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FIXED POINTS OF EXPANSIVE TYPE MAPPINGS IN 2-BANACH SPACES

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Abstract. In present paper, we define expansive mappings in 2-Banach space and prove some common unique fixed point theorems which are the extension of results of Wang et al. [12] and Rhoades [9] in 2-Banach space.

1. Introduction

The research about fixed points of expansive mapping was initiated by Machuca (see [6]). Later Jungck discussed fixed points for other forms of expansive mapping (see [5]). In 1982, Wang et al. (see [12]) presented some interesting work on expansive mappings in metric spaces which correspond to some contractive mapping in [10]. Also, Zhang has done considerable work in this field. In order to generalize the results about fixed point theory, Zhang (See [14]) published his work Fixed Point Theory and Its Applications, in which the fixed point problem for expansive mapping is systematically presented in a chapter. As applications, he also investigated the existence of solutions of equations for locally condensing mapping and locally accretive mapping. On the other hand Gahler ([2],[3]) investigated the idea of 2-metric and 2-Banach spaces and proved same results. Subsequently several authors including Iseki [4], Rhoades [8], White [13], Panja and Baisnab [7] and Saha et al [11] studied various aspects of the fixed point theory and proved fixed point theorems in 2-metric spaces and 2-Banach spaces. Recently, the study about fixed point theorem for expansive mapping is deeply explored and has extended too many others directions. Motivated and inspired by the above work, in this paper we investigate fixed point for expansive mapping in 2-Banach spaces. The presented theorems extend, generalize and improve many existing results in the literature [9] [12].

2. Preliminaries

Definition 2.1. Let X be a real linear space and $\| .,. \|$ be a non-negative real valued function defined on $X \times X$ satisfying the following conditions:
i) $\|x, y\| = 0$ if and only if x and y are linearly dependent in X,
ii) $\|x, y\| = \|y, x\|$ for all $x, y \in X$
iii) $\|x, ay\| =\mid a \mid \|x, y\|$ a being real, $x, y \in X$
iv) $\|x, y + z\| = \|x, y\| + \|x, z\|$ for all $x, y, z \in X$

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Then $\| .,. \|$ *is called a 2-norm and the pair* $(X, \| .,. \|)$ *is called a linear 2-normed space.*

Some of the basic properties of 2-norms are that they are non negative satisfying $\|x, y + ax\| = \|x, y\|$, for all $x, y \in X$ and all real numbers a.

Definition 2.2. A sequence $\{x_n\}$ in a linear 2-normed space $(X, \| .,. \|)$ is called Cauchy sequence if

$$\lim_{m,n\to\infty} \| x_m - x_n, y \| = 0 \quad \text{for all } y \text{ in } X.$$

Definition 2.3. A sequence $\{x_n\}$ in a linear 2-normed space $(X, \| .,. \|)$ is said to be convergent if there is a point x in X such that

$$\lim_{n\to\infty} \| x_n - x, y \| = 0 \quad \text{for all } y \text{ in } X.$$

If $\{x_n\}$ *converges to* x*, we write* $\{x_n\} \to x$ *as* $n \to \infty$*.*

Definition 2.4. A linear 2-normed space X is said to be complete if every Cauchy sequence is convergent to an element of X. We then call X to be a 2- Banach space.

Definition 2.5. Let X be a 2-Banach space and T be a self-mapping of X. T is said to be continuous at x if for every sequence $\{x_n\}$ in X, $\{x_n\} \to x$ as $n \to \infty$ implies $\{T(x_n)\} \to T(x)$ as $n \to \infty$.

Example 2.6. Let X is R^3 and consider the following 2-norm on X as

$$\| \mathbf{x}, \mathbf{y} \| = \left| det \begin{pmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \right|,$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. Then $(X, \| .,. \|)$ is a 2-Banach space.

Example 2.7. Let P_n denotes the set of all real polynomials of degree $\leq n$, on the interval [0,1]. By considering usual addition and scalar multiplication, P_n is a linear vector space over the reals. Let $\{x_o, x_1, ..., x_{2n}\}$ be distinct fixed points in [0,1] and define the following 2-norm on P_n:$\| f, g \| = \sum_{k=0}^{2n} | f(x_k)\, g(x_k) |$, whenever f and g are linearly independent and $\| f, g \| = 0$, if f, g are linearly dependent. Then $(P_n, \| .,. \|)$ is a 2-Banach space.

Example 2.8. Let X is Q^3, the field of rational number and consider the following 2-norm on X as:

$$\| \mathbf{x}, \mathbf{y} \| = \left| det \begin{pmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \right|,$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. Then $(X, \| .,. \|)$ is not a 2-Banach space but a 2-normed space.

3. Main Results

Definition 3.1. Let $(X, \| .,. \|)$ be a 2-Banach space with 2-norm $\| .,. \|$. A mapping T of X into itself is said to be expansive if there exists a constant $h > 1$ such that

$$\| Tx - Ty, a \| \geq h \| x - y, a \| \; for \; all \; x, y \in X$$

Example 3.2. Let $X = R^2$ and consider the following 2-norm on X as $\| x, y \| = | x_1 y_2 - x_2 y_1 |$ where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then $(X, \| .,. \|)$ is a 2-Banach space. Define a self map T on X as follows $Tx = \beta x$ where $\beta > 1$ for all $x \in X$, clearly T is an expansive mapping.

Theorem 3.3. Let $(X, \| .,. \|)$ be a 2-Banach space and S, T be surjective mapping of X into itself satisfying.

$$\begin{aligned} \| Sx - Ty, a \| + k\Big[\| x - Ty, a \| + \| y - Sx, a \| \Big] &\geq a_1 \| x - Sx, a \| \\ &+ b_1 \| y - Ty, a \| + c_1 \| x - y, a \| \end{aligned} \tag{3.3.1}$$

for all $x, y, a \in X$, $x \neq y$ where $a_1, b_1, c_1, k \geq 0$, satisfy $a_1 < k + 1$, $b_1 < k + 1, c_1 > 2k + 1$. Then S and T have a common unique fixed point in X.

Proof: We define a sequence $\{x_n\}$ as follows for $n = 0, 1, 2, 3, ...$

$$x_{2n} = Sx_{2n+1}, \quad x_{2n+1} = Tx_{2n+2} \tag{3.3.2}$$

If $x_{2n} = x_{2n+1} = x_{2n+2}$ for some n then we see that x_{2n} is a fixed point of S and T. Therefore, we suppose that no two consecutive terms of sequence $\{x_n\}$ are equal.

Now we put $x = x_{2n+1} \; and \; y = x_{2n+2}$ in (3.3.1) we get

$$\begin{aligned} &\| Sx_{2n+1} - Tx_{2n+2}, a \| + k\Big[\| x_{2n+1} - Tx_{2n+2}, a \| + \| x_{2n+2} - Sx_{2n+1}, a \| \Big] \\ &\geq a_1 \| x_{2n+1} - Sx_{2n+1}, a \| + b_1 \| x_{2n+2} - Tx_{2n+2}, a \| + c_1 \| x_{2n+1} - x_{2n+2}, a \| \end{aligned}$$

$$\begin{aligned} \Rightarrow \| x_{2n} - x_{2n+1}, a \| + k\Big[\| x_{2n+1} - x_{2n+1}, a \| + \| x_{2n+2} - x_{2n}, a \| \Big] &\geq a_1 \| x_{2n+1} - x_{2n}, a \| \\ &+ b_1 \| x_{2n+2} - x_{2n+1}, a \| + c_1 \| x_{2n+1} - x_{2n+2}, a \| \end{aligned}$$

$$\begin{aligned} \Rightarrow \| x_{2n} - x_{2n+1}, a \| + k\Big[\| x_{2n+2} - x_{2n}, a \| \Big] &\geq a_1 \| x_{2n+1} - x_{2n}, a \| + b_1 \| x_{2n+2} - x_{2n+1}, a \| \\ &+ c_1 \| x_{2n+1} - x_{2n+2}, a \| \end{aligned}$$

$$\begin{aligned} \Rightarrow \| x_{2n} - x_{2n+1}, a \| + k\Big[\| x_{2n+2} - x_{2n+1}, a \| + \| x_{2n+1} - x_{2n}, a \| \Big] &\geq a_1 \| x_{2n+1} - x_{2n}, a \| \\ &+ b_1 \| x_{2n+2} - x_{2n+1}, a \| + c_1 \| x_{2n+1} - x_{2n+2}, a \| \end{aligned}$$

$$\Rightarrow (1 + k - a_1) \| x_{2n} - x_{2n+1}, a \| \geq (b_1 + c_1 - k) \| x_{2n+1} - x_{2n+2}, a \|$$

$$\Rightarrow \| x_{2n+1} - x_{2n+2}, a \| \leq \tfrac{(1+k-a_1)}{(b_1+c_1-k)} \| x_{2n} - x_{2n+1}, a \|$$

$$\Rightarrow \| x_{2n+1} - x_{2n+2}, a \| \leq k_1 \| x_{2n} - x_{2n+1}, a \|$$

where $k_1 = \frac{(1+k-a_1)}{(b_1+c_1-k)} < 1 \; (As \quad a_1 + b_1 + c_1 > 1 + 2k)$

Similarly, we can calculate

$$\| x_{2n+2} - x_{2n+3}, a \| \leq k_2 \| x_{2n+1} - x_{2n+2}, a \| \text{ for } n = 0, 1, 2, ...$$
$$\text{where } k_2 = \frac{(1+k-a_1)}{(b_1+c_1-k)} < 1 \; (As \quad a_1 + b_1 + c_1 > 1 + 2k)$$

and so on

So, in general

$$\Rightarrow \| x_n - x_{n+1}, a \| \leq k \| x_{n-1} - x_n, a \| \text{ for } n = 1, 2, 3...$$
$$\text{where } k = \max\{k_1, k_2\} \text{ then } k < 1$$

(3.3.3) $$\Rightarrow \| x_n - x_{n+1}, a \| \leq k^n \| x_0 - x_1, a \|$$

We can prove that $\{x_n\}$ is a Cauchy sequence (using (3.3.3)). So there exists a point x in X such that

(3.3.4) $$\{x_n\} \to x \; as \; n \to \infty$$

Existence of fixed point: Since S and T are surjective maps, so there exist two points y and y' in X such that

(3.3.5) $$x = Sy \; and \; x = Ty'$$

Consider

$$\| x_{2n} - x, a \| = \| Sx_{2n+1} - Ty', a \|$$
$$\geq -k\Big[\| x_{2n+1} - Ty', a \| + \| y' - Sx_{2n+1}, a \| \Big] + a_1 \| x_{2n+1} - Sx_{2n+1}, a \| + b_1 \| y' - Ty', a \|$$
$$+ c_1 \| x_{2n+1} - y', a \|$$

$$\Rightarrow \| x_{2n} - x, a \| \geq -k\Big[\| x_{2n+1} - Ty', a \| + \| y' - x_{2n}, a \| \Big] + a_1 \| x_{2n+1} - x_{2n}, a \|$$
$$+ b_1 \| y' - Ty', a \| + c_1 \| x_{2n+1} - y', a \|$$

As $\{x_{2n}\}$, $\{x_{2n+1}\}$ are subsequences of $\{x_n\}$ as $n \to \infty$, $\{x_{2n}\} \to x$, $\{x_{2n+1}\} \to x$ (Using 3.3.4)

Therefore

$$\| x - x, a \| \geq -k\Big[\| x - x, a \| + \| y' - x, a \| \Big] + a_1 \| x - x, a \| + b_1 \| y' - x, a \|$$
$$+ c_1 \| x - y', a \|$$
$$\Rightarrow 0 \geq (b_1 + c_1 - k) \| x - y', a \|$$

$$\Rightarrow \| x - y', a \| = 0 \; (As \; 2k + 1 < a_1 + b_1 + c_1 < k + 1 + b_1 + c_1, \; so \; that \; k < b_1 + c_1)$$

(3.3.6) $$\Rightarrow x = y'$$

In an exactly similar way (Using $b_1 < k + 1$) we can prove that,

(3.3.7) $$x = y$$

The fact (3.3.5) along with (3.3.6) and (3.3.7) shows that x is a common fixed point of S and T.

Uniqueness: Let z be another common fixed point of S and T, that is

$$Sz = z \; and \; Tz = z$$

$$\begin{aligned}\| x - z, a \| &= \| Sx - Tz, a \| \\ &\geq -k\Big[\| x - Tz, a \| + \| z - Sx, a \| \Big] + a_1 \| x - Sx, a \| + b_1 \| z - Tz, a \| \\ &+ c_1 \| x - z, a \|\end{aligned}$$

$$\Rightarrow \| x - z, a \| \geq -k\Big[\| x - z, a \| + \| z - x, a \| \Big] + a_1 \| x - x, a \| + b_1 \| z - z, a \| + c_1 \| x - z, a \|$$
$$\Rightarrow \| x - z, a \| \geq (-2k + c_1) \| x - z, a \|$$
$$\Rightarrow (1 + 2k - c_1) \| x - z, a \| \geq 0$$
$$\Rightarrow \| x - z, a \| = 0 \quad (As \quad c_1 > 2k + 1)$$

$$\Rightarrow x = z$$

This completes the proof of the theorem 3.3

Corollary 3.4. Let $(X, \| .,. \|)$ be a 2-Banach space and S and T be two surjective mappings of X into itself such that for every $x, y, a \in X$

$$\| Sx - Ty, a \| \geq \; a_1 \| x - y, a \| \tag{3.4.1}$$

where $a_1 > 1$. Then S and T have a common unique fixed point in X.

Proof: If we put $k, a_1, b_1 = 0$ and $c_1 = a_1$ in theorem 3.3 then we get above corollary 3.4.

Corollary 3.5. Let $(X, \| .,. \|)$ be a 2-Banach space and T be a surjective mapping of X into itself such that for every $x, y, a \in X$

$$\| Tx - Ty, a \| \geq \; a_1 \| x - y, a \| \tag{3.5.1}$$

where $a_1 > 1$. Then T has a unique fixed point in X.

Proof: If we put $S = T$ in corollary 3.4 then we get above corollary 3.5 which is an extension of theorem 1 of Wang et al. [12] in 2-Banach space.

Corollary 3.6. Let $(X, \| .,. \|)$ be a 2-Banach space and S, T be two surjective mappings of X into itself satisfying.

$$\| Sx - Ty, a \| \geq a_1 \| x - Sx, a \| + a_2 \| y - Ty, a \| + a_3 \| x - y, a \| \tag{3.6.1}$$

for each $x, y \in X$, with $x \neq y$ where $a_1, a_2, a_3 \geq 0$ and $a_3 > 1$. Then S and T have a common unique fixed point in X.

Proof: If we put $k = 0$ and $b_1 = a_2$, $c_1 = a_3$ in theorem 3.3 then we get above corollary 3.6.

Corollary 3.7. Let $(X, \| .,. \|)$ be a 2-Banach space and T be a surjective mapping of X into itself satisfying.

$$(3.7.1) \qquad \| Tx - Ty, a \| \geq a_1 \| x - Tx, a \| + a_2 \| y - Ty, a \| + a_3 \| x - y, a \|$$

for each $x, y \in X$ with $x \neq y$ where $a_1, a_2, a_3 \geq 0$ and $a_3 > 1$. Then T has a unique fixed point in X.

Proof: If we put $S = T$ in corollary 3.6 then we get above corollary 3.7 which is an extension of theorem 2 of Wang et al. [12] in 2-Banach space.

The following theorem 3.8 is an example in which the fixed point need not be unique and continuity of self maps are required and it extend the theorem 3 of Rhoades [9] in 2-Banach spaces.

Theorem 3.8. Let $(X, \| .,. \|)$ be a 2-Banach space and S, T be two continuous mappings of X into itself satisfying.

$$(3.8.1) \qquad \| Sx - Ty, a \| \geq a_1 \min\{\| x - y, a \|, \| x - Sx, a \|, \| y - Ty, a \|\}$$

for every $x, y \in X$, $x \neq y$ Where $a_1 > 1$. Then S and T have a common fixed point in X.

Proof: We define a sequence $\{x_n\}$ as follows for $n = 0, 1, 2, 3, ...$

$$x_{2n} = Sx_{2n+1}, \quad x_{2n+1} = Tx_{2n+2}$$

Now we put $x = x_{2n+1}$ and $y = x_{2n+2}$ in (3.8.1) we get

$$\|Sx_{2n+1} - Tx_{2n+2}, a\| \geq a_1 \min\left\{ \|x_{2n+1} - x_{2n+2}, a\| , \|x_{2n+1} - Sx_{2n+1}, a\| , \|x_{2n+2} - Tx_{2n+2}, a\| \right\}$$

$$= a_1 \min\left\{ \|x_{2n+1} - x_{2n+2}, a\| , \|x_{2n+1} - x_{2n}, a\| , \|x_{2n+2} - x_{2n+1}, a\| \right\}$$

$$\Rightarrow \|x_{2n} - x_{2n+1}, a\| \geq a_1 \min\left\{ \|x_{2n+1} - x_{2n+2}, a\| , \|x_{2n} - x_{2n+1}, a\| \right\}$$

Case I

$$\| x_{2n} - x_{2n+1}, a \| \geq a_1 \| x_{2n+1} - x_{2n}, a \|$$

$$\Rightarrow 1 \geq a_1$$

Which is contradiction.

Case II

$$\| x_{2n+1} - x_{2n+2}, a \| \leq \frac{1}{a_1} \| x_{2n} - x_{2n+1}, a \|$$

$$\| x_{2n+1} - x_{2n+2}, a \| \leq k \| x_{2n} - x_{2n+1}, a \|$$

where $k = \frac{1}{a_1} < 1 \, (As \quad a_1 > 1)$

So, in general

$\Rightarrow \| x_n - x_{n+1}, a \| \leq k \| x_{n-1} - x_n, a \|$ for $n = 1, 2, 3...$

(3.8.2) $$\Rightarrow \| x_n - x_{n+1}, a \| \leq k^n \| x_0 - x_1, a \|$$

We can prove that $\{x_n\}$ is a Cauchy sequence (using (3.8.2). So there exists a point x in X such that

(3.8.3) $$\{x_n\} \to x \; as \; n \to \infty$$

Existence of Fixed Point: If S and T are continuous then existence part follows very easily. As shown below

$$x = \lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} Sx_{2n+1} = S \lim_{n\to\infty} x_{2n+1} = Sx \; (as \; n \to \infty \; \{x_{2n+1}\} \to x)$$

Similarly

$$x = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} Tx_{2n+2} = T \lim_{n\to\infty} x_{2n+2} = Tx \; (as \; n \to \infty \; \{x_{2n+2}\} \to x)$$

This completes the proof of the theorem 3.8

Corollary 3.9. Let $(X, \| .,. \|)$ be a 2-Banach space and T be a continuous mapping of X into itself satisfying.

(3.9.1) $$\| Tx - Ty, a \| \geq a_1 \min\{\| x - y, a \|, \| x - Tx, a \|, \| y - Ty, a \|\}$$

for every $x, y \in X$, $x \neq y$ Where $a_1 > 1$. Then T has a fixed point in X.

Proof: If we put $S = T$ in theorem 3.8 then we get above corollary 3.9 which is an extension of theorem 3 of Wang et al. [12] in 2-Banach space.

Remark 1: If mappings are continuous in theorem 3.3 then existence of fixed point follows very easily as proved in theorem 3.8.

Remark 2: In corollary 3.9, we proved the fixed point is unique by using only $a_3 > 1$ and there is no need of $a_1 < 1$ and $a_2 < 1$, so it extend and unify the theorem 2 of Wang et al. [12].

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