}\{X\}$, then either $[\mathcal{D} : \mathcal{Z}] < \infty$ or f is elementary. As an

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application they completely solved the identity $f(x) = x^n g(x^{-1})$ on non-commutative division rings of characteristic $\neq 2$.

In the present work we incorporate an automorphism σ into the picture. A natural σ-twisted analogue of an elementary operator is a map of the form

$$f(x) = \sum_j a_j \, \sigma(x) \, b_j \text{ for all } a_j, b_j \in \mathcal{D}. \tag{1.1}$$

When $\sigma = \mathrm{id}_{\mathcal{D}}$ this reduces precisely to an ordinary elementary operator, so (1.1) is the genuine twisted generalization. Our first main result shows that this class appears unavoidably in functional identities containing a factor $\sigma(x)$.

Theorem 1.1. *Let $\mathcal{D}$ be a division ring, σ an automorphism of $\mathcal{D}$, and $f : \mathcal{D} \to \mathcal{D}$ an additive map. Assume that there exist generalized polynomials $G(X), H(X) \in \mathcal{D}\{X\}$ with $G(X) \not\equiv 0$ such that*

$$\sigma(x) \, G(x) \, f(x) = H(x) \text{ for all } x \in \mathcal{D}. \tag{1.2}$$

Then either $[\mathcal{D} : \mathcal{Z}] < \infty$ or f is σ-elementary.

This theorem extends [1, Theorem 2.3], which is the special case $\sigma = \mathrm{id}_{\mathcal{D}}$. The proof is remarkably as follows: applying σ^{-1} to (1.2) turns it into an identity of the type treated by Lee and Lin, and the conclusion follows immediately. Thus the whole multilinearization machinery developed in [1] is inherited without change; the automorphism merely twists the final form of the solution.

As an application we consider the concrete functional equation

$$f(x) = \sigma(x)^n \, g(x^{-1}) \text{ for all } x \in \mathcal{D}^{\times}, \; n > 2, \tag{1.3}$$

where $f, g : \mathcal{D} \to \mathcal{D}$ are additive. By the same reduction technique we obtain a complete classification, which generalizes [1, Theorem 4.1].

Theorem 1.2. *Let $\mathcal{D}$ be a non-commutative division ring with* $\operatorname{char} \mathcal{D} \neq 2$ *and let $n > 2$ be an integer. Let σ be an automorphism of $\mathcal{D}$. If additive maps $f, g : \mathcal{D} \to \mathcal{D}$ satisfy* (1.3) *for all $x \in \mathcal{D}^{\times}$, then either $f = g = 0$, or* $\operatorname{char} \mathcal{D} = p > 0$, *$p - 1 \mid n - 2$, and n is a sum of two powers of p; in the latter case there exist $\alpha, \beta \in \mathcal{D}$ and integers $\ell, m \geq 0$ with $n = p^{\ell} + p^m$ such that*

$$f(x) = \sigma(\alpha) \, \sigma(x)^{p^{\ell}} + \sigma(\beta) \, \sigma(x)^{p^m}, \; g(x) = \sigma(\alpha) \, \sigma(x)^{p^m} + \sigma(\beta) \, \sigma(x)^{p^{\ell}} \text{ for all } x \in \mathcal{D}.$$

The paper is organized as follows. Section 2 fixes notation and recalls Lee–Lin's theorem. The proof of Theorem 1.1 is given in Section 3. Section 4 contains the proof of Theorem 1.2. We close with a few concrete examples in Section 5 and some remarks on further directions.

2. Preliminaries

We denote by $\mathcal{D}\{X\}$ the free product of $\mathcal{D}$ and the polynomial ring $\mathcal{Z}[X]$. Its elements are finite sums of monomials $a_0 X a_1 X \cdots a_{k-1} X a_k$ with $a_i \in \mathcal{D}$. The *degree* of such a monomial is the number of occurrences of X. A generalized polynomial is *homogeneous* if all its monomials have the same degree.

For an additive map $f : \mathcal{D} \to \mathcal{D}$ and a generalized polynomial $G(X) \in \mathcal{D}\{X\}$, the expression $G(x)f(x)$ is evaluated by substituting x for X and multiplying on the right by $f(x)$. We shall use the following fundamental result.

Theorem 2.1 (Lee–Lin [1], Theorem 2.3)**.** *Let $\mathcal{D}$ be a division ring and $f : \mathcal{D} \to \mathcal{D}$ an additive map. Suppose there exist generalized polynomials $G(X), H(X) \in \mathcal{D}\{X\}$ with $G(X) \not\equiv 0$ such that*

$$G(x)f(x) = H(x) \text{ for all } x \in \mathcal{D}.$$

Then either $[\mathcal{D} : \mathcal{Z}] < \infty$ or f is an elementary operator, i.e. $f(x) = \sum_i a_i x b_i$ for some $a_i, b_i \in \mathcal{D}$.

An automorphism σ of $\mathcal{D}$ acts on generalized polynomials coefficient-wise: for $G(X) = \sum a_0 X a_1 \cdots X a_k$ we set $\sigma(G)(X) = \sum \sigma(a_0) X \sigma(a_1) \cdots X \sigma(a_k)$. Clearly $\sigma(G) \in \mathcal{D}\{X\}$ and $\sigma(G(x)) = \sigma(G)(\sigma(x))$ for all $x \in \mathcal{D}$. In the sequel we use the inverse automorphism σ^{-1} in the same way.

Definition 2.1. *An additive map $f : \mathcal{D} \to \mathcal{D}$ is called* σ-elementary *if there exist finitely many $a_i, b_i \in \mathcal{D}$ such that*

$$f(x) = \sum_i a_i \, \sigma(x) \, b_i \text{ for all } x \in \mathcal{D}.$$

When $\sigma = \mathrm{id}$, an σ-elementary map is simply an elementary operator, so the definition extends the classical one.

3. Proof of the σ-twisted structure theorem

Proof of Theorem 1.1. Let $f : \mathcal{D} \to \mathcal{D}$ be an additive map satisfying

$$\sigma(x) \, G(x) \, f(x) = H(x) \text{ for all } x \in \mathcal{D}, \tag{3.1}$$

where $G(X), H(X) \in \mathcal{D}\{X\}$ and $G(X) \not\equiv 0$.

Reduction via the inverse automorphism. Apply the automorphism σ^{-1} to both sides of (3.1). Since σ^{-1} is a ring automorphism of $\mathcal{D}$, it is additive and multiplicative, and thus

$$\sigma^{-1}\Big(\sigma(x)G(x)f(x)\Big) = \sigma^{-1}\Big(\sigma(x)\Big)\sigma^{-1}\Big(G(x)\Big)\sigma^{-1}\Big(f(x)\Big)$$

for all $x \in \mathcal{D}$. Hence

$$x \cdot \sigma^{-1}\Big(G(x)\Big) \cdot (\sigma^{-1} \circ f)(x) = \sigma^{-1}\Big(H(x)\Big).$$

Define an additive map $g : \mathcal{D} \to \mathcal{D}$ by

$$g := \sigma^{-1} \circ f,$$

and define generalized polynomials

$$\widetilde{G}(X) := X \cdot \sigma^{-1}(G)(X), \ \widetilde{H}(X) := \sigma^{-1}(H)(X).$$

Then the above identity can be rewritten as

$$\widetilde{G}(x) \, g(x) = \widetilde{H}(x) \text{ for all } x \in \mathcal{D}. \tag{3.2}$$

Non-vanishing of $\widetilde{G}$. We claim that $\widetilde{G}(X) \not\equiv 0$ in $\mathcal{D}\{X\}$.

Since $G(X) \not\equiv 0$ and σ^{-1} is an automorphism of $\mathcal{D}$, it follows that $\sigma^{-1}(G)(X) \not\equiv 0$. It therefore suffices to show that left multiplication by X is injective on $\mathcal{D}\{X\}$.

Recall that $\mathcal{D}\{X\}$ is the free product of $\mathcal{D}$ and $\mathcal{Z}[X]$ over $\mathcal{Z}(\mathcal{D})$. Every nonzero element of $\mathcal{D}\{X\}$ admits a unique representation as a finite $\mathcal{Z}(\mathcal{D})$-linear combination of reduced monomials of the form

$$a_0 X a_1 X \cdots X a_k \ (a_i \in \mathcal{D},\, k \geq 0),$$

and these reduced monomials are linearly independent over $\mathcal{Z}(\mathcal{D})$.

Let $P(X) \in \mathcal{D}\{X\}$ be nonzero. Then $P(X)$ contains at least one nonzero reduced monomial. Multiplication on the left by X transforms each reduced monomial

$$a_0 X a_1 X \cdots X a_k$$

into the reduced monomial

$$X a_0 X a_1 X \cdots X a_k,$$

which has strictly greater degree. In particular, distinct reduced monomials in $P(X)$ are sent to distinct reduced monomials in $XP(X)$, and no cancellation can occur. Hence $XP(X) \neq 0$.

Applying this to $P(X) = \sigma^{-1}(G)(X)$ yields $\widetilde{G}(X) \neq 0$.

Application of the Lee–Lin theorem. Equation (3.2) is now a functional identity of the standard form

$$\widetilde{G}(x)\, g(x) = \widetilde{H}(x) \text{ for all } x \in \mathcal{D},$$

with $\widetilde{G}(X) \not\equiv 0$ and g additive. By Theorem 2.1, it follows that either $[\mathcal{D} : \mathcal{Z}(\mathcal{D})] < \infty$ or g is an elementary operator. In the latter case, there exist elements $a_i, b_i \in \mathcal{D}$ such that

$$g(x) = \sum_i a_i x b_i \text{ for all } x \in \mathcal{D}.$$

Reconstruction of f. Applying σ to the above identity and using multiplicativity, we obtain

$$f(x) = \sigma(g(x)) = \sum_i \sigma(a_i)\, \sigma(x)\, \sigma(b_i) \text{ for all } x \in \mathcal{D}.$$

Thus f is a σ-elementary operator.

This completes the proof. □

Remark 3.1. *The argument shows that the factor $\sigma(x)$ in the original identity is completely harmless: it can be removed by applying σ^{-1}, reducing the problem to the well-understood case $\sigma = \mathrm{id}$. The appearance of $\sigma(x)$ in the solution* (1.1) *is the sole remnant of the automorphism.*

4. Application to $f(x) = \sigma(x)^n g(x^{-1})$

We now turn to the concrete identity (1.3). The same reduction technique transforms it into the equation solved in [1], yielding Theorem 1.2 without any additional labour.

Proof of Theorem 1.2. Apply σ^{-1} to both sides of $f(x) = \sigma(x)^n g(x^{-1})$. For every $x \in \mathcal{D}^\times$ we obtain

$$(\sigma^{-1} \circ f)(x) = x^n \, (\sigma^{-1} \circ g)(x^{-1}).$$

Set $F := \sigma^{-1} \circ f$ and $G := \sigma^{-1} \circ g$; these are additive maps from $\mathcal{D}$ to $\mathcal{D}$. The identity becomes

$$F(x) = x^n \, G(x^{-1}) \text{ for all } x \in \mathcal{D}^\times,$$

which is precisely the functional equation studied in [1, Theorem 4.1]. Because $\mathcal{D}$ is non-commutative and $\operatorname{char} \mathcal{D} \neq 2$, that theorem gives a complete classification:

- either $F = G = 0$;
- or $\operatorname{char} \mathcal{D} = p > 0$, $p - 1 \mid n - 2$, and $n = p^\ell + p^m$ for some integers $\ell, m \geq 0$. In this case there exist $\alpha, \beta \in \mathcal{D}$ such that

$$F(x) = \alpha x^{p^\ell} + \beta x^{p^m}, \; G(x) = \alpha x^{p^m} + \beta x^{p^\ell} \text{ for all } x \in \mathcal{D}.$$

If $F = G = 0$, then clearly $f = g = 0$. In the exceptional case, applying σ to the expressions for F and G yields

$$f(x) = \sigma(F(x)) = \sigma(\alpha) \, \sigma(x)^{p^\ell} + \sigma(\beta) \, \sigma(x)^{p^m},$$

$$g(x) = \sigma(G(x)) = \sigma(\alpha) \, \sigma(x)^{p^m} + \sigma(\beta) \, \sigma(x)^{p^\ell},$$

which is exactly the stated form. This finishes the proof. □

Remark 4.1. *The original theorem of Lee–Lin required* $\operatorname{char} \mathcal{D} \neq 2$ *and that* $\mathcal{D}$ *be non-commutative; these hypotheses are inherited without change. No additional restriction on* σ *(such as* $\sigma|_{\mathcal{Z}} = \mathrm{id}_{\mathcal{Z}}$*) is needed because the reduction removes* σ *from the equation before the classification is invoked.*

5. Examples and further remarks

Example 5.1. *If* $\sigma = \mathrm{id}_{\mathcal{D}}$*, Theorem 1.1 recovers the original Lee–Lin theorem, and Theorem 1.2 reproduces the classical solutions* $f(x) = \alpha x^{p^\ell} + \beta x^{p^m}$.

Example 5.2. *Let* $\mathcal{D}$ *be a non-commutative division ring of characteristic* $p > 0$ *and let* σ *be an automorphism that is not the identity. Choose* $n = p^\ell + p^m$ *with* $p - 1 \mid n - 2$*. For any* $\alpha, \beta \in \mathcal{D}$*, the pair*

$$f(x) = \sigma(\alpha)\sigma(x)^{p^\ell} + \sigma(\beta)\sigma(x)^{p^m}, \; g(x) = \sigma(\alpha)\sigma(x)^{p^m} + \sigma(\beta)\sigma(x)^{p^\ell}$$

satisfies (1.3)*. For instance, take* $\mathcal{D}$ *to be the division ring of quaternions over an infinite field of characteristic* p *and* σ *conjugation by a non-central element; then* $\sigma(x)$ *is not a polynomial in* x*, and the maps are truly* σ*-twisted.*

Example 5.3. *The* σ*-elementary form* (1.1) *appears naturally when one studies derivations and automorphisms simultaneously. For an inner automorphism* $\sigma(x) = uxu^{-1}$*, an* σ*-elementary map is* $f(x) = \sum a_i u x u^{-1} b_i = \sum (a_i u) x (u^{-1} b_i)$*, which is again an ordinary elementary operator. Thus the novel phenomenon occurs only for outer automorphisms.*

Remark 5.1. *The reduction technique used in this paper applies to any functional identity of the form*

$$\sigma_1(x)^{e_1} G_1(x) f_1(x) + \cdots = H(x),$$

where each term starts with a power of an automorphism applied to x. By composing with an appropriate automorphism one can always reduce to the case where all outer automorphisms are replaced by the identity, after which the classical theory of functional identities takes over. This observation opens the way to a unified treatment of twisted identities on division rings.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] T.-K. Lee, J.-H. Lin, Certain Functional Identities on Division Rings, J. Algebra 647 (2024), 492–514. https://doi.org/10.1016/j.jalgebra.2024.03.002.

[2] W.S. Martindale III, Prime Rings Satisfying a Generalized Polynomial Identity, J. Algebra 12 (1969), 576–584. https://doi.org/10.1016/0021-8693(69)90029-5.

[3] M. Brešar, Introduction to Noncommutative Algebra, Springer International Publishing, 2014. https://doi.org/10.1007/978-3-319-08693-4.

[4] M. Brešar, M.A. Chebotar, W.S. Martindale III, Functional Identities, Birkhäuser Basel, 2007. https://doi.org/10.1007/978-3-7643-7796-0.

[5] L. Catalano, T. Merchán, On Rational Functional Identities, Commun. Algebra 52 (2024), 717–722. https://doi.org/10.1080/00927872.2023.2247488.

[6] T.Y. Lam, A First Course in Noncommutative Rings, Springer New York, 2001. https://doi.org/10.1007/978-1-4419-8616-0.

[7] N. Jacobson, Structure of Rings, revised ed., American Mathematical Society, 1964.