

Some New Oscillation Criteria for a Class of Nonlinear Sub Elliptic P-Laplacian Equations on the Heisenberg Group

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Abstract. In this article, obtain the some new oscillation criteria for a class of nonlinear sub elliptic p-Laplacian equations on the Heisenberg group. By employing a Riccati transformation and establishing suitable integral conditions, we derive both Fite-type and Philo's-type oscillation criteria for second-order sub-elliptic partial differential equations. The obtained results extend several known oscillation theorems for elliptic and quasilinear equations to the sub-elliptic setting. Examples are provided to illustrate the effectiveness of the main results.

1. INTRODUCTION

The oscillation theory for differential and partial differential equations has been extensively developed due to its fundamental importance in qualitative analysis and its wide range of applications in mathematical physics and engineering. Classical oscillation criteria for second-order ordinary differential equations were initiated by the pioneering works of Fite [1] and Wintner [2], and later significantly extended by many other authors, see [3–8].

In recent decades, considerable attention has been devoted to oscillation properties of elliptic and quasilinear elliptic equations involving the p - Laplacian operator. Such equations arise

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naturally in the study of non-Newtonian fluids, nonlinear elasticity, glaciology, and other applied fields [9]. Important contributions in this direction include the works of Noussair and Swanson [10], Usami [11], Marik [12] and Xu [13] among others [14–19].

The Heisenberg group provides a natural non-Euclidean framework in which hypoelliptic operators replace classical elliptic operators. Sub-elliptic partial differential equations on Carnot groups, particularly on the Heisenberg group, play a central role in harmonic analysis, holder regularity, sub-Riemannian geometry, and several complex variables. Fundamental analytical properties of the Heisenberg group have been studied by Folland and Stein [20], Garofalo and Lanconelli [21], and many others [22–28] and existence results on the Heisenberg group cited by [29–31].

Motivated by these developments, several authors have recently investigated oscillation behavior of nonlinear equations on $\mathbb{H}^n$. In particular, Wu & Niu [32] extended oscillation theorems to weighted p sub-Laplacian equations and also Balamani et.al. [33] studied mixed powers on the Heisenberg group. Motivated by above all works, we constructed the following subelliptic p -Laplacian equations of the form,

$$\begin{aligned} \nabla_{\mathbb{H}^n} (\|\nabla_{\mathbb{H}^n} w\|^{p-2} A(z, t) \nabla_{\mathbb{H}^n} w + \langle B^T(z, t), \|\nabla_{\mathbb{H}^n} w\|^{p-2} \nabla_{\mathbb{H}^n} w \rangle \\ + Q(z, t) |w|^{p-2} w + \sum_{i=1}^m Q_i(z, t) |w|^{p_i-2} w = f(z, t). \end{aligned} \quad (1.1)$$

Here $A = (a_{ij}) = (a_{ji}) > 0$ with $a_{ij} \in C^2(\Omega, \mathbb{R})$, $\nabla_{\mathbb{H}^n}$ denotes the Heisenberg gradient, $B^T(z, t) = (b_1, b_2, \dots, b_n, b_{n+1}, \dots, b_{2n})$ with $b_i \in C(\Omega, \mathbb{R})$, $Q, Q_i \in C(\Omega, \mathbb{R})$ and $p, p_i > 1$, $(i=1, 2, \dots, m)$, where $m \in \mathbb{N}$ are mutually different real numbers.

Assume the following notations: The number p^* is a conjugate of p , ie., $p^* = \frac{p}{p-1}$, λ be the smallest eigen value of A and there exists $\lambda(\|(z, t)\|_{\mathbb{H}^n}) \in C(\mathbb{R}^+, \mathbb{R}^+)$ defined by

$$\lambda(\|(z, t)\|_{\mathbb{H}^n}) \leq \min_{\|(z, t)\|_{\mathbb{H}^n} = r} \frac{\lambda}{\|A\|^{\frac{p^*}{p^*-1}}}, r > a_0 > 0, \quad (1.2)$$

where $\|A\| = \left(\sum_{i,j=1}^{2n} a_{ij}^2 \right)^{1/2}$.

The following sets are defined as,

$$\begin{aligned} \mathcal{A}(a_1, a_2) &= \{(z, t) \in \mathbb{H}^n : a_1 \leq \|(z, t)\|_{\mathbb{H}^n} \leq a_2\}, \\ \mathcal{A}[a_1, \infty) &= \{(z, t) \in \mathbb{H}^n : \|(z, t)\|_{\mathbb{H}^n} \geq a_1\} \subset \Omega, \\ B_{\mathbb{H}^n} &= \{(z, t) \in \mathbb{H}^n; \|(z, t)\|_{\mathbb{H}^n} < r\}, \\ S_{\mathbb{H}^n} &= \{(z, t) \in \mathbb{H}^n; \|(z, t)\|_{\mathbb{H}^n} = r\}, \end{aligned}$$

and $\rho \alpha_\rho r^{2n+1}$ is the area of $S_{\mathbb{H}^n}(o, r)$ with $\rho = 2n + 2$ denotes the homogeneous dimension of $\mathbb{H}^n$, α_ρ be volume of $B_{\mathbb{H}^n}(0, r)$ and Hausdorff measure with a $2n$ dimension $d\mathbb{H}^n$ is $\mathbb{R}^{2n+1}$. We denote $B_{\mathbb{H}^n}(0, r)$ and $S_{\mathbb{H}^n}(0, r)$ by B_r and S_r respectively, where $r = \|(z, t)\|_{\mathbb{H}^n}$.

The aim of this paper studying Fite-type and Philo's-type oscillation theory to a general class of nonlinear sub-elliptic p -Laplacian equations. By using Riccati-type transformations and appropriate integral inequalities, we establish new sufficient conditions for oscillation. The obtained criteria generalize several known results in the Euclidean case and provide new insights into oscillatory behavior in the sub-elliptic differential equations.

The following is the structure of the paper. We provide some fundamental lemmas and definitions in Section 2. We have discussed the Fite and Philo's type oscillation criterion in Section 3. Section 4 presents illustrative examples that demonstrate the validity of the main results.

2. DEFINITIONS AND LEMMAS

The nilpotent Lie group, or $\mathbb{H}^n$ of real dimension $2n + 1$, is a manifold that is the product $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ with the group law followed,

$$\bar{\eta} \circ \eta = \left(x + \bar{x}, y + \bar{y}, t + \bar{t} + 2 \sum_{i=1}^n (x_i \bar{y}_i - y_i \bar{x}_i) \right), \quad (2.1)$$

where $\eta = (x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n, t) := (x, y, t) := (z, t) \in \mathbb{C}^{2n} \times \mathbb{R}$, $\bar{\eta} = (\bar{z}, \bar{t})$. The horizontal vectors on $\mathbb{H}^n$ is given by

$$\mathbb{X}_i = \partial x_i + 2y_i \partial t, \quad \mathbb{Y}_i = \partial y_i - 2x_i \partial t, \quad T = \partial t,$$

where $i = (1, 2, \dots, n)$. The Heisenberg gradient is given by

$$\nabla_{\mathbb{H}^n} = (\mathbb{X}_1, \dots, \mathbb{X}_n, \mathbb{Y}_1, \dots, \mathbb{Y}_n),$$

and the sub-Laplacian on the Heisenberg group, defined by

$$\langle \nabla_{\mathbb{H}^n}, \nabla_{\mathbb{H}^n} \rangle = \Delta_{\mathbb{H}^n} = \sum_{i=1}^n \mathbb{X}_i^2 + \mathbb{Y}_i^2,$$

is hypoelliptic by Hormander's theorem [34]. The norm is defined on $\mathbb{H}^n$

$$\|\eta\|_{\mathbb{H}^n} = \left((x^2 + y^2)^4 + t^2 \right)^{\frac{1}{4}}. \quad (2.2)$$

We have provided some fundamental definitions and lemmas before presenting of our main theorems in the next section.

Definition 2.1. If $w(\eta) = w(z, t)$ is said to be nontrivial solution of (1.1), then $w(\eta)$ satisfies

$$\sup\{|w(\eta)| : (\eta) \in \Omega\} > 0. \quad (2.3)$$

Definition 2.2. A nontrivial solution w of (1.1) is considered oscillatory if it contains arbitrarily large zeros; otherwise, it is referred to as nonoscillatory. The equation (1.1) is considered oscillatory if all of its solutions are oscillatory.

If $w \neq 0$ in Ω with based on Riccati type partial differential inequality, we obtain following lemma.

Lemma 2.1. Suppose a function $D(\eta) = D$ such that

$$D \leq Q(\eta) + \sum_{i=1}^m Q_i(\eta) |w|^{p_i-p} w - \frac{f(\eta)}{|w|^{p_i-2} w}. \quad (2.4)$$

Define

$$U = \frac{\|\nabla_{\mathbb{H}^n} w\|^{p-2} A(\eta) \nabla_{\mathbb{H}^n} w}{|w|^{p-2} w}. \quad (2.5)$$

The function U satisfies the following inequality

$$\nabla_{\mathbb{H}^n} U + (p-1) \frac{\lambda(\eta)}{\|A(\eta)\|^{p^*}} |U|^{p^*} + \langle B^T(\eta) A^{-1}(\eta), U \rangle + D \leq 0. \quad (2.6)$$

Proof. Dividing (1.1) by $|w|^{p-2} w$ on both sides, we get

$$\begin{aligned} \nabla_{\mathbb{H}^n} U &= \frac{f(\eta)}{|w|^{p-2} w} - Q(\eta) - \frac{\sum_{i=1}^m Q_i(\eta) |w|^{p_i-2} w}{|w|^{p-2} w} \\ &\quad - (p-1) \frac{\|\nabla_{\mathbb{H}^n} w\|^{p-2} \nabla_{\mathbb{H}^n} w^T A(\eta) \nabla_{\mathbb{H}^n} w}{|w|^p} - \langle B^T(\eta) A^{-1}(\eta), U \rangle. \end{aligned} \quad (2.7)$$

Using the hypothesis (2.4), we obtain the following inequality

$$\nabla_{\mathbb{H}^n} U - (p-1) \frac{\|\nabla_{\mathbb{H}^n} w\|^{p-2} \nabla_{\mathbb{H}^n} w^T A(\eta) \nabla_{\mathbb{H}^n} w}{|w|^p} - \langle B^T(\eta) A^{-1}(\eta), U \rangle + D \leq 0. \quad (2.8)$$

From (2.5),

$$\|U\| \leq \frac{\|\nabla_{\mathbb{H}^n} w\|^{p-1} \|A(\eta)\|}{|w|^{p-1}}. \quad (2.9)$$

It yields

$$\begin{aligned} \|\nabla_{\mathbb{H}^n} w\|^{p-2} (\nabla_{\mathbb{H}^n} w)^T A(\eta) \nabla_{\mathbb{H}^n} w &\geq \lambda(\eta) \frac{\|\nabla_{\mathbb{H}^n} w\|^p}{|w|^p} \\ &= \frac{\lambda(\eta)}{\|A(\eta)\|^{\frac{p}{p-1}}} \|U\|^{\frac{p}{p-1}}, \end{aligned} \quad (2.10)$$

substitute (2.10) in (2.8), we get the required inequality (2.6). $\square$

The following lemma that ensues a vital role in our main results.

Lemma 2.2. Assume that (2.4)-(2.6) of Lemma 2.1 holds. Also $U(\eta)$ satisfies

$$\nabla_{\mathbb{H}^n} U + D_c + 2k \frac{\lambda(\eta)}{\|A(\eta)\|^{p^*}} \|U\|^{p^*} \leq 0, \quad (2.11)$$

where

$$D_c = D - \frac{1}{p} \left(\frac{c}{p} \right)^{p-1} \frac{\|A(\eta)\|^p}{\lambda^{p-1}(\eta)} \|B^T(\eta) A^{-1}(\eta)\|^p, \quad (2.12)$$

$$\text{and } k = \frac{(p-1)(c-1)}{2c}.$$

Proof. Consider

$$(p-1) \frac{\lambda(\eta)}{\|A(\eta)\|^{p^*}} \|U\|^{p^*} + \langle B^T(\eta)A^{-1}(\eta), U \rangle \\ = \left(\frac{p}{c}\right) \frac{\lambda(\eta)}{\|A(\eta)\|^{p^*}} \left[\frac{1}{p^*} \|U\|^{p^*} + \frac{c\|A(\eta)\|^{p^*}}{p\lambda(\eta)} \langle B^T(\eta)A^{-1}(\eta), U \rangle + \frac{c-1}{p^*} \|U\|^{p^*} \right]. \quad (2.13)$$

Using Young's inequality

$$\frac{1}{p^*} \|U\|^{p^*} + \frac{c\|A(\eta)\|^{p^*}}{p\lambda(\eta)} \langle B^T(\eta)A^{-1}(\eta), U \rangle \\ \geq -\frac{1}{p} \left(\frac{c}{p}\right)^p \frac{\|A(\eta)\|^{p^*}}{\lambda^p(\eta)} \|B^T(\eta)A^{-1}(\eta)\|^p,$$

which implies

$$(p-1) \frac{\lambda(\eta)}{\|A(\eta)\|^{p^*}} \|U\|^{p^*} + \langle B^T(\eta)A^{-1}(\eta), U \rangle \\ \geq -\frac{1}{p} \left(\frac{c}{p}\right)^{p-1} \frac{\|A(\eta)\|^p}{\lambda^p(\eta)} \|B^T(\eta)A^{-1}(\eta)\|^p + 2k \frac{\lambda(\eta)}{\|A(\eta)\|^{p^*}} \|U\|^{p^*}. \quad (2.14)$$

Adding bothsides by $D(\eta)$ in the above inequality,

$$(p-1) \frac{\lambda(\eta)}{\|A(\eta)\|^{p^*}} \|U\|^{p^*} + \langle B^T(\eta)A^{-1}(\eta), U \rangle + D \\ \geq -\frac{1}{p} \left(\frac{c}{p}\right)^{p-1} \frac{\|A(\eta)\|^p}{\lambda^p(\eta)} \|B^T(\eta)A^{-1}(\eta)\|^p + 2k \frac{\lambda_{\min}(\eta)}{\|A(\eta)\|^{p^*}} \|U\|^{p^*} + D. \quad (2.15)$$

In the view of (2.8), (2.12) and (2.15), we get desired result (2.11). Hence the work is done. $\square$

From [35], the first lemma is known as divergence formula on $\mathbb{H}^n$ and obtain Ricatti inequality on $\mathbb{H}^n$ by using second lemma.

Lemma 2.3. For any $\mathbf{C}'(\Omega_0)$ vector field $V = (V_1, V_2, \dots, V_n)$, we have

$$\int_{\Omega_0} \nabla_{\mathbb{H}^n} \cdot V dz dt = \int_{\partial\Omega_0} \mathbb{M} V \cdot \nu d\mathbb{H} \quad (2.16)$$

where

$$\mathbb{M} = \begin{pmatrix} I_n & 0 \\ 0 & I_n \\ 2y & -2x \end{pmatrix}_{(2n+1) \times 2n}. \quad (2.17)$$

Lemma 2.4. Let vector field $V = (V_1, V_2, \dots, V_{2n})$, and $p > 1$, we have

$$\|\mathbb{M} V \cdot \nu\|_{L^1(S_r)}^2 \leq (\rho \alpha_\rho r^{2n+1})^{\frac{2}{q}} (4r^2 + 1) \|V\|_{L^{\frac{p}{p-1}}(S_r)}^2. \quad (2.18)$$

In the following section, we provide Fite-type and Philo's-type oscillatory criteria of (1.1).

3. MAIN RESULTS

First, we obtain oscillation criteria for (1.1) by using Fite-type in below.

3.1. Fite type.

Theorem 3.1. Let $\phi \in C'([r_0, \infty), \mathbb{R})$ and $c > 1$ hold. If

$$\lim_{r \rightarrow \infty} \int_{r_0}^r D_{\phi,c}(s) ds = \infty \quad \text{and} \quad (3.1)$$

$$\lim_{r \rightarrow \infty} \int_{r_0}^r \lambda(s) g(s) ds = \infty, \quad (3.2)$$

where

$$D_{\phi,c}(r) = \phi \int_{S_r} D_c(\eta) d\mathbb{H} - \frac{1}{p} (p^* k)^{1-p} (\lambda g)^{1-p}(r) \left\| \frac{\phi'}{\phi} \right\|^p, \quad (3.3)$$

$$g(r) = \left((4r^2 + 1)^{p/2} \phi(r) \rho \alpha_\rho r^{2n+1} \right)^{\frac{1}{1-p}}, \quad (3.4)$$

then (1.1) is oscillatory.

Proof. Suppose that, $w > 0$ on $\Omega(r_1)$, $r_1 \geq r_0$ and define

$$V(r) = \phi \int_{S_r} \mathbb{M}U(\eta) \cdot \nu d\mathbb{H}. \quad (3.5)$$

Both sides differentiate with respect to r ,

$$V' = \frac{\phi'}{\phi} V + \phi \int_{S_r} \nabla_{\mathbb{H}^n} U d\mathbb{H}.$$

The value of (2.11) and in view of (1.2),

$$V' \leq \frac{\phi'}{\phi} V - \phi \int_{S_r} D_c(\eta) d\mathbb{H} - 2k\lambda(r) \phi \int_{S_r} \|U\|^{p^*} d\mathbb{H}. \quad (3.6)$$

By using Lemma 2.4 implies that,

$$\begin{aligned} \|V\| &= \left\| \int_{S_r} \mathbb{M}U(\eta) \cdot \nu d\mathbb{H} \right\|, \\ &\leq \left((4r^2 + 1) \right)^{1/2} \phi \left(\rho \alpha_\rho r^{2n+1} \right)^{1/p} \left(\int_{S_r} \|U\|^{p^*} d\mathbb{H} \right)^{\frac{1}{p^*}}, \\ \int_{S_r} \|U\|^{p^*} d\mathbb{H} &\geq \left(4r^2 + 1 \right)^{\frac{p}{2(1-p)}} \left(\phi^p \rho \alpha_\rho r^{2n+1} \right)^{\frac{1}{1-p}} \|V\|^{p^*}. \end{aligned}$$

It follows,

$$V' \leq \frac{\phi'}{\phi} V - \phi \int_{S_r} D_c(\eta) d\mathbb{H} - 2k\lambda(r) g(r) \|V\|^{p^*}. \quad (3.7)$$

Applying Young's inequality,

$$\left\| \frac{\phi'}{\phi} V \right\| \leq k\lambda(r)g(r) \|V\|^{p^*} + \frac{1}{p} (kp^*)^{1-p} (\lambda g)^{1-p}(r) \left\| \frac{\phi'}{\phi} \right\|^p.$$

Using the above inequality and (2.5) in (3.7),

$$V'(r) \leq -D_{\phi,c}(r) - k\lambda(r)g(r) \|V(r)\|^{p^*}. \quad (3.8)$$

Integrate from r_1 to r in equation (3.8), we have

$$V(r) \leq V(r_1) - \int_{r_1}^r D_{\phi,c}(s) ds - \int_{r_1}^r k\lambda(s)g(s) \|V(s)\|^{p^*} ds. \quad (3.9)$$

By (3.1), implies

$$V(r_1) - \int_{r_1}^r D_{\phi,c}(s) ds \leq 0, \quad r \geq r_2 \geq r_1.$$

From (3.8) and above inequality,

$$\begin{aligned} \|V(r)\| &\geq k \int_{r_1}^r \lambda(s)g(s) \|V(s)\|^{q^*} ds = J(r), \\ \left(\frac{J'}{J^{p^*}} \right)(r) &\geq kg(r). \end{aligned}$$

Integration of the above inequality over $[r_2, \infty]$, the L.H.S becomes convergent but R.H.S (by hypothesis (3.2) gives divergent, which is contradicts the proof. $\square$

In Theorem 3.1, we take $\phi(r) = 1$, $A(\eta) = I_{2n \times 2n}$ by using (1.2) implies $\|A(\eta)\|^2 = 2n$ and minimum eigen value of $A(\eta)$ is 1, then the following known as Fite type theorem.

Corollary 3.1. *Let $p \geq \rho - 2$ and $c \geq 1$ holds. If*

$$\int_{\Omega(r_0)} \left[D - \left(\frac{\sqrt{2n}}{q} \right)^p c^{p-1} \|B^T(\eta)\|^p \right] d\mathbb{H}ds = \infty, \quad (3.10)$$

then (1.1) is oscillatory.

Next obtain the following Corollary, again we take $A(z, t) = I_{2n \times 2n}$ and $\phi(r) = r^{p-(\rho-2)}$, implies $\left\| \frac{\phi'(r)}{\phi(r)} \right\| = \|p - \rho + 2\| r^{-1}$ in Theorem 3.1,

Corollary 3.2. *If*

$$\int_{\Omega(r_0)} \left[D - \frac{1}{p} \left(\frac{2}{p-1} \right)^{p-1} \|p - \rho + 2\|^p (4s^2 + 1)^{p/2} s^{-p} - \left(\frac{\sqrt{2n}}{p} \right)^p \|B^T(\eta)\|^p \right] s^{p-\rho+2} d\mathbb{H}ds = \infty, \quad (3.11)$$

then (1.1) is oscillatory.

In the next section, by using [36], we discuss Philo's type oscillation criteria.

3.2. Philo's type oscillation criteria. Let $\mathcal{B}(r, s, l) = \mathcal{B} \in \mathcal{Y}$, if $\mathcal{B} \in \mathbb{C}(\mathcal{D}_*; \mathbb{R})$, where $\mathcal{D}_* = \{(r, s, l) : r \geq s \geq l \geq r_0 \geq 0\}$, with $\mathcal{B}(r, r, l) = 0 = \mathcal{B}(r, l, l)$, $\mathcal{B}(r, s, l) \geq 0$ for $l < s < r$ and has the partial derivative $\frac{\partial \mathcal{B}}{\partial s}$ in $\mathcal{D}_*$. Also define $\mathcal{B} = \mathcal{B}_1(r, s)\mathcal{B}_2(s, l)$. Here $\mathcal{B}_1(r, s) = \mathcal{B}_1$ and $\mathcal{B}_2(s, l) = \mathcal{B}_2 \in \mathcal{X}$, if $\mathcal{B}_1(r, s), \mathcal{B}_2(s, l) \in \mathbb{C}(\mathcal{D}; \mathbb{R})$ where $\mathcal{D} = \{(r, s) : r \geq s \geq r_0 \geq 0\}$, with

$$\frac{\partial \mathcal{B}_1}{\partial s} = -h_1(r, s) \sqrt{\mathcal{B}_1}, \quad \text{and} \quad \frac{\partial \mathcal{B}_2}{\partial s} = h_2(s, l) \sqrt{\mathcal{B}_2},$$

where h_1, h_2 are locally integrable functions in $\mathcal{D}$. Using the above, we get the following theorem.

Theorem 3.2. For each $\phi \in \mathbb{C}'([r_0, \infty), \mathbb{R}^+)$, there exists $\mathcal{B} \in \mathcal{Y}$ such that

$$\limsup_{r \rightarrow \infty} \frac{1}{\mathcal{B}(r, r_1, l)} \int_{r_1}^r \mathcal{B} \left[D_{\phi, c}(s) - \frac{(p^*k)^{1-p}}{p} (\lambda g)^{1-p}(s) \|b\|^q \right] ds = \infty, \quad (3.12)$$

where

$$b = \frac{h_2}{\sqrt{\mathcal{B}_2}} - \frac{h_1}{\sqrt{\mathcal{B}_1}}, \quad (3.13)$$

then (1.1) is oscillatory.

Proof. Assume that the contrary, $w > 0$ on $\Omega(r_1)$ for some $r_1 \geq r_0$. Multiply both sides by $\mathcal{B}$ on (3.7) and taking integration over $[r_1, r]$, we get

$$\begin{aligned} \int_{r_1}^r \mathcal{B} D_{\phi, c}(s) ds &\leq - \int_{r_1}^r \mathcal{B} V'(s) ds - k \int_{r_1}^r \lambda(s) g(s) \mathcal{B} \|V(s)\|^{p^*} ds, \\ &\leq \mathcal{B}(r, r_1, l) V(r_1) + \int_{r_1}^r \mathcal{B} b V(s) ds - k \int_{r_1}^r \lambda(s) g(s) \mathcal{B} \|V(s)\|^{p^*} ds. \end{aligned} \quad (3.14)$$

Using Young's inequality in following,

$$\|bV(s)\| \leq \frac{1}{p} (p^*k)^{1-p} (\lambda g)^{1-p}(s) \|b\|^p + k \lambda(s) g(s) \|V(s)\|^{p^*}.$$

The above inequality (3.14) implies,

$$\int_{r_1}^r \mathcal{B} \left[D_{\phi, c}(s) - \frac{1}{p} (p^*k)^{1-p} (\lambda g)^{1-p}(s) \|b\|^p \right] ds \leq \mathcal{B}(r, r_1, l) V(r_1).$$

Taking $\limsup_{r \rightarrow \infty}$ and dividing $\mathcal{B}(r, r_1, l)$ in above equation, we get

$$\limsup_{r \rightarrow \infty} \frac{1}{\mathcal{B}(r, r_1, l)} \int_{r_1}^r \mathcal{B} \left[D_{\phi, c}(s) - \frac{1}{p} (p^*k)^{1-p} (\lambda g)^{1-p}(s) \|b\|^p \right] ds \leq \infty.$$

Which is contradiction for (3.8). Hence the proof. $\square$

The following corollary obtain by choose $\mathcal{B} = (r-s)^m (s-l)^n$ for $m, n > \frac{1}{2}$, ie., $\mathcal{B}_1 = (r-s)^m$, $\mathcal{B}_2 = (s-l)^n$, then $\|b(r, s, l)\|^p = \frac{|nr - (m+n)s - ml|^p}{(r-s)^p (s-l)^p}$ in above Theorem 3.2.

Corollary 3.3. If

$$\limsup_{r \rightarrow \infty} \frac{1}{r^m} \int_{r_1}^r (r-s)^m (s-l)^n D_{\phi, c}(s) - \frac{(p^*k)^{1-p}}{p} (\lambda g)^{1-p}(s) \frac{|nr - (m+n)s - ml|^p}{(r-s)^{p-m} (s-l)^{p-n}} ds = \infty, \quad (3.15)$$

then (1.1) is oscillatory.

Examples are illustrate by using Theorem 3.1.

Example 3.1. Consider (1.1) with

$$\begin{aligned} A(\eta) &= \text{diag} \left(\|\eta\|^3 (4\|\eta\|^2 + 1), \|\eta\|^3 (4\|\eta\|^2 + 1), \|\eta\|^3 (4\|\eta\|^2 + 1) \right), \\ B^T(\eta) &= (1, 1, 1), \quad Q(\eta) = \frac{1 + \sin \|\eta\|}{\|\eta\|^4}, \quad Q_i(\eta) = 0, f(\eta) = 0, \end{aligned} \quad (3.16)$$

where $\eta \in \Omega(1)$, $n = 1$, $p = 2$ and $c = 2$. put $\phi(r) = 1$. Using the above values and apply the conditions (3.1) - (3.4) of Theorem 3.1, we get

$$D_{\phi,c}(r) = \frac{\rho\alpha_\rho(1 + \sin r)}{r} - \left(\frac{27}{2}\right) \frac{\rho\alpha_\rho}{r^6(4r^2 + 1)^3}, \quad g(r) = \frac{1}{(4r^2 + 1)\rho\alpha_\rho r^3}.$$

Also,

$$\lim_{r \rightarrow \infty} \int_{r_0}^r D_{\phi,c}(s) ds = \rho\alpha_\rho \lim_{r \rightarrow \infty} \int_{r_0}^r \left(\frac{(1 + \sin s)}{s} - \left(\frac{27}{2}\right) \frac{1}{s^6(4s^2 + 1)^3} \right) ds = \infty,$$

and

$$\lim_{r \rightarrow \infty} \int_{r_0}^r \lambda(s) g(s) ds = \rho\alpha_\rho \lim_{r \rightarrow \infty} \int_{r_0}^r ds = \infty.$$

Hence (3.1), (3.2) of Theorem 3.1 are satisfied. Therefore equation (1.1) is oscillatory.

Example 3.2. Consider (1.1) with

$$A(\eta) = \text{diag}(1, 1, 1), B^T(\eta) = (1, 1, 1), Q(\eta) = \frac{1}{\|\eta\|}, Q_1(\eta) = Q_2(\eta) = 2\|\eta\|, f(\eta) = 0, \quad (3.17)$$

where $\eta \in \Omega(1)$, $n = 1$, $p = 2$ and $c = 2$. Also $\phi(r) = \|\eta\|$. Using the above values and apply the conditions (3.1) - (3.4) of Theorem 3.1, we get

$$D_{\phi,c}(r) = \frac{2 - 3\sqrt{3}}{2} \rho\alpha_\rho r^2, \quad \lim_{r \rightarrow \infty} \int_{r_0}^r g(r) = \lim_{r \rightarrow \infty} \int_{r_0}^r \frac{1}{(4r^2 + 1)\rho\alpha_\rho r^3}.$$

Also

$$\lim_{r \rightarrow \infty} \int_{r_0}^r D_{\phi,c}(s) ds = \left(\frac{2 - 3\sqrt{3}}{2} \right) \rho\alpha_\rho \lim_{r \rightarrow \infty} \int_{r_0}^r s^2 ds = \infty,$$

and

$$\lim_{r \rightarrow \infty} \int_{r_0}^r \lambda(s) g(s) ds = \rho\alpha_\rho \lim_{r \rightarrow \infty} \int_{r_0}^r \frac{1}{(4s^2 + 1)s^3} ds < \infty.$$

Here one of the condition (3.2) is not satisfied. Therefore equation (1.1) is not oscillatory.

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REFERENCES

- [1] W.B. Fite, Concerning the Zeros of the Solutions of Certain Differential Equations, *Trans. Am. Math. Soc.* 19 (1918), 341–352. <https://doi.org/10.1090/s0002-9947-1918-1501107-2>.
- [2] A. Wintner, A Criterion of Oscillatory Stability, *Q. Appl. Math.* 7 (1949), 115–117. <https://doi.org/10.1090/qam/28499>.
- [3] R.P. Agarwal, S.R. Grace, D. O'Regan, *Oscillation Theory for Second Order Linear, Half-Linear, Superlinear and Sublinear Dynamic Equations*, Springer Netherlands, 2002. <https://doi.org/10.1007/978-94-017-2515-6>.
- [4] P. Hartman, *Ordinary Differential Equations*, Wiley, 1964.
- [5] E. Hille, Non-Oscillation Theorems, *Trans. Am. Math. Soc.* 64 (1948), 234–252. <https://doi.org/10.1090/s0002-9947-1948-0027925-7>.
- [6] M. Kirane, Y.V. Rogovchenko, Oscillation Results for a Second Order Damped Differential Equation with Non-monotonous Nonlinearity, *J. Math. Anal. Appl.* 250 (2000), 118–138. <https://doi.org/10.1006/jmaa.2000.6975>.
- [7] Y.G. Sun, New Kamenev-Type Oscillation Criteria for Second-Order Nonlinear Differential Equations with Damping, *J. Math. Anal. Appl.* 291 (2004), 341–351. <https://doi.org/10.1016/j.jmaa.2003.11.008>.
- [8] C.A. Swanson, *Comparison and Oscillation Theory of Linear Differential Equations*, Elsevier, 1968. [https://doi.org/10.1016/s0076-5392\(08\)x6131-4](https://doi.org/10.1016/s0076-5392(08)x6131-4).
- [9] J.I. Díaz, *Nonlinear Partial Differential Equations and Free Boundaries, Vol. I: Elliptic Equations*, Pitman, London, 1985. <https://openlibrary.org/works/OL122307W>.
- [10] E.S. Noussair, C.A. Swanson, Oscillation of Semilinear Elliptic Inequalities by Riccati Transformations, *Can. J. Math.* 32 (1980), 908–923. <https://doi.org/10.4153/cjm-1980-069-8>.
- [11] H. Usami, Some Oscillation Theorems for a Class of Quasilinear Elliptic Equations, *Ann. Mat. Pura Appl.* 175 (1998), 277–283. <https://doi.org/10.1007/bf01783687>.
- [12] R. Mařík, Conjugacy Criteria for Half-Linear ODE in the Theory of PDE with Generalized p -Laplacian and Mixed Powers, *Nonlinear Anal. Theory Methods Appl.* 73 (2010), 500–509. <https://doi.org/10.1016/j.na.2010.03.045>.
- [13] Z. Xu, Oscillation Criteria for Certain Damped PDE's with p -Laplacian, *Glasg. Math. J.* 50 (2008), 129–142. <https://doi.org/10.1017/s0017089507004004>.
- [14] L.C. Evans, *Partial Differential Equations*, American Mathematical Society, 1998. <https://doi.org/10.1090/gsm/019>.
- [15] K. Kreith, *Oscillation Theory*, Springer Berlin Heidelberg, 1973. <https://doi.org/10.1007/BFb0067537>.
- [16] R. Mařík, Oscillation Criteria for PDE with p -Laplacian via the Riccati Technique, *J. Math. Anal. Appl.* 248 (2000), 290–308. <https://doi.org/10.1006/jmaa.2000.6901>.
- [17] S. Priyadarshini, V. Sadhasivam, K.K. Viswanathan, D.S. Sankar, Picone Identities of a Certain Class of Conformable Half Linear Anisotropic Biharmonic Equations, *Malays. J. Fundam. Appl. Sci.* 21 (2025), 1719–1725. <https://doi.org/10.11113/mjfas.v21n1.3549>.
- [18] N. Yoshida, Forced Oscillations of Solutions of Second Order Elliptic Equations, *Appl. Anal.* 25 (1987), 149–155. <https://doi.org/10.1080/00036818708839679>.
- [19] N. Yoshida, *Oscillation Theory of Partial Differential Equations*, World Scientific, Singapore, 2008. <https://doi.org/10.1142/7046>.
- [20] G.B. Folland, E.M. Stein, Estimates for the $\bar{\partial}_b$ Complex and Analysis on the Heisenberg Group, *Commun. Pure Appl. Math.* 27 (1974), 429–522. <https://doi.org/10.1002/cpa.3160270403>.
- [21] N. Garofalo, E. Lanconelli, Frequency Functions on the Heisenberg Group, the Uncertainty Principle and Unique Continuation, *Ann. Inst. Fourier* 40 (1990), 313–356. <https://doi.org/10.5802/aif.1215>.
- [22] S. Bordoni, R. Filippucci, P. Pucci, Nonlinear Elliptic Inequalities with Gradient Terms on the Heisenberg Group, *Nonlinear Anal. Theory Methods Appl.* 121 (2015), 262–279. <https://doi.org/10.1016/j.na.2015.02.012>.
- [23] L. Capogna, D. Danielli, S.D. Pauls, J.T. Tyson, *An Introduction to the Heisenberg Group and the Sub-Riemannian Isoperimetric Problem*, Birkhäuser Basel, 2007. <https://doi.org/10.1007/978-3-7643-8133-2>.

- [24] G. Dwivedi, J. Tyagi, Picone's Identity for Biharmonic Operators on Heisenberg Group and Its Applications, *Nonlinear Differ. Equ. Appl.* 23 (2016), 10. <https://doi.org/10.1007/s00030-016-0376-z>.
- [25] B. Franchi, E. Lanconelli, Hölder Regularity Theorem for a Class of Linear Nonuniformly Elliptic Operators with Measurable Coefficients, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* 10 (1983), 523–541.
- [26] P. Ganguly, S. Thangavelu, On the Lacunary Spherical Maximal Function on the Heisenberg Group, *J. Funct. Anal.* 280 (2021), 108832. <https://doi.org/10.1016/j.jfa.2020.108832>.
- [27] A.K. Nandakumaran, A. Sufian, High Contrasting Diffusion in Heisenberg Group: Homogenization of Optimal Control via Unfolding, *SIAM J. Control Optim.* 61 (2023), 1350–1374. <https://doi.org/10.1137/22M1488582>.
- [28] S. Thangavelu, *Harmonic Analysis on the Heisenberg Group*, Birkhäuser Boston, 1998. <https://doi.org/10.1007/978-1-4612-1772-5>.
- [29] N. Garofalo, E. Lanconelli, Existence and Nonexistence Results for Semilinear Equations on the Heisenberg Group, *Indiana Univ. Math. J.* 41 (1992), 71–98. <https://doi.org/10.1512/iumj.1992.41.41005>.
- [30] F. Safari, A. Razani, Existence of Radial Solutions of the Kohn-Laplacian Problem, *Complex Var. Elliptic Equ.* 67 (2022), 259–273. <https://doi.org/10.1080/17476933.2020.1818733>.
- [31] J.H. Zhang, P. Niu, Existence Results for the Positive Solutions of Semilinear Equations on the Heisenberg Group, *Nonlinear Anal. Theory Methods Appl.* 31 (1998), 181–189. [https://doi.org/10.1016/S0362-546X\(96\)00303-3](https://doi.org/10.1016/S0362-546X(96)00303-3).
- [32] D. Wu, P. Niu, Interval Oscillation Theorems for the Weighted p -Sub-Laplacian Equation in the Heisenberg Group, *Adv. Math. Phys.* 2022 (2022), 6231658. <https://doi.org/10.1155/2022/6231658>.
- [33] S. Balamani, S. Priyadharshini, K.K. Viswanathan, V. Sadhasivam, Oscillatory Behavior of Certain Class of Mixed Nonlinear Sub-Elliptic Equations in the Heisenberg Group, *Math. Stat.* 13 (2025), 302–311. <https://doi.org/10.13189/ms.2025.130505>.
- [34] L. Hörmander, Hypoelliptic Second Order Differential Equations, *Acta Math.* 119 (1967), 147–171. <https://doi.org/10.1007/BF02392081>.
- [35] D. Wu, P. Niu, An Oscillation Theorem for the Nonlinear Degenerate Elliptic Equation in the Heisenberg Group, *Bound. Value Probl.* 2021 (2021), 40. <https://doi.org/10.1186/s13661-021-01516-7>.
- [36] Ch.G. Philos, Oscillation Theorems for Linear Differential Equations of Second Order, *Arch. Math.* 53 (1989), 482–492. <https://doi.org/10.1007/BF01324723>.