

Dynamical Analysis of a Rumor Propagation Model With Saturated Recruitment and Incorporated Control Strategies

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Abstract. Unverified information can spread quickly and influence the public. In this work, we suggest and analyze a nonlinear rumor propagation model incorporating saturated incidence and optimal control. In this model, the qualitative properties of the model are rigorously analyzed. The positivity and boundedness of the solutions are established to ensure biological feasibility. The threshold dynamics is governed, equilibrium points are analyzed and their stability. Bifurcation analysis shows that forward bifurcations occur at a reproduction number equal to one. Sensitivity analysis reveals that rumor propagation is strongly influenced by contact rate with rumor, the direct probability of rumor, and the transition from rumor spreaders to lurkers. To design mitigation strategies, an optimal control is produced with two controls: educational control and truth-defender control. The optimality conditions are derived using Pontryagin's and numerically utilizing the forward-backward sweep method. Numerical simulations demonstrate that truth dissemination is the dominant intervention in reducing rumor spreaders and lurkers, while contact reduction provides strong support in the early stage. The combined control strategy produces the most robust outcome by simultaneously minimizing the rumor and strengthening the corrective spread of information.

1. INTRODUCTION

In the big data era, the propagation of information has become an increasingly essential concern, as unverified issues can be propagated and amplified at unprecedented speed. Rumors

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are unverified claims, their interpretation, or assessment concerning public topics whose legitimacy remains unconfirmed [1]. Now, rumor propagation is not geographically constrained, since social networking platforms allow users to spread information widely among heterogeneous humans [2–4]. Consequently, the harmful impacts of the propagation of rumors can lead to an increase in misinformation, heightened public interest, social instability, and a decrease in public trust, particularly when unreliable information spreads before corrective or verified information becomes available [5,6]. These phenomena indicate that the diffusion of rumors is a complex and unpredictable process influenced by individual psychological reactions, behavioral diversity, factual information, public exploration, and social interactions between individuals [7,8].

Due to its similarity to the modeling of disease transmission, mathematical models are recognized as a powerful tool for investigating, describing, predicting, and controlling the propagation of unconfirmed information [9–11]. Many compartments have been offered to portray the behavior of rumor propagation. By tracking, the basic of classical models to describe the rumor propagation was studied by Daley and Kendall in 1965 [11]. This model provides the basic analysis and proposes the categories of the model compartment as *SIR*. Subsequently, the deep model of rumor analysis conducted by Maki and Thomson [12], they interpreted the diffusion of rumor by including the stochastic process. The current literature develops rumor models by dividing the population into ignorant, rumor spreaders, hesitant, lurkers, recovered, truth spreaders, and counter rumor [5,8]. These models provide a rigorous analytical basis for the behavioral dynamics to represent the rumor phenomenon. Some researchers such as [13–15] constructed a mathematical model of rumor propagation that is interesting to latent groups, and also developed a model of rumor propagation that contains the educational level of individuals [16]. The findings contribute that the prior of educational level influences the spread of rumors among individuals. Jie in 2016 examined the diffusion of two rumors, and the results showed that the two circulating rumors can positively or negatively affect individuals [17]. In addition, the worker set up a mathematical model of rumor propagation that interests the wisdom level, analysis results showing that the wisdom level influences the speed of rumor distribution within a population [18,19]. Using the work of Liu [20], we show that the amplification of truth spreader in populations is effective in reducing rumors infection.

In subsequent research, the examination of rumor propagation using delay has been developed. For example, Zhu [21] expressed the rumor transmission by incorporating the force effort to stop rumors through time-delayed, considering the results showing that efforts to stop rumors were effective in reducing rumors. In addition, scientific research collaboration to develop mathematical models of rumors with time delays, as well as these findings, contributes that the degree of time delay in countering rumors influences the threshold value of the secondary reproduction number [9,21]. In 2021, Wang et al. pointed out in their research that incorporating a time delay component into rumor propagation models is effective in slowing and mitigating the spread of rumors [22]. In the other time, several scholars established and considered mitigation of rumor

propagation by using control approaches. Chen and friends recommended the role of controlling rumor diffusion by reducing interactions directly between susceptible individuals and rumor spreaders [23], and also controlling rumor by reducing the mobility of rumor spreaders studied in [24]. These investigations show that the optimality condition suggests that limiting the mobility of rumor spreaders can minimize rumors. In a further analysis by Cheng et al. in 2021, they develop an optimal control problem with the removal of rumors from the media as strategies [25], and other control are given through educational by protecting the susceptible individual and encouraging them to reject invalid information [4]. In addition, we know the work of [26] where they explored a strategy to mitigate rumors by addressing the factors that trigger them, and in [27] also governed a strategy by enhancing individual awareness to prevent individuals from easily disseminating unconfirmed information.

In furtherance, recent work on rumor propagation has highlighted several important aspects of the mechanisms underlying rumor spread, thereby enhancing our ability to realistically model the rumor propagation process. For example, rumor systems that involve skeptical individuals and rumor refutation strategies demonstrate that skeptical individuals can significantly influence the persistence of rumors [1]. Other studies have considered the roles of truth spreaders [6], counter rumor [7], media reports [3], and heterogeneous roles [9, 28]. Furthermore, research findings indicate that a realistic rumor model must not only depict the unclear spread of information but also address how to verify such information, as well as provide guidance for public opinion. In view of behavioral heterogeneity, the choice of the incidence rate plays a fundamental role in identifying the qualitative properties of rumor propagation. Many standard rumor models rely on a bilinear incidence rate to describe the direct interaction between susceptible individuals and rumor spreaders [29]. Although this approach is mathematically acceptable, it assumes that the occurrence of a rumor distribution grows proportionally with the size of the rumor. Such assumptions might not be realistic on a large scale, since in real life the rumor process is constrained by short term attention, repeated rumor again and again, skepticism, information exhaustion, and social resistance. Therefore, the nonlinear incidence rate, particularly the saturated function, provides a more realistic approach to depicting the diffusion of rumors over high rumor infection [30]. In dynamic models, the nonlinear incidence rate has also been utilized to represent nonlinear inhibition, behavioral responses, and psychological effects under high infection [31]. A highly relevant recent contribution is the *IRLTS* rumor propagation model developed by Yu and his team [29], which combines lurking psychology and truth behavior. In this model, lurkers correspond to individuals who are familiar with the rumor but do not directly join in its dissemination, whereas truth spreaders represent individuals who counter the rumor by disseminating verified information. This framework is useful, although the rumor mechanism is still formulated using a bilinear incidence rate [29].

Based upon the aforementioned facts, it is clear that the process of rumor propagation cannot be sufficiently understood by considering only the psychological aspects of human behavior related

to rumors or by addressing exclusively the spreaders of truth, without addressing the limitations of propagation. Therefore, it is crucial to examine these limitations so that the developed model can accurately reflect the reality of rumor propagation. Driven by this gap, future research could focus on nonlinear compartmentalized rumor propagation models that incorporate a saturation incidence rate and dual optimal control strategies. The saturated incidence rate is introduced to describe a threshold that occurs when there is massive exposure to rumors. In addition, two strategies are introduced as rumor mitigation strategies: educational control, designed to reduce interactions between susceptible individuals and rumor spreaders, and truth dissemination control, designed to strengthen the effect of verified information enforcement [8,24]. Therefore, the novelty of this work lies in the reconstruction and analysis of a rumor propagation system that integrates lurker psychology, truth spread, saturation threshold, and optimal control.

As discussed above, this work is organized into several sections. Section 2 explains the formulation of the model that incorporates a saturation effect. Section 3 presents the beneficial qualitative dynamics of the model, including the non-negativity of solution and its boundedness, the basic reproduction number analysis, the derivation of equilibrium points, stability analysis, bifurcation analysis and sensitivity analysis. Section 4 formulates the optimal control problem and the optimality is derived by applying Pontryagin's Maximum Principle. Section 5 explores the numerical simulations using a forward backward sweep scheme to support and validate the theoretical results, and also to define the best strategies under different scenarios. Finally, Section 6 summarizes the main findings and discusses possible directions for further work.

2. MODEL FORMULATION

To investigate the changes in time of rumor propagation in a wide range of people, we created a compartmental model that explicitly captures the effects of harmful spreaders through a saturated recruitment and combines the counteracting effect of truth disseminators. At time t , let $N(t)$ stand for the total number of people. Using a standard compartmental modeling approach, the population is categorized into five distinct categories. The susceptible class, represented by $S(t)$, comprises those who have not yet been infected with the rumor but are more likely to accept it upon exposure. The infected class, depicted by $I(t)$, comprises people who understand the rumor and aggressively spread it to other individuals. The lurker class, characterized by $L(t)$, refers to individuals who hear the rumor but are still inactive and are not actively involved in its propagation. The truth disseminator class, denoted by $T(t)$, consists of individuals who possess a reservoir of reliable information, recognize the distinction between misinformation and factual subject matter, and actively distribute corrective information. The recovered class, represented by $R(t)$, comprises those who have abandoned interest in the rumor and stopped participating in its propagation. Hence, all population sizes under the time t are expressed as

$$N(t) = S(t) + I(t) + L(t) + T(t) + R(t). \quad (2.1)$$

In order to clarify the dynamics of the rumor, we introduce that the number of people entering the system at time is π . These values represent the recruitment of people who have never heard the rumor before. Then, the susceptible individual over the time allows to become rumor spreader. Suppose that the direct contact between the susceptible probability q , then the transmission of phenomena is $q\alpha S(t)I(t)$. This phenomenon illustrates that the incidence of exposure to rumor increases without any limitation, and this condition does not correspond to real life. Therefore, to neglect this phenomenon, we introduce the saturated inhibitor as $\frac{1}{1+\eta I(t)}$, with η representing saturation. Then, the new expression of the transmission rumor in this paper is $\frac{q\alpha S(t)I(t)}{1+\eta I(t)}$. The susceptible subclass was further reduced when the individual never heard rumor contact with the lurker individual at the rate $\frac{(1-q)\alpha S(t)I(t)}{1+\eta I(t)}$, or when the susceptible became a truth spreader at the rate γ_1 , or when the individuals never again interested in the same rumor and the other time at the rate ν_1 , and due to the natural mortality rate in μ . Then, the susceptible changes in unit time as follows:

$$\frac{dS(t)}{dt} = \pi - \frac{\alpha S(t)I(t)}{1 + \eta I(t)} - \gamma_1 S(t)T(t) - (\nu_1 + \mu)S(t). \quad (2.2)$$

At unit time, and following that, some susceptible individuals can change into the rumor spreaders. We obtain the infected subgroup by adding when the susceptible individual contacts an infected individual at the rate $\frac{q\alpha S(t)I(t)}{1+\eta I(t)}$, and when the lurker individual spontaneously turns into a rumor spreader at δ_2 . On the other hand, the infected individual will move into another compartment when the infected turn into lurkers spreaders with δ_1 , or infected contact with the truth spreaders at the rate γ_3 , respectively. In other conditions, the infected individual recovers with a probability ν_2 , and the natural death as μ . Thus, the evolution of the infected individual over time on the dynamics as

$$\frac{dI(t)}{dt} = \frac{q\alpha S(t)I(t)}{1 + \eta I(t)} + \delta_2 L(t) - \gamma_3 I(t)T(t) - (\delta_1 + \nu_2 + \mu)I(t). \quad (2.3)$$

In the unit over time, we have the lurkers individual into the systems. The lurker individual comes from the interaction between susceptible and rumor spreaders with the probability proportion $(1 - q)$, and the rate of transmission is α , and when the infected rumor may not spontaneously propagate the rumor with probability δ_1 . The lurker individual further reduces with the condition if some individual becomes more active to spread the rumor at the rate δ_2 , or the lurker individual never becomes interested again in the rumor or they die normally, respectively, at rates ν_3 and μ . Upon contact of the lurker with the individual with good information, the lurker comes forward to the truth spreader, the probability value being γ_2 . The dynamics of the lurker individual can be shown as

$$\frac{dL(t)}{dt} = \frac{(1 - q)\alpha S(t)I(t)}{1 + \eta I(t)} + \delta_1 I(t) - \gamma_2 L(t)T(t) - (\delta_2 + \nu_3 + \mu)L(t). \quad (2.4)$$

After a period at the time, the susceptible individual, the lurker individual, and the infected rumor are converted into immunizes people when in contact with the truth spreader at rate γ_1 whose numbers are $\gamma_1 S(t)T(t)$, utilizing the rate γ_2 whose numbers are $\gamma_2 L(t)T(t)$, or at the rate

γ_3 whose number is $\gamma_3 I(t)L(t)$, respectively. In unit time, the truth spreader becomes a death over time at a rate μ . Then, the people of the truth spreaders per unit time is

$$\frac{dT(t)}{dt} = \gamma_1 S(t)T(t) + \gamma_2 L(t)T(t) + \gamma_3 I(t)T(t) - \mu T(t). \quad (2.5)$$

According to the information provided, the susceptible, the infected, and the lurker all contribute to the recovery cycle at different rates, denoted by ν_1, ν_2, ν_3 , respectively, while the natural death rate of the recovery subgroup allows the outflow, denoted by μ . Hence, the recovered class can be presented as

$$\frac{dR(t)}{dt} = \nu_1 S(t) + \nu_2 I(t) + \nu_3 L(t) - \mu R(t). \quad (2.6)$$

Based on the foregoing model formulation and the underlying assumptions, the dynamics of rumor propagation incorporating a saturated transmission mechanism and truth-spreading behavior, as described in (2.2)–(2.6), are governed by the following nonlinear ODE equation

$$\begin{aligned} \frac{dS(t)}{dt} &= \pi - \frac{\alpha S(t)I(t)}{1 + \eta I(t)} - \gamma_1 S(t)T(t) - (\nu_1 + \mu)S(t), \\ \frac{dI(t)}{dt} &= \frac{q\alpha S(t)I(t)}{1 + \eta I(t)} + \delta_2 L(t) - \gamma_3 I(t)T(t) - (\delta_1 + \nu_2 + \mu)I(t), \\ \frac{dL(t)}{dt} &= \frac{(1-q)\alpha S(t)I(t)}{1 + \eta I(t)} + \delta_1 I(t) - \gamma_2 L(t)T(t) - (\delta_2 + \nu_3 + \mu)L(t), \\ \frac{dT(t)}{dt} &= \gamma_1 S(t)T(t) + \gamma_2 L(t)T(t) + \gamma_3 I(t)T(t) - \mu T(t), \\ \frac{dR(t)}{dt} &= \nu_1 S(t) + \nu_2 I(t) + \nu_3 L(t) - \mu R(t), \end{aligned} \quad (2.7)$$

with the initial condition

$$S(0) > 0, I(0) \geq 0, L(0) \geq 0, T(0) \geq 0, R(0) \geq 0. \quad (2.8)$$

Table 1 shows all the parameters in the system (2.7), including their descriptions and physical interpretations. Furthermore, Figure 1 illustrates the representation of the blueprint and the compartmental framework of the proposed model in this work.

3. THE DYNAMICS ANALYSIS OF SYSTEMS

This section presents the qualitative analysis of the system (2.7). We specifically demonstrate the positivity and boundedness of solutions, identify the equilibrium points and the basic reproduction number, and examine the stability of equilibrium.

3.1. Positivity and boundedness of solutions. It is important to emphasize that the system is meaningful, we consider the non-negativity and boundedness of solutions.

Theorem 3.1. *The solutions of the nonlinear systems (2.7) under the initial conditions (2.8) remain positive for all $t \geq 0$.*

TABLE 1. Parameters interpretation

Symbol	Description
π	Rate of individuals entering the population who have never heard the rumor
α	Directly contact rate between susceptible and rumor spreader individuals
η	The level of inhibitory effect representing saturation of rumor transmission
q	The proportion that a susceptible becomes a rumor spreader upon contact
δ_1	Transition rate from rumor spreaders to lurkers
δ_2	Transition rate from lurkers to rumor spreaders
γ_1	Contact rate between susceptible individuals and truth spreaders
γ_2	Contact rate between lurkers and truth spreaders
γ_3	Interaction rate between infected rumor and truth defenders
ν_1	Rate at which susceptible individuals lose interest and become remover
ν_2	Rate at which rumor spreaders lose interest and become remover
ν_3	Rate at which lurkers lose interest and become remover
μ	The natural mortality rate in the population

Proof. Let $S(t), I(t), L(t), T(t), R(t)$ be any solution of system (2.7) under initial condition (2.8), and let the right-hand side of system (2.7) be continuously differentiable in the feasible region, then the solutions exist and are positive. From the first equation of system (2.7), we have

$$\frac{dS(t)}{dt} = \pi - \frac{\alpha S(t)I(t)}{1 + \eta I(t)} - \gamma_1 S(t)T(t) - (\nu_1 + \mu)S(t) \geq -\left(\frac{\alpha I(t)}{1 + \eta I(t)} + \gamma_1 T(t) + \nu_1 + \mu\right)S(t). \quad (3.1)$$

By applying Gronwall's inequality as shown in [32] and the initial condition (2.8), we get the solution of equation (3.1) in the following way

$$S(t) \geq S(0) \exp \left\{ - \int_0^t \left(\frac{\alpha I(\tau)}{1 + \eta I(\tau)} + \gamma_1 T(\tau) \right) d\tau - (\nu_1 + \mu)t \right\}.$$

Hence, $S(t) \geq 0$ for all $t \geq 0$. Exactly, in an identical schema before, we obtain

$$\begin{aligned} I(t) &\geq I(0) \exp \left\{ -\gamma_3 \int_0^t T(\tau) d\tau - (\delta_1 + \nu_2 + \mu)t \right\} \geq 0, \\ L(t) &\geq L(0) \exp \left\{ -\gamma_2 \int_0^t T(\tau) d\tau - (\delta_2 + \nu_3 + \mu)t \right\} \geq 0, \\ T(t) &= T(0) \exp \left\{ \int_0^t (\gamma_1 S(\tau) + \gamma_2 L(\tau) + \gamma_3 I(\tau)) d\tau - \mu t \right\} \geq 0, \end{aligned}$$

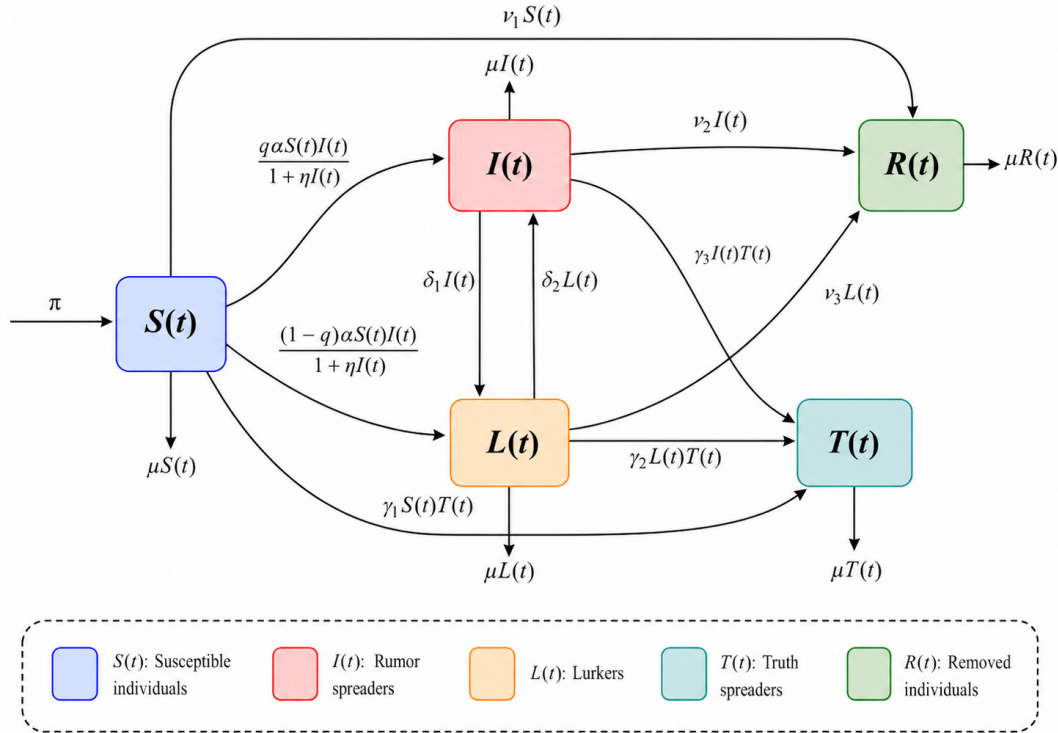


FIGURE 1. The transition diagram of the SILTR model with saturated function.

$$R(t) \geq R(0)e^{-\mu t} \geq 0.$$

If one of the initial values is zero, the vector field on the boundary of the nonnegative still indicates that the non-negativity is correct. In fact,

$$\begin{aligned} \left. \frac{dS}{dt} \right|_{S=0} &= \pi \geq 0, & \left. \frac{dI}{dt} \right|_{I=0} &= \delta_2 L \geq 0, & \left. \frac{dL}{dt} \right|_{L=0} &= \frac{(1-q)\alpha SI}{1+\eta I} + \delta_1 I \geq 0, \\ \left. \frac{dT}{dt} \right|_{T=0} &= 0, & \left. \frac{dR}{dt} \right|_{R=0} &= \nu_1 S + \nu_2 I + \nu_3 L \geq 0. \end{aligned}$$

Therefore, no solution trajectory can leave the nonnegative region due to its boundary. Consequently, all solutions of the system (2.7) with a nonnegative initial condition (2.8) remain nonnegative for all $t \geq 0$. This shows that the nonnegative is always positive. $\square$

Theorem 3.2. *The solution of system (2.7) under the initial condition (2.8) is bounded to the field*

$$\Omega = \left\{ (S(t), I(t), L(t), T(t), R(t)) \in \mathbb{R}^5 : 0 \leq N(t) \leq \frac{\pi}{\mu} \right\}.$$

Proof. According to equation (2.1) and by adding all of the equation in (2.7) yields

$$\frac{dN(t)}{dt} = \pi - \mu N(t). \quad (3.2)$$

The general solution of (3.2) with $t = 0$ is $N(t) = \frac{\pi}{\mu} + \left(N(0) - \frac{\pi}{\mu}\right)e^{-\mu t}$. When time $t \rightarrow \infty$ is provided by $\lim_{t \rightarrow \infty} N(t) = \frac{\pi}{\mu}$. Thus, We have to state that the set of positive invariance of the dynamics in (2.7) with reference to the following set

$$\Omega = \left\{ (S(t), I(t), L(t), T(t), R(t)) \in \mathbb{R}^5 : 0 \leq N(t) \leq \frac{\pi}{\mu} \right\}.$$

Obviously, the solution of the system (2.7) is bounded. $\square$

3.2. Equilibrium points. In this subsection, we discuss deeply the steady-state of the rumor propagation model incorporating a saturated effect and truth spreaders. The equilibrium points of equation (2.7) are reached by

$$\frac{dS}{dt} = \frac{dI}{dt} = \frac{dL}{dt} = \frac{dT}{dt} = \frac{dR}{dt} = 0.$$

By solving the corresponding algebraic equations, the system (2.7) has resulted in four equilibrium points that are biologically feasible. These points can be detailed as follows

1. The free equilibrium point

In this condition, both rumor spreaders and truth spreaders are absent from the population. This equilibrium always exists and is displayed by

$$E_0 = (S_0, I_0, L_0, T_0, R_0) = \left(\frac{\pi}{v_1 + \mu}, 0, 0, 0, \frac{\pi v_1}{\mu(v_1 + \mu)} \right).$$

Theorem 3.3. *The reproductive number of system (2.7) at free equilibrium E_0 is defined by*

$$\mathcal{R}_0 = \max \{ \mathcal{R}_{0I}, \mathcal{R}_{0T} \},$$

with $\mathcal{R}_{0I}$ represents the rumor-spreading reproduction number and $\mathcal{R}_{0T}$ represents the truth-spreading reproduction number. These quantities are considered by

$$\mathcal{R}_{0I} = \frac{q\alpha\pi(\delta_2 + v_3 + \mu) + (1-q)\alpha\pi\delta_2}{(v_1 + \mu)[\delta_1(v_3 + \mu) + (v_2 + \mu)(\delta_2 + v_3 + \mu)]}, \text{ and } \mathcal{R}_{0T} = \frac{\gamma_1\pi}{\mu(v_1 + \mu)}.$$

Proof. We conduct the basic reproduction number through the next-generation matrix approach [33]. From (2.7), the compartments associated with rumor propagation and truth defenders are I , L , and T . Let $\mathbf{x} = (I, L, T)^T$, then the subsystem corresponding to these compartments can be conveyed in

$$\frac{d\mathbf{x}}{dt} = \mathcal{F}(\mathbf{x}) - \mathcal{V}(\mathbf{x}),$$

with

$$\mathcal{F}(\mathbf{x}) = \begin{pmatrix} \frac{q\alpha SI}{1 + \eta I} \\ \frac{(1-q)\alpha SI}{1 + \eta I} \\ \gamma_1 ST + \gamma_2 LT + \gamma_3 IT \end{pmatrix}, \quad \mathcal{V}(\mathbf{x}) = \begin{pmatrix} -\delta_2 L + \gamma_3 IT + (\delta_1 + v_2 + \mu)I \\ -\delta_1 I + \gamma_2 LT + (\delta_2 + v_3 + \mu)L \\ \mu T \end{pmatrix}.$$

At E_0 , we have the Jacobian matrices of $\mathcal{F}$ and $\mathcal{V}$ with respect to (I, L, T) as below, respectively

$$D\mathcal{F}(E_0) = \begin{pmatrix} \frac{q\alpha\pi}{v_1 + \mu} & 0 & 0 \\ \frac{(1-q)\alpha\pi}{v_1 + \mu} & 0 & 0 \\ 0 & 0 & \frac{\gamma_1\pi}{v_1 + \mu} \end{pmatrix}, \quad D\mathcal{V}(E_0) = \begin{pmatrix} \delta_1 + v_2 + \mu & -\delta_2 & 0 \\ -\delta_1 & \delta_2 + v_3 + \mu & 0 \\ 0 & 0 & \mu \end{pmatrix}.$$

Thus, the next-generation matrix is presented by $K = (D\mathcal{F}(E_0))(D\mathcal{V}(E_0))^{-1}$, and it is the spectral radius of K , namely $\mathcal{R}_0 = \rho(K)$. By direct equating, we get the characteristic equation of K as

$$\lambda \left(\lambda - \frac{\gamma_1\pi}{\mu(v_1 + \mu)} \right) \left(\lambda - \frac{q\alpha\pi(\delta_2 + v_3 + \mu) + (1-q)\alpha\pi\delta_2}{(v_1 + \mu)[\delta_1(v_3 + \mu) + (v_2 + \mu)(\delta_2 + v_3 + \mu)]} \right) = 0,$$

and the eigenvalues of K are 0, $\mathcal{R}_{0T} = \frac{\gamma_1\pi}{\mu(v_1 + \mu)}$, and $\mathcal{R}_{0I} = \frac{q\alpha\pi(\delta_2 + v_3 + \mu) + (1-q)\alpha\pi\delta_2}{(v_1 + \mu)[\delta_1(v_3 + \mu) + (v_2 + \mu)(\delta_2 + v_3 + \mu)]}$. Therefore, $\mathcal{R}_0 = \rho(K) = \max\{\mathcal{R}_{0I}, \mathcal{R}_{0T}\}$. Here, $\mathcal{R}_{0I}$ represents the rumor-spreading reproduction number, whereas $\mathcal{R}_{0T}$ represents the truth-spreading reproduction number. This completes the proof. $\square$

2. The equilibrium point E_1 is no rumor, but truth spreaders exist

This state describes the case in which rumor spreaders are absent, while truth spreaders persist. Equilibrium E_1 is biologically feasible when $T_1 > 0$ iff $\mathcal{R}_{0T} > 1$. E_1 is given by

$$E_1 = \left(\frac{\mu}{\gamma_1}, 0, 0, \frac{v_1 + \mu}{\gamma_1} (\mathcal{R}_{0T} - 1), \frac{v_1}{\gamma_1} \right).$$

3. The equilibrium point E_2 is without true information but the rumor exist

This equilibrium describes the situation in which rumor spreaders persist, whereas truth spreaders are extinct. This equilibrium E_2 become biologically feasible with $I_2 > 0$ if and only if $\mathcal{R}_{0I} > 1$, and the part of E_2 are detailed by

$$S_2 = \frac{\pi(1 + \eta I_2)}{\alpha I_2 + (v_1 + \mu)(1 + \eta I_2)}, \quad I_2 = \frac{v_1 + \mu}{\alpha + \eta(v_1 + \mu)} (\mathcal{R}_{0I} - 1),$$

$$L_2 = \frac{I_2}{\delta_2 + v_3 + \mu} \left(\frac{(1-q)\alpha S_2}{1 + \eta I_2} + \delta_1 \right), \quad T_2 = 0, \quad R_2 = \frac{v_1 S_2 + v_2 I_2 + v_3 L_2}{\mu}.$$

4. The coexistence equilibrium point E_3

Assuming that all subclass persist, the system (2.7) admits a coexistence equilibrium point $E_3 = (S_3, I_3, L_3, T_3, R_3)$, whose components are considered by

$$S_3 = \frac{\pi(1 + \eta I_3)}{\alpha I_3 + \gamma_1 T_3(1 + \eta I_3) + r_1(1 + \eta I_3)}, \quad (3.3)$$

$$I_3 = \frac{1}{\alpha + \eta\gamma_1 T_3 + \eta r_1} \left(\frac{q\alpha\pi\gamma_2 T_3 + q\alpha\pi r_3 + (1-q)\alpha\pi\delta_2}{\gamma_2\gamma_3 T_3^2 + (\gamma_3 r_3 + \gamma_2 r_2) T_3 + r_2 r_3 - \delta_1 \delta_2} - (\gamma_1 T_3 + r_1) \right), \quad (3.4)$$

$$L_3 = \frac{I_3}{\gamma_2 T_3 + r_3} \left(\frac{(1-q)\alpha S_3}{1 + \eta I_3} + \delta_1 \right), \quad (3.5)$$

$$R_3 = \frac{v_1 S_3 + v_2 I_3 + v_3 L_3}{\mu}, \quad (3.6)$$

with $r_1 = v_1 + \mu$, $r_2 = \delta_1 + v_2 + \mu$, $r_3 = \delta_2 + v_3 + \mu$. On substituting the equation (3.3)–(3.5) into the truth-spreader equation in (2.7) yields the seventh-degree polynomial

$$\sum_{n=0}^7 Q_n T_3^n = 0, \quad (3.7)$$

with the coefficients $Q_0, Q_1, \dots, Q_7$ are listed in **Appendix A**. Therefore, existence E_3 analysis consists of two steps: first, determining the equation (3.7) admits a positive real root, and second, identifying that a positive root generates biologically meaning.

Theorem 3.4. *The polynomial equation (3.7) has at least one positive real root whenever $Q_0 Q_7 < 0$.*

Proof. Following the procedure outlined in [34], detailed proof is deferred to **Appendix B**. $\square$

To verify the infected individual feasibility, equation (3.4) can be reformed as

$$I_3 = \frac{\Phi(T_3)}{(\alpha + \eta\gamma_1 T_3 + \eta r_1)(\gamma_2 \gamma_3 T_3^2 + (\gamma_3 r_3 + \gamma_2 r_2)T_3 + r_2 r_3 - \delta_1 \delta_2)}, \quad (3.8)$$

with $\Phi(T_3) = q\alpha\pi\gamma_2 T_3 + q\alpha\pi r_3 + (1-q)\alpha\pi\delta_2 - (\gamma_1 T_3 + r_1)(\gamma_2 \gamma_3 T_3^2 + (\gamma_3 r_3 + \gamma_2 r_2)T_3 + r_2 r_3 - \delta_1 \delta_2)$. We immerse in equation (3.8), specifically $r_2 r_3 - \delta_1 \delta_2 > 0$, that the denominator in (3.8) is nonnegative for every $T_3 \geq 0$, and it directly allow that $I_3 \geq 0$ if only if $\Phi(T_3) \geq 0$. Furthermore, the result shows that the coexistence feasibility is provided by the sign of $\Phi(T_3)$.

Theorem 3.5. *Let T_3 be a positive real root of (3.7), then T_3 generates a biologically feasible coexistence equilibrium E_3 iff $\Phi(T_3) \geq 0$.*

Proof. Utilizing the condition $\Phi(T_3) \geq 0$ is equivalent to archiving the feasibility $I_3 \geq 0$. If $I_3 \geq 0$, the term in (3.3) immediately gives $S_3 > 0$. Moreover, (3.5) implies $L_3 \geq 0$, and (3.6) yields $R_3 \geq 0$. Therefore, all component of E_3 are nonnegative, it is biologically feasible when $\Phi(T_3) \geq 0$. $\square$

Theorem 3.5 shows that the existence of a positive real root of (3.7) is only an algebraic process for admitting a candidate value of T_3 . The biological admissibility of the interior equilibrium is provided by the sign of the infected subpopulation, or by the sign of $\Phi(T_3)$. Thus, the existence of E_3 is structured not only by the observability of (3.7), but also by the positivity of the equilibrium point. Furthermore, the feasibility of the infected compartment is closely to the reproduction threshold $\mathcal{R}_{0I}$. Obviously, by observing $\Phi(T_3)$

at $T_3 = 0$ guaranties

$$\Phi(0) = q\alpha\pi r_3 + (1-q)\alpha\pi\delta_2 - r_1(r_2r_3 - \delta_1\delta_2) = r_1(r_2r_3 - \delta_1\delta_2)(\mathcal{R}_{0I} - 1).$$

Since $r_1 > 0$ and $r_2r_3 - \delta_1\delta_2 > 0$, the sign of $\Phi(0)$ is addressed by $\mathcal{R}_{0I}$. Then, the condition $\mathcal{R}_{0I} > 1$ supports the positivity of I_3 , and persistence of the infected class at the coexistence equilibrium.

3.3. Local stability of equilibrium point. To achieve more insights into the dynamics of the model (2.7), this subsection presents a local stability analysis around the equilibrium points.

Theorem 3.6. *The free equilibrium point E_0 is locally asymptotically stable (LAS) provided that $\mathcal{R}_{0I} < 1$, $\mathcal{R}_{0T} < 1$, and $\alpha < \frac{(v_1+\mu)(\delta_1+\delta_2+v_3+v_2+2\mu)}{q\pi}$.*

Proof. By equating the simple algebra, we find that the Jacobian matrix of the system (2.7) at E_0 is

$$\mathcal{J}(E_0) = \begin{bmatrix} -r_1 & -\alpha S_0 & 0 & -\gamma_1 S_0 & 0 \\ 0 & q\alpha S_0 - r_2 & \delta_2 & 0 & 0 \\ 0 & (1-q)\alpha S_0 + \delta_1 & -r_3 & 0 & 0 \\ 0 & 0 & 0 & \gamma_1 S_0 - \mu & 0 \\ v_1 & v_2 & v_3 & 0 & -\mu \end{bmatrix}. \quad (3.9)$$

Then, the characteristic equation of matrix (3.9) can be conducted by

$$\det(\mathcal{J}(E_0) - \lambda I) = \begin{vmatrix} -r_1 - \lambda & -\alpha S_0 & 0 & -\gamma_1 S_0 & 0 \\ 0 & q\alpha S_0 - r_2 - \lambda & \delta_2 & 0 & 0 \\ 0 & (1-q)\alpha S_0 + \delta_1 & -r_3 - \lambda & 0 & 0 \\ 0 & 0 & 0 & \gamma_1 S_0 - \mu - \lambda & 0 \\ v_1 & v_2 & v_3 & 0 & -\mu - \lambda \end{vmatrix} = 0. \quad (3.10)$$

According to equation (3.10), we get the corresponding characteristic structure as

$$(-\mu - \lambda)(-r_1 - \lambda)(\gamma_1 S_0 - \mu - \lambda)(\lambda^2 + a_1\lambda + a_2) = 0, \quad (3.11)$$

with $a_1 = \delta_1 + \delta_2 + v_3 + v_2 + 2\mu - \frac{q\alpha\pi}{v_1+\mu}$, $a_2 = (\delta_1(v_3 + \mu) + (v_2 + \mu)(\delta_2 + v_3 + \mu))(1 - \mathcal{R}_{0I})$. Brightly, from the equation (3.11), we readily obtain $\lambda_1 = -\mu$, and $\lambda_2 = -r_1 = -(v_1 + \mu)$. When $\mathcal{R}_{0T} < 1$, then $\lambda_3 = \gamma_1 S_0 - \mu = \mu \left(\frac{\gamma_1}{\mu(v_1+\mu)} - 1 \right) = \mu(\mathcal{R}_{0T} - 1) < 0$. Based on the Routh-Hurwitz criteria accordance in [35], the quadratic equation part of (3.11) has roots with negative real parts when $\mathcal{R}_{0I} < 1$, and $\alpha < \frac{(v_1+\mu)(\delta_1+\delta_2+v_3+v_2+2\mu)}{q\pi}$. Therefore, the equilibrium E_0 is locally asymptotically stable if $\mathcal{R}_{0I} < 1$, $\mathcal{R}_{0T} < 1$ and $\alpha < \frac{(v_1+\mu)(\delta_1+\delta_2+v_3+v_2+2\mu)}{q\pi}$. Consequently, the equilibrium point E_0 is LAS. Biologically, this result indicates that, under the stated threshold conditions, neither rumor spreaders nor truth spreaders can persist in the population near E_0 . $\square$

Theorem 3.7. *The equilibrium E_1 is LAS if $\mathcal{R}_{0T} > 1$ and $\frac{q\alpha\mu}{\delta_1+\delta_2+v_2+v_3+2\mu} < \gamma_1 < \frac{q\mu\gamma_2(v_1+\mu)}{q\pi\gamma_2+q\mu(v_3+\mu)+\mu\delta_2}$.*

Proof. Taking into account the Jacobian matrix at E_1 and the detailed calculation of $\det(\mathcal{J}(E_1) - \lambda I) = 0$, we obtain the corresponding characteristic form as follows

$$(-\mu - \lambda)(\lambda^2 + b_1\lambda + b_2)(\lambda^2 + c_1\lambda + c_2) = 0, \quad (3.12)$$

with

$$\begin{aligned} b_1 &= r_1\mathcal{R}_{0T}, b_2 = \mu r_1(\mathcal{R}_{0T} - 1), c_1 = \frac{r_1(\gamma_2 + \gamma_3)}{\gamma_1}(\mathcal{R}_{0T} - 1) + \delta_1 + \delta_2 + v_2 + v_3 + 2\mu - \frac{q\alpha\mu}{\gamma_1}, \\ c_2 &= \gamma_2\gamma_3\left(\frac{r_1}{\gamma_1}(\mathcal{R}_{0T} - 1)\right)^2 + \frac{r_1(\gamma_2r_2 + \gamma_3r_3)}{\gamma_1}(\mathcal{R}_{0T} - 1) + \delta_1(v_3 + \mu) + r_2(v_2 + \mu) + \frac{q\alpha\mu\gamma_2r_1}{\gamma_1^2} \\ &\quad - \frac{q\alpha\pi\gamma_2 + \alpha\mu(v_3 + \mu) + \alpha\mu\delta_2}{\gamma_1}. \end{aligned}$$

Clearly, the characteristic equation (3.12) has an eigenvalue $\lambda_1 = -\mu$. Furthermore, by applying the Routh-Hurwitz criteria and lengthy calculations, the root of the quadratic equation $\lambda^2 + b_1\lambda + b_2 = 0$ has a negative real part if $\mathcal{R}_{0T} > 1$, and the quadratic equation $\lambda^2 + c_1\lambda + c_2 = 0$ has a negative real part if $\mathcal{R}_{0T} > 1$, provided that $\frac{q\alpha\mu}{\delta_1 + \delta_2 + v_2 + v_3 + 2\mu} < \gamma_1 < \frac{q\mu\gamma_2(v_1 + \mu)}{q\pi\gamma_2 + q\mu(v_3 + \mu) + \mu\delta_2}$. $\square$

Theorem 3.8. If $\mu > \gamma_1 S_2 + \gamma_2 L_2 + \gamma_3 I_2$, and $q < \min\left\{\frac{I_2(1 + \eta I_2)}{S_2}, \frac{I_2(\delta_1 + \delta_2 + v_2 + v_3 + 2\mu)(1 + \eta I_2)}{S_2(\delta_2 + \mu v_1 + v_3 + 2\mu)}\right\}$, then the model (2.7) is LAS at the point E_2 .

Proof. By simultaneous equating $\det(\mathcal{J}(E_2) - \lambda I) = 0$, the characteristic equation is acquired

$$(-\mu - \lambda)(\gamma_1 S_2 + \gamma_2 L_2 + \gamma_3 I_2 - \mu - \lambda)(\lambda^3 + d_1\lambda^2 + d_2\lambda + d_3) = 0, \quad (3.13)$$

with

$$\begin{aligned} d_1 &= r_1 + r_2 + r_3 + \frac{\alpha}{1 + \eta I_2}\left(I_2 - \frac{qS_2}{1 + \eta I_2}\right), \\ d_2 &= r_1r_3 + r_2r_3 + \delta_1\delta_2 + \frac{(1 - q)\alpha\delta_2S_2}{(1 + \eta I_2)^2} + \frac{\alpha}{1 + \eta I_2}\left(I_2(r_2 + r_3) - \frac{qS_2(r_1 + r_3)}{1 + \eta I_2}\right), \\ d_3 &= \frac{\alpha\delta_1\delta_2I_2}{1 + \eta I_2} + r_1\left(\frac{(1 - q)\alpha\delta_2S_2}{(1 + \eta I_2)^2} + \delta_1\delta_2\right) + \frac{\alpha r_2r_3}{1 + \eta I_2}\left(I_2 - \frac{qS_2}{1 + \eta I_2}\right). \end{aligned}$$

Based on the characteristic equation (3.13), $\lambda_1 = -\mu < 0$, and $\lambda_2 < 0$ or become negative if satisfy $\mu > \gamma_1 S_2 + \gamma_2 L_2 + \gamma_3 I_2$. Besides, the remaining cubic equation on the equation (3.13) will be observed following the Routh-Hurwitz criteria. The cubic equation has a negative real part if these conditions hold

$$\Delta_1 = |d_1| > 0, \quad \Delta_2 = \begin{vmatrix} d_1 & 1 \\ 0 & d_2 \end{vmatrix} > 0, \quad \Delta_3 = \begin{vmatrix} d_1 & 1 & 0 \\ d_3 & d_2 & d_1 \\ 0 & 0 & d_3 \end{vmatrix} > 0. \quad (3.14)$$

This condition implies that all determinants (3.14) become positive values ($\Delta_1 > 0$, $\Delta_2 > 0$, and $\Delta_3 > 0$) when $q < \min\left\{\frac{I_2(1 + \eta I_2)}{S_2}, \frac{I_2(\delta_1 + \delta_2 + v_2 + v_3 + 2\mu)(1 + \eta I_2)}{S_2(\delta_2 + v_1 + v_3 + 2\mu)}\right\}$. Consequently, E_2 is LAS. $\square$

Theorem 3.9. *The coexistence equilibrium E_3 of the system (2.7) is local asymptotic stable if $e_1 > 0$, $e_1 e_2 > 0$, $e_3(e_1 e_2 - e_3) > 0$, and $e_3(e_1 e_2 - e_3) - e_1^2 e_4 > 0$.*

Proof. The complete proof is provided in the **Appendix C**. $\square$

To enhance the analytical stability for the equilibrium E_3 , we validate the Routh–Hurwitz condition numerically. This verification is especially beneficial due to the algebraically challenging for stability E_3 . To achieve this goal, we set the parameter in Table 2, except $\pi = 2$. Underlying this parameter, we see that the threshold values are $\mathcal{R}_{0I} = 64.44 > 1$, $\mathcal{R}_{0T} = 6 > 1$, and the associated coexistence equilibrium is $E_3 = (1.39, 1.73, 0.38, 15.84, 0.61)$. Moreover, we evaluate the coefficients of the quartic factor of the characteristic polynomial given in **Appendix C**. The resulting coefficients are $e_1 = 2.41$, $e_2 = 1.77$, $e_3 = 0.415$, $e_4 = 0.025$. These values satisfy the Routh–Hurwitz inequalities stated in Theorem 3.9. Indeed, $e_1 = 2.41 > 0$, $e_1 e_2 = 4.28 > 0$, $e_3(e_1 e_2 - e_3) = 1.61 > 0$, $e_3(e_1 e_2 - e_3) - e_1^2 e_4 = 1.45 > 0$. Hence, all roots associated with the quartic factor of the characteristic polynomial have negative real parts. Therefore, the coexistence equilibrium E_3 is LAS by a selected parameter set. The phase portrait presented in Figure 2 further supports this stability result, and this convergence behavior suggests that E_3 acts as an attractor.

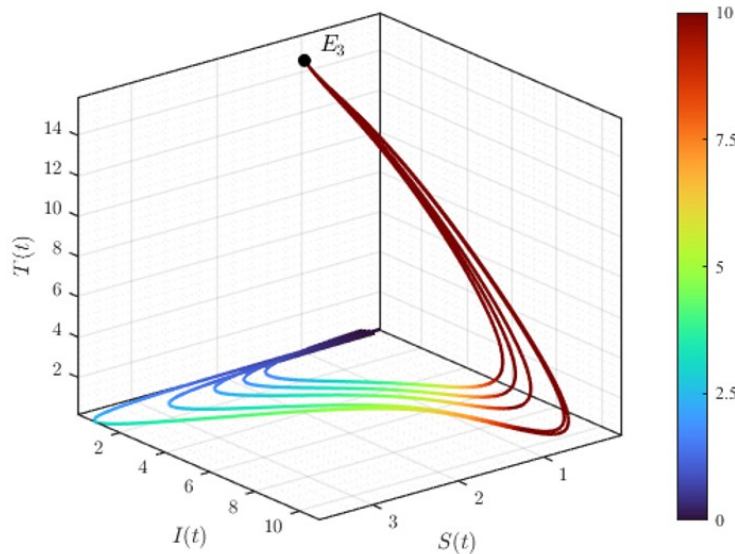


FIGURE 2. Phase diagram for coexistence equilibrium stability.

3.4. Global stability of equilibrium point. In this passage, we stay and focus on the global stability analysis of the propagation of the rumor model with the saturation effect (2.7).

Theorem 3.10. *The free equilibrium point E_0 is globally asymptotically stable (GAS) in the feasible region Ω if $\mathcal{R}_{0I} < 1$ and $\mathcal{R}_{0T} < 1$.*

Proof. Consider the Lyapunov function

$$\mathcal{L}_0(I, L, T) = (\delta_2 + v_3 + \mu)I + \delta_2(L + T). \quad (3.15)$$

Clearly, $\mathcal{L}_0 \geq 0$ for all $(I, L, T) \in \mathbb{R}_+^3$, and $\mathcal{L}_0 = 0$ if and only if $I = L = T = 0$ (i.e., at the free equilibrium E_0). Differentiating (3.15) along solutions of system (2.7) yields

$$\begin{aligned} \frac{d\mathcal{L}_0}{dt} &= (\delta_2 + \nu_3 + \mu) \left(\frac{q\alpha SI}{1 + \eta I} + \delta_2 L - \gamma_3 IT - (\delta_1 + \nu_2 + \mu) I \right) \\ &\quad + \delta_2 \left(\frac{(1-q)\alpha SI}{1 + \eta I} + \delta_1 I - \gamma_2 LT - (\delta_2 + \nu_3 + \mu) L \right) + \delta_2 (\gamma_1 ST + \gamma_2 LT + \gamma_3 IT - \mu T). \end{aligned}$$

Since $S_0 \leq N \leq \frac{\pi}{\mu}$ for all t in the invariant region Ω , and $\frac{1}{1+\eta I} \leq 1$, we obtain the estimate

$$\begin{aligned} \frac{d\mathcal{L}_0}{dt} &\leq \alpha S_0 (q(\delta_2 + \nu_3 + \mu) + \delta_2(1-q)) I - (\nu_3 + \mu) \gamma_3 IT + (\gamma_1 S - \mu) \delta_2 T \\ &= \left(\delta_1(\nu_3 + \mu) + (\nu_2 + \mu)(\delta_2 + \nu_3 + \mu) \right) (\mathcal{R}_{0I} - 1) I - (\nu_3 + \mu) \gamma_3 IT + \mu \delta_2 (\mathcal{R}_{0T} - 1) T. \end{aligned}$$

Hence, if $\mathcal{R}_{0I} < 1$ and $\mathcal{R}_{0T} < 1$, then $\dot{\mathcal{L}}_0 < 0$ in Ω . Moreover, $\dot{\mathcal{L}}_0 = 0$ implies $I = 0$ and $T = 0$. Therefore, the largest invariant subset of $\{(S, I, L, T) \in \Omega : \dot{\mathcal{L}}_0 = 0\}$ is the singleton $\{E_0\}$. By LaSalle's invariance principle [36], E_0 is globally asymptotically stable in feasible Ω . $\square$

Theorem 3.11. *The rumor-free equilibrium with truth spreaders E_1 of the system (2.7) exists whenever $\mathcal{R}_{0T} > 1$. Moreover, E_1 is globally asymptotically stable if $\alpha < \frac{\nu_2 + \mu + \gamma_3 T_1}{S_1}$.*

Proof. The proof is associated with the direct Lyapunov method and follows the same approach as the proof for E_0 . For readability, the detailed algebraic derivation is given in **Appendix D**. $\square$

Theorem 3.12. *The rumor-persistent equilibrium without truth spreaders E_2 of system (2.7) exists while $\mathcal{R}_{0I} > 1$. Moreover, E_2 is GAS in the feasible region if $\mu > \gamma_1 S_2 + \gamma_2 L_2 + \gamma_3 I_2$.*

Proof. The proof is given in **Appendix E**. $\square$

Moving forward to the essential part of the coexistence point, we explore the interior global stability E_3 using the geometric approach of Li and Muldowney [37, 38]. Since the variable $R(t)$ does not appear in other equations, the reduced subsystem S, I, L, T with the feasible region

$$\Omega_4 = \left\{ (S, I, L, T) \in \mathbb{R}_+^4 : S + I + L + T \leq \frac{\pi}{\mu} \right\}$$

is positively invariant. Initially, we begin with some preliminaries on the geometric approach to declare a global stability criterion.

Definition 3.1. *The reduced subsystem S, I, L, T becomes uniformly persistent in Ω_4 if there exists a constant $\hat{k} > 0$ such that every solution $(S(t), I(t), L(t), T(t))$ of the reduced system with initial value satisfies*

$$\min \left\{ \liminf_{t \rightarrow \infty} S(t), \liminf_{t \rightarrow \infty} I(t), \liminf_{t \rightarrow \infty} L(t), \liminf_{t \rightarrow \infty} T(t) \right\} \geq \hat{k}.$$

Remark 3.1. *If the reduced subsystem is uniformly persistent in Ω_4 , then there exists a compact absorbing set $\mathcal{K} = \{(S, I, L, T) \in \Omega_4 : \delta_S \leq S \leq M_S, \delta_I \leq I \leq M_I, \delta_L \leq L \leq M_L, \delta_T \leq T \leq M_T\} \subset \Omega$, with $\delta > 0$.*

Assume that the reduce subsystem *SILT* has unique equilibrium and compact absorbing is exist. Remaining that $P(x)$ is $\binom{4}{2} \times \binom{4}{2}$ matrix-valued function of subsystem, and suppose $\mathcal{B} = P_f P^{-1} + P J^{[2]} P^{-1}$, with $J = Df(x)$ is the Jacobian matrix of reduced subsystem, and $J^{[2]}$ denotes about the second additive compound matrix that related with subsystem. Let $\mu(\mathcal{B})$ be the Lozinskii measure of $\mathcal{B}$ with respect to a norm $\|\cdot\|$ in $\mathbb{R}^6$ [38], and define as follows

$$\hat{q} = \limsup_{t \rightarrow \infty} \sup_{x_0 \in K} \frac{1}{t} \int_0^t \sigma(\mathcal{B}(x(s, x_0))) ds.$$

Moreover, the Li-Muldowney criteria could possibly be demonstrated through the following result.

Lemma 3.1. *Suppose that system (2.7) has a unique equilibrium and a compact absorbing set in Ω , then the unique point E_3 of (2.7) is globally asymptotically stable (GAS) in Ω if $\hat{q} < 0$.*

Now we apply the Lemma 3.1 to investigate the global stability of system (2.7) through the theorem statement below.

Theorem 3.13. *Assume that system (2.7) is uniformly persistent in Ω and admits a unique endemic equilibrium E_3 . Then the endemic equilibrium E_3 is globally asymptotically stable in Ω .*

Proof. The proof is provided in **Appendix F**. □

4. BIFURCATION ANALYSIS

An interesting phenomenon in a dynamical system is the occurrence of a bifurcation at the critical point, while the threshold equals one. To examine the existence of a bifurcation for system (2.7), we use the center manifold theory as discussed by Castillo-Chavez and Song [39].

Theorem 4.1. *If $\mathcal{R}_{0I} = 1$, then the model (2.7) exhibits forward bifurcation.*

Proof. Let the variable set in (2.7) be redefined as g_i for $i = 1, 2, 3, 4, 5$. In the next step, we choose $\alpha = \alpha^*$ as a bifurcation parameter because the criteria α show the interaction between the susceptible individual and the rumor spreaders, which contribute the most to the persistence of the rumor propagation. Equating with the threshold $\mathcal{R}_{0I} = 1$, we omit the value of α^* as follows

$$\alpha^* = \frac{(v_1 + \mu) \left(\delta_1(v_3 + \mu) + (v_2 + \mu)(\delta_2 + v_3 + \mu) \right)}{\pi \left(q(\delta_2 + v_3 + \mu) + (1 - q)\delta_2 \right)}. \quad (4.1)$$

The Jacobian matrix of (2.7) around the bifurcation parameter (4.1) and the equilibrium E_0 is shown

$$\mathcal{J}(E_0, \alpha^*) = \begin{pmatrix} -(v_1 + \mu) & -\alpha^* S_0 & 0 & -\gamma_1 S_0 & 0 \\ 0 & q\alpha^* S_0 - (\delta_1 + v_2 + \mu) & \delta_2 & 0 & 0 \\ 0 & (1 - q)\alpha^* S_0 + \delta_1 & -(\delta_2 + v_3 + \mu) & 0 & 0 \\ 0 & 0 & 0 & \gamma_1 S_0 - \mu & 0 \\ v_1 & v_2 & v_3 & 0 & -\mu \end{pmatrix}. \quad (4.2)$$

The characteristic equation associated with (4.2) is given by $\det(\mathcal{J}(E_0, \alpha^*) - \lambda I) = 0$, which provides a simple zero eigenvalue, while the other eigenvalues have a negative real part. These eigenvalues that correspond to the equation $\det(\mathcal{J}(E_0, \alpha^*) - \lambda I) = 0$, are $\lambda_1 = -\mu$, $\lambda_2 = -(v_1 + \mu)$, $\lambda_3 = \mu(R_{0T} - 1) < 0$ if $R_{0T} < 1$, $\lambda_4 = \delta_1 + \delta_2 + v_3 + v_2 + 2\mu - \frac{q\alpha\pi}{v_1 + \mu} < 0$ if $\alpha < \frac{(\delta_1 + \delta_2 + v_3 + v_2 + 2\mu)(v_1 + \mu)}{q\pi}$, and $\lambda_5 = 0$. This means that the condition is in line with the Center Manifold criteria. Furthermore, we compute the right eigenvector corresponding to the zero eigenvalue $\lambda_5 = 0$, which is expressed as

$$\begin{pmatrix} -(v_1 + \mu) & -\alpha^* S_0 & 0 & -\gamma_1 S_0 & 0 \\ 0 & q\alpha^* S_0 - (\delta_1 + v_2 + \mu) & \delta_2 & 0 & 0 \\ 0 & (1 - q)\alpha^* S_0 + \delta_1 & -(\delta_2 + v_3 + \mu) & 0 & 0 \\ 0 & 0 & 0 & \gamma_1 S_0 - \mu & 0 \\ v_1 & v_2 & v_3 & 0 & -\mu \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (4.3)$$

Based on the equal matrices (4.3), we obtain the right eigenvector $\vec{w} = (w_1, w_2, w_3, w_4, w_5)$ is

$$w_1 = -\frac{\alpha^* \pi}{(v_1 + \mu)^2}, \quad w_2 = 1, \quad w_3 = \frac{(v_1 + \mu)(\delta_2 + v_3 + \mu) - q\alpha^* \pi}{\delta_2(v_1 + \mu)},$$

$$w_4 = 0, \quad w_5 = \frac{1}{\mu} \left(v_2 - \frac{\alpha^* \pi v_1}{(v_1 + \mu)^2} - \frac{v_3((v_1 + \mu)(\delta_2 + v_3 + \mu) - q\alpha^* \pi)}{\delta_2(v_1 + \mu)} \right).$$

On the other hand, the left eigenvector corresponding to (4.2) is $\vec{h} = (h_1, h_2, h_3, h_4, h_5)$ with $h_1 = h_4 = h_5 = 0$, $h_3 = 1$, and $h_2 = \frac{\delta_2}{\delta_2 + v_3 + \mu}$. Utilizing the Center Manifold theorem, and since $h_1 = h_4 = h_5 = 0$, we only need to arrange the second partial derivative of g_2 , and g_3 . Then we get the values a and b as follow

$$a = h_2 \left(2w_1 w_2 \frac{\partial^2 g_2(E_0, \alpha^*)}{\partial S \partial I} + w_2^2 \frac{\partial^2 g_2(E_0, \alpha^*)}{\partial I^2} \right) + h_3 \left(2w_1 w_2 \frac{\partial^2 g_3(E_0, \alpha^*)}{\partial S \partial I} + w_2^2 \frac{\partial^2 g_3(E_0, \alpha^*)}{\partial I^2} \right)$$

$$= -2\pi\alpha^* \left(\eta + \frac{\alpha^*}{v_1 + \mu} \right) \left(\frac{q}{v_1 + \mu} + \frac{(1 - q)\delta_2}{(v_1 + \mu)(\delta_2 + v_3 + \mu)} \right).$$

$$b = h_2 \left(w_1 \frac{\partial^2 g_2(E_0, \alpha^*)}{\partial S \partial \alpha^*} + w_2 \frac{\partial^2 g_2(E_0, \alpha^*)}{\partial I \partial \alpha^*} \right) + h_3 \left(w_1 \frac{\partial^2 g_3(E_0, \alpha^*)}{\partial S \partial \alpha^*} + w_2 \frac{\partial^2 g_3(E_0, \alpha^*)}{\partial I \partial \alpha^*} \right)$$

$$= \frac{q\pi}{v_1 + \mu} + \frac{(1 - q)\pi\delta_2}{(v_1 + \mu)(\delta_2 + v_3 + \mu)}.$$

Furthermore, it is shown that, by achieving all positive parameters, we have a always negative and b always positive. These findings imply that the system (2.7) undergoes a forward bifurcation at $\mathcal{R}_{0I} = 1$. $\square$

Theorem 4.2. *If $\mathcal{R}_{0T} = 1$, then the system (2.7) exhibits forward bifurcation.*

Proof. Suppose that the values $\gamma_1 = \gamma_1^*$ are a bifurcation parameter corresponding to the threshold of the truth spreader. While $\mathcal{R}_{0T} = 1$, we have $\gamma_1^* = \frac{\mu(v_1 + \mu)}{\pi}$. By straightforward terms, the Jacobian

matrix of the system (2.7) around E_0 and γ^* gives

$$\mathcal{J}(E_0, \gamma_1^*) = \begin{bmatrix} -(v_1 + \mu) & -\alpha S_0 & 0 & -\gamma_1^* S_0 & 0 \\ 0 & q\alpha S_0 - (\delta_1 + v_2 + \mu) & \delta_2 & 0 & 0 \\ 0 & (1-q)\alpha S_0 + \delta_1 & -(\delta_2 + v_3 + \mu) & 0 & 0 \\ 0 & 0 & 0 & \gamma_1^* S_0 - \mu & 0 \\ v_1 & v_2 & v_3 & 0 & -\mu \end{bmatrix}. \quad (4.4)$$

If $\mathcal{R}_{0T} = 1$, and directly computed (4.4), we get $\lambda_3 = 0$, and this condition implies that bifurcation occurs while $\gamma_1 = \gamma_1^*$. Furthermore, a straightforward calculation, we obtain the right eigenvector as $\vec{w} = (\hat{w}_1, \hat{w}_2, \hat{w}_3, \hat{w}_4, \hat{w}_5) = \left(-\frac{\pi\gamma_1^*}{(v_1+\mu)^2}, 0, 0, 1, -\frac{\pi\gamma_1^*v_1}{\mu(v_1+\mu)^2}\right)$. Through direct calculation, we obtain the left eigenvector, such as $\vec{h} = (\hat{h}_1, \hat{h}_2, \hat{h}_3, \hat{h}_4, \hat{h}_5) = (0, 0, 0, 1, 0)$. Consequently, we only need to calculate the second partial derivative of g_4 to determine the values $\hat{a}$ and $\hat{b}$. The values $\hat{a}$ and $\hat{b}$ display as

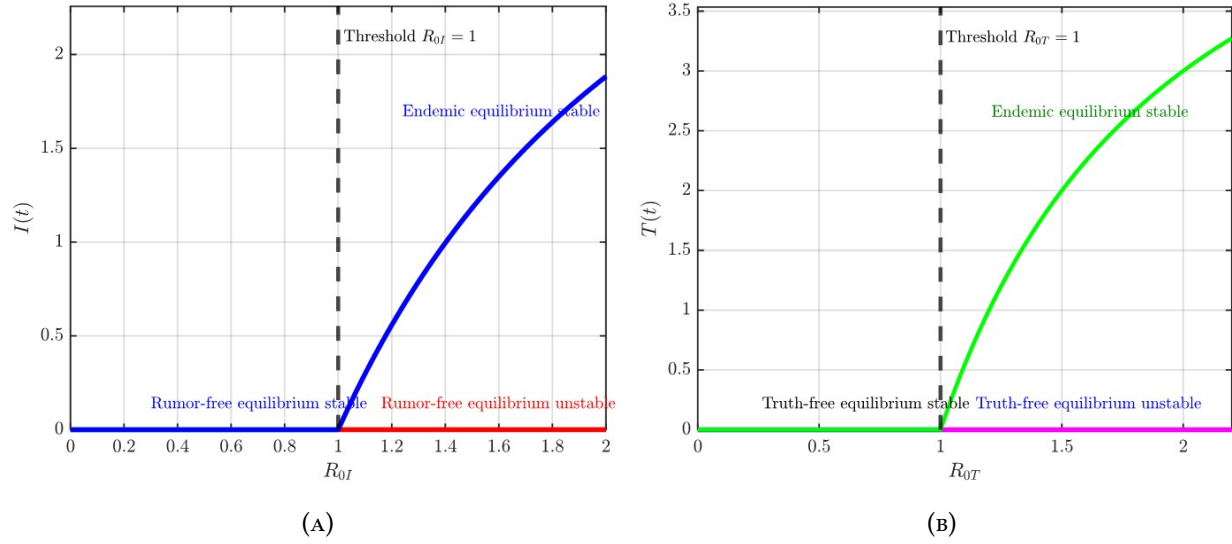
$$\hat{a} = \hat{h}_4 \left(2\hat{w}_1\hat{w}_4 \frac{\partial^2 g_4(E_0, \gamma_1^*)}{\partial S \partial T} \right) = -\frac{2\pi(\gamma_1^*)^2}{(v_1 + \mu)^2}, \quad \hat{b} = \hat{h}_4 \left(\hat{w}_1 \frac{\partial^2 g_4(E_0, \gamma_1^*)}{\partial S \partial \gamma_1^*} + \hat{w}_4 \frac{\partial^2 g_4(E_0, \gamma_1^*)}{\partial T \partial \gamma_1^*} \right) = \frac{\pi}{(v_1 + \mu)}.$$

Furthermore, it is shown that with all positive parameters, we always obtain $\hat{a}$ negative and $\hat{b}$ positive. This implies that the system (2.7) undergoes a forward bifurcation at $\mathcal{R}_{0T} = 1$. $\square$

Given the parameter values listed in Table 2, bifurcation simulations are shown to support the analytical results. In the first simulation, all parameters are fixed as given in Table 2, except for the parameter α , which is varied to obtain different values of $\mathcal{R}_{0I}$. Figure 3(A) shows that when $\mathcal{R}_{0I} < 1$, the rumor-free equilibrium is stable, as indicated by the stable branch along $I(t) = 0$. However, when $\mathcal{R}_{0I}$ passes through the critical value $\mathcal{R}_{0I} = 1$, the free equilibrium becomes unstable and a stable endemic equilibrium appears. This result confirms that the rumor persists when $\mathcal{R}_{0I} > 1$. Therefore, the numerical result agrees with Theorem 4.1, which states that the system undergoes a forward bifurcation at $\mathcal{R}_{0I} = 1$. Similarly, the bifurcation diagram with respect to $\mathcal{R}_{0T}$ is obtained by using the parameter values in Table 2, while varying the truth transmission parameter γ_1 . As shown in Figure 3(B), the truth-free equilibrium is stable when $\mathcal{R}_{0T} < 1$. After $\mathcal{R}_{0T}$ exceeds the threshold value $\mathcal{R}_{0T} = 1$, the truth-free equilibrium loses its stability, and the stable truth-spreading equilibrium emerges. This behavior indicates that the population of truth-spreading persists only when $\mathcal{R}_{0T} > 1$. Hence, the numerical bifurcation diagram supports Theorem 4.2.

5. SENSITIVITY ANALYSIS

Another important aspect of a dynamical system is sensitivity analysis. The goal of the local sensitivity of R_0 is to explore the impact of parameters on the reproductive number R_0 . According to Theorem 3.3, we note that the threshold consist of $\mathcal{R}_{0I}$, and $\mathcal{R}_{0T}$. The formulas of the local sensitivity are depicted to measure the relative impact of each parameter on both $\mathcal{R}_{0I}$, and $\mathcal{R}_{0T}$,

FIGURE 3. Forward bifurcation at $\mathcal{R}_{0I}$ and $\mathcal{R}_{0T}$.

respectively,

$$C_{\varrho}^{\mathcal{R}_{0I}} = \frac{\partial \mathcal{R}_{0I}}{\partial \varrho} \cdot \frac{\varrho}{\mathcal{R}_{0I}}, \quad (5.1)$$

$$C_{\varrho}^{\mathcal{R}_{0T}} = \frac{\partial \mathcal{R}_{0T}}{\partial \varrho} \cdot \frac{\varrho}{\mathcal{R}_{0T}}, \quad (5.2)$$

with ϱ is a parameters involved in $\mathcal{R}_{0I}$, or $\mathcal{R}_{0T}$. Utilizing the expression of $\mathcal{R}_{0I}$ and equation (5.1) gives

$$C_q^{\mathcal{R}_{0I}} = \frac{\partial \mathcal{R}_{0I}}{\partial q} \cdot \frac{q}{\mathcal{R}_{0I}} = \frac{q(v_3 + \mu)}{q(v_3 + \mu) + \delta_2}.$$

The above calculation illustrates the derivation of the sensitivity index. The remaining sensitivity index associated with $\mathcal{R}_{0I}$ can be obtained in the same manner by differentiating $\mathcal{R}_{0I}$ with respect to each parameter. Similarly, for the threshold $\mathcal{R}_{0T}$, the sensitivity index is derived by applying equation (5.2) to the parameters involving in $\mathcal{R}_{0T}$. Therefore, instead of listing all analytical expressions explicitly, the sensitivity results are evaluated using the parameter values provided in Table 2. Based on these derivations and the value of baseline parameters, the corresponding sensitivity indices for both $\mathcal{R}_{0I}$ and $\mathcal{R}_{0T}$ are visualized in Figures 4(A) and 4(B), respectively.

In Figure 4(A), the parameters α , π , q , and δ_2 have positive sensitivity, while δ_1 , v_1 , v_2 , v_3 , and μ have negative sensitivity. This means that the $\mathcal{R}_{0I}$ can be minimized by decreasing the values of the parameters with positive sensitivity, or by increasing the values of parameters with negative sensitivity. Among these parameters, μ has the largest negative influence, indicating that μ is the most effective parameter in reducing $\mathcal{R}_{0I}$. Similarly, Fig.4(B) shows that γ_1 and π have positive sensitivity indices for $\mathcal{R}_{0T}$, whereas v_1 and μ have negative sensitivity indices. Therefore, the truth-dissemination threshold $\mathcal{R}_{0T}$ increases with the contact rate γ_1 and the recruitment rate π , but decreases with the recovery rate v_1 and the natural death rate μ .

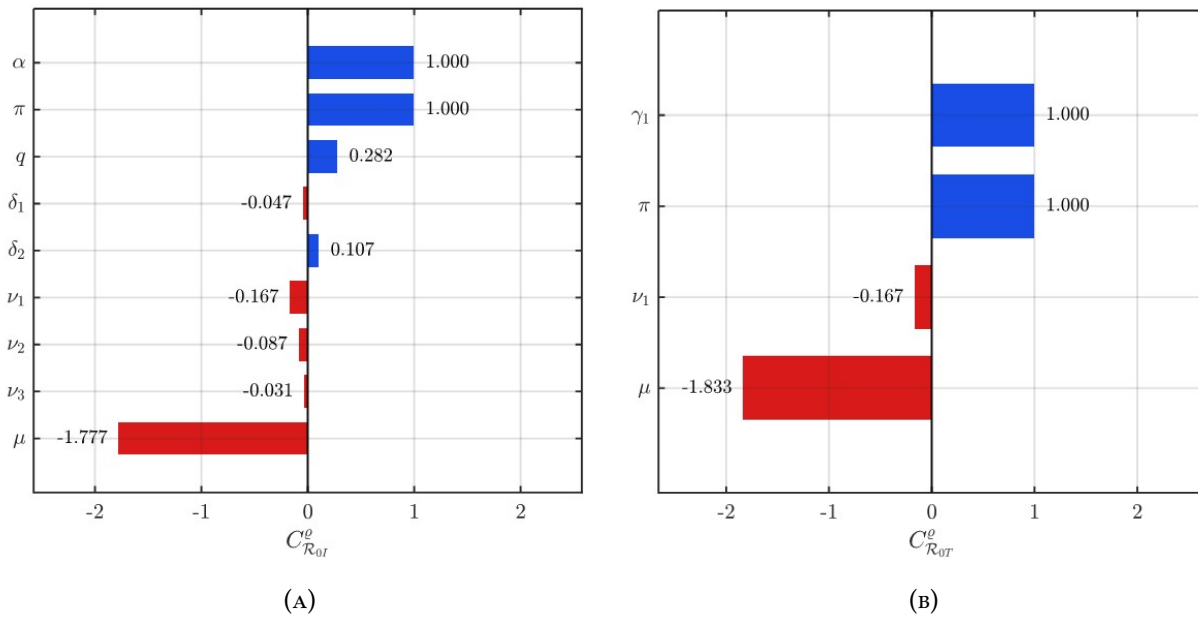


FIGURE 4. Sensitivity index of R_{0I} and R_{0T} under parameter set in Table 2.

Figure 5 describes and illustrates the effects of several parameters involved in basic reproduction numbers R_{0I} and R_{0T} . The blue surfaces illustrate the values of the reproduction numbers, while the yellow plane figures the threshold value equal to one. When the surface lies above this plane, the rumor or truth spreading process can persist. Conversely, when it lies below the plane, the spreading process tends to die out. Overall, R_{0I} increases as the rumor contact parameter α increases, indicating that stronger interaction with rumor spreaders promotes rumor transmission. On the other hand, parameters such as ν_1 , μ , q , and δ_2 tend to reduce R_{0I} , which means that they help suppress rumor propagation. Similarly, R_{0T} increases with the truth contact parameter γ_1 , showing that stronger interaction with truth spreaders supports the dissemination of truthful information. However, increasing parameters such as ν_1 , μ , and π can decrease R_{0T} . Therefore, these three-dimensional plots confirm that rumor spreading can be controlled by reducing rumor-related contact and increasing parameters that weaken rumor transmission. At the same time, the dissemination of truth can be enhanced by strengthening interactions between susceptible individuals and truth spreaders.

6. OPTIMAL CONTROL PROBLEMS

In this part, we talk about the role of optimal control in reducing the rumors propagation and strengthening the spread of truthful information. Under the proposed rumor propagation framework, we consider two control strategies in a limited time interval. The first control, which is denoted by $u_1(t)$, stands for the educational effort that aims to reduce the proportion of susceptible individuals transformed into rumor propagators. As individuals might have limited capacity to distinguish accurate information and misinformation, the educational control is used to enhance

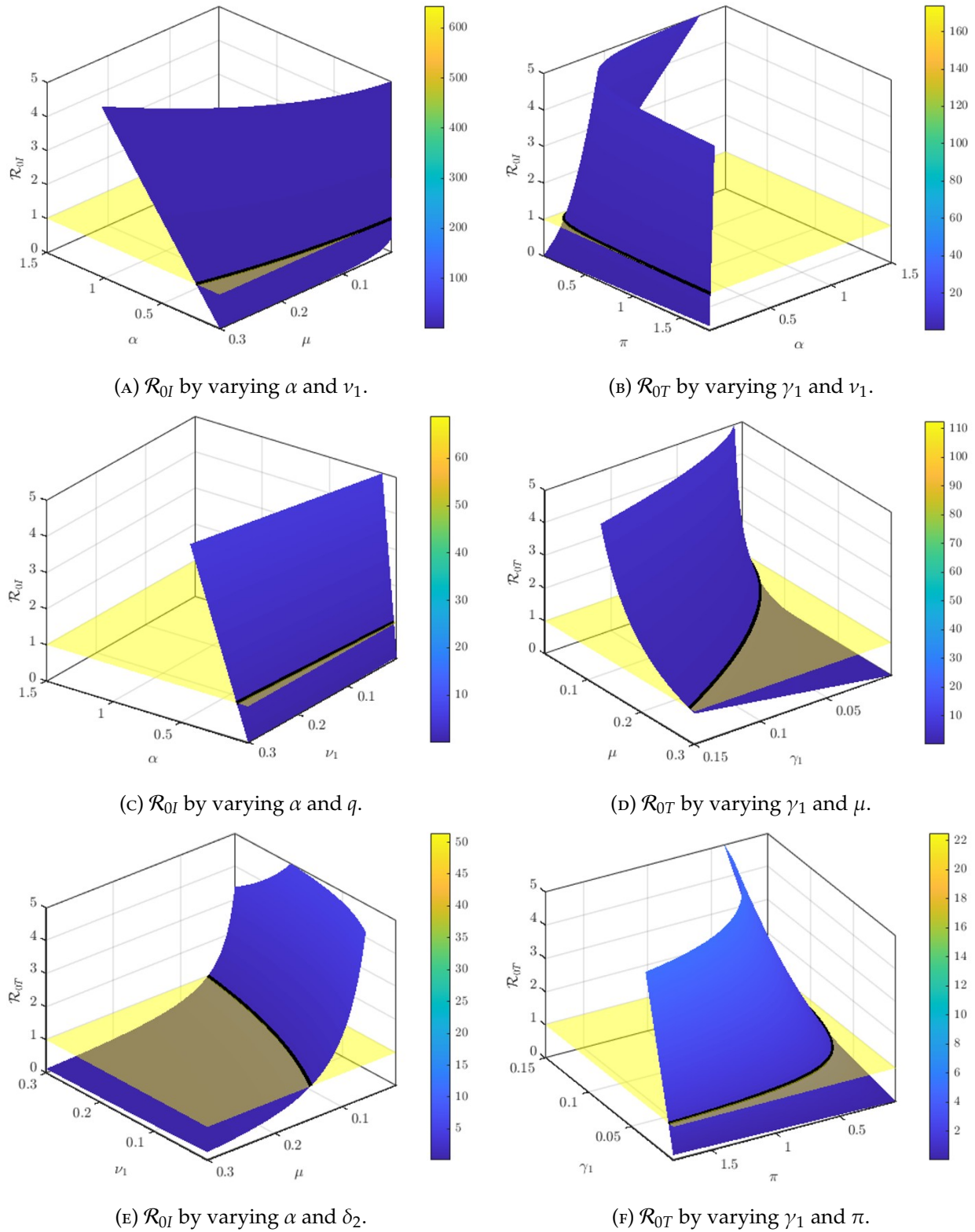


FIGURE 5. The 3D plots of reproduction numbers $\mathcal{R}_{0I}$ and $\mathcal{R}_{0T}$ under parameter variations.

their skills in identifying rumors and refusing false statements. From a biological and social perspective, this control includes interventions such as filtering rumors, reducing susceptibility to misinformation, or any mechanism that effectively reduces the possibility of susceptible individuals being exposed to rumors. The second control, designated as $u_2(t)$, reflects efforts to increase the circulation of factual information, encouraging susceptible individuals, rumor spreaders, and observers to become truth spreaders. In an informative environment, individuals can be selective in accepting information without neglecting the facts that support it. Therefore, truth-spreading controls, such as motivating individuals to actively spread accurate information, or official clarification, are implemented to increase the role of truth-spreaders and reduce the rumors prevalence.

Based on the information above, our goal is to minimize the number of $I(t)$ and $L(t)$, then also to maximize the number of truth defenders ($T(t)$). Furthermore, the functional objective is derived by

$$\min J = \int_0^{t_f} \left(A_1 I + A_2 L - A_3 T + \frac{1}{2} A_4 u_1^2 + \frac{1}{2} A_5 u_2^2 \right) dt. \quad (6.1)$$

Related to the functional objective (6.1), A_1 , A_2 , and A_3 represent the weight constants corresponding to $I(t)$, $L(t)$, and $T(t)$, respectively, whereas A_4 and A_5 are the weight constants corresponding to the control $u_1(t)$ and $u_2(t)$. Hence, we chose that all weight parameters are positive and that the control time lies in the interval $t_0 \leq t \leq t_f$. The quadratic function of the control describes the education as a preventive measure at the interaction between the susceptible and the rumor spreader, and also the truth campaign. Then, we derive the best control u_1^*, u_2^* satisfy $J(u_1^*, u_2^*) = \min\{J(u_1, u_2) \mid (u_1, u_2) \in \mathcal{U}\}$.

Through the implementation of these two controls in the model (2.7), the following control system is established

$$\begin{aligned} \frac{dS}{dt} &= \pi - \frac{(1-u_1)\alpha SI}{1+\eta I} - (\gamma_1 + u_2)ST - (v_1 + \mu)S, \\ \frac{dI}{dt} &= \frac{(1-u_1)q\alpha SI}{1+\eta I} + \delta_2 L - (\gamma_3 + u_2)IT - (\delta_1 + v_2 + \mu)I, \\ \frac{dL}{dt} &= \frac{(1-u_1)(1-q)\alpha SI}{1+\eta I} + \delta_1 I - (\gamma_2 + u_2)LT - (\delta_2 + v_3 + \mu)L, \\ \frac{dT}{dt} &= (\gamma_1 + u_2)ST + (\gamma_2 + u_2)LT + (\gamma_3 + u_2)IT - \mu T, \\ \frac{dR}{dt} &= v_1 S + v_2 I + v_3 L - \mu R, \end{aligned} \quad (6.2)$$

with the initial condition (2.8), and the feasible regions of $u_1(t)$, and $u_2(t)$ are $\mathcal{U} = \{u_1, u_2, \mid 0 \leq u_1 \leq 1, 0 \leq u_2 \leq 1\}$. As we know, the popular methods commonly used in optimal control include Pontryagin's Maximum Principle, which many researchers use [40–42]. This method displays how to convert the optimal problem into a minimisation problem of the Hamiltonian with respect to the variables u_1 and u_2 . To this end, we introduce the Hamiltonian function, which

combines the integrand of the objective function defined as

$$\begin{aligned}\mathcal{H} = & A_1 I + A_2 L - A_3 T + \frac{A_4}{2} u_1^2 + \frac{A_5}{2} u_2^2 + \xi_S \left(\pi - \frac{(1-u_1)\alpha SI}{1+\eta I} - (\gamma_1 + u_2)ST - (v_1 + \mu)S \right) \\ & + \xi_I \left(\frac{(1-u_1)q\alpha SI}{1+\eta I} + \delta_2 L - (\gamma_3 + u_2)IT - (\delta_1 + v_2 + \mu)I \right) \\ & + \xi_L \left(\frac{(1-u_1)(1-q)\alpha SI}{1+\eta I} + \delta_1 I - (\gamma_2 + u_2)LT - (\delta_2 + v_3 + \mu)L \right) \\ & + \xi_T ((\gamma_1 + u_2)ST + (\gamma_2 + u_2)LT + (\gamma_3 + u_2)IT - \mu T) + \xi_R (v_1 S + v_2 I + v_3 L - \mu R).\end{aligned}\quad (6.3)$$

Here, ξ_S , ξ_I , ξ_L , ξ_T , and ξ_R in (6.3) denote the adjoint (co-state) variables associated with the state variables S , I , L , T , and R , respectively. Based on the Hamiltonian $\mathcal{H}$, the necessary conditions for optimality and the characterization of the optimal controls are given in the following theorem.

Theorem 6.1. Let $(u_1^*, u_2^*) \in \mathcal{U}$ be a best control pair with associated optimal state $(S^*, I^*, L^*, T^*, R^*)$ solving the controlled system

$$\begin{aligned}\dot{S} &= \pi - \frac{(1-u_1^*)\alpha SI}{1+\eta I} - (\gamma_1 + u_2^*)ST - (v_1 + \mu)S, \\ \dot{I} &= \frac{(1-u_1^*)q\alpha SI}{1+\eta I} + \delta_2 L - (\gamma_3 + u_2^*)IT - (\delta_1 + v_2 + \mu)I, \\ \dot{L} &= \frac{(1-u_1^*)(1-q)\alpha SI}{1+\eta I} + \delta_1 I - (\gamma_2 + u_2^*)LT - (\delta_2 + v_3 + \mu)L, \\ \dot{T} &= (\gamma_1 + u_2^*)ST + (\gamma_2 + u_2^*)LT + (\gamma_3 + u_2^*)IT - \mu T, \\ \dot{R} &= v_1 S + v_2 I + v_3 L - \mu R.\end{aligned}$$

Then there exist continuous adjoint (co-state) variables $\xi_S, \xi_I, \xi_L, \xi_T, \xi_R$ such that $\xi_S(t_f) = \xi_I(t_f) = \xi_L(t_f) = \xi_T(t_f) = \xi_R(t_f) = 0$, and, for $t \in [0, t_f]$, they satisfy

$$\begin{aligned}\dot{\xi}_S &= \xi_S \left(\frac{(1-u_1^*)\alpha I^*}{1+\eta I^*} + (\gamma_1 + u_2^*)T^* + (v_1 + \mu) \right) - \xi_I \left(\frac{(1-u_1^*)q\alpha I^*}{1+\eta I^*} \right) - \xi_L \left(\frac{(1-u_1^*)(1-q)\alpha I^*}{1+\eta I^*} \right) \\ &\quad - \xi_T (\gamma_1 + u_2^*)T^* - \xi_R v_1, \\ \dot{\xi}_I &= -A_1 + \xi_S \left(\frac{(1-u_1^*)\alpha S^*}{(1+\eta I^*)^2} \right) - \xi_I \left(\frac{(1-u_1^*)q\alpha S^*}{(1+\eta I^*)^2} - (\gamma_3 + u_2^*)T^* - (\delta_1 + v_2 + \mu) \right) \\ &\quad - \xi_L \left(\frac{(1-u_1^*)(1-q)\alpha S^*}{(1+\eta I^*)^2} + \delta_1 \right) - \xi_T (\gamma_3 + u_2^*)T^* - \xi_R v_2, \\ \dot{\xi}_L &= -A_2 - \xi_I \delta_2 + \xi_L ((\gamma_2 + u_2^*)T^* + (\delta_2 + v_3 + \mu)) - \xi_T (\gamma_2 + u_2^*)T^* - \xi_R v_3, \\ \dot{\xi}_T &= A_3 + \xi_S (\gamma_1 + u_2^*)S^* + \xi_I (\gamma_3 + u_2^*)I^* + \xi_L (\gamma_2 + u_2^*)L^* - \xi_T ((\gamma_1 + u_2^*)S^* + (\gamma_2 + u_2^*)L^* \\ &\quad + (\gamma_3 + u_2^*)I^* - \mu T^*) - \xi_R \mu.\end{aligned}$$

$$+ (\gamma_3 + u_2^*)I^* - \mu),$$

$$\dot{\xi}_R = \mu \xi_R.$$

Moreover, the optimal controls minimize the Hamiltonian and are given by the control set below

$$u_1^* = \min \left\{ 1, \max \left\{ 0, \frac{\alpha S^* I^*}{A_4(1 + \eta I^*)} (q \xi_I + (1 - q) \xi_L - \xi_S) \right\} \right\},$$

$$u_2^* = \min \left\{ 1, \max \left\{ 0, \frac{S^* T^* (\xi_S - \xi_T) + I^* T^* (\xi_I - \xi_T) + L^* T^* (\xi_L - \xi_T)}{A_5} \right\} \right\}.$$

Proof. By Pontryagin's Minimum Principle, there exist adjoint variables $\xi_S, \xi_I, \xi_L, \xi_T, \xi_R$ such that $\dot{\xi}_i = -\partial H / \partial x_i$ with transversality conditions $\xi_i(t_f) = 0, i \in \{S, I, L, T, R\}$. Since the Hamiltonian is strictly convex in u_1, u_2 due to the quadratic terms $\frac{A_4}{2} u_1^2 + \frac{A_5}{2} u_2^2$, the pointwise minimizer is obtained from $\partial H / \partial u_1 = 0$ and $\partial H / \partial u_2 = 0$, giving the unconstrained controls. Imposing the bounds $0 \leq u_1, u_2 \leq 1$ gives the projection formulas stated above. $\square$

7. NUMERICAL SIMULATION

Here, we carry out a numerical analysis to support the analysis of the effects of two control variables on reducing rumor propagation. The optimal control problem is addressed by employing Pontryagin's Maximum Principle, and the derived adjoint state system is solved numerically using the forward-backward sweep method. To assess the impact of each intervention, we consider four simulation settings: (i) a control-free case, (ii) a strategy using only u_1 , (iii) a strategy using only u_2 , and (iv) a combined strategy using both u_1 and u_2 . Since the model variables are normalized as proportions of the population, the initial conditions are chosen as $(S(0), I(0), L(0), T(0), R(0)) = (0.80, 0.15, 0.03, 0.01, 0.01)$, satisfying $S(0) + I(0) + L(0) + T(0) + R(0) = 1$.

These values are chosen to represent the relative importance of each element in the optimal control problem. The weight A_1 is greater than A_2 because active rumor spreaders play a stronger role in spreading rumors than lurker individuals. Therefore, suppression of active rumor spreaders is prioritized. The lower weight A_3 is applied to truth spreaders, who are rewarded in the objective functional to encourage the spread of truth without dominating the main goal of rumor minimization. The control cost $A_4 < A_5$ indicates that the first control, u_1 , is expected to be cheaper or more practical to implement than the second control, u_2 . Therefore, u_2 is assigned a larger penalty to prevent excessive usage. Generally, these weights provide a balance between reducing rumor propagation, promoting the spread of truth, and maintaining reasonable control efforts. Numerical simulations are performed using the parameter values listed in Table 2. This set of parameters is used as the baseline configuration to model a representative rumor-spreading framework.

As depicted in Figure 6(A), the control u_1 increases and maintains the susceptible subpopulation at a significantly higher level than in the case without control, particularly during the early and

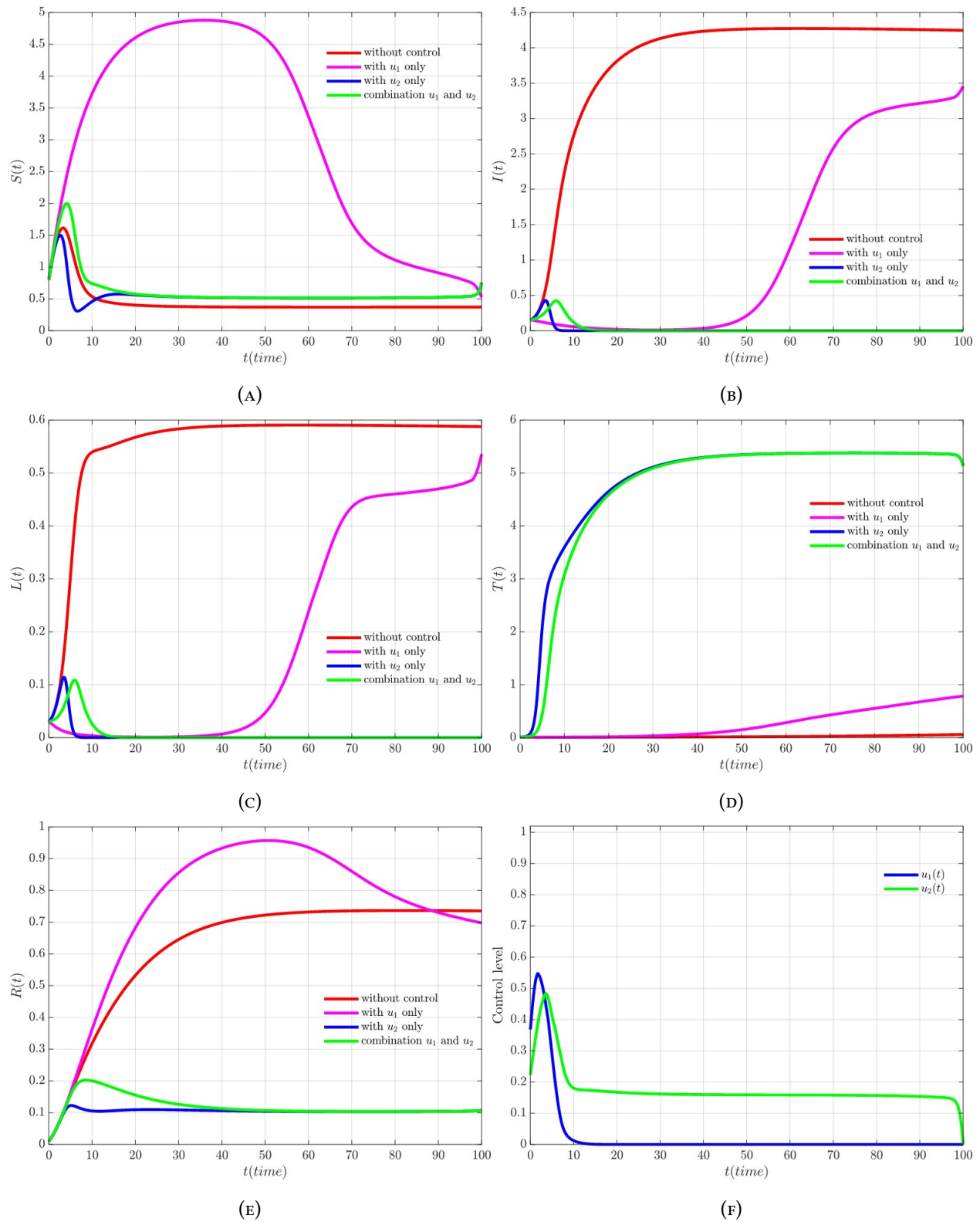


FIGURE 6. Comparative dynamics of the optimal control strategies: (A) susceptible $S(t)$, (B) infected $I(t)$, (C) lurkers $L(t)$, (D) truth spreaders $T(t)$, (E) recovered $R(t)$, and (F) optimal control variable levels.

TABLE 2. Baseline parameter set

Parameter	Value	References	Parameter	Value	References
π	0.60	[43]	γ_2	0.030	[29,50]
α	0.50	[29,44]	γ_3	0.022	[49]
η	0.10	[45]	ν_1	0.02	[43,51]
q	0.70	[46]	ν_2	0.01	[50]
δ_1	0.015	[29,47]	ν_3	0.04	[51]
δ_2	0.25	[43,48]	μ	0.10	[29,44]
γ_1	0.036	[49]			

middle stages of the simulation. In the absence of control, the susceptible subpopulation initially increases slowly, reaches a peak around $t \approx 5$, and subsequently decreases gradually to a stable level. However, under u_1 , $S(t)$ increases significantly and reaches its peak during the middle time interval, approximately at $t \approx 35\text{--}45$, before declining toward the end of the simulation. This implies that u_1 plays a protective role by reducing rumor contact and delaying the transition of susceptible individuals to rumor spreaders. Meanwhile, strategies involving u_2 combined control (u_1, u_2) also contribute to protecting the susceptible group by suppressing rumor propagation more effectively.

Figure 6(B) illustrates the dynamics of the infected population. Without control, the infected population grows rapidly in the early period and peaks around $t \approx 20\text{--}30$, suggesting sustained rumor propagation in the absence of control. Only under u_1 , the infected population remains at a low level, but rapidly increases after approximately $t \approx 50$ and then peaks toward the end of the simulation. This result shows that limiting the contact rate can delay the peak of rumor spread, but is not sufficient to eliminate rumor propagation over the time interval. Conversely, under only u_2 , the infected population exhibits a small initial peak and then rapidly falls to a near-zero level. The combined control (u_1, u_2) also produces a small initial peak, followed by a rapid decline of $I(t)$ to a near-zero level. These results confirm that truth-dissemination control is more effective in reducing both the peak size and the persistence time of active rumor spreaders.

Figure 6(C) describes the dynamics of the lurker population $L(t)$. Without control, $L(t)$ quickly increases and reaches a high stable level after the initial period, indicating that many individuals remain passively exposed to rumor-related information. Only under u_1 , the growth of $L(t)$ is slowed down for a long period, but the lurker population increases dramatically after approximately $t \approx 50$ and peaks near the end of the simulation period. This confirms that u_1 can delay the growth of lurkers but cannot completely suppress it. In contrast, the control u_2 produces only a small initial peak in $L(t)$, followed by a rapid decline to a level near zero. Therefore, u_2 is more effective in preventing individuals from remaining in a passive rumor-exposed state.

Figure 6(D) shows the behavior of the truth-spreading population $T(t)$. In the absence of control, $T(t)$ remains very low, which implies that correct information does not spread effectively without intervention. Under u_1 only, $T(t)$ increases slowly and remains relatively small throughout the simulation. A stronger effect is observed under u_2 only and the combined control (u_1, u_2) . In both cases, the truth-spreader population grows rapidly, reaches a high level around $t \approx 30-40$, and then remains at a relatively high level. This rapid increase suggests that u_2 accelerates the spread of truth information and acts as the dominant factor in expanding the population of truth seekers. The similar behavior of the u_2 -only strategy and the combined strategy further confirms that u_2 is the primary factor in the growth of truth spreaders, while u_1 provides additional support.

The recovered class $R(t)$, shown in Figure 6(E), represents individuals who are no longer active in the rumor. Without control, $R(t)$ gradually rises and converges on a moderately stable level. Only under u_1 does $R(t)$ reach a peak, roughly in the middle of the simulation, and then decays slightly toward the end. This reveals that reducing rumor contact primarily occurs from the active rumor into the recovered or uninvolved class. Conversely, only under u_2 and the combined control (u_1, u_2) does the recovered subpopulation reach only a low peak. These results do not imply that these strategies are less effective. On the contrary, it indicates that truth defenders direct some individuals to the truth-spreading class, rather than just moving them to the recovered class. Thus, u_1 mainly acts as a recovery-oriented control, while u_2 serves as a truth-spreading control.

The optimal control profile is depicted in Figure 6(F). The control $u_1(t)$ peaks at the beginning of the simulation and then drops rapidly to a negligible level after around $t \approx 8-10$. This means that reducing contact with the rumor is most effective during the early stage of its spread. Meanwhile, $u_2(t)$ also begins at high intensity, peaks early, and then gradually decreases, remaining positive throughout most of the simulation period. This highlights that the dissemination of truth should be implemented early and sustained over time. The early peak of $u_2(t)$ contributes to suppressing the growth of rumor spreaders and lurkers, while its continued application supports the maintenance of a truth-dominated information environment.

In general, the peak-time analysis reveals that u_1 delays the peaks of the infected and lurker populations, while u_2 reduces the peak magnitude and the duration of rumor persistence. The combined control yields the most effective result because it suppresses the initial growth of rumor spreaders, prevents the accumulation of lurkers, and drives the rapid growth of truth spreaders. Therefore, the most effective control strategy is to combine both controls.

8. DISCUSSION AND CONCLUSION

In this paper, we analyze a nonlinear rumor propagation model that combines a saturated incidence function, lurkers, truth spreaders, and an optimal control strategy. This model categorizes the population into susceptible individuals, rumor spreaders, lurkers, truth spreaders, and recovered individuals. A saturated incidence function is used to describe the limitation of rumor

propagation under high exposure. The theoretical analysis confirms that the proposed model has a strong mathematical and biological basis. In particular, the solutions are nonnegative and bounded in a well-posedness region. This model yields several equilibrium points, including a rumor-free and truth-free equilibrium, a rumor-free equilibrium with truth spreaders, a persistent rumor equilibrium without truth spreaders, and a coexistence equilibrium. The reproduction threshold of this model is determined by two components: the reproduction number for rumor spread, $\mathcal{R}_{0I}$ and the reproduction number for truth spread, $\mathcal{R}_{0T}$. The value of $\mathcal{R}_{0I}$ represents the potential for rumor propagation, while $\mathcal{R}_{0T}$ represents the potential for truth spreaders to persist. Therefore, the dynamics are not governed by a single threshold, but by the interaction between rumor amplification and the spread of truth.

The analytical results show that the rumor-free and truth-free equilibrium is asymptotically stable locally and globally when $\mathcal{R}_{0I} \leq 1$ and $\mathcal{R}_{0T} \leq 1$. Other stability properties also depend on the reproduction numbers and the relevant conditions. From a biological and social perspective, this means that if rumor spread is too weak and truth spread cannot sustain itself, both rumor spreaders and truth spreaders will eventually disappear from the system. However, practical considerations suggest that the more desirable goal is not the disappearance of truth spreaders, but rather the reduction of rumor spreaders. This discussion highlights the importance of interpreting the thresholds in relation to the specific roles of each compartment in the system. In addition, bifurcation analysis is discussed to provide important insights into the qualitative behavior of the model. By applying the center manifold approach, the system is shown that the system undergoes forward bifurcations at $\mathcal{R}_{0I} = 1$ and $\mathcal{R}_{0T} = 1$. These results imply that lowering $\mathcal{R}_{0I}$ can reduce or mitigate the spread of rumors. Conversely, increasing the basic reproduction number associated with truth spreaders can strengthen the ability of verified information to compete with circulating rumors.

A sensitivity analysis was conducted to identify the parameters that most influence $\mathcal{R}_{0I}$ and $\mathcal{R}_{0T}$. The number $\mathcal{R}_{0I}$ increases with the rumor contact rate α , the recruitment rate π , the proportion q and the transition rate from lurker to rumor spreader δ_2 . Therefore, rumor persistence is facilitated by stronger contact with rumor spreaders. Conversely, parameters such as δ_1 , ν_1 , ν_2 , ν_3 , and μ decrease $\mathcal{R}_{0I}$, indicating that rumors spread can be weakened by enhancing the transitions from the active rumor spreading and by increasing the recovery from rumors. For the truth-spreading threshold $\mathcal{R}_{0T}$, the parameters γ_1 and π have a positive influence, while ν_1 and μ have a negative influence. This means that truth-spreading is stronger when susceptible individuals interact more effectively with truth-spreaders. Practically, these results suggest that public communication strategies should be improved to control rumors in society. Therefore, rumor mitigation requires not only reducing contact with rumors but also maintaining constructive engagement with verified information.

Simulations of optimal control further demonstrate that these two strategies play different but complementary roles. The first control, $u_1(t)$, reduces effective contact between susceptible

individuals and rumor spreaders. It is especially effective in the early stages of rumor spread, when transmission rates are highest. The numerical results show that $u_1(t)$ is heavily activated at the beginning and then decreases rapidly until it reaches a negligible level. This suggests that reducing rumor contact is more valuable as a short-term response. Conversely, the second control $u_2(t)$, which enhances the spread of truth, functions as a corrective mechanism. Simulations show that $u_2(t)$ remains active over a longer period of time, indicating that the spread of truth must be maintained beyond the initial period of the outbreak. In practice, this aligns with visible fact-checking and clarification.

A comparison between scenarios without control, a single control, and a combination of controls shows that the truth-spreading intervention u_2 has a significant effect in reducing rumor spreaders and lurkers while simultaneously increasing the population of truth-spreaders. Only u_1 can delay the escalation of rumors and protect sensitive populations for a short time, but it is not powerful enough to eliminate the persistence of rumors over the long term. In contrast, a strategy involving u_2 , alone or in combination with u_1 , rapidly reduces the infected and lurker class, and also leads to a strong increase in the truth-spreader class. The combined strategy yields the strongest results because it simultaneously reduces rumor infection and strengthens the truth spreaders. Together, the results of the analysis and numerical simulation support that the most effective rumor mitigation should not be based solely on limiting exposure to rumors. Instead, a successful strategy must combine early efforts to reduce the intensity of rumor exposure with stronger, more sustained campaigns to disseminate the truth. Thus, this study supports a combined intervention framework in which preventive and corrective strategies are applied simultaneously, with an emphasis on enhancing verified information.

Despite the insights obtained from this study, several limitations must be acknowledged. First, the model assumes homogeneous interactions among individuals, although real-world online and offline information networks are highly heterogeneous. Second, this system is deterministic and explicitly excludes random variation. Third, the parameter values used in the numerical simulations are baseline values taken from previous studies. Future empirical fitting to actual rumor-spread data would improve the quantitative performance of the model. Finally, future work may extend with time-delay effects to describe delays in rumor spreading or to describe delays in fact-checking. Furthermore, future studies may include data-driven parameter estimation to evaluate the practical applicability of the rumor model on real digital platforms.

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APPENDIX A

The coefficients $Q_0, Q_1, \dots, Q_7$ in equation (3.7) are provided in the complexity form below

$$\begin{aligned}
 r_1 &= v_1 + \mu, \quad r_2 = \delta_1 + v_2 + \mu, \quad r_3 = \delta_2 + v_3 + \mu, \quad r_4 = \gamma_2 r_2 + \gamma_3 r_3, \quad r_5 = \delta_1(v_3 + \mu) + r_3(v_2 + \mu), \\
 r_6 &= (\gamma_2 \gamma_3 r_1 + \gamma_1 r_4)^2 - 2\gamma_1 \gamma_2 \gamma_3 (\alpha \pi \gamma_2 - (r_1 r_4 + \gamma_1 r_5)), \\
 r_7 &= -2(\gamma_2 \gamma_3 r_1 + \gamma_1 r_4) (q \alpha \pi \gamma_2 - (r_1 r_4 + \gamma_1 r_5)) - 2\gamma_1 \gamma_2 \gamma_3 r_1 r_5 (\mathcal{R}_{0I} - 1), \\
 r_8 &= (q \alpha \pi \gamma_2 - (r_1 r_4 + \gamma_1 r_5))^2 - 2(\gamma_2 \gamma_3 r_1 + \gamma_1 r_4) r_1 r_5 (\mathcal{R}_{0I} - 1), \\
 r_9 &= 2r_1 r_5 (\mathcal{R}_{0I} - 1) (q \alpha \pi \gamma_2 - (r_1 r_4 + \gamma_1 r_5)), \\
 r_{10} &= \gamma_1 \gamma_2 (\pi \eta + \delta_1) + \gamma_3 (\gamma_1 r_3 + \gamma_2 r_1) - \mu \gamma_2 (\alpha + \eta r_1) - \mu \eta \gamma_1 r_3, \\
 r_{11} &= \mu \eta r_1 r_3 (\mathcal{R}_{0T} - 1) + \mu \eta r_1 r_3 + (1 - q) \alpha \pi \gamma_2 + r_1 (\delta_1 \gamma_2 + \gamma_3 r_3) - \mu r_3 (\alpha + \eta r_1), \\
 r_{12} &= \eta^2 \gamma_1^2 r_4^2 + \gamma_2^2 \gamma_3^2 (\alpha + \eta r_1)^2 + 2\eta \gamma_1 \gamma_2 \gamma_3 (\eta \gamma_1 r_5 + 2r_4 (\alpha + \eta r_1)), \\
 r_{13} &= 2 \left(r_4 (\eta^2 \gamma_1^2 r_5 + \gamma_2 \gamma_3 (\alpha + \eta r_1)^2) + \eta \gamma_1 (\alpha + \eta r_1) (2\gamma_2 \gamma_3 r_5 + r_4^2) \right), \\
 r_{14} &= \eta^2 \gamma_1^2 r_5^2 + r_4^2 (\alpha + \eta r_1)^2 + 2r_5 (\alpha + \eta r_1) (2\eta \gamma_1 r_4 + \gamma_2 \gamma_3 (\alpha + \eta r_1)), \\
 Q_7 &= 2\eta \gamma_1^2 \gamma_2^2 \gamma_3^2 (\gamma_2 \gamma_3 r_1 + \gamma_1 r_4) + \gamma_1^2 \gamma_2^2 \gamma_3^2 (\eta \gamma_1 (\delta_1 \gamma_2 + \gamma_3 r_3) + \gamma_2 \gamma_3 (\alpha + \eta r_1)) - \gamma_1 \gamma_2 (\gamma_3 - \mu \eta) \\
 &\quad \times \left[\gamma_1 \gamma_2 \gamma_3 (\eta \gamma_1 r_4 + \gamma_2 \gamma_3 (\alpha + \eta r_1)) + \eta \gamma_1 \gamma_2 \gamma_3 (\gamma_2 \gamma_3 r_1 + \gamma_1 r_4) \right] - \eta \gamma_1^2 \gamma_2^2 \gamma_3^2 r_{10} \\
 &\quad - 2\mu \eta \gamma_1^2 \gamma_2^2 \gamma_3 (\eta \gamma_1 r_4 + \gamma_2 \gamma_3 (\alpha + \eta r_1)) + \eta^2 \gamma_1^2 \gamma_2^2 \gamma_3^2 ((\pi \gamma_1 - \mu r_1) \gamma_2 - \mu \gamma_1 r_3), \\
 Q_6 &= \eta \gamma_1 \gamma_2 \gamma_3 r_6 + 2\gamma_1 \gamma_2 \gamma_3 (\gamma_2 \gamma_3 r_1 + \gamma_1 r_4) (\eta \gamma_1 (\delta_1 \gamma_2 + \gamma_3 r_3) + \gamma_2 \gamma_3 (\alpha + \eta r_1)) \\
 &\quad + \gamma_1^2 \gamma_2^2 \gamma_3^2 (\alpha + \eta r_1) (\delta_1 \gamma_2 + \gamma_3 r_3) - \gamma_1^2 \gamma_2^2 \gamma_3 (\gamma_3 - \mu \eta) (\eta \gamma_1 r_5 + r_4 (\alpha + \eta r_1)) \\
 &\quad + \eta \gamma_1 r_4 - \gamma_2 \gamma_3 (\alpha + \eta r_1) \left[\gamma_1 \gamma_2 (\gamma_3 - \mu \eta) (\gamma_2 \gamma_3 r_1 + \gamma_1 (\gamma_3 r_3 + \gamma_2 r_2)) + \gamma_1 \gamma_2 \gamma_3 r_{10} \right] \\
 &\quad + \eta \gamma_1 \gamma_2 \gamma_3 \left[\gamma_1 \gamma_2 (\gamma_3 - \mu \eta) (q \alpha \pi \gamma_2 - (r_1 r_4 + \gamma_1 r_5)) - (\gamma_2 \gamma_3 r_1 + \gamma_1 r_4) r_{10} - \gamma_1 \gamma_2 \gamma_3 r_{11} \right] \\
 &\quad - \mu \gamma_1 \gamma_2 r_{12} + 2\eta \gamma_1 \gamma_2 \gamma_3 ((\pi \gamma_1 - \mu r_1) \gamma_2 - \mu \gamma_1 r_3) (\eta \gamma_1 r_4 + \gamma_2 \gamma_3 (\alpha + \eta r_1)) + \eta^2 \gamma_1^2 \gamma_2^2 \gamma_3^2 r_3 (\pi \gamma_1 - \mu r_1), \\
 Q_5 &= \eta \gamma_1 \gamma_2 \gamma_3 r_7 + r_6 (\eta \gamma_1 (\delta_1 \gamma_2 + \gamma_3 r_3) + \gamma_2 \gamma_3 (\alpha + \eta r_1)) + 2\gamma_1 \gamma_2 \gamma_3 (\alpha + \eta r_1) (\gamma_2 \gamma_3 r_1 + \gamma_1 r_4) \\
 &\quad \times (\delta_1 \gamma_2 + \gamma_3 r_3) - \gamma_1^2 \gamma_2^2 \gamma_3 r_5 (\alpha + \eta r_1) (\gamma_3 - \mu \eta) + \eta \gamma_1 r_5 - r_4 (\alpha + \eta r_1) \left[\gamma_1 \gamma_2 (\gamma_3 - \mu \eta) \right. \\
 &\quad \times (\gamma_2 \gamma_3 r_1 + \gamma_1 r_4) + \gamma_1 \gamma_2 \gamma_3 r_{10} \left. \right] + (\eta \gamma_1 r_4 + \gamma_2 \gamma_3 (\alpha + \eta r_1)) \left[\gamma_1 \gamma_2 (\gamma_3 - \mu \eta) (q \alpha \pi \gamma_2 - (r_1 r_4 \right. \\
 &\quad \left. + \gamma_1 r_5)) - (\gamma_2 \gamma_3 r_1 + \gamma_1 r_4) r_{10} - \gamma_1 \gamma_2 \gamma_3 r_{11} \right] + \eta \gamma_1 \gamma_2 \gamma_3 \left[\gamma_1 \gamma_2 (\gamma_3 - \mu \eta) (\alpha \pi (q r_3 + (1 - q) \delta_2) \right. \\
 &\quad \left. - r_1 r_5) + r_{10} (q \alpha \pi \gamma_2 - (r_1 r_4 + \gamma_1 r_5)) - r_{11} (\gamma_2 \gamma_3 r_1 + \gamma_1 r_4) \right] - \mu \gamma_1 \gamma_2 r_{13} + r_{12} ((\pi \gamma_1 - \mu r_1) \gamma_2 \\
 &\quad - \mu \gamma_1 r_3) + 2\eta \gamma_1 \gamma_2 \gamma_3 r_3 (\pi \gamma_1 - \mu r_1) (\eta \gamma_1 r_4 + \gamma_2 \gamma_3 (\alpha + \eta r_1)),
 \end{aligned}$$

$$\begin{aligned}
Q_4 = & \eta\gamma_1\gamma_2\gamma_3r_8 + r_7\left(\eta\gamma_1(\delta_1\gamma_2 + \gamma_3r_3) + \gamma_2\gamma_3(\alpha + \eta r_1)\right) + r_6(\alpha + \eta r_1)(\delta_1\gamma_2 + \gamma_3r_3) - r_5(\alpha + \eta r_1) \\
& \times \left[\gamma_1\gamma_2(\gamma_3 - \mu\eta)(\gamma_2\gamma_3r_1 + \gamma_1r_4) + \gamma_1\gamma_2\gamma_3r_{10}\right] + (\eta\gamma_1r_5 + r_4(\alpha + \eta r_1))\left[\gamma_1\gamma_2(\gamma_3 - \mu\eta)\right. \\
& \times (q\alpha\pi\gamma_2 - (r_1r_4 + \gamma_1r_5)) - r_{10}(\gamma_2\gamma_3r_1 + \gamma_1r_4) - \gamma_1\gamma_2\gamma_3r_{11}\left. \right] + (\eta\gamma_1r_4 + \gamma_2\gamma_3(\alpha + \eta r_1)) \\
& \times \left[\gamma_1\gamma_2(\gamma_3 - \mu\eta)(q\alpha\pi r_3 + (1 - q)\alpha\pi\delta_2 - r_1r_5) + r_{10}(q\alpha\pi\gamma_2 - (r_1r_4 + \gamma_1r_5))\right. \\
& \left. - r_{11}(\gamma_2\gamma_3r_1 + \gamma_1r_4)\right] + \eta\gamma_1\gamma_2\gamma_3\left[r_{10}(q\alpha\pi r_3 + (1 - q)\alpha\pi\delta_2 - r_1r_5) + r_{11}(q\alpha\pi\gamma_2\right. \\
& \left. - (r_1r_4 + \gamma_1r_5))\right] - \mu\gamma_1\gamma_2r_{14} + r_{13}\left((\pi\gamma_1 - \mu r_1)\gamma_2 - \mu\gamma_1r_3\right) + r_3r_{12}(\pi\gamma_1 - \mu r_1),
\end{aligned}$$

$$\begin{aligned}
Q_3 = & \eta\gamma_1\gamma_2\gamma_3r_9 + r_8\left(\eta\gamma_1(\delta_1\gamma_2 + \gamma_3r_3) + \gamma_2\gamma_3(\alpha + \eta r_1)\right) + r_7(\alpha + \eta r_1)(\delta_1\gamma_2 + \gamma_3r_3) + r_5(\alpha + \eta r_1) \\
& \times \left[\gamma_1\gamma_2(\gamma_3 - \mu\eta)(q\alpha\pi\gamma_2 - (r_1r_4 + \gamma_1r_5)) - r_{10}(\gamma_2\gamma_3r_1 + \gamma_1r_4) - \gamma_1\gamma_2\gamma_3r_{11}\right] + (\eta\gamma_1r_5 \\
& + r_4(\alpha + \eta r_1))\left[\gamma_1\gamma_2(\gamma_3 - \mu\eta)(q\alpha\pi r_3 + (1 - q)\alpha\pi\delta_2 - r_1r_5) + r_{10}(q\alpha\pi\gamma_2 - (r_1r_4 + \gamma_1r_5))\right. \\
& \left. - r_{11}(\gamma_2\gamma_3r_1 + \gamma_1r_4)\right] + (\eta\gamma_1r_4 + \gamma_2\gamma_3(\alpha + \eta r_1))\left[r_{10}(q\alpha\pi r_3 + (1 - q)\alpha\pi\delta_2 - r_1r_5)\right. \\
& \left. + r_{11}(q\alpha\pi\gamma_2 - (r_1r_4 + \gamma_1r_5))\right] + \eta\gamma_1\gamma_2\gamma_3r_{11}(q\alpha\pi r_3 + (1 - q)\alpha\pi\delta_2 - r_1r_5) - 2\mu\gamma_1\gamma_2r_5 \\
& \times (\alpha + \eta r_1)(\eta\gamma_1r_5 + r_4(\alpha + \eta r_1)) + r_{14}\left((\pi\gamma_1 - \mu r_1)\gamma_2 - \mu\gamma_1r_3\right) + r_3r_{13}(\pi\gamma_1 - \mu r_1),
\end{aligned}$$

$$\begin{aligned}
Q_2 = & \eta\gamma_1\gamma_2\gamma_3(q\alpha\pi\gamma_2 - r_1r_5)^2 + r_9\left(\eta\gamma_1(\delta_1\gamma_2 + \gamma_3r_3) + \gamma_2\gamma_3(\alpha + \eta r_1)\right) + r_8(\alpha + \eta r_1)(\delta_1\gamma_2 + \gamma_3r_3) \\
& + r_5(\alpha + \eta r_1)\left[\gamma_1\gamma_2(\gamma_3 - \mu\eta)(q\alpha\pi r_3 + (1 - q)\alpha\pi\delta_2 - r_1r_5) + r_{10}(q\alpha\pi\gamma_2 - (r_1r_4 + \gamma_1r_5))\right. \\
& \left. - r_{11}(\gamma_2\gamma_3r_1 + \gamma_1r_4)\right] + (\eta\gamma_1r_5 + r_4(\alpha + \eta r_1))\left[r_{10}(q\alpha\pi r_3 + (1 - q)\alpha\pi\delta_2 - r_1r_5) + r_{11}(q\alpha\pi\gamma_2\right. \\
& \left. - (r_1r_4 + \gamma_1r_5))\right] + r_{11}(q\alpha\pi r_3 + (1 - q)\alpha\pi\delta_2 - r_1r_5)(\eta\gamma_1r_4 + \gamma_2\gamma_3(\alpha + \eta r_1)) - \mu\gamma_1\gamma_2r_5^2(\alpha + \eta r_1)^2 \\
& + 2r_5(\alpha + \eta r_1)\left((\pi\gamma_1 - \mu r_1)\gamma_2 - \mu\gamma_1r_3\right)(\eta\gamma_1r_5 + r_4(\alpha + \eta r_1)) + r_3r_{14}(\pi\gamma_1 - \mu r_1),
\end{aligned}$$

$$\begin{aligned}
Q_1 = & (q\alpha\pi\gamma_2 - r_1r_5)^2\left(\eta\gamma_1(\delta_1\gamma_2 + \gamma_3r_3) + \gamma_2\gamma_3(\alpha + \eta r_1)\right) + r_9(\alpha + \eta r_1)(\delta_1\gamma_2 + \gamma_3r_3) + r_5(\alpha + \eta r_1) \\
& \times \left[r_{10}(\alpha\pi(qr_3 + (1 - q)\delta_2) - r_1r_5) + r_{11}(q\alpha\pi\gamma_2 - (r_1r_4 + \gamma_1r_5))\right] + r_{11}(q\alpha\pi r_3 + (1 - q)\alpha\pi\delta_2 \\
& - r_1r_5)(\eta\gamma_1r_5 + r_4(\alpha + \eta r_1)) + r_5^2(\alpha + \eta r_1)^2\left((\pi\gamma_1 - \mu r_1)\gamma_2 - \mu\gamma_1r_3\right) + 2r_3r_5(\alpha + \eta r_1) \\
& \times (\pi\gamma_1 - \mu r_1)(\eta\gamma_1r_5 + r_4(\alpha + \eta r_1)),
\end{aligned}$$

$$Q_0 = r_1r_5^2(\alpha + \eta r_1)\left[(\delta_1\gamma_2 + \gamma_3r_3)r_1(\mathcal{R}_{0I} - 1)^2 + r_{10}(\mathcal{R}_{0I} - 1) + \mu r_3(\alpha + \eta r_1)(\mathcal{R}_{0T} - 1)\right].$$

APPENDIX B

Assumed that the seventh-degree polynomial (3.7) consists of three pairs of conjugate non-real complex roots, symbolized by $\vartheta, \vartheta^*, \kappa, \kappa^*, \psi, \psi^*$ and one real root associated with the linear factor $(T + m)$. The first conjugate pair, we understand that $(T - \vartheta)(T - \vartheta^*) = T^2 - 2\operatorname{Re}(\vartheta)T + |\vartheta|^2$. Thus, it can be rewritten in the form $T^2 + x_1T + x_2$, with $x_1 = -2\operatorname{Re}(\vartheta)$, $x_2 = |\vartheta|^2$. Similarly, the conjugate pairs κ, κ^* and ψ, ψ^* generate the quadratic factors $T^2 + y_1T + y_2$ and $T^2 + z_1T + z_2$, respectively, with $y_1 = -2\operatorname{Re}(\kappa)$, $y_2 = |\kappa|^2$ and $z_1 = -2\operatorname{Re}(\psi)$, $z_2 = |\psi|^2$. Since these roots are non-real complex conjugates, then the corresponding quadratic factors have negative discriminants. Consequently, the polynomial in (3.7) can be factorized as

$$\sum_{n=0}^7 Q_n T^n = Q_7(T + m)(T^2 + x_1T + x_2)(T^2 + y_1T + y_2)(T^2 + z_1T + z_2). \quad (\text{B.1})$$

By expanding and equating the coefficients on both sides of (B.1) gives the equality term below

$$\frac{Q_6}{Q_7} = m + x_1 + y_1 + z_1,$$

$$\frac{Q_5}{Q_7} = x_2 + y_2 + z_2 + x_1y_1 + x_1z_1 + y_1z_1 + m(x_1 + y_1 + z_1),$$

$$\frac{Q_4}{Q_7} = x_1y_2 + x_2y_1 + x_1z_2 + y_1z_2 + x_2z_1 + y_2z_1 + x_1y_1z_1 + m(x_2 + y_2 + z_2 + x_1y_1 + x_1z_1 + y_1z_1),$$

$$\begin{aligned} \frac{Q_3}{Q_7} = & x_2y_2 + x_2z_2 + y_2z_2 + x_1y_1z_2 + x_1y_2z_1 + x_2y_1z_1 + m(x_1y_2 + x_2y_1 + x_1z_2 + y_1z_2 + x_2z_1 + y_2z_1 \\ & + x_1y_1z_1), \end{aligned}$$

$$\frac{Q_2}{Q_7} = x_2y_2z_1 + x_1y_2z_2 + x_2y_1z_2 + m(x_2y_2 + x_2z_2 + y_2z_2 + x_1y_1z_2 + x_1y_2z_1 + x_2y_1z_1),$$

$$\frac{Q_1}{Q_7} = x_2y_2z_2 + m(x_2y_2z_1 + x_1y_2z_2 + x_2y_1z_2),$$

$$\frac{Q_0}{Q_7} = mx_2y_2z_2.$$

Henceforth, it follows from the coefficient Q_0/Q_7 term, we obtain the root form m as

$$m = \frac{Q_0}{Q_7x_2y_2z_2} = \frac{Q_0}{Q_7|\vartheta|^2|\rho|^2|\psi|^2}.$$

Since $x_2y_2z_2 = |\vartheta|^2|\rho|^2|\psi|^2 > 0$, the sign of m is determined only by the sign of Q_0/Q_7 . The linear factor $T + m$ gives the real root $T^* = -m$. Therefore, this root is positive iff $m < 0$, or equivalently $\frac{Q_0}{Q_7} < 0$. Finally, there is a real positive root that exists under the following conditions $Q_0 < 0$ and $Q_7 > 0$, or $Q_0 > 0$ and $Q_7 < 0$.

APPENDIX C

The characteristic equation of equation (2.7) in E_3 can be governed and expressed as

$$(-\mu - \lambda)(\lambda^4 + e_1\lambda^3 + e_2\lambda^2 + e_3\lambda + e_4) = 0, \quad (\text{C.1})$$

with

$$\begin{aligned} e_1 &= -(J_{11} + J_{22} + J_{33} + J_{44}), \\ e_2 &= (\gamma_1^2 S_3 + \gamma_2^2 L_3 + \gamma_3^2 I_3)T_3 + J_{11}(J_{22} + J_{44}) + J_{22}(J_{33} + J_{44}) + J_{33}(J_{11} + J_{44}) - J_{12}J_{21} - \delta_2 J_{32}, \\ e_3 &= J_{12}J_{21}(J_{33} + J_{44}) - J_{11}J_{22}(J_{33} + J_{44}) - J_{33}J_{44}(J_{11} + J_{22}) + T_3(\gamma_3(\gamma_1 J_{21}S_3 + \gamma_2 J_{32}I_3) \\ &\quad + \gamma_1(\gamma_2 J_{31}S_3 + \gamma_3 J_{12}I_3) - \gamma_1^2 S_3(J_{22} + J_{33}) - \gamma_2^2 L_3(J_{11} + J_{22}) - \gamma_3^2 I_3(J_{11} + J_{33})) \\ &\quad + \delta_2(J_{32}(J_{11} + J_{44}) - J_{12}J_{31} + \gamma_2\gamma_3 L_3 T_3), \\ e_4 &= (J_{33}J_{44} + \gamma_2^2 L_3 T_3)(J_{11}J_{22} - J_{12}J_{21}) + T_3((\gamma_2 J_{32} - \gamma_3 J_{33})(\gamma_1 J_{21}S_3 - \gamma_3 J_{11}I_3) + (\gamma_2 J_{31} - \gamma_1 J_{33}) \\ &\quad \times (\gamma_3 J_{12}I_3 - \gamma_1 J_{22}S_3)) + \delta_2 T_3(\gamma_1 S_3(\gamma_3 J_{31} - \gamma_1 J_{32}) + \gamma_2 L_3(\gamma_1 J_{12} - \gamma_3 J_{11})) \\ &\quad + \delta_2 J_{44}(J_{12}J_{31} - J_{11}J_{32}). \end{aligned}$$

Deeply discuss and correspond to the Routh-Hurwitz criteria that the quartic equation (C.1) will have negative real parts of all the roots if these conditions hold, i.e $e_1 > 0$, $e_1 e_2 > 0$, $e_3(e_1 e_2 - e_3) > 0$, and $e_3(e_1 e_2 - e_3) - e_1^2 e_4 > 0$. This implies that the equilibrium E_3 is locally asymptotically stable.

APPENDIX D

Suppose that we conduct the suitable Lyapunov function for E_1 as below

$$\mathcal{L}_1 = S - S_1 - S_1 \ln \frac{S}{S_1} + I + L + T - T_1 - T_1 \ln \frac{T}{T_1}. \quad (\text{D.1})$$

If $S = S_1, I = I_1, L = L_1, T = T_1$ ensures that $\mathcal{L}_1 = 0$, and $\mathcal{L}_1 > 0$, when $S \neq S_1, I \neq I_1, L \neq L_1, T \neq T_1$. Injecting the steady state E_1 , $\pi = \gamma_1 S_1 T_1 + (v_1 + \mu)S_1$, and $\mu = \gamma_1 S_1$ into The partial derivatives of equation (D.1), such that

$$\begin{aligned} \frac{d\mathcal{L}_1}{dt} &= \left(1 - \frac{S_1}{S}\right) \left(\gamma_1 S_1 T_1 + (v_1 + \mu)S_1 - \frac{\alpha SI}{1 + \eta I} - \gamma_1 ST - (v_1 + \mu)S \right) + \frac{q\alpha SI}{1 + \eta I} + \delta_2 L - \gamma_3 IT \\ &\quad - (\delta_1 + v_2 + \mu)I + \frac{(1 - q)\alpha SI}{1 + \eta I} + \delta_1 I - \gamma_2 LT - (\delta_2 + v_3 + \mu)L \\ &\quad + \left(1 - \frac{T_1}{T}\right) (\gamma_1 ST + \gamma_2 LT + \gamma_3 IT - \gamma_1 S_1 T) \\ &= -\frac{(v_1 + \mu)\mathcal{R}_{0T}}{S} (S - S_1)^2 + (\alpha S_1 - (v_2 + \mu + \gamma_3 T_1))I - \left((v_3 + \mu) + \frac{\gamma_2(v_1 + \mu)}{\gamma_1} (\mathcal{R}_{0T} - 1) \right)L. \end{aligned}$$

Clearly, $\frac{d\mathcal{L}_1}{dt} < 0$, if $R_{0T} > 1$, and $\alpha < \frac{\nu_2 + \mu + \gamma_3 T_1}{S_1}$. According to the LaSalle invariance principle, the equilibrium E_1 is globally asymptotically stable if $R_{0T} > 1$ and $\alpha < \frac{\nu_2 + \mu + \gamma_3 T_1}{S_1}$.

APPENDIX E

Employing the Goh-Volterra Lyapunov type, we construct the Lyapunov type to correspond to the equilibrium E_2 being

$$\mathcal{L}_2 = S - S_2 - S_2 \ln \frac{S}{S_2} + I - I_2 - I_2 \ln \frac{I}{I_2} + L - L_2 - L_2 \ln \frac{L}{L_2} + T. \quad (\text{E.1})$$

Obviously, $\mathcal{L}_2 = 0$ when $S = S_2, I = I_2, L = L_2, T = T_2$, and $\mathcal{L}_2 > 0$ when $S \neq S_2, I \neq I_2, L \neq L_2, T \neq T_2$. According to the steady state E_2 that corresponds to the system (2.7), we have the following equation

$$\pi = \frac{\alpha S_2 I_2}{1 + \eta I_2} + (\nu_1 + \mu) S_2, \quad \delta_1 + \nu_2 + \mu = \frac{q \alpha S_2}{1 + \eta I_2} + \frac{\delta_2 L_2}{I_2}, \quad \delta_2 + \nu_3 + \mu = \frac{(1 - q) \alpha S_2 I_2}{L_2 (1 + \eta I_2)} + \frac{\delta_1 I_2}{L_2}. \quad (\text{E.2})$$

Substituting and carrying out the equation (E.2) into the partial derivative of (E.1) produces

$$\begin{aligned} \frac{d\mathcal{L}_2}{dt} &= \left(1 - \frac{S_2}{S}\right) \left(\frac{\alpha S_2 I_2}{1 + \eta I_2} + (\nu_1 + \mu) S_2 - \frac{\alpha S I}{1 + \eta I} - \gamma_1 S T - (\nu_1 + \mu) S \right) \\ &\quad + \left(1 - \frac{I_2}{I}\right) \left(\frac{q \alpha S I}{1 + \eta I} + \delta_2 L - \gamma_3 I T - \left(\frac{q \alpha S_2}{1 + \eta I_2} + \frac{\delta_2 L_2}{I_2} \right) I \right) \\ &\quad + \left(1 - \frac{L_2}{L}\right) \left(\frac{(1 - q) \alpha S I}{1 + \eta I} + \delta_1 I - \gamma_2 L T - \left(\frac{(1 - q) \alpha S_2 I_2}{L_2 (1 + \eta I_2)} + \frac{\delta_1 I_2}{L_2} \right) L \right) \\ &\quad + (\gamma_1 S T + \gamma_2 L T + \gamma_3 I T - \mu T), \\ &= (\nu_1 + \mu) S_2 \left(2 - \frac{S_2}{S} - \frac{S}{S_2} \right) + \frac{\alpha S_2 I_2}{1 + \eta I_2} \left(2 - \frac{S_2}{S} - \frac{S}{S_2} \right) + \left(\frac{q \alpha S_2 I_2}{1 + \eta I_2} + \delta_2 L_2 \right) \left(2 - \frac{I}{I_2} - \frac{I_2}{I} \right) \\ &\quad + \left(\frac{(1 - q) \alpha S_2 I_2}{(1 + \eta I_2)} + \delta_1 I_2 \right) \left(1 - \frac{L}{L_2} - \frac{L_2}{L} \right) + (\gamma_1 S_2 + \gamma_2 L_2 + \gamma_3 I_2 - \mu) T. \end{aligned}$$

Hence, the information of the arithmetic mean is greater than or equal to the geometric mean. Finally, we arrive at the inequalities which hold

$$\left(2 - \frac{S_2}{S} - \frac{S}{S_2} \right) \leq 0, \quad \left(2 - \frac{I}{I_2} - \frac{I_2}{I} \right) \leq 0, \quad \left(2 - \frac{L}{L_2} - \frac{L_2}{L} \right) \leq 0. \quad (\text{E.3})$$

Furthermore, we have $\frac{d\mathcal{L}_2}{dt} \leq 0$ when the conditions (E.3) and $\mu > \gamma_1 S_2 + \gamma_2 L_2 + \gamma_3 I_2$ are valid. On the other hand, we ensure that $\frac{d\mathcal{L}_2}{dt} = 0$ if only if $S = S_2, I = I_2, L = L_2, T = T_2$, and $R = R_2$. According to LaSalle's invariance principle, the equilibrium point E_2 is globally asymptotically stable.

APPENDIX F

From the reduced system of model (2.7), the corresponding Jacobian matrix can be derived. Hence, the associated additive compound matrix is obtained as follows

$$J^{[2]} = \begin{pmatrix} M_{11} & \delta_2 & 0 & -\gamma_3 I & \gamma_1 S & 0 \\ \frac{(1-q)\alpha S}{(1+\eta I)^2} + \delta_1 & M_{22} & -\frac{\alpha S}{(1+\eta I)^2} & -\gamma_2 L & 0 & \gamma_1 S \\ -\frac{(1-q)\alpha I}{1+\eta I} & \frac{q\alpha I}{1+\eta I} & M_{33} & 0 & -\gamma_2 L & \gamma_3 I \\ \gamma_3 T & \gamma_2 T & 0 & M_{44} & -\frac{\alpha S}{(1+\eta I)^2} & 0 \\ -\gamma_1 T & 0 & \gamma_2 T & \frac{q\alpha I}{1+\eta I} & M_{55} & \delta_2 \\ 0 & -\gamma_1 T & -\gamma_3 T & \frac{(1-q)\alpha I}{1+\eta I} & \frac{(1-q)\alpha S}{(1+\eta I)^2} + \delta_1 & M_{66} \end{pmatrix}, \quad (\text{F.1})$$

with

$$M_{11} = \frac{q\alpha S}{(1+\eta I)^2} - \frac{\alpha I}{1+\eta I} - (\gamma_1 + \gamma_3)T - \delta_1 - v_1 - v_2 - 2\mu,$$

$$M_{22} = -\frac{\alpha I}{1+\eta I} - (\gamma_1 + \gamma_2)T - \delta_2 - v_1 - v_3 - 2\mu,$$

$$M_{33} = \frac{q\alpha S}{(1+\eta I)^2} - (\gamma_2 + \gamma_3)T - \delta_1 - \delta_2 - v_2 - v_3 - 2\mu,$$

$$M_{44} = -\frac{\alpha I}{1+\eta I} + \gamma_1 S + \gamma_2 L + \gamma_3 I - \gamma_1 T - v_1 - 2\mu,$$

$$M_{55} = \frac{q\alpha S}{(1+\eta I)^2} + \gamma_1 S + \gamma_2 L + \gamma_3 I - \gamma_3 T - \delta_1 - v_2 - 2\mu,$$

$$M_{66} = \gamma_1 S + \gamma_2 L + \gamma_3 I - \gamma_2 T - \delta_2 - v_3 - 2\mu.$$

In the next step, we try to have the suitable function as $P(S, I, L, T) = \text{diag}\left(\frac{1}{I}, \frac{1}{I}, \frac{1}{I}, \frac{1}{I}, \frac{1}{I}, \frac{1}{I}\right)$, and we have $P_f(S, I, L, T) = \text{diag}\left(-\frac{I'}{I^2}, -\frac{I'}{I^2}, -\frac{I'}{I^2}, -\frac{I'}{I^2}, -\frac{I'}{I^2}, -\frac{I'}{I^2}\right)$. Consequently, $P_f P^{-1}$ is easy to find. Hence, $\mathcal{B} = P_f P^{-1} + P J^{[2]} P^{-1}$ is provided by

$$\mathcal{B} = \begin{pmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{pmatrix},$$

with some expression as the following

$$B_{11} = M_{11} - \frac{I'}{I}, \quad B_{12} = (\delta_2 \quad 0), \quad B_{13} = (-\gamma_3 I \quad \gamma_1 S), \quad B_{14} = 0, \quad B_{21} = \begin{pmatrix} \frac{(1-q)\alpha S}{(1+\eta I)^2} + \delta_1 \\ -\frac{(1-q)\alpha I}{1+\eta I} \end{pmatrix},$$

$$\begin{aligned}
B_{22} &= \begin{pmatrix} M_{22} - \frac{I'}{I} & -\frac{\alpha S}{(1+\eta I)^2} \\ \frac{q\alpha I}{1+\eta I} & M_{33} - \frac{I'}{I} \end{pmatrix}, \quad B_{23} = \begin{pmatrix} -\gamma_2 L & 0 \\ 0 & -\gamma_2 L \end{pmatrix}, \quad B_{24} = \begin{pmatrix} \gamma_1 S \\ \gamma_3 I \end{pmatrix}, \quad B_{31} = \begin{pmatrix} \gamma_3 T \\ -\gamma_1 T \end{pmatrix}, \\
B_{32} &= \begin{pmatrix} \gamma_2 T & 0 \\ 0 & \gamma_2 T \end{pmatrix}, \quad B_{33} = \begin{pmatrix} M_{44} - \frac{I'}{I} & -\frac{\alpha S}{(1+\eta I)^2} \\ \frac{q\alpha I}{1+\eta I} & M_{55} - \frac{I'}{I} \end{pmatrix}, \quad B_{34} = \begin{pmatrix} 0 \\ \delta_2 \end{pmatrix}, \quad B_{41} = 0, \\
B_{42} &= \begin{pmatrix} -\gamma_1 T & -\gamma_3 T \end{pmatrix}, \quad B_{43} = \begin{pmatrix} \frac{(1-q)\alpha I}{1+\eta I} & \frac{(1-q)\alpha S}{(1+\eta I)^2} + \delta_1 \end{pmatrix}, \quad B_{44} = M_{66} - \frac{I'}{I}.
\end{aligned}$$

Let $z = (z_1, z_2, z_3, z_4, z_5, z_6) \in \mathbb{R}^6$ and then we have the norm of $\mathbb{R}^6$ as

$$\|z\| = \max\{|z_1|, |z_2| + |z_3|, |z_4| + |z_5|, |z_6|\}.$$

Suppose that $\sigma(\mathcal{B})$ represents the Lozinskii measure of $\mathcal{B}$ with respect to this norm, then this criterion applies

$$\sigma(\mathcal{B}) \leq \max\{\hat{g}_1, \hat{g}_2, \hat{g}_3, \hat{g}_4\},$$

with

$$\hat{g}_1 = \sigma_1(B_{11}) + |B_{12}| + |B_{13}| + |B_{14}|, \quad \hat{g}_2 = \sigma_1(B_{22}) + |B_{21}| + |B_{23}| + |B_{24}|,$$

$$\hat{g}_3 = \sigma_1(B_{33}) + |B_{31}| + |B_{32}| + |B_{34}|, \quad \hat{g}_4 = \sigma_1(B_{44}) + |B_{41}| + |B_{42}| + |B_{43}|.$$

Here, $|B_{ij}|$ ($i \neq j$) generates the matrix norm induced by the ℓ_1 -norm, while σ_1 denotes the Lozinskii measure with respect to the ℓ_1 -norm. Now, we approximate each term $\hat{g}_i$ for stability. Initially, we start from $\hat{g}_1$, since B_{11} is a scalar, which gives $\sigma_1(B_{11}) = M_{11} - \frac{I'}{I}$. Moreover, $|B_{12}| = \delta_2$, $|B_{13}| = \max\{\gamma_3 I, \gamma_1 S\}$, $|B_{14}| = 0$. Therefore, we have the value of $\hat{g}_1$ as $\hat{g}_1 = M_{11} - \frac{I'}{I} + \delta_2 + \max\{\gamma_3 I, \gamma_1 S\}$. Substituting the detail form of M_{11} into $\hat{g}_1$ and using the remark (3.1) gives

$$\hat{g}_1 = -\frac{I'}{I} + \frac{q\alpha S}{(1+\eta I)^2} - \frac{\alpha I}{1+\eta I} - (\gamma_1 + \gamma_3)T - \delta_1 - \nu_1 - \nu_2 - 2\mu + \delta_2 + \max\{\gamma_3 I, \gamma_1 S\} \leq -\frac{I'}{I} - \hat{b}_1.$$

with $\hat{b} = \frac{\alpha\delta_I}{1+\eta\delta_I} + (\gamma_1 + \gamma_3)\delta_T + \delta_1 + \nu_1 + \nu_2 + 2\mu - \delta_2 - \frac{q\alpha M_S}{(1+\eta\delta_I)^2} - \max\{\gamma_3 M_I, \gamma_1 M_S\}$.

Next, consider $\hat{g}_2$. Since $B_{22} = \begin{pmatrix} M_{22} - \frac{I'}{I} & -\frac{\alpha S}{(1+\eta I)^2} \\ \frac{q\alpha I}{1+\eta I} & M_{33} - \frac{I'}{I} \end{pmatrix}$, then the Lozinskii measure with respect to the ℓ_1 -norm yields the following result $\sigma_1(B_{22}) = -\frac{I'}{I} + \max\left\{M_{22} + \frac{q\alpha I}{1+\eta I}, M_{33} + \frac{\alpha S}{(1+\eta I)^2}\right\}$. Furthermore, $|B_{21}| = \frac{(1-q)\alpha S}{(1+\eta I)^2} + \delta_1 + \frac{(1-q)\alpha I}{1+\eta I}$, $|B_{23}| = \gamma_2 L$, $|B_{24}| = \gamma_1 S + \gamma_3 I$. Consequently, we have

$$\hat{g}_2 \leq -\frac{I'}{I} + \max\left\{M_{22} + \frac{q\alpha I}{1+\eta I}, M_{33} + \frac{\alpha S}{(1+\eta I)^2}\right\} + \frac{(1-q)\alpha S}{(1+\eta I)^2} + \delta_1 + \frac{(1-q)\alpha I}{1+\eta I} + \gamma_2 L + \gamma_1 S + \gamma_3 I.$$

Now, in the deep discussion, we have the equality form

$$\begin{aligned} M_{22} + \frac{q\alpha I}{1+\eta I} &= -\frac{(1-q)\alpha I}{1+\eta I} - (\gamma_1 + \gamma_2)T - \delta_2 - \nu_1 - \nu_3 - 2\mu, \\ M_{33} + \frac{\alpha S}{(1+\eta I)^2} &= \frac{(1+q)\alpha S}{(1+\eta I)^2} - (\gamma_2 + \gamma_3)T - \delta_1 - \delta_2 - \nu_2 - \nu_3 - 2\mu. \end{aligned} \quad (\text{F.2})$$

Based on the (F.2) and remark (3.1), we have $\hat{g}_2 \leq -\frac{I'}{I} - \hat{b}_2$, with

$$\begin{aligned} \hat{b}_2 &= \delta_2 + \nu_3 + 2\mu + \min \left\{ \frac{(1-q)\alpha\delta_I}{1+\eta\delta_I} + (\gamma_1 + \gamma_2)\delta_T + \nu_1, -\frac{(1+q)\alpha M_S}{(1+\eta\delta_I)^2} + (\gamma_2 + \gamma_3)\delta_T + \delta_1 + \nu_2 \right\} \\ &\quad - \frac{(1-q)\alpha M_S}{(1+\eta\delta_I)^2} - \delta_1 - \frac{(1-q)\alpha M_I}{1+\eta M_I} - \gamma_2 M_L - \gamma_1 M_S - \gamma_3 M_I. \end{aligned}$$

Similarly, for g_3 , we obtain $\hat{g}_3 \leq -\frac{I'}{I} - \hat{b}_3$, with

$$\begin{aligned} \hat{b}_3 &= (\gamma_1 M_S + \gamma_2 M_L + \gamma_3 M_I) + \min \left\{ \frac{(1-q)\alpha\delta_I}{1+\eta\delta_I} + \gamma_1\delta_T + \nu_1 + 2\mu, -\frac{(1+q)\alpha M_S}{(1+\eta\delta_I)^2} + \gamma_3\delta_T + \delta_1 + \nu_2 + 2\mu \right\} \\ &\quad - (\gamma_1 + \gamma_2 + \gamma_3)M_T - \delta_2. \end{aligned}$$

Finally, g_4 also given by $\hat{g}_4 \leq -\frac{I'}{I} - \hat{b}_4$, with

$$\begin{aligned} \hat{b}_4 &= \gamma_2\delta_T + \delta_2 + \nu_3 + 2\mu - \gamma_1 M_S - \gamma_2 M_L - \gamma_3 M_I - \max\{\gamma_1 M_T, \gamma_3 M_T\} \\ &\quad - \max \left\{ \frac{(1-q)\alpha M_I}{1+\eta M_I}, \frac{(1-q)\alpha M_S}{(1+\eta\delta_I)^2} + \delta_1 \right\}. \end{aligned}$$

Therefore, in combination with all of the criteria, we obtain the Lozinskii measure as

$$\sigma(\mathcal{B}) \leq \max\{g_1, g_2, g_3, g_4\} \leq \max \left\{ -\frac{I'}{I} - \hat{b}_1, -\frac{I'}{I} - \hat{b}_2, -\frac{I'}{I} - \hat{b}_3, -\frac{I'}{I} - \hat{b}_4 \right\} \leq -\frac{I'}{I} - \min\{\bar{b}\},$$

with $\bar{b} = \hat{b}_1, \hat{b}_2, \hat{b}_3, \hat{b}_4$. Integrating over $[0, t]$, we obtain the following.

$$\frac{1}{t} \int_0^t \sigma(\mathcal{B}(x(s))) ds \leq -\frac{1}{t} \int_0^t \left(\frac{I'(s)}{I(s)} ds - \bar{b} \right) = -\frac{1}{t} \ln \frac{I(t)}{I(0)} - \bar{b}.$$

Since $x(t) \in K$ for all sufficiently large t , it follows that $\delta_I \leq I(t) \leq M_I$, and hence $\lim_{t \rightarrow \infty} \frac{1}{t} \ln \frac{I(t)}{I(0)} = 0$. Consequently,

$$\hat{q} = \limsup_{t \rightarrow \infty} \sup_{x_0 \in K} \frac{1}{t} \int_0^t \sigma(\mathcal{B}(x(s, x_0))) ds \leq -\bar{b} < 0.$$

Therefore, by the Li–Muldowney geometric condition in (3.1), the equilibrium E_3 is GAS in Ω .

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