

Coefficient Bounds For Certain Subclasses Of Close-Convex Functions Associated With A Differential Operator

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Abstract. In this paper, we define a new subclass of close-to-convex functions influenced by the multiplicative derivative. Multiplicative derivative is defined in a domain which excludes zero, so the defined subclass did not involve replacing the ordinary derivative with a multiplicative derivative. But we have used the only motivation behind the purpose of such a restrictive calculus, to define a new family of analytic functions. Estimates involving the initial coefficients of the functions having Maclaurin series, which belong to the defined function class are our main results. Some examples along with graphs have been used to establish the inclusion and closure properties. Also, we obtain the logarithmic and inverse coefficients for the defined function class.

1. INTRODUCTION AND DEFINITION

Let $\mathbb{U}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{N}$ denote the unit disc, set of real numbers, set of complex numbers and set of natural numbers respectively. Let $\mathcal{H}(a, n)$ be the class of analytic functions having a series of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$. Also, let

$$\mathcal{A}_n = \{f \in \mathcal{H}, f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \dots\}$$

and $\mathcal{A} = \mathcal{A}_1$. Let $\mathcal{P}(\rho)$ ($0 \leq \rho \leq 1$) denote the class of the functions $p(z)$ analytic with $p(0) = 1$ and $\operatorname{Re}[p(z)] > \rho$. Also we let $\mathcal{S}$ to denote the subclass of $\mathcal{A}$ which are analytic in $\mathbb{U}$. Famous

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subclasses of $\mathcal{S}$ are the so-called classes of starlike and convex function will be denoted by $\mathcal{S}^*$ and $\mathcal{C}$ respectively. Refer to [24], for formal definitions of various subclasses of $\mathcal{S}$. Gao and Zhou [16] introduced and studied a class of analytic functions satisfying the condition

$$\operatorname{Re} \left(\frac{z^2 f'(z)}{g(z)g(-z)} \right) < 0, \quad (z \in \mathbb{U}; g \in \mathcal{S}^*(1/2)), \quad (1.1)$$

where $\mathcal{S}^*(1/2)$ denotes the class of starlike functions of order $1/2$. The class of functions satisfying the above condition will be denoted as $\mathcal{K}_s$. Since $G(z) = g(z)g(-z)/z$ is starlike if $g \in \mathcal{S}^*(1/2)$, the class $\mathcal{K}_s$ would belong to the family of close-to-convex functions. Hence functions in $\mathcal{K}_s$ are univalent in $\mathbb{U}$. Using the principle of subordination, Wang, Gao and Yuan [37] (also see [14, 38]) extended the class $\mathcal{K}_s$ to $\mathcal{K}_s(\psi)$ by replacing the condition (1.1) with the following condition

$$-\frac{z^2 f'(z)}{g(z)g(-z)} < \psi(z), \quad (z \in \mathbb{U}; g \in \mathcal{S}^*(1/2)). \quad (1.2)$$

The class was further generalized and studied by various authors (see [12, 15, 17, 20, 25, 27–29]). In this paper, we will introduce a new class by replacing f' with $e^{\frac{zf'(z)}{f(z)}}$ in (1.2). The term $e^{\frac{zf'(z)}{f(z)}}$ is motivated by the definition of non-Newtonian derivative popularly called as *multiplicative derivative*.

Bashirov, Kurpinar and Özyapın [5] (also see [6, 7, 26]) studied the properties of *Multiplicative calculus* which is not adaptable to many application problems when compared with the calculus due to Newton and Euler. But it has been a useful mathematical tool in economics and finance. The multiplicative derivative denoted by f^* whose definition is given explicitly by

$$f^*(x) = \lim_{h \rightarrow 0} \left(\frac{f(x+h)}{f(x)} \right)^{\frac{1}{h}} = e^{\frac{f'(x)}{f(x)}} = e^{[\ln f(x)]'} \quad (x \in \mathbb{R})$$

where $f'(x)$ is the ordinary derivative. The $*$ -derivative of f at z belonging to a small neighbourhood of a domain in a complex plane where f is non-vanishing differentiable, is given by

$$f^*(z) = e^{f'(z)/f(z)} \quad (1.3)$$

and $f^{*(n)}(z) = e^{[f'(z)/f(z)]^{(n)}}$, $n = 1, 2, \dots$. The multiplicative derivative is defined only on domain which omits zero. But the subclasses of univalent functions defined on a domain which admits zero. So we will only be able to use the idea and motivation behind the definition of a multiplicative calculus. Influenced by the definition of multiplicative derivative, recently Karthikeyan and Murugusundaramoorthy in [18] introduced and studied a class of analytic functions $\mathcal{R}(\psi)$ satisfying the subordination condition

$$\frac{ze^{\frac{zf'(z)}{f(z)}}}{f(z)} < \psi(z), \quad (1.4)$$

where $\psi \in \mathcal{P}$ is starlike symmetric with respect to horizontal axis and is of the form

$$\psi(z) = 1 + L_1 z + L_2 z^2 + L_3 z^3 + \dots, \quad (L_1 \in \mathbb{C}; z \in \mathbb{U}). \quad (1.5)$$

The class is non-empty and it does not reduce to the subclasses of $\mathcal{S}$. For the detailed analysis and closure properties of the class $\mathcal{R}(\psi)$, refer to [18].

Mittag-Leffler function, is a special transcendental function and it reduces to a well-known elementary functions. Refer to Srivastava [30,31] and Srivastava et al. [32–34] for detailed studies which involved Mittag-Leffler function. The Srivastava–Tomovski generalization of the Mittag-Leffler function [33], is given by

$$E_{\alpha,\beta}^{\gamma,k}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nk} z^n}{\Gamma(\alpha n + \beta) n!}, \quad z, \alpha, \beta, \gamma, k \in \mathbb{C}, \operatorname{Re}(\alpha) > 0, \operatorname{Re}(k) > 0. \quad (1.6)$$

where $(x)_n$ will be used denote the usual Pochhammer symbol. Using (1.6), Cang and Liu in [13] (also see [8,35,36]) introduced an operator which is given by

$$H_{\alpha,\beta}^{\gamma,k} f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\gamma + nk) \Gamma(\alpha + \beta)}{\Gamma(\gamma + k) \Gamma(\alpha n + \beta) n!} a_n z^n. \quad (1.7)$$

Motivated by (1.7), Breaz et al. [9] defined an operator $\mathfrak{J}_k^m(\alpha, \beta, \gamma) f(z) : \mathcal{A} \rightarrow \mathcal{A}$ by

$$\mathfrak{J}_{k,\lambda}^m(\alpha, \beta, \gamma) f(z) = z + \sum_{n=2}^{\infty} [1 - \lambda + \lambda n]^m \frac{\Gamma(\gamma + nk) \Gamma(\alpha + \beta)}{\Gamma(\gamma + k) \Gamma(\alpha n + \beta) n!} a_n z^n. \quad (1.8)$$

$$(0 \leq \lambda \leq 1; m \in \mathbb{N}_0 = \{0, 1, 2, \dots\}, \alpha, \beta, \gamma, k \in \mathbb{C}, \operatorname{Re}(\alpha) > 0, \operatorname{Re}(k) > 0).$$

Motivated by class $\mathcal{R}(\psi)$ and $\mathcal{K}_s(\psi)$, we introduce a new family influenced by the definition of multiplicative derivative.

Definition 1.1. For $0 \leq \lambda \leq 1; m \in \mathbb{N}_0$ and $\alpha, \beta, \gamma, k \in \mathbb{C}$ such that $\operatorname{Re}(\alpha) > 0, \operatorname{Re}(k) > 0$, a function f in $\mathcal{N}_t^m(\alpha, \beta, \gamma, k; \lambda; \psi)$ if it satisfies

$$\frac{tz^2 e^{\frac{z D_r^m(\theta, \vartheta, \rho) f'(z)}{D_r^m(\theta, \vartheta, \rho) f(z)} - 1}}{D_r^m(\theta, \vartheta, \rho) g(z) D_r^m(\theta, \vartheta, \rho) g(tz)} < \psi(z). \quad (1.9)$$

where $D_r^m(\theta, \vartheta, \rho) g(z) \in \mathcal{S}^*(1/2), |t| < 1, t \neq 0$ and $\psi(z) \in \mathcal{P}$ is of the form (1.5).

Remark 1.1. Letting $\alpha = 0 = m, t = -1$ and $\gamma = k = 1$ in Definition 1.1, then the class $\mathcal{N}_t^m(\alpha, \beta, \gamma, k; \lambda; \psi)$ reduces to the class

$$\mathcal{N}(\psi) = \left\{ f \in \mathcal{A} : -\frac{z^2 e^{\frac{z f'(z)}{f(z)}}}{e g(z) g(-z)} < \psi(z), z \in \mathbb{U} \right\}.$$

Note that $\mathcal{N}(\psi)$ is neither a subclass nor generalized class of $\mathcal{K}_s(\psi)$. If we let $\psi(z) = \frac{1+z}{1-z}$ in $\mathcal{N}(\psi)$, then the class reduces to

$$\mathcal{N} = \left\{ f \in \mathcal{A} : \operatorname{Re} \left(\frac{z^2 e^{\frac{z f'(z)}{f(z)}}}{e g(z) g(-z)} \right) < 0, z \in \mathbb{U} \right\}.$$

For other studies related to $\mathcal{N}_t^m(\alpha, \beta, \gamma, k; \lambda; \psi)$, refer to [10,11] and the references provided therein.

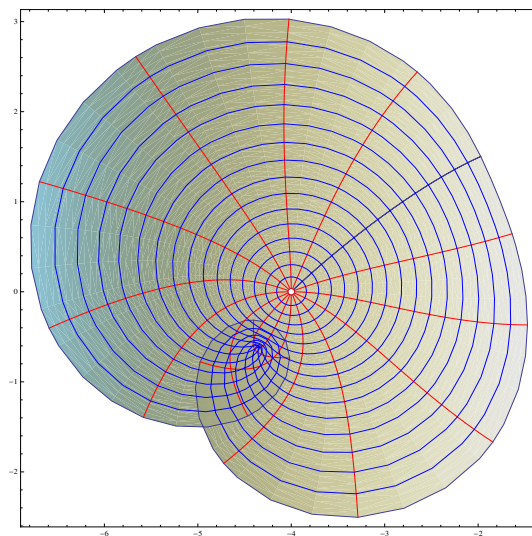


FIGURE 1. Mapping of $G(z) = -\frac{z^2}{\left(\frac{z}{(2-z)} \frac{z}{(2+z)}\right)} e^{\frac{\left(z - \frac{1+i}{\sqrt{2}} z^2\right)}{z - \frac{1+i}{2\sqrt{2}} z^2} - 1}$ if $|z| < 1$

Example 1.1. Let the fixed function be $g(z) = \frac{z}{(2-z)}$. It is well-known that $g(z) = \frac{z}{(2-z)}$ is convex univalent and maps the unit disc onto a circular shaped region in w -plane. Now let $f(z) = z - \frac{1+i}{2\sqrt{2}} z^2$, $|z| < 1$. Clearly, $f(z)$ is univalent in $\mathbb{U}$ and $\text{Area}(f(\mathbb{U})) = \pi \sum_{n=1}^{\infty} n |a_n|^2 = \pi(1 \cdot |1|^2 + 2 \cdot 1/4) = \frac{3\pi}{2}$. Now

$$G(z) = \frac{z^2 e^{\frac{zf'(z)}{f(z)}}}{eg(z)g(-z)} = -\frac{z^2}{\left(\frac{z}{(2-z)} \frac{z}{(2+z)}\right)} e^{\frac{\left(z - \frac{1+i}{\sqrt{2}} z^2\right)}{z - \frac{1+i}{2\sqrt{2}} z^2} - 1}.$$

We can see that the function $G(z) = -\frac{z^2}{\left(\frac{z}{(2-z)} \frac{z}{(2+z)}\right)} e^{\frac{\left(z - \frac{1+i}{\sqrt{2}} z^2\right)}{z - \frac{1+i}{2\sqrt{2}} z^2} - 1}$ maps the unit disc onto semi-cardioid region in the left half-plane (see Figure 1). Further, Figure 1 illustrates that the class $\mathcal{N}$ is non-empty.

Remark 1.2. Note that functions belonging to classes $\mathcal{N}$, $\mathcal{N}(\psi)$ and $\mathcal{N}_t^m(\alpha, \beta, \gamma, k; \lambda; \psi)$ need not be univalent. It is neither a subclass nor a generalized class of $\mathcal{S}$.

2. COEFFICIENT INEQUALITIES AND FEKETE-SZEGŐ INEQUALITY OF $\mathcal{N}_t^m(\alpha, \beta, \gamma, k; \lambda; \psi)$

We will need the following inequality obtained by Ma and Minda [21] for the class $\mathcal{P}$ and the inequality obtained by Koepf [19] for the class $\mathcal{S}^*$.

Lemma 2.1. [19,21] Let $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*$. For v to be a complex number, then we have the following inequalities

$$|p_2 - vp_1^2| \leq 2 \max \{1; |2v - 1|\},$$

and

$$|b_2 - vb_1^2| \leq \max \{1; |3 - 4v|\}.$$

The result is sharp.

Throughout this paper, we denote

$$\Gamma_n = \frac{\Gamma(\gamma + nk)\Gamma(\alpha + \beta)}{\Gamma(\gamma + k)\Gamma(\alpha n + \beta)n!}$$

which may be real or complex parameters.

Theorem 2.1. If $f(z) \in \mathcal{N}_t^m(\alpha, \beta, \gamma, k; \lambda; \psi)$, then we have

$$|a_2| \leq \frac{1}{|(1 + \lambda)^m \Gamma_2|} [|L_1| + 2|1 + t|]. \quad (2.1)$$

$$|a_3| \leq \frac{1}{|(1 + 2\lambda)^m \Gamma_3|} \left\{ \frac{|L_1|}{2} \max \left\{ 1; \left| \frac{L_2}{L_1} + \frac{L_1}{2} \right| \right\} + 2|L_1(1 + t)| \right. \\ \left. + \frac{|1 + t|^2}{2} \max \left\{ 1; \left| \frac{5t^2 + 14t + 5}{(1 + t)^2} \right| \right\} \right\}, \quad (2.2)$$

and for all $\rho \in \mathbb{C}$

$$|a_3 - \rho a_2^2| \leq \frac{1}{|(1 + 2\lambda)^m \Gamma_3|} \left\{ \frac{|L_1|}{2} \max \left\{ 1; \left| \frac{L_2}{L_1} + \frac{L_1}{2} \left(1 - \frac{4\rho(1 + 2\lambda)^m \Gamma_3}{(1 + \lambda)^{2m} \Gamma_2^2} \right) \right| \right\} \right. \\ \left. + 2 \left| L_1(1 + t) \left(1 - \frac{2\rho(1 + 2\lambda)^m \Gamma_3}{(1 + \lambda)^{2m} \Gamma_2^2} \right) \right| \right. \\ \left. + \frac{|1 + t|^2}{2} \max \left\{ 1, \left| 3 - 2 \left[\frac{4\rho(1 + 2\lambda)^m \Gamma_3}{(1 + \lambda)^{2m} \Gamma_2^2} - \frac{[(1 + t)^2 + 2t]}{(1 + t)^2} \right] \right| \right\} \right\}. \quad (2.3)$$

Proof. As $f \in \mathcal{N}_t^m(\alpha, \beta, \gamma, k; \lambda; \psi)$, by (1.9) we have

$$e^{\frac{z D_r^m(\theta, \vartheta, \rho) f'(z)}{D_r^m(\theta, \vartheta, \rho) f(z)} - 1} = \psi[w(z)] \frac{D_r^m(\theta, \vartheta, \rho) g(z) D_r^m(\theta, \vartheta, \rho) g(tz)}{tz^2}. \quad (2.4)$$

Thus, let $\vartheta \in \mathcal{P}$ be of the form $\vartheta(z) = 1 + \sum_{k=1}^{\infty} \vartheta_k z^k$ and defined by

$$\vartheta(z) = \frac{1 + w(z)}{1 - w(z)}, \quad z \in \mathbb{U}.$$

On computation, the right hand side of (2.4)

$$\psi[w(z)] \frac{D_r^m(\theta, \vartheta, \rho) g(z) D_r^m(\theta, \vartheta, \rho) g(tz)}{tz^2} = 1 + \left[\frac{\vartheta_1 L_1}{2} + b_2(1 + t) \right] z \quad (2.5)$$

$$+ \left\{ \frac{L_1}{2} \left[\vartheta_2 - \frac{\vartheta_1^2}{2} \left(1 - \frac{L_2}{L_1} \right) \right] + \frac{b_2 \vartheta_1 L_1}{2} (1 + t) + b_3(1 + t)^2 + b_2^2 t \right\} z^2 + \dots. \quad (2.6)$$

Expanding (2.4), we have

$$e^{\frac{z D_r^m(\theta, \vartheta, \rho) f'(z)}{D_r^m(\theta, \vartheta, \rho) f(z)} - 1} = 1 + (1 + \lambda)^m a_2 \Gamma_2 z + \frac{1}{2} \left(4a_3(1 + 2\lambda)^m \Gamma_3 - a_2^2(1 + \lambda)^{2m} \Gamma_2^2 \right) z^2 + \dots \quad (2.7)$$

From (2.7) and (2.6), we obtain

$$a_2 = \frac{1}{(1+\lambda)^m \Gamma_2} \left[\frac{\vartheta_1 L_1}{2} + b_2(1+t) \right] \quad (2.8)$$

and

$$a_3 = \frac{1}{(1+2\lambda)^m \Gamma_3} \left\{ \frac{L_1}{4} \left[\vartheta_2 - \frac{\vartheta_1^2}{2} \left(1 - \frac{L_2}{L_1} - \frac{L_1}{2} \right) \right] + \frac{b_2 \vartheta_1 L_1}{2} (1+t) \right. \\ \left. + \frac{b_3(1+t)^2}{2} + \frac{b_2^2}{4} [(1+t)^2 + 2t] \right\}. \quad (2.9)$$

Since $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*$ implies that $|b_n| \leq n$, $\forall n \geq 2$. Also it is well-known that $|\vartheta_1| \leq 2$. Equations (2.1) can be obtained by applying the well-known bounds of b_2 and ϑ_1 in (2.8). From (2.9), we have

$$|a_3| \leq \frac{1}{|(1+2\lambda)^m \Gamma_3|} \left\{ \frac{|L_1|}{4} \left| \vartheta_2 - \frac{\vartheta_1^2}{2} \left(1 - \frac{L_2}{L_1} - \frac{L_1}{2} \right) \right| + \frac{|b_2 \vartheta_1 L_1 (1+t)|}{2} \right. \\ \left. + \frac{(1+t)^2}{2} \left| b_3 - \left(-\frac{1}{2} \right) \left[1 + \frac{2t}{(1+t)^2} \right] b_2^2 \right| \right\}. \quad (2.10)$$

Applying Lemma 2.1 together with inequalities $|\vartheta_1| \leq 2$ and $|b_2| \leq 2$ in (2), we get (2.2).

Now to prove the Fekete-Szegő inequality for the class $\mathcal{N}_t^m(\alpha, \beta, \gamma, k; \lambda; \psi)$, we consider

$$|a_3 - \rho a_2^2| = \left| \frac{1}{(1+2\lambda)^m \Gamma_3} \left\{ \frac{L_1}{4} \left[\vartheta_2 - \frac{\vartheta_1^2}{2} \left(1 - \frac{L_2}{L_1} - \frac{L_1}{2} \right) \right] + \frac{b_2 \vartheta_1 L_1}{2} (1+t) + \frac{b_3(1+t)^2}{2} \right. \right. \\ \left. \left. + \frac{b_2^2}{4} [(1+t)^2 + 2t] \right\} - \frac{\rho}{(1+\lambda)^{2m} \Gamma_2^2} \left(\frac{\vartheta_1^2 L_1^2}{4} + \vartheta_1 L_1 b_2 (1+t) + b_2^2 (1+t)^2 \right) \right| \\ = \left| \frac{1}{(1+2\lambda)^m \Gamma_3} \left\{ \frac{L_1}{4} \left[\vartheta_2 - \frac{\vartheta_1^2}{2} \left(1 - \frac{L_2}{L_1} - \frac{L_1}{2} + \frac{2\rho L_1 (1+2\lambda)^m \Gamma_3}{(1+\lambda)^{2m} \Gamma_2^2} \right) \right] \right. \right. \\ \left. \left. + \frac{b_2 \vartheta_1 L_1}{2} (1+t) \left(1 - \frac{2\rho (1+2\lambda)^m \Gamma_3}{(1+\lambda)^{2m} \Gamma_2^2} \right) \right. \right. \\ \left. \left. + \frac{(1+t)^2}{2} \left[b_3 - \frac{b_2^2}{2} \left(\frac{4\rho (1+2\lambda)^m \Gamma_3}{(1+\lambda)^{2m} \Gamma_2^2} - \frac{[(1+t)^2 + 2t]}{(1+t)^2} \right) \right] \right\} \right|.$$

Using Lemma 2.1, we obtain the assertion (2.3). $\square$

Letting $\alpha = 0 = m$, $t = -1$ and $\gamma = k = 1$ in Theorem 2.1, we get the following result.

Corollary 2.1. *If $f(z) \in \mathcal{N}(\psi)$ (see Remark 1.1), then we have*

$$|a_2| \leq |L_1|.$$

$$|a_3| \leq \frac{|L_1|}{2} \max \left\{ 1; \left| \frac{L_2}{L_1} + \frac{L_1}{2} \right| \right\}.$$

and for all $\rho \in \mathbb{C}$

$$|a_3 - \rho a_2^2| \leq \frac{|L_1|}{2} \max \left\{ 1; \left| \frac{L_2}{L_1} + \frac{L_1}{2} (1 - 4\rho) \right| \right\}.$$

Letting $\psi(z) = \frac{1+z}{1-z}$ in Corollary 2.1, we get

Corollary 2.2. *If $f(z) \in \mathcal{N}$ (see Remark 1.1), then we have*

$$|a_2| \leq 2, \quad |a_3| \leq 2.$$

and for all $\rho \in \mathbb{C}$

$$|a_3 - \rho a_2^2| \leq \max \left\{ 1; |2(1 - 2\rho)| \right\}.$$

3. INVERSE FUNCTION COEFFICIENT ESTIMATES FOR FUNCTIONS BELONGING TO $\mathcal{N}_t^m(\alpha, \beta, \gamma, k; \lambda; \psi)$

We let $\mathcal{S}$ to denote the class of functions univalent in $\mathbb{U}$. It is well-known from Koebe 1/4-quarter theorem that every function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in $\mathcal{S}$ has an inverse f^{-1} , defined by $f^{-1}(f(z)) = z$, $z \in \mathbb{U}$ and $f(f^{-1}(w)) = w$, ($|w| < r$; $r \geq 1/4$) where

$$\chi(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^2 - 5a_2 a_3 + a_4)w^4 + \cdots. \quad (3.1)$$

Now we will obtain the inverse function coefficient bounds for functions in $\mathcal{N}_t^m(\alpha, \beta, \gamma, k; \lambda; \psi)$.

Theorem 3.1. *Let $f \in \mathcal{N}_t^m(\alpha, \beta, \gamma, k; \lambda; \psi)$ and let f^{-1} be the inverse of f defined by*

$$f^{-1}(w) = w + \sum_{k=2}^{\infty} d_k w^k, \quad (|w| < r; r \geq 1/4),$$

then

$$|d_2| \leq \frac{1}{|(1 + \lambda)^m \Gamma_2|} [|L_1| + 2|1 + t|]$$

and

$$\begin{aligned} |d_3| \leq & \frac{1}{|(1 + 2\lambda)^m \Gamma_3|} \left\{ \frac{|L_1|}{2} \max \left\{ 1; \left| \frac{L_2}{L_1} + \frac{L_1}{2} \left(1 - \frac{8(1 + 2\lambda)^m \Gamma_3}{(1 + \lambda)^{2m} \Gamma_2^2} \right) \right| \right\} \right. \\ & \left. + 2 \left| L_1 (1 + t) \left(1 - \frac{4(1 + 2\lambda)^m \Gamma_3}{(1 + \lambda)^{2m} \Gamma_2^2} \right) \right| \right. \\ & \left. + \frac{|1 + t|^2}{2} \max \left\{ 1, \left| 3 - 2 \left[\frac{8(1 + 2\lambda)^m \Gamma_3}{(1 + \lambda)^{2m} \Gamma_2^2} - \frac{[(1 + t)^2 + 2t]}{(1 + t)^2} \right] \right| \right\} \right\}. \end{aligned}$$

Also, for all $\tau \in \mathbb{C}$, we have

$$\begin{aligned} |d_3 - \tau d_2^2| \leq & \frac{1}{|(1 + 2\lambda)^m \Gamma_3|} \left\{ \frac{|L_1|}{2} \max \left\{ 1; \left| \frac{L_2}{L_1} + \frac{L_1}{2} \left(1 - \frac{4(\tau - 2)(1 + 2\lambda)^m \Gamma_3}{(1 + \lambda)^{2m} \Gamma_2^2} \right) \right| \right\} \right. \\ & \left. + 2 \left| L_1 (1 + t) \left(1 - \frac{2(\tau - 2)(1 + 2\lambda)^m \Gamma_3}{(1 + \lambda)^{2m} \Gamma_2^2} \right) \right| \right\} \end{aligned}$$

$$+ \frac{|1+t|^2}{2} \max \left\{ 1, \left| 3 - 2 \left[\frac{4(\tau-2)(1+2\lambda)^m \Gamma_3}{(1+\lambda)^{2m} \Gamma_2^2} - \frac{[(1+t)^2 + 2t]}{(1+t)^2} \right] \right| \right\}.$$

Proof. From $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and (3.1), we have

$$d_2 = -a_2 \quad \text{and} \quad d_3 = 2a_2^2 - a_3.$$

The estimate for $|d_2| = |a_2|$ follows immediately from (2.8). Letting $\rho = 2$ in (2.3), we get the estimate $|d_3|$. Now, consider

$$|d_3 - \tau d_2^2| = |2a_2^2 - a_3 - \tau a_2^2| = |a_3 - (\tau - 2)a_2^2|.$$

Changing $\rho = (\tau - 2)$ in the (2.3), we get the desired result. $\square$

Letting $\alpha = 0 = m$, $t = -1$ and $\gamma = k = 1$ in Theorem 3.1, we get the following result.

Corollary 3.1. *If $f(z) \in \mathcal{N}(\psi)$ (see Remark 1.1) and let f^{-1} be the inverse of f defined by*

$$f^{-1}(w) = w + \sum_{k=2}^{\infty} d_k w^k, \quad (|w| < r; r \geq 1/4),$$

then we have

$$|d_2| \leq |L_1|.$$

$$|d_3| \leq \frac{|L_1|}{2} \max \left\{ 1, \left| \frac{L_2}{L_1} - \frac{7L_1}{2} \right| \right\}.$$

and for all $\rho \in \mathbb{C}$

$$|d_3 - \tau d_2^2| \leq \frac{|L_1|}{2} \max \left\{ 1, \left| \frac{L_2}{L_1} + \frac{L_1}{2} (9 - 4\tau) \right| \right\}.$$

Letting $\psi(z) = \frac{1+z}{1-z}$ in Corollary 3.1, we get

Corollary 3.2. *If $f(z) \in \mathcal{N}$ (see Remark 1.1) and let f^{-1} be the inverse of f defined by*

$$f^{-1}(w) = w + \sum_{k=2}^{\infty} d_k w^k, \quad (|w| < r; r \geq 1/4),$$

then we have

$$|d_2| \leq 2, \quad |d_3| \leq 6.$$

and for all $\rho \in \mathbb{C}$

$$|d_3 - \tau d_2^2| \leq \max \left\{ 1, |(10 - 4\tau)| \right\}.$$

4. LOGARITHMIC COEFFICIENTS

Milin in [22] studied the properties of the logarithmic coefficients to obtain the bounds of the Taylor coefficients of univalent functions. The Milin conjecture about the inequalities of the logarithmic coefficients garnered the attention several researchers in those period of time, because proving Milin conjecture would imply proving Robertson conjecture and the Bieberbach conjecture. Refer to [1–4,23] for the detailed study on properties and significance of the logarithmic coefficients.

The logarithmic coefficients ℓ_n of a function $f \in \mathcal{A}$ such that $\frac{f(z)}{z} \neq 0$ for all $z \in \mathbb{U}$ is defined by

$$\log f(z) = 2 \sum_{n=1}^{\infty} \ell_n z^n. \quad (4.1)$$

Note that for all functions $f(z) \in \mathcal{N}_t^m(\alpha, \beta, \gamma, k; \lambda; \psi)$, the relation (4.1) is well-defined.

Theorem 4.1. *If $f(z) \in \mathcal{N}_t^m(\alpha, \beta, \gamma, k; \lambda; \psi)$, with the logarithmic coefficients given by (4.1), then,*

$$\begin{aligned} |\ell_1| &\leq \frac{1}{2|(1+\lambda)^m \Gamma_2|} [|L_1| + 2|1+t|], \\ |\ell_2| &\leq \frac{1}{2|(1+2\lambda)^m \Gamma_3|} \left\{ \frac{|L_1|}{2} \max \left\{ 1, \left| \frac{L_2}{L_1} + \frac{L_1}{2} \left(1 - \frac{2(1+2\lambda)^m \Gamma_3}{(1+\lambda)^{2m} \Gamma_2^2} \right) \right| \right\} \right. \\ &\quad \left. + 2 \left| L_1 (1+t) \left(1 - \frac{(1+2\lambda)^m \Gamma_3}{(1+\lambda)^{2m} \Gamma_2^2} \right) \right| \right. \\ &\quad \left. + \frac{|1+t|^2}{2} \max \left\{ 1, \left| 3 - 2 \left[\frac{2(1+2\lambda)^m \Gamma_3}{(1+\lambda)^{2m} \Gamma_2^2} - \frac{[(1+t)^2 + 2t]}{(1+t)^2} \right] \right| \right\} \right\}. \end{aligned}$$

For $\mu \in \mathbb{C}$ we have

$$\begin{aligned} |\ell_2 - \mu \ell_1^2| &\leq \frac{1}{2|(1+2\lambda)^m \Gamma_3|} \left\{ \frac{|L_1|}{2} \max \left\{ 1, \left| \frac{L_2}{L_1} + \frac{L_1}{2} \left(1 - \frac{2(1+\mu)(1+2\lambda)^m \Gamma_3}{(1+\lambda)^{2m} \Gamma_2^2} \right) \right| \right\} \right. \\ &\quad \left. + 2 \left| L_1 (1+t) \left(1 - \frac{(1+\mu)(1+2\lambda)^m \Gamma_3}{(1+\lambda)^{2m} \Gamma_2^2} \right) \right| \right. \\ &\quad \left. + \frac{|1+t|^2}{2} \max \left\{ 1, \left| 3 - 2 \left[\frac{2(1+\mu)(1+2\lambda)^m \Gamma_3}{(1+\lambda)^{2m} \Gamma_2^2} - \frac{[(1+t)^2 + 2t]}{(1+t)^2} \right] \right| \right\} \right\}. \quad (4.2) \end{aligned}$$

Proof. From $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and equating the first two coefficients of the relation (4.1) we get

$$\ell_1 = \frac{a_2}{2}, \quad \ell_2 = \frac{1}{2} \left(a_3 - \frac{a_2^2}{2} \right).$$

Using (2.1) and (2.3) in the above expression, we can find the estimates for ℓ_1 and ℓ_2 . To find the estimate (4.1), consider

$$|\ell_2 - \mu \ell_1^2| = \frac{1}{2} \left[a_3 - \frac{(1+\mu)}{2} a_2^2 \right].$$

Changing $\rho = \frac{1+\mu}{2}$ in (2.3), we get the desired result. $\square$

Remark 4.1. For different choices of ψ and parameters involved in the Definition 1.1, the class $\mathcal{N}_t^m(\alpha, \beta, \gamma, k; \lambda; \psi)$ reduces to classes impacted by conic regions but classes like starlike, convex, and spirallike are not special case of $\mathcal{N}_t^m(\alpha, \beta, \gamma, k; \lambda; \psi)$. Hence main results have lots of applications, here we pointed out only few of them.

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REFERENCES

- [1] D. Alimohammadi, N.E. Cho, E.A. Adegani, A. Motamednezhad, Argument and Coefficient Estimates for Certain Analytic Functions, *Mathematics* 8 (2020), 88. <https://doi.org/10.3390/math8010088>.
- [2] D. Alimohammadi, E.A. Adegani, T. Bulboacă, N.E. Cho, Logarithmic Coefficients for Classes Related to Convex Functions, *Bull. Malays. Math. Sci. Soc.* 44 (2021), 2659–2673. <https://doi.org/10.1007/s40840-021-01085-z>.
- [3] E.A. Adegani, N.E. Cho, M. Jafari, Logarithmic Coefficients for Univalent Functions Defined by Subordination, *Mathematics* 7 (2019), 408. <https://doi.org/10.3390/math7050408>.
- [4] E.A. Adegani, T. Bulboacă, N.H. Mohammed, P. Zaprawa, Solution of Logarithmic Coefficients Conjectures for Some Classes of Convex Functions, *Math. Slovaca* 73 (2023), 79–88. <https://doi.org/10.1515/ms-2023-0009>.
- [5] A.E. Bashirov, E.M. Kurpinar, A. Özyapıcı, Multiplicative Calculus and Its Applications, *J. Math. Anal. Appl.* 337 (2008), 36–48. <https://doi.org/10.1016/j.jmaa.2007.03.081>.
- [6] A.E. Bashirov, E. Misirlı, Y. Tandoğdu, A. Özyapıcı, On Modeling with Multiplicative Differential Equations, *Appl. Math. J. Chinese Univ.* 26 (2011), 425–438. <https://doi.org/10.1007/s11766-011-2767-6>.
- [7] A.E. Bashirov, M. Riza, On Complex Multiplicative Differentiation, *TWMS J. Appl. Eng. Math.* 1 (2011), 75–85.
- [8] D. Breaz, K.R. Karthikeyan, E. Umadevi, Non-Carathéodory Analytic Functions with Respect to Symmetric Points, *Math. Comput. Model. Dyn. Syst.* 30 (2024), 266–283. <https://doi.org/10.1080/13873954.2024.2341691>.
- [9] D. Breaz, K.R. Karthikeyan, E. Umadevi, A. Senguttuvan, Some Properties of Bazilevič Functions Involving Srivastava-Tomovski Operator, *Axioms* 11 (2022), 687. <https://doi.org/10.3390/axioms11120687>.
- [10] D. Breaz, K.R. Karthikeyan, G. Murugusundaramoorthy, Applications of Mittag-Leffler Functions on a Subclass of Meromorphic Functions Influenced by the Definition of a Non-Newtonian Derivative, *Fractal Fract.* 8 (2024), 509. <https://doi.org/10.3390/fractalfract8090509>.
- [11] D. Breaz, K.R. Karthikeyan, A. Senguttuvan, D. Mohankumar, Properties of a Subclass of Starlike Functions Involving the Quantum Derivative Operator, *MethodsX* 15 (2025), 103740. <https://doi.org/10.1016/j.mex.2025.103740>.
- [12] S. Bulut, Coefficient Estimates for Functions Associated with Vertical Strip Domain, *Commun. Korean Math. Soc.* 37 (2022), 537–549. <https://doi.org/10.4134/CKMS.c210138>.
- [13] Y.-L. Cang, J.-L. Liu, A Family of Multivalent Analytic Functions Associated with Srivastava-Tomovski Generalization of the Mittag-Leffler Function, *Filomat* 32 (2018), 4619–4625. <https://doi.org/10.2298/fil1813619c>.
- [14] N.E. Cho, O.S. Kwon, V. Ravichandran, Coefficient, Distortion and Growth Inequalities for Certain Close-to-Convex Functions, *J. Inequal. Appl.* 2011 (2011), 100. <https://doi.org/10.1186/1029-242X-2011-100>.
- [15] N.E. Cho, A. Ebadian, S. Bulut, E.A. Adegani, Subordination Implications and Coefficient Estimates for Subclasses of Starlike Functions, *Mathematics* 8 (2020), 1150. <https://doi.org/10.3390/math8071150>.
- [16] C. Gao, S. Zhou, On a Class of Analytic Functions Related to the Starlike Functions, *Kyungpook Math. J.* 45 (2005), 123–130.
- [17] K.R. Karthikeyan, S. Varadharajan, S. Lakshmi, Some Properties of Close-to-Convex Functions Associated with a Strip Domain, *Sahand Commun. Math. Anal.* 21 (2024), 105–124. <https://doi.org/10.22130/scma.2023.1987052.1233>.

- [18] K.R. Karthikeyan, G. Murugusundaramoorthy, Properties of a Class of Analytic Functions Influenced by Multiplicative Calculus, *Fractal Fract.* 8 (2024), 131. <https://doi.org/10.3390/fractalfract8030131>.
- [19] W. Koepf, On the Fekete-Szegő Problem for Close-to-Convex Functions, *Proc. Amer. Math. Soc.* 101 (1987), 89–95. <https://doi.org/10.2307/2046556>.
- [20] O.S. Kwon, Y.J. Sim, N.E. Cho, H.M. Srivastava, Some Radius Problems Related to a Certain Subclass of Analytic Functions, *Acta Math. Sin. Engl. Ser.* 30 (2014), 1133–1144. <https://doi.org/10.1007/s10114-014-3100-0>.
- [21] W.C. Ma, D. Minda, A Unified Treatment of Some Special Classes of Univalent Functions, in: *Proceedings of the Conference on Complex Analysis (Tianjin, 1992)*, Conf. Proc. Lecture Notes Anal. I, International Press, Cambridge, MA, pp. 157–169, (1994).
- [22] I.M. Milin, *Univalent Functions and Orthonormal Systems*, Nauka, Moscow, 1971 (in Russian); English translation: Transl. Math. Monogr. vol. 49, American Mathematical Society, Providence, 1977.
- [23] M. Obradović, S. Ponnusamy, K.-J. Wirths, Logarithmic Coefficients and a Coefficient Conjecture for Univalent Functions, *Monatsh. Math.* 185 (2018), 489–501. <https://doi.org/10.1007/s00605-017-1024-3>.
- [24] C. Pommerenke, *Univalent Functions*, *Studia Mathematica/Mathematische Lehrbücher*, Band XXV, Vandenhoeck & Ruprecht, Göttingen, 1975.
- [25] J.K. Prajapat, A New Subclass of Close-to-Convex Functions, *Surv. Math. Appl.* 11 (2016), 11–19.
- [26] M. Riza, A. Özyapıcı, E. Mısırlı, Multiplicative Finite Difference Methods, *Quart. Appl. Math.* 67 (2009), 745–754. <https://doi.org/10.1090/S0033-569X-09-01158-2>.
- [27] B. Şeker, S.S. Eker, On Subclasses of Bi-Close-to-Convex Functions Related to the Odd-Starlike Functions, *Palest. J. Math.* 6 (2017), Special Issue II, 215–221.
- [28] Y.J. Sim, O.S. Kwon, On Certain Classes of Convex Functions, *Int. J. Math. Math. Sci.* 2013 (2013), 294378. <https://doi.org/10.1155/2013/294378>.
- [29] A. Soni, S. Kant, A New Subclass of Close-to-Convex Functions with Fekete-Szegő Problem, *J. Rajasthan Acad. Phys. Sci.* 12 (2013), 125–138.
- [30] H.M. Srivastava, A Survey of Some Recent Developments on Higher Transcendental Functions of Analytic Number Theory and Applied Mathematics, *Symmetry* 13 (2021), 2294. <https://doi.org/10.3390/sym13122294>.
- [31] H.M. Srivastava, An Introductory Overview of Fractional-Calculus Operators Based upon the Fox-Wright and Related Higher Transcendental Functions, *J. Adv. Eng. Comput.* 5 (2021), 135–166. <https://doi.org/10.55579/jaec.202153.340>.
- [32] H.M. Srivastava, M. Bansal, P. Harjule, A Study of Fractional Integral Operators Involving a Certain Generalized Multi-Index Mittag-Leffler Function, *Math. Methods Appl. Sci.* 41 (2018), 6108–6121. <https://doi.org/10.1002/mma.5122>.
- [33] H.M. Srivastava, Z. Tomovski, Fractional Calculus with an Integral Operator Containing a Generalized Mittag-Leffler Function in the Kernel, *Appl. Math. Comput.* 211 (2009), 198–210. <https://doi.org/10.1016/j.amc.2009.01.055>.
- [34] H.M. Srivastava, On an Extension of the Mittag-Leffler Function, *Yokohama Math. J.* 16 (1968), 77–88.
- [35] E. Umadevi, K.R. Karthikeyan, A Subclass of Close-to-Convex Functions Involving Srivastava-Tomovski Operator, in: H.H. Leung, R. Sivaraj, F. Kamalov, (eds) *Recent Developments in Algebra and Analysis*, Trends in Mathematics, Springer, Cham, pp. 257–266, (2024). https://doi.org/10.1007/978-3-031-37538-5_25.
- [36] E. Umadevi, K.R. Karthikeyan, D. Mohankumar, Properties of Analytic Functions Involving a Differintegral Operator Defined Using R-Function, *Arab J. Basic Appl. Sci.* 33 (2026), 186–198. <https://doi.org/10.1080/25765299.2026.2670798>.
- [37] Z.-G. Wang, C. Gao, S. Yuan, On Certain Subclass of Close-to-Convex Functions, *Acta Math. Acad. Paedagog. Nyházi. (N.S.)* 22 (2006), 171–177.
- [38] Z.-G. Wang, D.-Z. Chen, On a Subclass of Close-to-Convex Functions, *Hacet. J. Math. Stat.* 38 (2009), 95–101.