

On New Refinement of Convexity via Interval Calculus with its Fractional Extensions with Applications in Information Theory

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Abstract. This paper introduces, for the first time, a novel class of center-radius order P-superquadratic interval-valued functions and systematically examines their fundamental structural properties. Leveraging these properties, we establish new Jensen and Hermite-Hadamard ($\mathcal{H}\mathcal{H}$) type inequalities, together with their fractional extensions

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via Riemann-Liouville ($\mathcal{R}\mathcal{L}$) fractional integral operators within the framework of interval calculus. The validity and sharpness of our results are demonstrated through numerical examples and graphical illustrations. Furthermore, we enrich the theoretical developments with applications in information theory, yielding significant generalizations and improvements over several existing results. Future research directions include extending these inequalities to other fractional operators, exploring multidimensional settings, and investigating potential applications in optimization and machine learning under uncertainty.

1. INTRODUCTION

This notion of inequality in mathematics goes far beyond the initial concept of comparison of magnitude and encompasses a sophisticated vocabulary of inequalities, bounds, and constraints. This vocabulary is found to be absolutely indispensable in all areas of the mathematical spectrum, ranging from convergence theory in calculus, through the accuracy of approximations in numerical mathematics, topological properties in functional analysis, and physical limitations in engineering sciences. The fundamental role of inequalities in quantifying error bounds has been the driving force behind continued research into various numerical integration techniques, with special emphasis on Newton-Cotes quadrature methods and their generalizations. Modern applied mathematics views inequalities not only as tools but as fundamental concepts that shape the way we pose problems and solve them. The profoundest connection between inequalities and mathematics is through the theory of convex functions, in which the geometric properties of these functions lead to powerful inequalities such as Jensen's inequality, $\mathcal{H}\mathcal{H}$ bounds, and their latest generalizations, while inequalities, in turn, lead to the theory of convex functions. This connection is constantly evolving, as seen in the latest developments in fractional calculus, interval mathematics, and non-convex functions.

Definition 1.1. [1] A function $\mathcal{G} : \mathcal{J} \subset \mathcal{R} \rightarrow \mathcal{R}$ is convex, if

$$\mathcal{G}(\wp\sigma + (1-\wp)\vartheta) \leq \wp\mathcal{G}(\sigma) + (1-\wp)\mathcal{G}(\vartheta), \quad (1.1)$$

holds $\forall \wp \in [0, 1]$ and $\sigma, \vartheta \in \mathcal{J}$.

Remark 1.1. Interchanging the inequality sign in (1.1) leads to the corresponding definition for concave functions.

The $\mathcal{H}\mathcal{H}$ inequality represents one of the most celebrated results in convex analysis, providing sharp lower and upper estimates for the integral mean of convex functions on compact intervals. First established independently by Hermite and Hadamard [2], this inequality has become indispensable across diverse mathematical disciplines, including numerical integration, optimization, and approximation theory. The classical formulation states that for a convex function $\mathcal{G} : \mathcal{J} \rightarrow \mathcal{R}$ on the interval $\mathcal{J}$ of real numbers and $\sigma, \vartheta \in \mathcal{J}$ with $\sigma < \vartheta$, then

$$\mathcal{G}\left(\frac{\sigma + \vartheta}{2}\right) \leq \frac{1}{\vartheta - \sigma} \int_{\sigma}^{\vartheta} \mathcal{G}(\mathcal{V}) d\mathcal{V} \leq \frac{\mathcal{G}(\sigma) + \mathcal{G}(\vartheta)}{2} \quad (1.2)$$

For the literature, see the references [3–16]

Among the most fascinating areas of research in the theory of inequality is the refinement and modification of classical inequalities involving convex functions. These are usually done in an attempt to improve such inequalities using the detailed analysis of generalized convexity. Of all the extensions that have been developed in this area, the theory of superquadratic functions is perhaps the most important and powerful tool in deriving sharper bounds and estimates.

Abramovich, Jameson, and Sinnamon [17], have been the pioneers in the idea of superquadratics, and their work on this powerful generalization has greatly enriched the theory of integral inequalities. Superquadratics provide tighter bounds than are possible with the use of convexity and fill an important gap in applied mathematics. In many applications, there is a need to improve the accuracy of approximation in mathematical modeling and optimization problems.

If the function $\mathcal{G}$ is superquadratic on $[0, \infty)$, then

$$\mathcal{G}(\vartheta) \geq \mathcal{G}(\sigma) + \mathfrak{C}_\sigma(\vartheta - \sigma) + \mathcal{G}(|\vartheta - \sigma|), \quad (1.3)$$

holds $\forall \vartheta \geq 0$, where $\mathfrak{C}_\sigma \in \mathcal{R}$. Moreover the function $\mathcal{G}$ is subquadratic, if the inequality (1.3) is reversed.

Let us consider a function $\mathcal{G}(\mathcal{V}) = \mathcal{V}^{\mathfrak{k}}$ defined on $[0, \infty)$. This function is superquadratic for $\mathfrak{k} \geq 2$, and subquadratic for $0 \leq \mathfrak{k} < 2$. In the case $\mathfrak{C}_\mathcal{V}$ is set equal to $\mathfrak{k}\mathcal{V}^{\mathfrak{k}-1}$, equality holds in (1.3) at $\mathfrak{k} = 2$.

We observe that every superquadratic function necessarily satisfies three supplementary conditions, which we enumerate in the lemma below:

Lemma 1.1. *If the function $\mathcal{G}$ is superquadratic, then*

- $\mathcal{G}(0) \leq 0$.
- $\mathcal{G}'(\mathcal{V}) = \mathfrak{C}_\mathcal{V}$ for $\mathcal{G}$ being a differentiable function at $\mathcal{V} > 0$ and $\mathcal{G}(0) = \mathcal{G}'(0) = 0$.
- $\mathcal{G}$ is convex, in the case $\mathcal{G}$ is positive and $\mathcal{G}(0) = \mathcal{G}'(0) = 0$.

The following lemma delineates the structural relationship between classical convexity and superquadraticity, revealing how these two fundamental concepts intersect and diverge:

Lemma 1.2. *The function $\mathcal{G}$ is superquadratic, if $\mathcal{G}'$ is convex and $\mathcal{G}(0) = \mathcal{G}'(0) = 0$. Its converse is false*

Superquadraticity admits multiple equivalent formulations. The line-of-support definition is recorded in equation (1.3), and an alternative characterization due to Abramovich et al. [17] is stated below:

Definition 1.2. *If the function $\mathcal{G}$ is superquadratic, then*

$$\begin{aligned} \mathcal{G}((1-\wp)\mathcal{V}_1 + \wp\mathcal{V}_2) &\leq (1-\wp)\mathcal{G}(\mathcal{V}_1) + \wp\mathcal{G}(\mathcal{V}_2) - \wp\mathcal{G}((1-\wp)|\mathcal{V}_1 - \mathcal{V}_2|) \\ &\quad - (1-\wp)\mathcal{G}(\wp|\mathcal{V}_1 - \mathcal{V}_2|), \end{aligned} \quad (1.4)$$

holds for every $\wp \in (0, 1)$ and $\mathcal{V}_1, \mathcal{V}_2 \geq 0$.

Remark 1.2. *Reversing the inequality in (1.4) yields the definition of a subquadratic function $\mathcal{G}$.*

Two inequalities have proven particularly instrumental in propelling advances within superquadraticity theory—namely, the Jensen and $\mathcal{H}\mathcal{H}$ type inequalities. These theorems constitute the most significant and broadly employed results in the investigation of superquadratic functions.

Theorem 1.1. *If the function $\mathcal{G}$ is superquadratic, then*

$$\sum_{i=1}^n \wp_i \mathcal{G}(\mathcal{V}_i) \geq \mathcal{G}(\overline{\mathcal{V}}) + \sum_{i=1}^n \wp_i \mathcal{G}(|\mathcal{V}_i - \overline{\mathcal{V}}|), \quad (1.5)$$

holds $\forall \mathcal{V}_i \geq 0$, and $0 \leq \wp_i \leq 1$, where $\overline{\mathcal{V}} = \sum_{i=1}^n \wp_i \mathcal{V}_i$, and $\sum_{i=1}^n \wp_i = 1$.

Banić et al. [18] extended the theory of superquadraticity by deriving the $\mathcal{H}\mathcal{H}$ -type inequalities that follow:

Theorem 1.2. *Let the function $\mathcal{G}$ is superquadratic on $J = [\sigma, \vartheta] \subset [0, \infty$, then*

$$\begin{aligned} \mathcal{G}\left(\frac{\sigma + \vartheta}{2}\right) + \frac{1}{\vartheta - \sigma} \int_{\sigma}^{\vartheta} \mathcal{G}\left(\left|\mathcal{V} - \frac{\sigma + \vartheta}{2}\right|\right) d\mathcal{V} &\leq \frac{1}{\vartheta - \sigma} \int_{\sigma}^{\vartheta} \mathcal{G}(\mathcal{V}) d\mathcal{V} \\ &\leq \frac{\mathcal{G}(\sigma) + \mathcal{G}(\vartheta)}{2} - \frac{1}{(\vartheta - \sigma)^2} \int_{\sigma}^{\vartheta} [(\vartheta - \mathcal{V})\mathcal{G}(\mathcal{V} - \sigma) + (\mathcal{V} - \sigma)\mathcal{G}(\vartheta - \mathcal{V})] d\mathcal{V}. \end{aligned} \quad (1.6)$$

Extending this groundwork, Li et al. [19] made significant advances by incorporating superquadraticity into fractional calculus. Specifically, they developed fractional versions of $\mathcal{H}\mathcal{H}$ -type inequalities utilizing $\mathcal{R}\mathcal{L}$ integral operators.

Theorem 1.3. *Let $\mathcal{G}: J \subset [0, \infty) \rightarrow \mathcal{R}$ be a superquadratic function on $J = [\sigma, \vartheta]$ then*

$$\begin{aligned} \mathcal{G}\left(\frac{\sigma + \vartheta}{2}\right) + \frac{\gamma}{2(\vartheta - \sigma)^{\gamma}} \int_{\sigma}^{\vartheta} \mathcal{G}\left(\left|\frac{\sigma + \vartheta}{2} - \mathcal{V}\right|\right) ((\vartheta - \mathcal{V})^{\gamma-1} + (\mathcal{V} - \sigma)^{\gamma-1}) d\mathcal{V} \\ \leq \frac{\Gamma(1 + \gamma)}{2(\vartheta - \sigma)^{\gamma}} (\mathfrak{I}_{\sigma+}^{\gamma} \mathcal{G}(\vartheta) + \mathfrak{I}_{\vartheta-}^{\gamma} \mathcal{G}(\sigma)) \\ \leq \frac{\mathcal{G}(\sigma) + \mathcal{G}(\vartheta)}{2} - \frac{\gamma}{2(\vartheta - \sigma)^{\gamma}} \int_{\sigma}^{\vartheta} \left(\left(\frac{\mathcal{V} - \sigma}{\vartheta - \sigma}\right) \mathcal{G}(\vartheta - \mathcal{V}) + \left(\frac{\vartheta - \mathcal{V}}{\vartheta - \sigma}\right) \mathcal{G}(\mathcal{V} - \sigma) \right) \\ \times ((\vartheta - \mathcal{V})^{\gamma-1} + (\mathcal{V} - \sigma)^{\gamma-1}) d\mathcal{V}, \end{aligned} \quad (1.7)$$

where $\mathfrak{I}_{\sigma+}^{\gamma}$ and $\mathfrak{I}_{\vartheta-}^{\gamma}$ are given by:

$$\mathfrak{I}_{\sigma+}^{\gamma} \mathcal{G}(\mathcal{V}) = \frac{1}{\Gamma(\gamma)} \int_{\sigma}^{\mathcal{V}} (\mathcal{V} - x)^{\gamma-1} \mathcal{G}(x) dx, \quad (\mathcal{V} > \sigma),$$

and

$$\mathfrak{I}_{\vartheta-}^{\gamma} \mathcal{G}(\mathcal{V}) = \frac{1}{\Gamma(\gamma)} \int_{\mathcal{V}}^{\vartheta} (x - \mathcal{V})^{\gamma-1} \mathcal{G}(x) dx, \quad (\mathcal{V} < \vartheta).$$

The notations $\mathfrak{I}_{\sigma+}^{\gamma} \mathcal{G}(\mathcal{V})$ and $\mathfrak{I}_{\vartheta-}^{\gamma} \mathcal{G}(\mathcal{V})$ are the left and right sided operators. Where Γ is stated as a gamma function and given by $\Gamma(\gamma) = \int_0^{\infty} \mathcal{V}^{\gamma-1} e^{-\mathcal{V}} d\mathcal{V}$.

Superquadratic functions have recently attracted significant attention due to their rich theoretical structure and numerous applications, particularly in the domain of information theory [20]. Building on this foundation, Alomari et al. [21] introduced the concept of h-superquadratic functions, systematically exploring their structural properties and establishing associated inequalities. Extending these ideas further, Mario Krnić and collaborators [22] developed the framework of multiplicatively superquadratic functions, providing novel insights and generalizations that have broadened the applicability of superquadratic analysis in both pure and applied contexts. These developments highlight the ongoing evolution of the theory, motivating further research into more generalized and fractional extensions of superquadratic functions. For the literature related to inequalities and superquadracity, see the references [23–31].

Definition 1.3. *If a function $\mathcal{G}$ is multiplicatively superquadratic on $(0, \infty)$, then*

$$\mathcal{G}((1 - \wp)\sigma + \wp\vartheta) \leq \frac{\mathcal{G}^{1-\wp}(\sigma)\mathcal{G}^{\wp}(\vartheta)}{\mathcal{G}^{\wp}((1 - \wp)|\sigma - \vartheta|)\mathcal{G}^{1-\wp}(\wp|\sigma - \vartheta|)} \quad (1.8)$$

holds $\forall \sigma, \vartheta \in \mathbb{J}$ and $\wp \in [0, 1]$.

Remark 1.3. *If the “ $\leq$ ” in (1.8) reverses then the function $\mathcal{G}$ is called multiplicatively subquadratic.*

Khan et al. [32] advanced this line of research by introducing the P-superquadratic function, a more general and versatile framework that broadens the functional landscape of superquadracity. Their formulation not only captures a wide range of specific instances and key structural properties but also facilitates the derivation of new integral inequalities and enables meaningful applications in various real-world contexts.

Definition 1.4. *If $\mathcal{G}$ is P-superquadratic function on $\mathbb{J} = [0, \mathfrak{k}]$ for $\mathfrak{k} > 0$, then*

$$\mathcal{G}(\wp\sigma + (1 - \wp)\vartheta) \leq [\mathcal{G}(\sigma) - \mathcal{G}((1 - \wp)|\sigma - \vartheta|)] + [\mathcal{G}(\vartheta) - \mathcal{G}(\wp|\sigma - \vartheta|)],$$

holds $\forall \sigma, \vartheta \in \mathbb{J}$ and $\wp \in [0, 1]$. *The function $\mathcal{G}$ is P-subquadratic, provided $-\mathcal{G}$ is P-superquadratic.*

Butt and Khan [33] introduced a novel form of superquadracity in the context of interval-valued functions, termed superquadratic $\mathcal{IVF}$, and explored its associated inequalities along with applications in information theory within the framework of fractional interval calculus. For readers seeking a deeper understanding of convexity and superquadratic functions, we refer to [34–38] and the references cited therein, which provide a comprehensive treatment of structural properties, applications, and their role in the broader context of inequality theory.

Uncertainty is one of the key issues in many problems. When problems are represented using precise numerical values, there may be substantial errors in the representation, leading to the creation of unreliable results. Therefore, the need to develop efficient techniques to minimize the occurrence of such errors remained an important issue. In this regard, the idea of interval analysis was first introduced by Moore [39] in 1969 as a technique for automated error estimation. Interval analysis was successful in solving the problem by increasing the accuracy of the results

and gaining significant research attention. In interval analysis, interval numbers are treated as variables, and arithmetic operations are performed using intervals instead of precise numbers due to the existence of uncertainty.

Several researchers have made significant contributions to the extension of classical inequalities to interval calculus and fuzzy calculus. In particular, Chalco-Cano et al. [40], Flores-Franulić et al. [41], and Costa et al. [42] have explored the generalization of classical inequalities in interval calculus and fuzzy calculus. Following this direction, Zhao et al. [43] have successfully introduced the idea of h -convex $\mathcal{IVF}$ and established inequalities based on the interval inclusion relation. Further, in 2021, Khan et al. [44] have refined the idea of h -convex $\mathcal{IVF}$ based on the Kulisch-Miranker order and have established a number of integral and discrete inequalities involving different classes of convex functions.

However, inequalities based on partial order relations, such as fuzzy order, inclusion order, or pseudo-order relations, are less accurate since two intervals may not always be comparable. Thus, finding a suitable order framework for exploring inequalities for interval-valued functions is still a challenging task. To overcome this problem, Bhunia et al. [45] introduced the concept of α -order in 2014. This framework established a complete ranking relation such that any two intervals are comparable. Inspired by this concept, Khan and Butt [46] introduced a significant class of inequalities by developing α -superquadratic $\mathcal{IVF}$ s and their integer and fractional version inequalities. They showed the effectiveness of their results by providing various applications. To explore more on interval-based inequalities, we recommend the interested reader to look at some recent results in [47–50].

We first discuss the fundamental mathematical background related to the order relation α . We then use this to discuss the definition of α -superquadratic functions.

As defined in Moore [39], interval numbers extend the concept of real numbers, and interval arithmetic extends the operations of real numbers. An interval number $[\varphi]$ is defined as:

$$[\varphi] = [\underline{\varphi}, \overline{\varphi}] = \{\mathcal{V} : \underline{\varphi} \leq \mathcal{V} \leq \overline{\varphi}, \mathcal{V} \in \mathcal{R}\}.$$

Each real number $\eta \in \mathcal{R}$ can be represented as an interval number $[\eta, \eta]$ having zero width. Interval numbers can alternatively be represented in the α form. In this representation, an interval $[\varphi] = [\underline{\varphi}, \overline{\varphi}]$ is expressed as:

$$[\varphi] = \langle \varphi_c, \varphi_r \rangle = \{\mathcal{V} : \varphi_c - \varphi_r \leq \mathcal{V} \leq \varphi_c + \varphi_r, \mathcal{V} \in \mathcal{R}\}.$$

where $\varphi_c = \frac{\varphi + \overline{\varphi}}{2}$ is the formula for computing the centre and $\varphi_r = \frac{\overline{\varphi} - \varphi}{2}$ is the formula for computing the radius of $[\underline{\varphi}, \overline{\varphi}]$.

Definition 1.5. [48] The addition, subtraction and multiplication of α -intervals are defined as follows:

$[\varphi] = \langle \varphi_c, \varphi_r \rangle$ and $[\tau] = \langle \tau_c, \tau_r \rangle$ are defined as follows:

- $[\varphi] + [\tau] = \langle \varphi_c, \varphi_r \rangle + \langle \tau_c, \tau_r \rangle = \langle \varphi_c + \tau_c, \varphi_r + \tau_r \rangle$

- $[\varphi] - [\tau] = \langle \varphi_c, \varphi_r \rangle - \langle \tau_c, \tau_r \rangle = \langle \varphi_c, \varphi_r \rangle + \langle -\tau_c, -\tau_r \rangle$
 $= \langle \varphi_c - \tau_c, \varphi_r - \tau_r \rangle$
- For any $\wp \in \mathcal{R}$ we have

$$\wp[\varphi] = \wp \langle \varphi_c, \varphi_r \rangle = \begin{cases} \langle \wp \varphi_c, \wp \varphi_r \rangle, & \text{if } \wp \geq 0 \\ \langle \wp \varphi_c, -\wp \varphi_r \rangle, & \text{if } \wp < 0. \end{cases}$$

or

$$\wp[\varphi] = \wp \langle \varphi_c, \varphi_r \rangle = \langle \wp \varphi_c, |\wp| \varphi_r \rangle.$$

Definition 1.6. [48] The positive n th power of $[\varphi] = [\underline{\varphi}, \overline{\varphi}]$ is denoted by $[\varphi]^n$ and defined as follows:

$$[\varphi]^n = [\underline{\varphi}^n, \overline{\varphi}^n], \quad \text{for } \underline{\varphi} \geq 0.$$

The α version $[\varphi]^n = [\underline{\varphi}^n, \overline{\varphi}^n]$ is represented as:

$$[\varphi]^n = \langle \varphi_c^n, \varphi_r^n \rangle.$$

Where $\varphi_c^n = \frac{\varphi^n + \overline{\varphi}^n}{2}$ and $\varphi_r^n = \frac{\overline{\varphi}^n - \varphi^n}{2}$.

Definition 1.7. [48] The positive n th root of $[\varphi] = [\underline{\varphi}, \overline{\varphi}]$, n is expressed as $[\varphi]^{\frac{1}{n}}$ and given by:

$$[\varphi]^{\frac{1}{n}} = [\underline{\varphi}, \overline{\varphi}]^{\frac{1}{n}} = \sqrt[n]{[\underline{\varphi}, \overline{\varphi}]} = [\sqrt[n]{\underline{\varphi}}, \sqrt[n]{\overline{\varphi}}], \quad \text{for } \underline{\varphi} \geq 0.$$

The α version of $[\varphi]^{\frac{1}{n}} = [\sqrt[n]{\underline{\varphi}}, \sqrt[n]{\overline{\varphi}}]$ is given by:

$$[\varphi]^{\frac{1}{n}} = \langle \sqrt[n]{\varphi_c}, \sqrt[n]{\varphi_r} \rangle.$$

Where $\sqrt[n]{\varphi_c} = \frac{\sqrt[n]{\underline{\varphi}} + \sqrt[n]{\overline{\varphi}}}{2}$ and $\sqrt[n]{\varphi_r} = \frac{\sqrt[n]{\overline{\varphi}} - \sqrt[n]{\underline{\varphi}}}{2}$.

Definition 1.8. [48] The absolute of an interval $[\varphi]$ is denoted by $||[\varphi]||$ and given by:

$$\begin{aligned} ||[\varphi]|| &= ||[\underline{\varphi}, \overline{\varphi}]|| = [\underline{\varphi}, \overline{\varphi}], \quad \text{if } \underline{\varphi} \geq 0 \\ &= [|\overline{\varphi}|, |\underline{\varphi}|], \quad \text{if } \overline{\varphi} \leq 0. \end{aligned}$$

The α version of $||[\varphi]|| = ||[\underline{\varphi}, \overline{\varphi}]||$ is given by:

$$\begin{aligned} ||[\varphi]|| &= \langle |\varphi_c|, |\varphi_r| \rangle = \left\langle \frac{\varphi + \overline{\varphi}}{2}, \frac{\overline{\varphi} - \varphi}{2} \right\rangle, \quad \text{if } \underline{\varphi} \geq 0 \\ &= \left\langle \frac{|\underline{\varphi}| + |\overline{\varphi}|}{2}, \frac{|\overline{\varphi}| - |\underline{\varphi}|}{2} \right\rangle, \quad \text{if } \overline{\varphi} \leq 0 \end{aligned}$$

Definition 1.9. [45] Let $\varphi = [\underline{\varphi}, \overline{\varphi}] = \langle \varphi_c, \varphi_r \rangle, \Theta = [\underline{\Theta}, \overline{\Theta}] = \langle \Theta_c, \Theta_r \rangle \in \mathcal{R}_I$, then the α order relation is defined as follows:

$$\varphi \leq_{cr} \Theta \Leftrightarrow \begin{cases} \varphi_c < \Theta_c, & \text{if } \varphi_c \neq \Theta_c \\ \varphi_r \leq \Theta_r, & \text{if } \varphi_c = \Theta_c. \end{cases} \quad (1.9)$$

For any $\varphi, \Theta \in \mathcal{R}_I$, either $\varphi \leq_{cr} \Theta$ or $\Theta \leq_{cr} \varphi$.

The relation $\leq_{cr}$ satisfies properties of total order relation for every $\varphi = [\underline{\varphi}, \overline{\varphi}] = \langle \varphi_c, \varphi_r \rangle, \Theta = [\underline{\Theta}, \overline{\Theta}] = \langle \Theta_c, \Theta_r \rangle$ and $z = [\underline{z}, \overline{z}] = \langle z_c, z_r \rangle$:

- **Reflexivity:** $\varphi \leq_{cr} \varphi$.
- **Anti-symmetry:** $\varphi \leq_{cr} \Theta$ and $\Theta \leq_{cr} \varphi$.
- **Comparability:** $\varphi \leq_{cr} \Theta$ or $\Theta \leq_{cr} \varphi$.

Theorem 1.4. [47] Let the function $\mathcal{G} : [\sigma, \vartheta] \rightarrow \mathcal{R}_I$ be an interval valued such that $\mathcal{G} = [\underline{\mathcal{G}}, \overline{\mathcal{G}}]$, then $\mathcal{G}$ is interval Riemann integrable on $[\sigma, \vartheta]$ if and only if $\underline{\mathcal{G}}$ and $\overline{\mathcal{G}}$ are Riemann integrable on $[\sigma, \vartheta]$ such that

$$(IR) \int_{\sigma}^{\vartheta} \mathcal{G}(\mathcal{V}) d\mathcal{V} = \left[(R) \int_{\sigma}^{\vartheta} \underline{\mathcal{G}}(\mathcal{V}) d\mathcal{V}, (R) \int_{\sigma}^{\vartheta} \overline{\mathcal{G}}(\mathcal{V}) d\mathcal{V} \right]. \quad (1.10)$$

In the framework of the α -order relation, the integral maintains the order of intervals.

Theorem 1.5. Let the functions $\mathcal{G}, g : [\sigma, \vartheta] \rightarrow \mathcal{R}_I$ be interval valued such that $\mathcal{G} = [\underline{\mathcal{G}}, \overline{\mathcal{G}}]$ and $g = [\underline{g}, \overline{g}]$. If $\mathcal{G}(\mathcal{V}) \leq_{cr} g(\mathcal{V}), \forall \mathcal{V} \in [\sigma, \vartheta]$, then

$$\int_{\sigma}^{\vartheta} \mathcal{G}(\mathcal{V}) d\mathcal{V} \leq_{cr} \int_{\sigma}^{\vartheta} g(\mathcal{V}) d\mathcal{V}. \quad (1.11)$$

This paper is unique in using the ideas of P-superquadratic functions and their fractional counterparts to prove inequalities in the framework of the α -ordered relation, which has not been studied in any paper in the literature. This paper also uses classical integral inequalities such as Jensen's inequality, $\mathcal{H}\mathcal{H}$ -type inequalities, and their fractional counterparts in the framework of the α -interval-valued functions, which is a new direction in inequalities. It is interesting to note that the α -ordered interval-valued analysis is quite distinct from conventional interval-valued analysis in terms of representing intervals in terms of centers and radii.

The structure of the paper is as follows: The background information and necessary conditions for interval valued analysis and inequalities are discussed in Section 1. With the aid of numerical estimations and graphical representations, we discuss the concept of α -ordered interval valued P-superquadratic functions in Section 2. We also derive some new inequalities in interval-valued Jensen and $\mathcal{H}\mathcal{H}$ types for α -superquadratic functions. In Section 3, we extend the results in the fractional domain by deriving fractional $\mathcal{H}\mathcal{H}$ type inequalities for α -interval valued superquadratic functions. We present various examples, graphical representations, and numerical estimations to support the results. We discuss the information theory results in Section 4. Finally, in Section 5, we present some closing thoughts and propose some possible areas for further investigation.

2. α -P-SUPERQUADRATIC $\mathcal{IVF}$ AND ITS INTEGER ORDER INEQUALITIES

In this section, we first introduce the concept of the center-radius order P-superquadratic $\mathcal{IVF}$, discuss its properties in detail, and derive the associated inequalities.

Definition 2.1. An $\mathcal{IVF}$ $\mathcal{G} : J \subseteq \mathbb{R} \rightarrow \mathbb{R}_J$, such that $\mathcal{G} = [\underline{\mathcal{G}}, \overline{\mathcal{G}}]$, is called center-radius order P-superquadratic $\mathcal{IVF}$ on $J = [0, t]$, where $t > 0$, if

$$\mathcal{G}(\wp\sigma + (1-\wp)\vartheta) \leq_c^r [\mathcal{G}(\sigma) - \mathcal{G}((1-\wp)|\sigma - \vartheta|)] + [\mathcal{G}(\vartheta) - \mathcal{G}(\wp|\sigma - \vartheta|)]. \quad (2.1)$$

holds $\forall \sigma, \vartheta \in J$, and $\wp \in (0, 1)$.

Remark 2.1. When the inequality in (2.1) holds in the opposite direction, the function $\mathcal{G}$ is called center-radius order P-subquadratic $\mathcal{IVF}$.

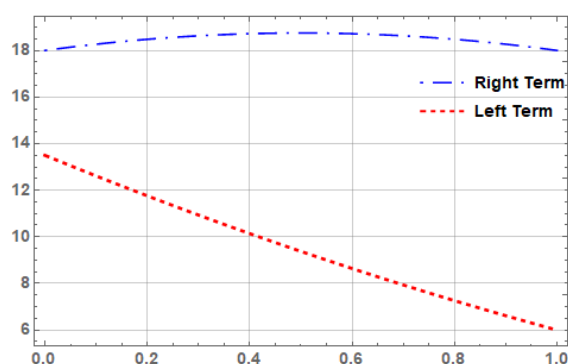
Using the definition of the center-radius order P-superquadratic $\mathcal{IVF}$, we proceed to derive several of its characteristics.

We now present a simple example of center-radius order P-superquadratic $\mathcal{IVF}$ and center-radius order P-subquadratic $\mathcal{IVF}$ to illustrate the underlying concept of center-radius order P-superquadraticity. It should be observed that if $\mathcal{G}$ is center-radius order P-subquadratic $\mathcal{IVF}$, then $-\mathcal{G}$ becomes center-radius order P-superquadratic $\mathcal{IVF}$. Furthermore, for a center-radius order P-subquadratic $\mathcal{IVF}$ $\mathcal{G}$, the direction of the order symbol $\leq_c^r$ in (2.1) is reversed.

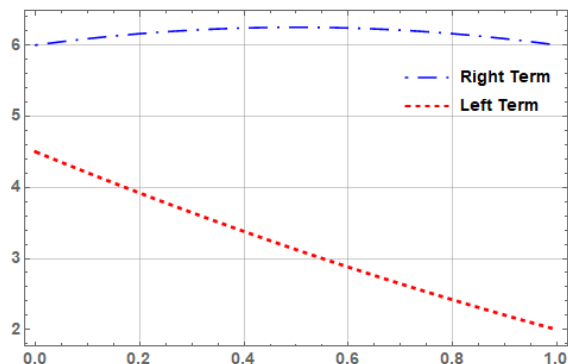
Example 2.1. The below mentioned function is a center-radius order P-superquadratic $\mathcal{IVF}$ on $[\sigma, \vartheta]$ if and only if $\vartheta - \sigma = 1$, for every $\mathcal{V} \geq 2$ and $\mathfrak{p} \geq 2$.

$$\mathcal{G}(\mathcal{V}) = [-\mathcal{V}^{\mathfrak{p}}, 2\mathcal{V}^{\mathfrak{p}}]. \quad (2.2)$$

The center of the interval is $\mathcal{G}_c(\mathcal{V}) = \frac{\mathcal{G}(\mathcal{V}) + \overline{\mathcal{G}}(\mathcal{V})}{2} = \frac{3\mathcal{V}^{\mathfrak{p}}}{2}$, while the radius of the interval is $\mathcal{G}_r(\mathcal{V}) = \frac{\overline{\mathcal{G}}(\mathcal{V}) - \underline{\mathcal{G}}(\mathcal{V})}{2} = \frac{\mathcal{V}^{\mathfrak{p}}}{2}$. The Graphs 1a and 1b given by Figure 1 validate the above mentioned function for $\mathfrak{p} = 2$ by employing Definition 2.1 for various values of σ , ϑ and $\wp$.



(A) σ and ϑ are fixed while $\wp$ varies such that $\sigma = 2$, $\vartheta = 3$, and $\wp \in (0, 1)$ for $\mathcal{G}_c$.



(B) $\wp$ varies, while σ and ϑ are fixed such that $\wp \in (0, 1)$, $\sigma = 2$ and $\vartheta = 3$ for $\mathcal{G}_r$.

FIGURE 1. Graphical description of $\mathcal{G}(\mathcal{V}) = \langle \mathcal{G}_c, \mathcal{G}_r \rangle$ via Definition 2.1.

In the subsequent section, we investigate the distinctive characteristics of center-radius order P-superquadratic $\mathcal{IVF}$.

Proposition 2.1. Let $\mathcal{G}$ and φ are center-radius order P-superquadratic $\mathcal{IVF}$ s such that $\mathcal{G} = [\underline{\mathcal{G}}, \overline{\mathcal{G}}]$ and $\varphi = [\underline{\varphi}, \overline{\varphi}]$ respectively then $\mathcal{G} + \varphi$ and $K\mathcal{G}$, where $K > 0$ are center-radius order P-superquadratic $\mathcal{IVF}$ s.

Proof. Suppose that $\mathcal{G}$ and φ are considered to be center-radius order P-superquadratic $\mathcal{IVF}$ s i.e., $\forall \sigma, \vartheta \in I$ and $\wp \in (0, 1)$, we have

$$\mathcal{G}(\wp\sigma + (1 - \wp)\vartheta) \leq_c^r \left(\mathcal{G}(\sigma) - \mathcal{G}((1 - \wp)|\sigma - \vartheta|) \right) + \left(\mathcal{G}(\vartheta) - \mathcal{G}(\wp|\sigma - \vartheta|) \right),$$

or

$$\begin{aligned} & \left\langle \mathcal{G}_c(\wp\sigma + (1 - \wp)\vartheta), \mathcal{G}_r(\wp\sigma + (1 - \wp)\vartheta) \right\rangle \\ & \leq_c^r \left\langle \left(\mathcal{G}_c(\sigma) - \mathcal{G}_c((1 - \wp)|\sigma - \vartheta|) \right) + \left(\mathcal{G}_c(\vartheta) - \mathcal{G}_c(\wp|\sigma - \vartheta|) \right), \right. \\ & \left. \left(\mathcal{G}_r(\sigma) - \mathcal{G}_r((1 - \wp)|\sigma - \vartheta|) \right) + \left(\mathcal{G}_r(\vartheta) - \mathcal{G}_r(\wp|\sigma - \vartheta|) \right) \right\rangle. \end{aligned} \quad (2.3)$$

and

$$\varphi(\wp\sigma + (1 - \wp)\vartheta) \leq_c^r \left(\varphi(\sigma) - \varphi((1 - \wp)|\sigma - \vartheta|) \right) + \left(\varphi(\vartheta) - \varphi(\wp|\sigma - \vartheta|) \right),$$

or

$$\begin{aligned} & \left\langle g_c(\wp\sigma + (1 - \wp)\vartheta), g_r(\wp\sigma + (1 - \wp)\vartheta) \right\rangle \\ & \leq_c^r \left\langle \left(g_c(\sigma) - g_c((1 - \wp)|\sigma - \vartheta|) \right) + \left(g_c(\vartheta) - g_c(\wp|\sigma - \vartheta|) \right), \right. \\ & \left. \left(g_r(\sigma) - g_r((1 - \wp)|\sigma - \vartheta|) \right) + \left(g_r(\vartheta) - g_r(\wp|\sigma - \vartheta|) \right) \right\rangle. \end{aligned} \quad (2.4)$$

Next adding (2.3) and (2.4), and using the properties of center and radius values, we achieve that

$$\begin{aligned} & (\mathcal{G} + \varphi)(\wp\sigma + (1 - \wp)\vartheta) = \mathcal{G}(\wp\sigma + (1 - \wp)\vartheta) + \varphi(\wp\sigma + (1 - \wp)\vartheta) \\ & \leq_c^r \left(\mathcal{G}(\sigma) - \mathcal{G}((1 - \wp)|\sigma - \vartheta|) \right) + \left(\mathcal{G}(\vartheta) - \mathcal{G}(\wp|\sigma - \vartheta|) \right) \\ & + \left(\varphi(\sigma) - \varphi((1 - \wp)|\sigma - \vartheta|) \right) + \left(\varphi(\vartheta) - \varphi(\wp|\sigma - \vartheta|) \right) \\ & = \left((\mathcal{G} + \varphi)(\sigma) - (\mathcal{G} + \varphi)((1 - \wp)|\sigma - \vartheta|) \right) + \left((\mathcal{G} + \varphi)(\vartheta) - (\mathcal{G} + \varphi)(\wp|\sigma - \vartheta|) \right). \end{aligned}$$

Thus

$$\begin{aligned} & (\mathcal{G} + \varphi)(\wp\sigma + (1 - \wp)\vartheta) \leq_c^r \left((\mathcal{G} + \varphi)(\sigma) - (\mathcal{G} + \varphi)((1 - \wp)|\sigma - \vartheta|) \right) \\ & + \left((\mathcal{G} + \varphi)(\vartheta) - (\mathcal{G} + \varphi)(\wp|\sigma - \vartheta|) \right). \end{aligned} \quad (2.5)$$

Similarly

$$K\mathcal{G}(\wp\sigma + (1 - \wp)\vartheta) \leq_c^r K \left(\left(\mathcal{G}(\sigma) - \mathcal{G}((1 - \wp)|\sigma - \vartheta|) \right) + \left(\mathcal{G}(\vartheta) - \mathcal{G}(\wp|\sigma - \vartheta|) \right) \right)$$

$$= (\mathcal{KG}(\sigma) - \mathcal{KG}((1-\wp)|\sigma - \vartheta|)) + (\mathcal{KG}(\vartheta) - \mathcal{KG}(\wp|\sigma - \vartheta|)).$$

Thus

$$\begin{aligned} & \mathcal{KG}(\wp\sigma + (1-\wp)\vartheta) \\ & \leq_c^r ((\mathcal{KG}(\sigma) - \mathcal{KG}((1-\wp)|\sigma - \vartheta|)) + (\mathcal{KG}(\vartheta) - \mathcal{KG}(\wp|\sigma - \vartheta|))). \end{aligned} \quad (2.6)$$

□

Thus, by (2.5) and (2.6), we infer that $\mathcal{G} + \varphi$ and $\mathcal{KG}$ are center-radius order P -superquadratic.

Proposition 2.2. Let $\mathcal{G}$ is a center-radius order P -superquadratic $\mathcal{IVF}$ such that $\mathcal{G} = [\underline{\mathcal{G}}, \overline{\mathcal{G}}]$ and φ is a center-radius order convex function such that $\varphi = [\underline{\varphi}, \overline{\varphi}]$ on J , then $\mathcal{G} \circ \varphi$ is a center-radius order P -superquadratic $\mathcal{IVF}$.

Proof. Since $\sigma, \vartheta \in J$ and $\wp \in (0, 1)$ then we have

$$\begin{aligned} & (\mathcal{G} \circ \varphi)(\wp\sigma + (1-\wp)\vartheta) \\ & \left\langle (\mathcal{G}_c \circ \varphi_c)(\wp\sigma + (1-\wp)\vartheta), (\mathcal{G}_r \circ \varphi_r)(\wp\sigma + (1-\wp)\vartheta) \right\rangle. \end{aligned}$$

It implies that

$$\begin{aligned} & (\mathcal{G}_c \circ \varphi_c)(\wp\sigma + (1-\wp)\vartheta) \\ & = \mathcal{G}_c(\varphi_c(\wp\sigma + (1-\wp)\vartheta)) \\ & \leq \mathcal{G}(\wp\varphi_c(\sigma) + (1-\wp)\varphi_c(\vartheta)) \\ & \leq (\mathcal{G}_c(\varphi_c(\sigma)) - \mathcal{G}_c((1-\wp)|\varphi_c(\sigma) - \varphi_c(\vartheta)|)) + (\mathcal{G}_c(\varphi_c(\vartheta)) - \mathcal{G}(\wp|\varphi_c(\sigma) - \varphi_c(\vartheta)|)). \end{aligned} \quad (2.7)$$

and

$$\begin{aligned} & (\mathcal{G}_r \circ \varphi_r)(\wp\sigma + (1-\wp)\vartheta) \\ & \leq (\mathcal{G}_r(\varphi_r(\sigma)) - \mathcal{G}_r((1-\wp)|\varphi_r(\sigma) - \varphi_r(\vartheta)|)) + (\mathcal{G}_r(\varphi_r(\vartheta)) - \mathcal{G}(\wp|\varphi_r(\sigma) - \varphi_r(\vartheta)|)). \end{aligned} \quad (2.8)$$

By Definition 1.9, the results (2.7) and (2.8) can be combined and written as:

$$\begin{aligned} & (\mathcal{G} \circ \varphi)(\wp\sigma + (1-\wp)\vartheta) \\ & \leq_c^r (\mathcal{G}(\varphi(\sigma)) - \mathcal{G}((1-\wp)|\varphi(\sigma) - \varphi(\vartheta)|)) + (\mathcal{G}(\varphi(\vartheta)) - \mathcal{G}(\wp|\varphi(\sigma) - \varphi(\vartheta)|)). \end{aligned}$$

This demonstrates that $\mathcal{G} \circ \varphi$ is a center-radius order P -superquadratic $\mathcal{IVF}$. □

Proposition 2.3. If $\mathcal{G}$ is a non-negative center-radius order superquadratic $\mathcal{IVF}$ function such that $\mathcal{G} = [\underline{\mathcal{G}}, \overline{\mathcal{G}}]$, then $\mathcal{G}$ is a center-radius order P -superquadratic.

Proof. Let $\mathcal{G}$ is a center-radius order superquadratic $\mathcal{IVF}$ function, so we have

$$\begin{aligned} & \mathcal{G}(\wp\sigma + (1-\wp)\vartheta) \\ & \leq_c^r \wp[\mathcal{G}(\sigma) - \mathcal{G}((1-\wp)|\sigma - \vartheta|)] + (1-\wp)[\mathcal{G}(\vartheta) - \mathcal{G}(\wp|\sigma - \vartheta|)], \end{aligned} \quad (2.9)$$

for any $\sigma, \vartheta \in J$.

By Definition 1.9, the results (2.9) can be written as:

$$\begin{aligned} & \mathcal{G}_c(\wp\sigma + (1-\wp)\vartheta) \\ & \leq \wp[\mathcal{G}_c(\sigma) - \mathcal{G}_c((1-\wp)|\sigma - \vartheta|)] + (1-\wp)[\mathcal{G}_c(\vartheta) - \mathcal{G}_c(\wp|\sigma - \vartheta|)]. \end{aligned} \quad (2.10)$$

and

$$\begin{aligned} & \mathcal{G}_r(\wp\sigma + (1-\wp)\vartheta) \\ & \leq \wp[\mathcal{G}_r(\sigma) - \mathcal{G}_r((1-\wp)|\sigma - \vartheta|)] + (1-\wp)[\mathcal{G}_r(\vartheta) - \mathcal{G}_r(\wp|\sigma - \vartheta|)]. \end{aligned} \quad (2.11)$$

First consider (2.10) and we have

$$\begin{aligned} & \mathcal{G}_c(\wp\sigma + (1-\wp)\vartheta) \\ & \leq [\mathcal{G}_c(\sigma) - \mathcal{G}_c((1-\wp)|\sigma - \vartheta|)] + [\mathcal{G}_c(\vartheta) - \mathcal{G}_c(\wp|\sigma - \vartheta|)]. \end{aligned} \quad (2.12)$$

Next we consider (2.11) and we have

$$\begin{aligned} & \mathcal{G}_r(\wp\sigma + (1-\wp)\vartheta) \\ & \leq [\mathcal{G}_r(\sigma) - \mathcal{G}_r((1-\wp)|\sigma - \vartheta|)] + [\mathcal{G}_r(\vartheta) - \mathcal{G}_r(\wp|\sigma - \vartheta|)]. \end{aligned} \quad (2.13)$$

Again by Definition 1.9, we can combine the (2.12) and (2.13)

$$\begin{aligned} & \mathcal{G}(\wp\sigma + (1-\wp)\vartheta) \\ & \leq_c^r [\mathcal{G}(\sigma) - \mathcal{G}((1-\wp)|\sigma - \vartheta|)] + [\mathcal{G}(\vartheta) - \mathcal{G}(\wp|\sigma - \vartheta|)]. \end{aligned}$$

Hence the proof. $\square$

Remark 2.2. The Proposition 2.3, states that center-radius order P -superquadratic $\mathcal{IVF}$ is center-radius order superquadratic $\mathcal{IVF}$ as well.

Proposition 2.4. Let $\mathcal{G}, \varphi : J \subset \mathcal{R} \rightarrow \mathcal{R}_J$ be the center-radius order P -superquadratic $\mathcal{IVF}$ s such that $\mathcal{G} = [\underline{\mathcal{G}}, \overline{\mathcal{G}}]$ and $\varphi = [\underline{\varphi}, \overline{\varphi}]$, then $\mathcal{G}(\mathcal{V}) = \max\{\mathcal{G}(\mathcal{V}), \varphi(\mathcal{V})\}$ is center-radius order P -superquadratic $\mathcal{IVF}$.

Proof. Since for every $\sigma, \vartheta \in J$ as well as $0 < \wp < 1$, one derives that

$$\begin{aligned} & \mathcal{G}(\wp\sigma + (1-\wp)\vartheta) \\ & = \max\{\mathcal{G}(\wp\sigma + (1-\wp)\vartheta), \varphi(\wp\sigma + (1-\wp)\vartheta)\} \\ & \leq_c^r \max\{[\mathcal{G}(\sigma) - \mathcal{G}((1-\wp)|\sigma - \vartheta|)] + [\mathcal{G}(\vartheta) - \mathcal{G}(\wp|\sigma - \vartheta|)] \\ & \quad , [\varphi(\sigma) - \varphi((1-\wp)|\sigma - \vartheta|)] + [\varphi(\vartheta) - \varphi(\wp|\sigma - \vartheta|)]\}. \end{aligned} \quad (2.14)$$

Again by Definition 1.9, the result (2.14) can be split as

$$\mathcal{G}_c(\wp\sigma + (1-\wp)\vartheta)$$

$$\begin{aligned} &\leq \max\{[\mathcal{G}_c(\sigma) - \mathcal{G}_c((1-\wp)|\sigma - \vartheta|)] + [\mathcal{G}_c(\vartheta) - \mathcal{G}_c(\wp|\sigma - \vartheta|)], \\ &[\varphi_c(\sigma) - \varphi_c((1-\wp)|\sigma - \vartheta|)] + [\varphi_c(\vartheta) - \varphi_c(\wp|\sigma - \vartheta|)]\}, \end{aligned} \quad (2.15)$$

and

$$\begin{aligned} &\mathcal{G}_r(\wp\sigma + (1-\wp)\vartheta) \\ &\leq \max\{[\mathcal{G}_r(\sigma) - \mathcal{G}_r((1-\wp)|\sigma - \vartheta|)] + [\mathcal{G}_r(\vartheta) - \mathcal{G}_r(\wp|\sigma - \vartheta|)], \\ &[\varphi_r(\sigma) - \varphi_r((1-\wp)|\sigma - \vartheta|)] + [\varphi_r(\vartheta) - \varphi_r(\wp|\sigma - \vartheta|)]\}. \end{aligned} \quad (2.16)$$

First consider (2.15), we get:

$$\begin{aligned} &\mathcal{G}_c(\wp\sigma + (1-\wp)\vartheta) \\ &\leq \max\{\mathcal{G}_c(\sigma), \varphi_c(\sigma)\} - \max\{\mathcal{G}_c((1-\wp)|\sigma - \vartheta|), \varphi_c((1-\wp)|\sigma - \vartheta|)\} \\ &+ \max\{\mathcal{G}_c(\vartheta), \varphi_c(\vartheta)\} - \max\{\mathcal{G}_c(\wp|\sigma - \vartheta|), \varphi_c(\wp|\sigma - \vartheta|)\} \\ &= [\mathcal{G}_c(\sigma) - \mathcal{G}_c((1-\wp)|\sigma - \vartheta|)] + [\mathcal{G}_c(\vartheta) - \mathcal{G}_c(\wp|\sigma - \vartheta|)]. \end{aligned} \quad (2.17)$$

Similarly by considering (2.16), we get:

$$\begin{aligned} &\mathcal{G}_r(\wp\sigma + (1-\wp)\vartheta) \\ &\leq \max\{\mathcal{G}_r(\sigma), \varphi_r(\sigma)\} - \max\{\mathcal{G}_r((1-\wp)|\sigma - \vartheta|), \varphi_r((1-\wp)|\sigma - \vartheta|)\} \\ &+ \max\{\mathcal{G}_r(\vartheta), \varphi_r(\vartheta)\} - \max\{\mathcal{G}_r(\wp|\sigma - \vartheta|), \varphi_r(\wp|\sigma - \vartheta|)\} \\ &= [\mathcal{G}_r(\sigma) - \mathcal{G}_r((1-\wp)|\sigma - \vartheta|)] + [\mathcal{G}_r(\vartheta) - \mathcal{G}_r(\wp|\sigma - \vartheta|)]. \end{aligned} \quad (2.18)$$

Again using Definition 1.9, the results (2.17) and (2.18) can be combined as follows:

$$\begin{aligned} &\mathcal{G}(\wp\sigma + (1-\wp)\vartheta) \\ &\leq_c^r [\mathcal{G}(\sigma) - \mathcal{G}((1-\wp)|\sigma - \vartheta|)] + [\mathcal{G}(\vartheta) - \mathcal{G}(\wp|\sigma - \vartheta|)]. \end{aligned}$$

Consequently, $\mathcal{G}(\mathcal{V})$ is termed as a center-radius order P-superquadratic $\mathcal{IVF}$. $\square$

Theorem 2.1. Let $\mathcal{G} : \mathcal{J} \subset \mathcal{R} \rightarrow (0, \infty)$ be a function for all points $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3 \in \mathcal{J}$ such that $\mathcal{V}_1 < \mathcal{V}_2 < \mathcal{V}_3$. If $\mathcal{G}$ is a center-radius order P-superquadratic $\mathcal{IVF}$ such that $\mathcal{G} = [\underline{\mathcal{G}}, \overline{\mathcal{G}}]$, then the inequality

$$\left| \begin{array}{cc} \mathcal{G}(\mathcal{V}_2) & 0 \\ 0 & 1 \end{array} \right| \leq_c^r \left| \begin{array}{cc} \mathcal{G}(\mathcal{V}_1) & 1 \\ \mathcal{G}(|\mathcal{V}_2 - \mathcal{V}_1|) & 1 \end{array} \right| + \left| \begin{array}{cc} \mathcal{G}(\mathcal{V}_3) & 1 \\ \mathcal{G}(|\mathcal{V}_3 - \mathcal{V}_2|) & 1 \end{array} \right|, \quad (2.19)$$

holds, $\forall m \in [0, 1]$.

Proof. Let $\mathcal{G}$ is a center-radius order P-superquadratic $\mathcal{IVF}$, then

$$\mathcal{G}(\wp\mathcal{V} + (1-\wp)\mathcal{V}_o) \leq_c^r (\mathcal{G}(\mathcal{V}) - \mathcal{G}((1-\wp)|\mathcal{V} - \mathcal{V}_o|)) + (\mathcal{G}(\mathcal{V}_o) - \mathcal{G}(\wp|\mathcal{V} - \mathcal{V}_o|)). \quad (2.20)$$

By Definition 1.9, the result (2.20) can be split as:

$$\mathcal{G}_c(\wp\mathcal{V} + (1-\wp)\mathcal{V}_o) \leq (\mathcal{G}_c(\mathcal{V}) - \mathcal{G}_c((1-\wp)|\mathcal{V} - \mathcal{V}_o|)) + (\mathcal{G}_c(\mathcal{V}_o) - \mathcal{G}_c(\wp|\mathcal{V} - \mathcal{V}_o|)). \quad (2.21)$$

and

$$\mathcal{G}_r(\wp \mathcal{V} + (1 - \wp) \mathcal{V}_o) \leq \left(\mathcal{G}_r(\mathcal{V}) - \mathcal{G}_r((1 - \wp)|\mathcal{V} - \mathcal{V}_o|) \right) + \left(\mathcal{G}_r(\mathcal{V}_o) - \mathcal{G}_r(\wp|\mathcal{V} - \mathcal{V}_o|) \right). \quad (2.22)$$

Setting

$$\mathcal{V}_2 = \wp \mathcal{V} + (1 - \wp) \mathcal{V}_o. \quad (2.23)$$

Next putting $\mathcal{V} = \mathcal{V}_1$ and $\mathcal{V}_o = \mathcal{V}_3$ in (2.23), we get

$$\begin{aligned} \mathcal{V}_2 &= \wp \mathcal{V}_1 + (1 - \wp) \mathcal{V}_3. \\ \Rightarrow \wp &= \frac{\mathcal{V}_3 - \mathcal{V}_2}{\mathcal{V}_3 - \mathcal{V}_1}. \end{aligned}$$

Substituting $\mathcal{V} = \mathcal{V}_1$, $\mathcal{V}_o = \mathcal{V}_3$ and $\wp = \frac{\mathcal{V}_3 - \mathcal{V}_2}{\mathcal{V}_3 - \mathcal{V}_1}$ in (2.21), we get:

$$\mathcal{G}_c(\mathcal{V}_2) \leq \mathcal{G}_c(\mathcal{V}_1) - \mathcal{G}_c\left(\left|\frac{(\mathcal{V}_2 - \mathcal{V}_1)(\mathcal{V}_3 - \mathcal{V}_1)}{\mathcal{V}_3 - \mathcal{V}_1}\right|\right) + \mathcal{G}_c(\mathcal{V}_3) - \mathcal{G}_c\left(\left|\frac{(\mathcal{V}_3 - \mathcal{V}_2)(\mathcal{V}_3 - \mathcal{V}_1)}{\mathcal{V}_3 - \mathcal{V}_1}\right|\right). \quad (2.24)$$

It implies that

$$\mathcal{G}_c(\mathcal{V}_1) - \mathcal{G}_c\left(\left|\frac{(\mathcal{V}_2 - \mathcal{V}_1)(\mathcal{V}_3 - \mathcal{V}_1)}{\mathcal{V}_3 - \mathcal{V}_1}\right|\right) + \mathcal{G}_c(\mathcal{V}_3) - \mathcal{G}_c\left(\left|\frac{(\mathcal{V}_3 - \mathcal{V}_2)(\mathcal{V}_3 - \mathcal{V}_1)}{\mathcal{V}_3 - \mathcal{V}_1}\right|\right) - \mathcal{G}_c(\mathcal{V}_2) \geq 0. \quad (2.25)$$

Inequality (2.25) can be written as follows

$$\begin{vmatrix} \mathcal{G}_c(\mathcal{V}_1) & 1 \\ \mathcal{G}_c\left(\left|\frac{(\mathcal{V}_2 - \mathcal{V}_1)(\mathcal{V}_3 - \mathcal{V}_1)}{\mathcal{V}_3 - \mathcal{V}_1}\right|\right) & 1 \end{vmatrix} + \begin{vmatrix} \mathcal{G}_c(\mathcal{V}_3) & 1 \\ \mathcal{G}_c\left(\left|\frac{(\mathcal{V}_3 - \mathcal{V}_2)(\mathcal{V}_3 - \mathcal{V}_1)}{\mathcal{V}_3 - \mathcal{V}_1}\right|\right) & 1 \end{vmatrix} \geq \begin{vmatrix} \mathcal{G}_c(\mathcal{V}_2) & 0 \\ 0 & 1 \end{vmatrix}. \quad (2.26)$$

Similarly by considering (2.22) and moving in the same fashion as before, we get:

$$\begin{vmatrix} \mathcal{G}_r(\mathcal{V}_1) & 1 \\ \mathcal{G}_r\left(\left|\frac{(\mathcal{V}_2 - \mathcal{V}_1)(\mathcal{V}_3 - \mathcal{V}_1)}{\mathcal{V}_3 - \mathcal{V}_1}\right|\right) & 1 \end{vmatrix} + \begin{vmatrix} \mathcal{G}_r(\mathcal{V}_3) & 1 \\ \mathcal{G}_r\left(\left|\frac{(\mathcal{V}_3 - \mathcal{V}_2)(\mathcal{V}_3 - \mathcal{V}_1)}{\mathcal{V}_3 - \mathcal{V}_1}\right|\right) & 1 \end{vmatrix} \geq \begin{vmatrix} \mathcal{G}_r(\mathcal{V}_2) & 0 \\ 0 & 1 \end{vmatrix}. \quad (2.27)$$

Again by Definition 1.9, the inequalities (2.26) and (2.27) can be merged and we get the required result. $\square$

Theorem 2.2. (Jensen inequality) Let $\mathcal{V}_i, \wp_i \geq 0$. $i = 1, \dots, n$, and $\mathcal{G}: \mathcal{J} \rightarrow \mathcal{R}$. If $\mathcal{G}$ is a center-radius order P -superquadratic $\mathcal{IVF}$ on $\mathcal{J}$ such that $\mathcal{G} = [\underline{\mathcal{G}}, \overline{\mathcal{G}}]$, then

$$\mathcal{G}\left(\sum_{i=1}^n \wp_i \mathcal{V}_i\right) \leq_c^r \sum_{i=1}^n \mathcal{G}(\mathcal{V}_i) - \sum_{i=1}^n \mathcal{G}\left(\left|\mathcal{V}_i - \sum_{i=1}^n \wp_i \mathcal{V}_i\right|\right), \quad (2.28)$$

where $\sum_{i=1}^n \wp_i = 1$.

Proof. Since by Definition 1.9, the result (2.28) can be split as:

$$\mathcal{G}_c\left(\sum_{i=1}^n \wp_i \mathcal{V}_i\right) \leq \sum_{i=1}^n \mathcal{G}_c(\mathcal{V}_i) - \sum_{i=1}^n \mathcal{G}_c\left(\left|\mathcal{V}_i - \sum_{i=1}^n \wp_i \mathcal{V}_i\right|\right), \quad (2.29)$$

and

$$\mathcal{G}_r\left(\sum_{i=1}^n \wp_i \mathcal{V}_i\right) \leq \sum_{i=1}^n \mathcal{G}_r(\mathcal{V}_i) - \sum_{i=1}^n \mathcal{G}_r\left(\left|\mathcal{V}_i - \sum_{i=1}^n \wp_i \mathcal{V}_i\right|\right), \quad (2.30)$$

First considering (2.29) and (2.30) and setting $n = 2$, we get

$$\mathcal{G}_c(\wp_1 \mathcal{V}_1 + \wp_2 \mathcal{V}_2) \leq \mathcal{G}_c(\mathcal{V}_2) - \mathcal{G}_c(\wp_1 |\mathcal{V}_2 - \mathcal{V}_1|) + \mathcal{G}_c(\mathcal{V}_1) - \mathcal{G}_c(\wp_2 |\mathcal{V}_2 - \mathcal{V}_1|), \quad (2.31)$$

and

$$\mathcal{G}_r(\wp_1 \mathcal{V}_1 + \wp_2 \mathcal{V}_2) \leq \mathcal{G}_r(\mathcal{V}_2) - \mathcal{G}_r(\wp_1 |\mathcal{V}_2 - \mathcal{V}_1|) + \mathcal{G}_r(\mathcal{V}_1) - \mathcal{G}_r(\wp_2 |\mathcal{V}_2 - \mathcal{V}_1|), \quad (2.32)$$

respectively. Again by Definition 1.9, the results (2.31) and (2.32) can be combined as follows:

$$\mathcal{G}(\wp_1 \mathcal{V}_1 + \wp_2 \mathcal{V}_2) \leq \mathcal{G}(\mathcal{V}_2) - \mathcal{G}(\wp_1 |\mathcal{V}_2 - \mathcal{V}_1|) + \mathcal{G}(\mathcal{V}_1) - \mathcal{G}(\wp_2 |\mathcal{V}_2 - \mathcal{V}_1|), \quad (2.33)$$

Thus the statement 2.28 is true for $n = 2$.

Let us assume that the relation (2.28) holds true for $n - 1$. Consequently, we get

$$\mathcal{G}\left(\sum_{i=1}^{n-1} \wp_i \mathcal{V}_i\right) \leq_c^r \sum_{i=1}^{n-1} \mathcal{G}(\mathcal{V}_i) - \sum_{i=1}^{n-1} \mathcal{G}\left(\left|\mathcal{V}_i - \sum_{i=1}^{n-1} \wp_i \mathcal{V}_i\right|\right). \quad (2.34)$$

Now, let us demonstrate that (2.28) holds for n , so we first consider the center value of the function

$$\mathcal{G}_c\left(\sum_{i=1}^n \wp_i \mathcal{V}_i\right) = \mathcal{G}_c\left(\wp_n \mathcal{V}_n + \wp \sum_{i=1}^{n-1} \frac{\wp_i \mathcal{V}_i}{\wp}\right).$$

Where $\wp = \sum_{i=1}^{n-1} \wp_i$, then it implies that

$$\begin{aligned} \mathcal{G}_c\left(\sum_{i=1}^n \wp_i \mathcal{V}_i\right) &\leq \mathcal{G}_c(\mathcal{V}_n) + \mathcal{G}_c\left(\sum_{i=1}^{n-1} \left(\frac{\wp_i \mathcal{V}_i}{\wp}\right)\right) \\ &- \mathcal{G}_c\left(\wp \left|\mathcal{V}_n - \sum_{i=1}^{n-1} \left(\frac{\wp_i \mathcal{V}_i}{\wp}\right)\right|\right) - \mathcal{G}_c\left(\wp_n \left|\mathcal{V}_n - \sum_{i=1}^{n-1} \left(\frac{\wp_i \mathcal{V}_i}{\wp}\right)\right|\right). \end{aligned} \quad (2.35)$$

Using the center value result of (2.34) in (2.35), we obtain

$$\begin{aligned} \mathcal{G}_c\left(\sum_{i=1}^n \wp_i \mathcal{V}_i\right) &\leq \mathcal{G}_c(\mathcal{V}_n) + \sum_{i=1}^{n-1} \mathcal{G}_c(\mathcal{V}_i) - \sum_{i=1}^{n-1} \mathcal{G}_c\left(\left|\mathcal{V}_i - \sum_{i=1}^{n-1} \wp_i \mathcal{V}_i\right|\right) \\ &- \mathcal{G}_c\left(\wp \left|\mathcal{V}_n - \sum_{i=1}^{n-1} \left(\frac{\wp_i \mathcal{V}_i}{\wp}\right)\right|\right) - \mathcal{G}_c\left(\wp_n \left|\mathcal{V}_n - \sum_{i=1}^{n-1} \left(\frac{\wp_i \mathcal{V}_i}{\wp}\right)\right|\right) \\ &= \mathcal{G}_c(\mathcal{V}_n) + \sum_{i=1}^{n-1} \mathcal{G}_c(\mathcal{V}_i) - \sum_{i=1}^{n-1} \mathcal{G}_c\left(\left|\mathcal{V}_i - \sum_{i=1}^{n-1} \wp_i \mathcal{V}_i\right|\right) \end{aligned}$$

$$\begin{aligned}
& -\mathcal{G}_c\left(\varphi\left|\mathcal{V}_n-\sum_{i=1}^{n-1}\left(\frac{\varphi_i\mathcal{V}_i}{\varphi}\right)\right|\right)-\mathcal{G}_c\left(\varphi_n\left|\mathcal{V}_n-\sum_{i=1}^{n-1}\left(\frac{\varphi_i\mathcal{V}_i}{\varphi}\right)\right|\right) \\
& \leq \sum_{i=1}^n \mathcal{G}_c(\mathcal{V}_i)-\sum_{i=1}^{n-1} \mathcal{G}_c\left(\left|\mathcal{V}_i-\sum_{i=1}^{n-1} \varphi_i\mathcal{V}_i\right|\right)-\mathcal{G}_c\left(\varphi\left|\mathcal{V}_n-\sum_{i=1}^{n-1}\left(\frac{\varphi_i\mathcal{V}_i}{\varphi}\right)\right|\right) \\
& = \sum_{i=1}^n \mathcal{G}_c(\mathcal{V}_i)-\sum_{i=1}^{n-1} \mathcal{G}_c\left(\left|\mathcal{V}_i-\sum_{i=1}^{n-1} \varphi_i\mathcal{V}_i\right|\right)-\mathcal{G}_c\left(\left|\mathcal{V}_n-\varphi_n\mathcal{V}_n-\sum_{i=1}^{n-1}\left(\varphi_i\mathcal{V}_i\right)\right|\right) \\
& = \sum_{i=1}^n \mathcal{G}_c(\mathcal{V}_i)-\sum_{i=1}^{n-1} \mathcal{G}_c\left(\left|\mathcal{V}_i-\sum_{i=1}^{n-1} \varphi_i\mathcal{V}_i\right|\right)-\mathcal{G}_c\left(\left|\mathcal{V}_n-\varphi_n\mathcal{V}_n-\sum_{i=1}^{n-1}\left(\varphi_i\mathcal{V}_i\right)\right|\right) \\
& = \sum_{i=1}^n \mathcal{G}_c(\mathcal{V}_i)-\sum_{i=1}^{n-1} \mathcal{G}_c\left(\left|\mathcal{V}_i-\sum_{i=1}^{n-1} \varphi_i\mathcal{V}_i\right|\right)-\mathcal{G}_c\left(\left|\mathcal{V}_n-\sum_{i=1}^n\left(\varphi_i\mathcal{V}_i\right)\right|\right) \\
& = \sum_{i=1}^n \mathcal{G}_c(\mathcal{V}_i)-\sum_{i=1}^n \mathcal{G}_c\left(\left|\mathcal{V}_i-\sum_{i=1}^n \varphi_i\mathcal{V}_i\right|\right). \tag{2.36}
\end{aligned}$$

Likewise, taking the radius values into account and following the same steps, we attain

$$\mathcal{G}_r\left(\sum_{i=1}^n \varphi_i\mathcal{V}_i\right) \leq \sum_{i=1}^n \mathcal{G}_r(\mathcal{V}_i)-\sum_{i=1}^n \mathcal{G}_r\left(\left|\mathcal{V}_i-\sum_{i=1}^n \varphi_i\mathcal{V}_i\right|\right). \tag{2.37}$$

Again by Definition 1.9, the results (2.36) and (2.37) can be combined as follows:

$$\mathcal{G}\left(\sum_{i=1}^n \varphi_i\mathcal{V}_i\right) \leq_c^r \sum_{i=1}^n \mathcal{G}(\mathcal{V}_i)-\sum_{i=1}^n \mathcal{G}\left(\left|\mathcal{V}_i-\sum_{i=1}^n \varphi_i\mathcal{V}_i\right|\right)$$

Hence the proof. $\square$

Next, we provide the proof of $\mathcal{H}\mathcal{H}$ -type inequalities for center-radius order P-superquadratic $\mathcal{IVF}$.

Theorem 2.3. Let $\mathcal{G} : \mathbb{J} \subset \mathbb{R} \rightarrow (0, \infty)$ be a center-radius order P-superquadratic $\mathcal{IVF}$ such that $\mathcal{G} = [\underline{\mathcal{G}}, \overline{\mathcal{G}}]$. If $\underline{\mathcal{G}}, \overline{\mathcal{G}} \in \mathcal{J}_{\mathcal{V}}[\sigma, \vartheta]$ with $\sigma < \vartheta$, then

$$\begin{aligned}
& \frac{1}{2}\mathcal{G}\left(\frac{\sigma+\vartheta}{2}\right)+\frac{1}{\vartheta-\sigma}\int_{\sigma}^{\vartheta}\mathcal{G}\left(\left|\frac{(\sigma+\vartheta)-2\mathcal{V}}{2}\right|\right)d\mathcal{V} \\
& \leq_c^r \frac{1}{\vartheta-\sigma}\int_{\sigma}^{\vartheta}\mathcal{G}(\mathcal{V})d\mathcal{V} \\
& \leq_c^r \mathcal{G}(\sigma)+\mathcal{G}(\vartheta)-\frac{1}{\vartheta-\sigma}\left\{\int_{\sigma}^{\vartheta}\left(\mathcal{G}(|\mathcal{V}-\sigma|)+\mathcal{G}(|\vartheta-\mathcal{V}|)\right)d\mathcal{V}\right\}, \tag{2.38}
\end{aligned}$$

holds.

Proof. Since $\mathcal{G}$ is a center-radius order P-superquadratic $\mathcal{IVF}$, therefore we have

$$\mathcal{G}\left(\frac{\mathcal{V}+\mathcal{V}_o}{2}\right) \leq_c^r \mathcal{G}(\mathcal{V})+\mathcal{G}(\mathcal{V}_o)-2\mathcal{G}\left(\frac{1}{2}|\mathcal{V}-\mathcal{V}_o|\right). \tag{2.39}$$

By Definition 1.9, the result (2.39) can be split as follows:

$$\mathcal{G}_c\left(\frac{\mathcal{V} + \mathcal{V}_o}{2}\right) \leq \mathcal{G}_c(\mathcal{V}) + \mathcal{G}_c(\mathcal{V}_o) - 2\mathcal{G}_c\left(\frac{1}{2}|\mathcal{V} - \mathcal{V}_o|\right). \quad (2.40)$$

and

$$\mathcal{G}_r\left(\frac{\mathcal{V} + \mathcal{V}_o}{2}\right) \leq \mathcal{G}_r(\mathcal{V}) + \mathcal{G}_r(\mathcal{V}_o) - 2\mathcal{G}_r\left(\frac{1}{2}|\mathcal{V} - \mathcal{V}_o|\right). \quad (2.41)$$

First considering (2.40) and changing $\mathcal{V}$ by $\wp\sigma + (1 - \wp)\vartheta$ and $\mathcal{V}_o$ by $\wp\vartheta + (1 - \wp)\sigma$, we attain:

$$\begin{aligned} \mathcal{G}_c\left(\frac{\sigma + \vartheta}{2}\right) &\leq \mathcal{G}_c(\wp\sigma + (1 - \wp)\vartheta) + \mathcal{G}_c(\wp\vartheta + (1 - \wp)\sigma) \\ &\quad - 2\mathcal{G}_c(|(\sigma\wp + \vartheta(1 - \wp)) - (\sigma + \vartheta)|). \end{aligned} \quad (2.42)$$

Integrating (2.42) w.r.t $\wp$ over $[0, 1]$, we have

$$\begin{aligned} \mathcal{G}_c\left(\frac{\sigma + \vartheta}{2}\right) &\leq \int_0^1 \mathcal{G}_c(\wp\sigma + (1 - \wp)\vartheta)d\wp + \int_0^1 \mathcal{G}_c(\wp\vartheta + (1 - \wp)\sigma)d\wp \\ &\quad - 2 \int_0^1 \mathcal{G}_c(|(\sigma\wp + \vartheta(1 - \wp)) - (\sigma + \vartheta)|)d\wp. \end{aligned}$$

It implies that

$$\mathcal{G}_c\left(\frac{\sigma + \vartheta}{2}\right) \leq \frac{2}{\vartheta - \sigma} \int_{\sigma}^{\vartheta} \mathcal{G}_c(\mathcal{V})d\mathcal{V} - \frac{2}{\vartheta - \sigma} \int_{\sigma}^{\vartheta} \mathcal{G}_c(|(\sigma + \vartheta) - 2\mathcal{V}|)d\mathcal{V}.$$

Thus

$$\mathcal{G}_c\left(\frac{\sigma + \vartheta}{2}\right) + \frac{2}{\vartheta - \sigma} \int_{\sigma}^{\vartheta} \mathcal{G}_c(|(\sigma + \vartheta) - 2\mathcal{V}|)d\mathcal{V} \leq \frac{2}{\vartheta - \sigma} \int_{\sigma}^{\vartheta} \mathcal{G}_c(\mathcal{V})d\mathcal{V}. \quad (2.43)$$

Similarly considering (2.41), and moving in the same way, we get:

$$\mathcal{G}_r\left(\frac{\sigma + \vartheta}{2}\right) + \frac{2}{\vartheta - \sigma} \int_{\sigma}^{\vartheta} \mathcal{G}_r(|(\sigma + \vartheta) - 2\mathcal{V}|)d\mathcal{V} \leq \frac{2}{\vartheta - \sigma} \int_{\sigma}^{\vartheta} \mathcal{G}_r(\mathcal{V})d\mathcal{V}. \quad (2.44)$$

Again by Definition 1.9, the results (2.43) and (2.44) can be merged as follows:

$$\mathcal{G}\left(\frac{\sigma + \vartheta}{2}\right) + \frac{2}{\vartheta - \sigma} \int_{\sigma}^{\vartheta} \mathcal{G}(|(\sigma + \vartheta) - 2\mathcal{V}|)d\mathcal{V} \leq_c^r \frac{2}{\vartheta - \sigma} \int_{\sigma}^{\vartheta} \mathcal{G}(\mathcal{V})d\mathcal{V}. \quad (2.45)$$

As $\mathcal{G}_c$ is a P-superquadratic, it follows that

$$\begin{aligned} &\mathcal{G}_c(\wp\sigma + (1 - \wp)\vartheta) + \mathcal{G}_c(\wp\vartheta + (1 - \wp)\sigma) \\ &= \mathcal{G}_c(\wp\sigma + (1 - \wp)\vartheta) + \mathcal{G}_c((1 - \wp)\sigma + \wp(\vartheta)) \\ &\leq \mathcal{G}_c(\sigma) - \mathcal{G}_c((1 - \wp)|\sigma - \vartheta|) + \mathcal{G}_c(\vartheta) - \mathcal{G}_c(\wp|\sigma - \vartheta|) \\ &\quad + \mathcal{G}_c(\sigma) - \mathcal{G}_c(\wp|\sigma - \vartheta|) + \mathcal{G}_c(\vartheta) - \mathcal{G}_c((1 - \wp)|\sigma - \vartheta|). \end{aligned} \quad (2.46)$$

Integrating (2.44) w.r.t $\wp$ over $[0, 1]$, we have

$$\int_0^1 \mathcal{G}_c(\wp\sigma + (1 - \wp)\vartheta)d\wp + \int_0^1 \mathcal{G}_c(\wp\vartheta + (1 - \wp)\sigma)d\wp$$

$$\begin{aligned}
&\leq \int_0^1 \left(\mathcal{G}_c(\sigma) - \mathcal{G}_c((1-\wp)|\sigma - \wp|) \right) d\wp \\
&+ \int_0^1 \left(\mathcal{G}_c(\wp) - \mathcal{G}_c(\wp|\sigma - \wp|) \right) d\wp \\
&+ \int_0^1 \left(\mathcal{G}_c(\sigma) - \mathcal{G}_c(\wp|\sigma - \wp|) \right) d\wp \\
&+ \int_0^1 \left(\mathcal{G}_c(\wp) - \mathcal{G}_c((1-\wp)|\sigma - \wp|) \right) d\wp.
\end{aligned} \tag{2.47}$$

It implies

$$\begin{aligned}
&\frac{2}{\wp - \sigma} \int_{\sigma}^{\wp} \mathcal{G}_c(\mathcal{V}) d\mathcal{V} \leq 2[\mathcal{G}_c(\sigma) + \mathcal{G}_c(\wp)] \\
&- \frac{2}{\wp - \sigma} \left\{ \int_{\sigma}^{\wp} \left(\mathcal{G}_c(|\mathcal{V} - \sigma|) + \mathcal{G}_c(|\wp - \mathcal{V}|) \right) d\mathcal{V} \right\}.
\end{aligned} \tag{2.48}$$

Similarly considering $\mathcal{G}_r$ as P -superquadratic function, we get

$$\begin{aligned}
&\frac{2}{\wp - \sigma} \int_{\sigma}^{\wp} \mathcal{G}_r(\mathcal{V}) d\mathcal{V} \leq 2[\mathcal{G}_r(\sigma) + \mathcal{G}_r(\wp)] \\
&- \frac{2}{\wp - \sigma} \left\{ \int_{\sigma}^{\wp} \left(\mathcal{G}_r(|\mathcal{V} - \sigma|) + \mathcal{G}_r(|\wp - \mathcal{V}|) \right) d\mathcal{V} \right\}.
\end{aligned} \tag{2.49}$$

Again by Definition 1.9, the results (2.48) and (2.49) can be merged as follows:

$$\begin{aligned}
&\frac{2}{\wp - \sigma} \int_{\sigma}^{\wp} \mathcal{G}(\mathcal{V}) d\mathcal{V} \leq_c^r 2[\mathcal{G}_r(\sigma) + \mathcal{G}(\wp)] \\
&- \frac{2}{\wp - \sigma} \left\{ \int_{\sigma}^{\wp} \left(\mathcal{G}(|\mathcal{V} - \sigma|) + \mathcal{G}(|\wp - \mathcal{V}|) \right) d\mathcal{V} \right\}.
\end{aligned} \tag{2.50}$$

Merging (2.45) and (2.50), we get the required result. $\square$

Corollary 2.1. *Reversing the inequality in (2.38) yields the $\mathcal{H}.\mathcal{H}$ -inequalities for center-radius order P -subquadratic $I\mathcal{V}\mathcal{F}$ s.*

3. $\mathcal{R}.\mathcal{L}$ FRACTIONAL INTEGRAL INEQUALITIES OF $\mathcal{H}.\mathcal{H}'$ 'S TYPE FOR CENTER-RADIUS ORDER P -SUPERQUADRATIC $I\mathcal{V}\mathcal{F}$

Next, using the $\mathcal{R}.\mathcal{L}$ fractional integral operators developed in interval calculus, we construct the fractional extensions of $\mathcal{H}.\mathcal{H}$ -inequalities.

Theorem 3.1. *Let the function $\mathcal{G} : \mathbb{J} \subset \mathbb{R} \rightarrow (0, \infty)$ be a center-radius order P -superquadratic $I\mathcal{V}\mathcal{F}$ such that $\mathcal{G} = [\underline{\mathcal{G}}, \overline{\mathcal{G}}]$. If $\underline{\mathcal{G}}, \overline{\mathcal{G}} \in I_{\mathcal{V}}[\sigma, \wp]$ with $\sigma < \wp$, then*

$$\begin{aligned}
&\mathcal{G}\left(\frac{\sigma + \wp}{2}\right) + \frac{4\gamma}{(\wp - \sigma)^{\gamma}} \int_{\sigma}^{\frac{\sigma + \wp}{2}} (\wp - \mathcal{V})^{\gamma-1} \mathcal{G}\left(\left|\frac{(\sigma + \wp) - 2\mathcal{V}}{2}\right|\right) d\mathcal{V} \\
&\leq_c^r \frac{\Gamma(1 + \gamma)}{(\wp - \sigma)^{\gamma}} \left[\mathfrak{I}_{\sigma+}^{\gamma} \mathcal{G}(\wp) + \mathfrak{I}_{\wp-}^{\gamma} \mathcal{G}(\sigma) \right]
\end{aligned}$$

$$\begin{aligned} &\leq_c^r 2[\mathcal{G}(\sigma) + \mathcal{G}(\vartheta)] \\ &- \frac{2\gamma}{(\vartheta - \sigma)^\gamma} \left\{ \int_\sigma^\vartheta (\vartheta - \mathcal{V})^{\gamma-1} \left(\mathcal{G}(|\mathcal{V} - \sigma|) + \mathcal{G}(|\vartheta - \mathcal{V}|) \right) d\mathcal{V} \right\}, \end{aligned} \quad (3.1)$$

holds $\forall \gamma \in (0, 1)$.

Proof. Since the function $\mathcal{G}$ is cr- P -superquadratic therefore we attain:

$$\mathcal{G}\left(\frac{\mathcal{V} + \mathcal{V}_o}{2}\right) \leq_c^r \mathcal{G}(\mathcal{V}) + \mathcal{G}(\mathcal{V}_o) - 2\mathcal{G}\left(\frac{1}{2}|\mathcal{V} - \mathcal{V}_o|\right). \quad (3.2)$$

By Definition 1.9, the result (3.2) can be split as follows:

$$\mathcal{G}_c\left(\frac{\mathcal{V} + \mathcal{V}_o}{2}\right) \leq \mathcal{G}_c(\mathcal{V}) + \mathcal{G}_c(\mathcal{V}_o) - 2\mathcal{G}_c\left(\frac{1}{2}|\mathcal{V} - \mathcal{V}_o|\right). \quad (3.3)$$

and

$$\mathcal{G}_r\left(\frac{\mathcal{V} + \mathcal{V}_o}{2}\right) \leq \mathcal{G}_r(\mathcal{V}) + \mathcal{G}_r(\mathcal{V}_o) - 2\mathcal{G}_r\left(\frac{1}{2}|\mathcal{V} - \mathcal{V}_o|\right). \quad (3.4)$$

Considering (3.3) and changing $\mathcal{V}$ by $\wp\sigma + (1 - \wp)\vartheta$ and $\mathcal{V}_o$ by $\wp\vartheta + (1 - \wp)\sigma$, we attain:

$$\begin{aligned} \mathcal{G}_c\left(\frac{\sigma + \vartheta}{2}\right) &\leq \mathcal{G}_c(\wp\sigma + (1 - \wp)\vartheta) + \mathcal{G}_c(\wp\vartheta + (1 - \wp)\sigma) \\ &- 2\mathcal{G}_c\left(\left|\frac{2(\wp\sigma + \vartheta(1 - \wp)) - (\sigma + \vartheta)}{2}\right|\right). \end{aligned} \quad (3.5)$$

Multiplying (3.5) by $\wp^{\gamma-1}$ and then integrating it w.r.t $\wp$ over $[0, 1]$, we have

$$\begin{aligned} \frac{1}{\gamma} \mathcal{G}_c\left(\frac{\sigma + \vartheta}{2}\right) &\leq \int_0^1 \wp^{\gamma-1} \mathcal{G}_c(\wp\sigma + (1 - \wp)\vartheta) d\wp \\ &+ \int_0^1 \wp^{\gamma-1} \mathcal{G}_c(\wp\vartheta + (1 - \wp)\sigma) d\wp \\ &- 2 \int_0^1 \wp^{\gamma-1} \mathcal{G}_c\left(\left|\frac{2(\wp\sigma + \vartheta(1 - \wp)) - (\sigma + \vartheta)}{2}\right|\right) d\wp. \end{aligned}$$

It implies that

$$\begin{aligned} \frac{1}{\gamma} \mathcal{G}_c\left(\frac{\sigma + \vartheta}{2}\right) &\leq \frac{1}{(\vartheta - \sigma)^\gamma} \left\{ \int_\sigma^\vartheta (\vartheta - \mathcal{V})^{\gamma-1} \mathcal{G}_c(\mathcal{V}) d\mathcal{V} + \int_\sigma^\vartheta (\mathcal{V} - \sigma)^{\gamma-1} \mathcal{G}_c(\mathcal{V}) d\mathcal{V} \right\} \\ &- \frac{2}{(\vartheta - \sigma)^\gamma} \int_\sigma^\vartheta (\vartheta - \mathcal{V})^{\gamma-1} \mathcal{G}_c\left(\left|\frac{(\sigma + \vartheta) - 2\mathcal{V}}{2}\right|\right) d\mathcal{V}. \end{aligned}$$

Thus

$$\begin{aligned} \frac{1}{\gamma} \mathcal{G}_c\left(\frac{\sigma + \vartheta}{2}\right) &+ \frac{2}{(\vartheta - \sigma)^\gamma} \int_\sigma^\vartheta (\vartheta - \mathcal{V})^{\gamma-1} \mathcal{G}_c\left(\left|\frac{(\sigma + \vartheta) - 2\mathcal{V}}{2}\right|\right) d\mathcal{V} \\ &\leq \frac{\Gamma(\gamma)}{(\vartheta - \sigma)^\gamma} \left[\mathfrak{I}_{\sigma^+}^\gamma \mathcal{G}_c(\vartheta) + \mathfrak{I}_{\vartheta^-}^\gamma \mathcal{G}_c(\sigma) \right]. \end{aligned} \quad (3.6)$$

Similarly considering (3.4), and moving in the same fashion we attain

$$\begin{aligned} & \frac{1}{\gamma} \mathcal{G}_r\left(\frac{\sigma + \vartheta}{2}\right) + \frac{2}{(\vartheta - \sigma)^\gamma} \int_\sigma^\vartheta (\vartheta - \mathcal{V})^{\gamma-1} \mathcal{G}_r\left(\left|\frac{(\sigma + \vartheta) - 2\mathcal{V}}{2}\right|\right) d\mathcal{V} \\ & \leq \frac{\Gamma(\gamma)}{(\vartheta - \sigma)^\gamma} \left[\mathfrak{I}_{\sigma^+}^\gamma \mathcal{G}_r(\vartheta) + \mathfrak{I}_{\vartheta^-}^\gamma \mathcal{G}_r(\sigma) \right]. \end{aligned} \quad (3.7)$$

Again by Definition 1.9, the results (3.6) and (3.7) can be merged as follows:

$$\begin{aligned} & \frac{1}{\gamma} \mathcal{G}\left(\frac{\sigma + \vartheta}{2}\right) + \frac{2}{(\vartheta - \sigma)^\gamma} \int_\sigma^\vartheta (\vartheta - \mathcal{V})^{\gamma-1} \mathcal{G}\left(\left|\frac{(\sigma + \vartheta) - 2\mathcal{V}}{2}\right|\right) d\mathcal{V} \\ & \leq_c^r \frac{\Gamma(\gamma)}{(\vartheta - \sigma)^\gamma} \left[\mathfrak{I}_{\sigma^+}^\gamma \mathcal{G}(\vartheta) + \mathfrak{I}_{\vartheta^-}^\gamma \mathcal{G}(\sigma) \right]. \end{aligned} \quad (3.8)$$

As $\mathcal{G}_c$ is a P -superquadratic, therefore we attain:

$$\begin{aligned} & \mathcal{G}_c(\wp\sigma + (1 - \wp)\vartheta) + \mathcal{G}_c(\wp\vartheta + (1 - \wp)\sigma) \\ & = \mathcal{G}_c(\wp\sigma + (1 - \wp)\vartheta) + \mathcal{G}_c((1 - \wp)\sigma + \wp\vartheta) \\ & \leq (\mathcal{G}_c(\sigma) - \mathcal{G}_c((1 - \wp)|\sigma - \vartheta|)) + (\mathcal{G}_c(\vartheta) - \mathcal{G}_c(\wp|\sigma - \vartheta|)) \\ & \quad + (\mathcal{G}_c(\sigma) - \mathcal{G}_c(\wp|\sigma - \vartheta|)) + (\mathcal{G}_c(\vartheta) - \mathcal{G}_c((1 - \wp)|\sigma - \vartheta|)). \end{aligned} \quad (3.9)$$

Multiplying (3.9) by $\wp^{\gamma-1}$ and then integrating it w.r.t $\wp$ over $[0, 1]$, we have

$$\begin{aligned} & \int_0^1 \wp^{\gamma-1} \mathcal{G}_c(\wp\sigma + (1 - \wp)\vartheta) d\wp + \int_0^1 \wp^{\gamma-1} \mathcal{G}_c(\wp\vartheta + (1 - \wp)\sigma) d\wp \\ & \leq \int_0^1 \wp^{\gamma-1} (\mathcal{G}_c(\sigma) - \mathcal{G}_c((1 - \wp)|\sigma - \vartheta|)) d\wp \\ & \quad + \int_0^1 \wp^{\gamma-1} (\mathcal{G}_c(\vartheta) - \mathcal{G}_c(\wp|\sigma - \vartheta|)) d\wp \\ & \quad + \int_0^1 \wp^{\gamma-1} (\mathcal{G}_c(\sigma) - \mathcal{G}_c(\wp|\sigma - \vartheta|)) d\wp \\ & \quad + \int_0^1 \wp^{\gamma-1} (\mathcal{G}_c(\vartheta) - \mathcal{G}_c((1 - \wp)|\sigma - \vartheta|)) d\wp. \end{aligned}$$

It implies that

$$\begin{aligned} & \frac{1}{(\vartheta - \sigma)^\gamma} \left\{ \int_\sigma^\vartheta (\vartheta - \mathcal{V})^{\gamma-1} \mathcal{G}_c(\mathcal{V}) d\mathcal{V} + \int_\sigma^\vartheta (\mathcal{V} - \sigma)^{\gamma-1} \mathcal{G}_c(\mathcal{V}) d\mathcal{V} \right\} \\ & \leq \frac{2}{\gamma} [\mathcal{G}_c(\sigma) + \mathcal{G}_c(\vartheta)] \\ & \quad - \frac{1}{(\vartheta - \sigma)^\gamma} \left\{ \int_\sigma^\vartheta (\vartheta - \mathcal{V})^{\gamma-1} \left(\mathcal{G}_c(|\mathcal{V} - \sigma|) + \mathcal{G}_c(|\vartheta - \mathcal{V}|) \right) d\mathcal{V} \right\}. \end{aligned} \quad (3.10)$$

Thus

$$\frac{\Gamma(\gamma)}{(\vartheta - \sigma)^\gamma} \left[\mathfrak{I}_{\sigma^+}^\gamma \mathcal{G}_c(\vartheta) + \mathfrak{I}_{\vartheta^-}^\gamma \mathcal{G}_c(\sigma) \right]$$

$$\begin{aligned} &\leq \frac{2}{\gamma} [\mathcal{G}_c(\sigma) + \mathcal{G}_c(\vartheta)] \\ &- \frac{1}{(\vartheta - \sigma)^\gamma} \left\{ \int_\sigma^\vartheta (\vartheta - \mathcal{V})^{\gamma-1} (\mathcal{G}_c(|\mathcal{V} - \sigma|) + \mathcal{G}_c(|\vartheta - \mathcal{V}|)) d\mathcal{V} \right\}. \end{aligned} \quad (3.11)$$

Similarly for $\mathcal{G}_r$ we attain:

$$\begin{aligned} &\frac{\Gamma(\gamma)}{(\vartheta - \sigma)^\gamma} \left[\mathfrak{I}_{\sigma^+}^\gamma \mathcal{G}_r(\vartheta) + \mathfrak{I}_{\vartheta^-}^\gamma \mathcal{G}_r(\sigma) \right] \\ &\leq \frac{2}{\gamma} [\mathcal{G}_r(\sigma) + \mathcal{G}_r(\vartheta)] \\ &- \frac{1}{(\vartheta - \sigma)^\gamma} \left\{ \int_\sigma^\vartheta (\vartheta - \mathcal{V})^{\gamma-1} (\mathcal{G}_r(|\mathcal{V} - \sigma|) + \mathcal{G}_r(|\vartheta - \mathcal{V}|)) d\mathcal{V} \right\}. \end{aligned} \quad (3.12)$$

Again by Definition 1.9, the results (3.11) and (3.13) can be merged as follows:

$$\begin{aligned} &\frac{\Gamma(\gamma)}{(\vartheta - \sigma)^\gamma} \left[\mathfrak{I}_{\sigma^+}^\gamma \mathcal{G}(\vartheta) + \mathfrak{I}_{\vartheta^-}^\gamma \mathcal{G}(\sigma) \right] \\ &\leq_c^r \frac{2}{\gamma} [\mathcal{G}(\sigma) + \mathcal{G}(\vartheta)] \\ &- \frac{1}{(\vartheta - \sigma)^\gamma} \left\{ \int_\sigma^\vartheta (\vartheta - \mathcal{V})^{\gamma-1} (\mathcal{G}(|\mathcal{V} - \sigma|) + \mathcal{G}(|\vartheta - \mathcal{V}|)) d\mathcal{V} \right\}. \end{aligned} \quad (3.13)$$

The proof is completed by merging inequalities (3.8) and (3.13). $\square$

Corollary 3.1. Reversing the inequalities in (3.1) yields the fractional $\mathcal{H}.\mathcal{H}$ -inequalities for center-radius order subquadratic $\mathcal{IVF}$ s using $\mathcal{R}.\mathcal{L}$ fractional integrals.

Corollary 3.2. If the parameter γ is set to 1 in (3.1), it reduces to (2.38), termed as the $\mathcal{H}.\mathcal{H}$ inequalities for center-radius order P -superquadratic $\mathcal{IVF}$ s.

Example 3.1. Since $\mathcal{G}(\mathcal{V}) = [-\mathcal{V}^3, 2\mathcal{V}^3]$ is a center-radius order P -superquadratic function on $J = [0, \infty)$, then the Figure 2 and Table 1, describe the right, middle and left terms for the Theorem 3.1.

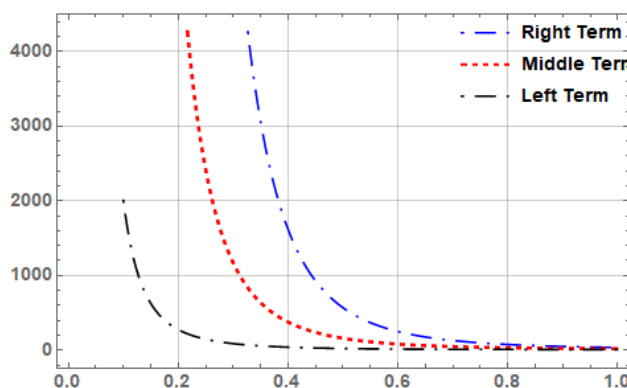


FIGURE 2. Graphical description of $\mathcal{G}_r(\mathcal{V}) = \frac{\mathcal{V}^3}{2}$ via Theorem 3.1 for $\sigma = 2, \vartheta = 3, \gamma = 0.6$ and $m \in (0, 1]$.

γ	Left term	Middle term	Right term
0.1	2006.66	93935.7	1.41657×10^6
0.2	275.859	5879.78	46401.2
0.3	88.4557	1168.97	6424.55
0.4	40.4631	376.29	1615.52
0.5	22.9163	159.672	568.462
0.6	15.1954	81.864	250.007
0.7	11.445	48.5097	129.599
0.8	9.54471	32.3212	76.4297
0.9	8.58633	23.7058	50.0053
1.0	8.12621	18.7826	35.5673

TABLE 1. Numerical description of $\mathcal{G}_r(\mathcal{V}) = \frac{\mathcal{V}^3}{2}$ via Theorem 3.1 for $\sigma = 2, \vartheta = 3$, and $\gamma \in [0, 1]$.

Similarly considering the function $\mathcal{G}_c(\mathcal{V}) = \frac{3\mathcal{V}^3}{2}$, we get the following Figure 3 and Table 2 showing the authenticity of Theorem (3.1).

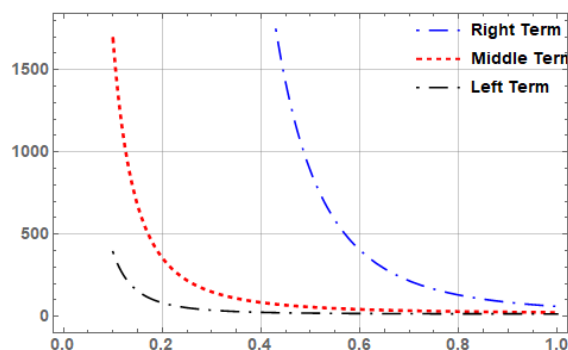


FIGURE 3. Graphical description of $\mathcal{G}_r(\mathcal{V}) = \frac{\mathcal{V}^3}{2}$ via Theorem 3.1 for $\sigma = 2, \vartheta = 3$, $\gamma=0.6$ and $m \in (0, 1]$.

m	Left term	Middle term	Right term
0.1	393.01	1690.54	2.09378×10^6
0.2	83.2462	356.236	68113.4
0.3	38.4606	150.049	9484.71
0.4	25.4242	85.2886	2433.29
0.5	20.3285	57.5737	884.
0.6	17.9986	43.4195	403.644
0.7	16.8351	35.3092	216.643
0.8	16.2282	30.2692	130.905
0.9	15.9096	26.9423	86.4187
1.0	15.7481	24.6413	61.

TABLE 2. Numerical description of $\mathcal{G}_c(\mathcal{V}) = \frac{3\mathcal{V}^3}{2}$ via Theorem 3.1 for $\sigma = 2, \vartheta = 3$, and $\gamma \in [0, 1]$.

4. INFORMATION-THEORETIC APPLICATIONS

The problem of distinguishing between two probability distributions is a basic problem in various fields such as statistics, information theory, and machine learning. The divergence measures provide a quantitative tool for measuring the difference between probability distributions. In 1991, a novel class of divergence measures based on Shannon entropy, which is a fundamental concept in information theory and is used to measure both uncertainty and information in a probability distribution, was proposed by Lin [51]. Lin proposed a divergence measure that provided a rational basis for measuring distributional discrepancy based on information theory principles. Continuing from this work, Shioya and Da-te [52] proposed a novel class of divergence measures based on Lin's divergence in 1995 and proposed a $\mathcal{H}\mathcal{H}\mathcal{G}$ -divergence measure. In this novel class of divergence measures, the $\mathcal{H}\mathcal{H}$ inequality, which is a well-known mathematical inequality associated with convex functions, was used to provide a more general and flexible framework for divergence measures. With this development in divergence measures, a wide variety of applications were made possible in analyzing probability distributions.

We restrict ourselves to the definitions that are pertinent to the outcomes found below in order to guarantee clarity.

Definition 4.1. *Csiszár $\mathcal{G}$ -divergence [53]:*

$$D_{\mathcal{G}}(p_o \| q_o) = \int_{\mathfrak{U}} q_o(\mathcal{V}) \mathcal{G}\left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})}\right) d\mu(\mathcal{V}), \quad p_o, q_o \in \mathbf{P}.$$

Here, $\mathbf{P}$ represents the class of all probability densities with respect to a σ -finite measure δ , $\mathfrak{U} \neq \emptyset$, and $\mathcal{G}$ is convex on $(0, +\infty)$.

$$\mathbf{P} = \left\{ p_o \| p_o : \mathfrak{U} \rightarrow \mathbb{R}, p_o(\mathcal{V}) \geq 0, \int_{\mathfrak{U}} p_o(\mathcal{V}) d\mu(\mathcal{V}) = 1 \right\}.$$

Remark 4.1. Taking $\mathcal{G}$ to be a P -superquadratic function yields the Csiszár $\mathcal{G}$ -divergence for P -superquadratic functions.

Definition 4.2. *$\mathcal{H}\mathcal{H}\mathcal{G}$ -divergence [54]:*

$$D_{HH}^{\mathcal{G}}(p_o \| q_o) = \int_{\mathfrak{U}} q_o(\mathcal{V}) \frac{\int_1^{\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})}} \mathcal{G}(\mathcal{V}) d\mathcal{V}}{\left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - 1\right)} d\mu(\mathcal{V}), \quad p_o, q_o \in \mathbf{P}. \quad (4.1)$$

Definition 4.3. *$\mathcal{R}\mathcal{L}$ fractional $\mathcal{H}\mathcal{H}\mathcal{G}$ -divergence [55]:*

$${}^{\gamma}D_{HH}^{\mathcal{G}}(p_o \| q_o) = \int_{\mathfrak{U}} q_o(\mathcal{V}) \frac{\left[(\mathfrak{I}_{1+}^{\gamma} \mathcal{G})\left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})}\right) + (\mathfrak{I}_{\left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})}\right)^{-}}^{\gamma} \mathcal{G}(1)) \right]}{2 \left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - 1\right)^{\gamma}} d\mu(\mathcal{V}), \quad p_o, q_o \in \mathbf{P}. \quad (4.2)$$

Where the Theorem 1.3, specifies the fractional integrals utilized in (4.2).

Remark 4.2. By taking $\gamma = 1$ in (4.2), we recover (4.1).

We now establish the results associated with fractional $\mathcal{H}\mathcal{H}$ -divergence and $\mathcal{H}\mathcal{H}$ -divergence for center-radius order P-superquadratic $\mathcal{IVF}$:

Theorem 4.1. Let $\mathcal{G} : J \times \Pi \rightarrow \mathfrak{R}$, on $J \subseteq (0, \infty)$, is a α -P-superquadratic $\mathcal{IVF}$ and $\mathcal{G}(1) = 0$, then

$$SD_{\mathcal{G}}\left(\frac{1}{2}q_o + \frac{1}{2}p_o \| q_o\right) + \eta_1 \leq_c^r 2SD_{HH}^{\mathcal{G}}(p_o \| q_o) \leq_c^r 2SD_{\mathcal{G}}(p_o \| q_o) - \eta_2. \quad (4.3)$$

$\forall p_o, q_o \in \mathbf{P}$, where

$$\begin{aligned} SD_{HH}^{\mathcal{G}}(p_o \| q_o) &= \int_{\mathcal{U}} \frac{q_o(\mathcal{V})}{\left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - 1\right)} \left(\int_1^{\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})}} \mathcal{G}(\mathcal{V}) d\mathcal{V} \right) d\mu(\mathcal{V}), \\ \eta_1 &= \int_{\mathcal{U}} \frac{2q_o(\mathcal{V})}{\left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - 1\right)} \left(\int_1^{\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})}} \mathcal{G}\left(\left|\frac{q_o(\mathcal{V}) + p_o(\mathcal{V})}{2q_o(\mathcal{V})} - \mathcal{V}\right|\right) d\mathcal{V} \right) d\mu(\mathcal{V}), \\ \text{and} \\ \eta_2 &= \int_{\mathcal{U}} \frac{2q_o(\mathcal{V})}{\left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - 1\right)} \left(\int_1^{\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})}} \left(\mathcal{G}(|\mathcal{V} - 1|) + \mathcal{G}\left(\left|\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - \mathcal{V}\right|\right) \right) d\mathcal{V} \right) d\mu(\mathcal{V}). \end{aligned}$$

Proof. The $\mathcal{H}\mathcal{H}$ -type inequalities for enter-radius order P-superquadratic $\mathcal{IVF}$, as stated in Theorem 2.3, are considered.

$$\begin{aligned} &\mathcal{G}\left(\frac{\sigma + \vartheta}{2}\right) + \frac{2}{\vartheta - \sigma} \int_{\sigma}^{\vartheta} \mathcal{G}\left(\left|\frac{(\sigma + \vartheta) - 2\mathcal{V}}{2}\right|\right) d\mathcal{V} \\ &\leq_c^r \frac{2}{\vartheta - \sigma} \int_{\sigma}^{\vartheta} \mathcal{G}(\mathcal{V}) d\mathcal{V} \\ &\leq_c^r 2[\mathcal{G}(\sigma) + \mathcal{G}(\vartheta)] - \frac{2}{\vartheta - \sigma} \left\{ \int_{\sigma}^{\vartheta} \left(\mathcal{G}(|\mathcal{V} - \sigma|) + \mathcal{G}(|\vartheta - \mathcal{V}|) \right) d\mathcal{V} \right\}. \end{aligned} \quad (4.4)$$

Now setting $\sigma = 1$, and $\vartheta = \frac{p_o(\mathcal{V})}{q_o(\mathcal{V})}$, in (4.4), we obtain

$$\begin{aligned} &\mathcal{G}\left(\frac{q_o(\mathcal{V}) + p_o(\mathcal{V})}{2q_o(\mathcal{V})}\right) + \frac{2}{\left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - 1\right)} \int_1^{\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})}} \mathcal{G}\left(\left|\frac{q_o(\mathcal{V}) + p_o(\mathcal{V})}{2q_o(\mathcal{V})} - \mathcal{V}\right|\right) d\mathcal{V} \\ &\leq_c^r \frac{2}{\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - 1} \left\{ \int_1^{\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})}} \mathcal{G}(\mathcal{V}) d\mathcal{V} \right\} \leq_c^r 2 \left[\mathcal{G}(1) + \mathcal{G}\left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})}\right) \right] \\ &\quad - \frac{2}{\left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - 1\right)} \left\{ \int_1^{\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})}} \left(\mathcal{G}(|\mathcal{V} - 1|) + \mathcal{G}\left(\left|\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - \mathcal{V}\right|\right) \right) d\mathcal{V} \right\}. \end{aligned} \quad (4.5)$$

Since $\mathcal{G}(1) = 0$, it follows that:

$$\mathcal{G}\left(\frac{q_o(\mathcal{V}) + p_o(\mathcal{V})}{2q_o(\mathcal{V})}\right) + \frac{2}{\left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - 1\right)} \int_1^{\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})}} \mathcal{G}\left(\left|\frac{q_o(\mathcal{V}) + p_o(\mathcal{V})}{2q_o(\mathcal{V})} - \mathcal{V}\right|\right) d\mathcal{V}$$

$$\begin{aligned} &\leq_c^r \frac{2}{\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - 1} \left\{ \int_1^{\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})}} \mathcal{G}(\mathcal{V}) d\mathcal{V} \right\} \leq_c^r 2\mathcal{G}\left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})}\right) \\ &- \frac{2}{\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - 1} \left\{ \int_1^{\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})}} \left(\mathcal{G}(|\mathcal{V} - 1|) + \mathcal{G}\left(\left|\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - \mathcal{V}\right|\right) \right) d\mathcal{V} \right\}. \end{aligned} \quad (4.6)$$

For all $\mathcal{V} \in \mathfrak{U}$, multiply both sides of (4.6) by $q_o(\mathcal{V}) \geq 0$ and integrate over $\mathfrak{U}$ to obtain:

$$\begin{aligned} &\int_{\mathfrak{U}} q_o(\mathcal{V}) \mathcal{G}\left(\frac{q_o(\mathcal{V}) + p_o(\mathcal{V})}{2q_o(\mathcal{V})}\right) d\mu(\mathcal{V}) + \int_{\mathfrak{U}} \frac{2q_o(\mathcal{V})}{\left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - 1\right)} \left(\int_1^{\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})}} \mathcal{G}\left(\left|\frac{q_o(\mathcal{V}) + p_o(\mathcal{V})}{2q_o(\mathcal{V})} - \mathcal{V}\right|\right) d\mathcal{V} \right) d\mu(\mathcal{V}) \\ &\leq_c^r \int_{\mathfrak{U}} \frac{2q_o(\mathcal{V})}{\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - 1} \left(\int_1^{\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})}} \mathcal{G}(\mathcal{V}) d\mathcal{V} \right) d\mu(\mathcal{V}) \leq_c^r 2 \int_{\mathfrak{U}} q_o(\mathcal{V}) \mathcal{G}\left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})}\right) d\mu(\mathcal{V}) \\ &- \int_{\mathfrak{U}} \frac{2q_o(\mathcal{V})}{\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - 1} \left(\int_1^{\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})}} \left(\mathcal{G}(|\mathcal{V} - 1|) + \mathcal{G}\left(\left|\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - \mathcal{V}\right|\right) \right) d\mathcal{V} \right) d\mu(\mathcal{V}). \end{aligned} \quad (4.7)$$

Hence, the desired conclusion follows from the definitions of divergence and $\mathcal{H}\mathcal{H}$ -divergence. $\square$

Theorem 4.2. Let $\mathcal{G} : J \times \mathbb{I} \rightarrow \mathfrak{R}$, on $J \subseteq (0, \infty)$, is a center-radius order superquadratic $\mathcal{IVF}$ and $\mathcal{G}(1) = 0$, then

$$\begin{aligned} &\frac{SD_{\mathcal{G}}\left(\frac{1}{2}q_o + \frac{1}{2}p_o\|q_o\right)}{\Gamma(1 + \gamma)} + \frac{4\gamma\tau_1}{\Gamma(1 + \gamma)} \leq_c^r 2^\gamma SD_{P_{HH}}^{\mathcal{G}}(p_o\|q_o) \\ &\leq_c^r \frac{1}{\Gamma(1 + \gamma)} \left\{ SD_{\mathcal{G}}(p_o\|q_o) + \gamma SD_{\mathcal{G}}(p_o\|q_o) \right\} - \frac{\gamma\tau_2}{\Gamma(1 + \gamma)}. \end{aligned}$$

$\forall p_o, q_o \in P$, and $\gamma > 0$, where

$${}^\gamma SD_{P_{HH}}^{\mathcal{G}}(p_o\|q_o) = \int_{\mathfrak{U}} \frac{q_o(\mathcal{V})}{2\left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - 1\right)^\gamma} \left[\mathfrak{I}_{1+}^\gamma \mathcal{G}\left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})}\right) \right] d\mu(\mathcal{V}),$$

$$\tau_1 = \int_{\mathfrak{U}} \left(\frac{q_o(\mathcal{V})}{\left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - 1\right)^\gamma} \int_1^{\frac{q_o(\mathcal{V}) + p_o(\mathcal{V})}{2q_o(\mathcal{V})}} \left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - \mathcal{V} \right)^{\gamma-1} \mathcal{G}\left(\left| \frac{(1 + \frac{p_o(\mathcal{V})}{q_o(\mathcal{V})}) - 2\mathcal{V}}{2} \right|\right) d\mathcal{V} \right) d\mu(\mathcal{V}),$$

and

$$\tau_2 = \int_{\mathfrak{U}} \frac{q_o(\mathcal{V})}{\left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - 1\right)^\gamma} \left(\int_1^{\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})}} \left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - \mathcal{V} \right)^{\gamma-1} \left(\mathcal{G}(|\mathcal{V} - 1|) + \mathcal{G}\left(\left|\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - \mathcal{V}\right|\right) \right) d\mathcal{V} \right) d\mu(\mathcal{V}).$$

Proof. Theorem 3.1 presents fractional $\mathcal{H}\mathcal{H}$ -type inequalities for center-radius order P-superquadratic $\mathcal{IVF}$ formulated using $\mathcal{R}\mathcal{L}$ integral operators of order $\gamma > 0$.

$$\mathcal{G}\left(\frac{\sigma + \vartheta}{2}\right) + \frac{4\gamma}{(\vartheta - \sigma)^\gamma} \int_\sigma^{\frac{\sigma + \vartheta}{2}} (\vartheta - \mathcal{V})^{\gamma-1} \mathcal{G}\left(\left| \frac{(\sigma + \vartheta) - 2\mathcal{V}}{2} \right|\right) d\mathcal{V}$$

$$\begin{aligned}
&\leq_c^r \frac{\Gamma(1+\gamma)}{(\vartheta-\sigma)^\gamma} \left[\mathfrak{I}_{\sigma+}^\gamma \mathcal{G}(\vartheta) + \mathfrak{I}_{\vartheta-}^\gamma \mathcal{G}(\sigma) \right] \\
&\leq_c^r 2 [\mathcal{G}(\sigma) + \mathcal{G}(\vartheta)] - \frac{\gamma}{(\vartheta-\sigma)^\gamma} \left\{ \int_\sigma^\vartheta (\vartheta-\mathcal{V})^{\gamma-1} (\mathcal{G}(|\mathcal{V}-\sigma|) + \mathcal{G}(|\vartheta-\mathcal{V}|)) d\mathcal{V} \right\}, \quad (4.8)
\end{aligned}$$

Now setting $\sigma = 1$, and $\vartheta = \frac{p_o(\mathcal{V})}{q_o(\mathcal{V})}$, in (4.8), we obtain

$$\begin{aligned}
&\mathcal{G}\left(\frac{q_o(\mathcal{V}) + p_o(\mathcal{V})}{2q_o(\mathcal{V})}\right) + \frac{4\gamma}{\left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - 1\right)^\gamma} \\
&\times \int_1^{\frac{q_o(\mathcal{V}) + p_o(\mathcal{V})}{2q_o(\mathcal{V})}} \left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - \mathcal{V}\right)^{\gamma-1} \mathcal{G}\left(\left|\frac{(1 + \frac{p_o(\mathcal{V})}{q_o(\mathcal{V})}) - 2\mathcal{V}}{2}\right|\right) d\mathcal{V} \\
&\leq_c^r \frac{\Gamma(1+\gamma)}{\left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - 1\right)^\gamma} \left[\mathfrak{I}_{1+}^\gamma \mathcal{G}\left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})}\right) + \mathfrak{I}_{\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})}-}^\gamma \mathcal{G}(1) \right] \leq_c^r 2 \left[\mathcal{G}(1) + \mathcal{G}\left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})}\right) \right] \\
&- \frac{2\gamma}{\left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - 1\right)^\gamma} \left\{ \int_1^{\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})}} \left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - \mathcal{V}\right)^{\gamma-1} \left(\mathcal{G}(|\mathcal{V}-1|) + \mathcal{G}\left(\left|\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - \mathcal{V}\right|\right) \right) d\mathcal{V} \right\}.
\end{aligned}$$

Given that $\mathcal{G}(1) = 0$, it implies that:

$$\begin{aligned}
&\mathcal{G}\left(\frac{q_o(\mathcal{V}) + p_o(\mathcal{V})}{2q_o(\mathcal{V})}\right) + \frac{4\gamma}{\left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - 1\right)^\gamma} \\
&\times \int_1^{\frac{q_o(\mathcal{V}) + p_o(\mathcal{V})}{2q_o(\mathcal{V})}} \left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - \mathcal{V}\right)^{\gamma-1} \mathcal{G}\left(\left|\frac{(1 + \frac{p_o(\mathcal{V})}{q_o(\mathcal{V})}) - 2\mathcal{V}}{2}\right|\right) d\mathcal{V} \\
&\leq_c^r \frac{\Gamma(1+\gamma)}{\left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - 1\right)^\gamma} \left[\mathfrak{I}_{1+}^\gamma \mathcal{G}\left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})}\right) + \mathfrak{I}_{\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})}-}^\gamma (0) \right] \leq_c^r 2\mathcal{G}\left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})}\right) \\
&- \frac{2\gamma}{\left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - 1\right)^\gamma} \left\{ \int_1^{\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})}} \left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - \mathcal{V}\right)^{\gamma-1} \left(\mathcal{G}(|\mathcal{V}-1|) + \mathcal{G}\left(\left|\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - \mathcal{V}\right|\right) \right) d\mathcal{V} \right\}. \quad (4.9)
\end{aligned}$$

Multiplying (4.9), both sides by $q_o(\mathcal{V}) \geq 0$, where $\mathcal{V} \in \mathfrak{U}$ and then integrating the result on $\mathfrak{U}$, we obtain:

$$\begin{aligned}
&\int_{\mathfrak{U}} q_o(\mathcal{V}) \mathcal{G}\left(\frac{q_o(\mathcal{V}) + p_o(\mathcal{V})}{2q_o(\mathcal{V})}\right) d\mu(\mathcal{V}) \\
&+ \int_{\mathfrak{U}} \left(\frac{4q_o(\mathcal{V})\gamma}{\left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - 1\right)^\gamma} \int_1^{\frac{q_o(\mathcal{V}) + p_o(\mathcal{V})}{2q_o(\mathcal{V})}} \left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - \mathcal{V}\right)^{\gamma-1} \mathcal{G}\left(\left|\frac{(1 + \frac{p_o(\mathcal{V})}{q_o(\mathcal{V})}) - 2\mathcal{V}}{2}\right|\right) d\mathcal{V} \right) d\mu(\mathcal{V}) \\
&\leq_c^r \int_{\mathfrak{U}} \frac{q_o(\mathcal{V})\Gamma(1+\gamma)}{\left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})} - 1\right)^\gamma} \left[\mathfrak{I}_{1+}^\gamma \mathcal{G}\left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})}\right) + \mathfrak{I}_{\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})}-}^\gamma (0) \right] d\mu(\mathcal{V}) \leq_c^r 2 \int_{\mathfrak{U}} \left(q_o(\mathcal{V}) \mathcal{G}\left(\frac{p_o(\mathcal{V})}{q_o(\mathcal{V})}\right) \right) d\mu(\mathcal{V})
\end{aligned}$$

$$- \int_{\mathcal{U}} \frac{2q_0\gamma}{\left(\frac{p_0(\mathcal{V})}{q_0(\mathcal{V})} - 1\right)^\gamma} \left[\int_1^{\frac{p_0(\mathcal{V})}{q_0(\mathcal{V})}} \left(\frac{p_0(\mathcal{V})}{q_0(\mathcal{V})} - \mathcal{V} \right)^{\gamma-1} \left(\mathcal{G}(|\mathcal{V} - 1|) + \mathcal{G}\left(\left| \frac{p_0(\mathcal{V})}{q_0(\mathcal{V})} - \mathcal{V} \right| \right) \right) d\mathcal{V} \right] d\mu(\mathcal{V}).$$

Employing the definitions of divergence, $\mathcal{H}\mathcal{H}$ -divergence and $\mathcal{R}\mathcal{L}$ fractional $\mathcal{H}\mathcal{H}$ -divergence, consequently, the stated result is established. $\square$

Remark 4.3. The result of Theorem 4.1 follows directly by taking $\gamma = 1$ in Theorem 4.2.

5. CONCLUSION

In this work, we introduced and systematically developed a new class of center-radius order P -superquadratic $\mathcal{IVF}$ and explored their fundamental structural properties within the framework of interval calculus. Utilizing these properties, we established novel Jensen- and $\mathcal{H}\mathcal{H}$ -type inequalities together with their fractional extensions involving the $\mathcal{R}\mathcal{L}$ fractional integral operators. The theoretical findings were supported by numerical computations and graphical illustrations, which confirmed the validity, stability, and broader applicability of the proposed results. Moreover, the applications presented in information theory demonstrate the practical significance and interdisciplinary relevance of the developed inequalities. Overall, the obtained results not only extend but also refine several known inequalities in the literature, thereby contributing to the advancement of modern inequality theory and fractional analysis. Future investigations may focus on deriving fractional inequalities for center-radius order P -superquadratic $\mathcal{IVF}$ via generalized proportional, Katugampola, Atangana-Baleanu, and conformable fractional operators, as well as exploring stochastic, fuzzy, or quantum variants of these inequalities. Such extensions, together with possible generalizations to multidimensional or operator-valued settings, may lead to further developments and applications in functional analysis, optimization, uncertainty modeling, and quantum information theory.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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