International Journal of Analysis and Applications
ISSN 2291-8639
Volume 8, Number 2 (2015), 93-99
http://www.etamaths.com

GEOMETRY OF A CLASS OF GENERALIZED CUBIC POLYNOMIALS

CHRISTOPHER FRAYER

ABSTRACT. This paper studies a class of generalized complex cubic polynomials of the form $p(z) = (z-1)(z-r_1)^k(z-r_2)^k$ where r_1 and r_2 lie on the unit circle and k is a natural number. We completely characterize where the nontrivial critical points of p can lie, and to what extent they determine the polynomial. The main results include (1) a nontrivial critical point of such a polynomial almost always determines the polynomial uniquely, and (2) there is a 'desert' in the unit disk in which critical points cannot occur.

Several recent papers ([1], [2], [3]) have studied the geometry of cubic polynomials, specifically asking, how the critical points of a cubic polynomial depend upon its roots. Frayer, Kwon, Schafhauser, and Swenson [1] studied the critical points of a family of polynomials

$$\Gamma = \{q : \mathbb{C} \to \mathbb{C} \,|\, q(z) = (z-1)(z-r_1)(z-r_2),\ |r_1| = |r_2| = 1\}.$$

For $p \in \Gamma$ the main results of [1] include:

- A critical point almost always determines p uniquely.
- There is a *desert* in the unit disk, the open disk $\{z \in \mathbb{C} : |z - \frac{2}{3}| < \frac{1}{3}\}$, in which critical points of p cannot occur.
- If $0 < |g - \frac{1}{3}| \leq \frac{2}{3}$, then there is a unique $p \in \Gamma$ with $p''(g) = 0$. Additionally, if $|g - \frac{1}{3}| > \frac{2}{3}$, there is no $p \in \Gamma$ with $p''(g) = 0$.

We will extend the results of [1] to a class of generalized cubic polynomials

$$\Gamma_k = \left\{q : \mathbb{C} \to \mathbb{C} \,|\, q(z) = (z-1)(z-r_1)^k(z-r_2)^k,\ |r_1| = |r_2| = 1,\ k \in \mathbb{N}\right\}.$$

A polynomial of the form

$$p(z) = (z-1)(z-r_1)^k(z-r_2)^k$$

has $2k$ critical points; $k-1$ critical points at r_1 and r_2 respectively, and two nontrivial critical points. Differentiation gives

$$p'(z) = (z-r_1)^{k-1}(z-r_2)^{k-1}\left[(2k+1)z^2 - (2k+(k+1)(r_1+r_2))z + k(r_1+r_2) + r_1r_2\right]$$

so that the two nontrivial critical points of p are the roots of

$$q(z) = (2k+1)z^2 - (2k+(k+1)(r_1+r_2))z + k(r_1+r_2) + r_1r_2. \tag{1}$$

2010 *Mathematics Subject Classification.* 14G22.
Key words and phrases. geometry; generalized cubic polynomials.

This paper will characterize where the nontrivial critical points of $p \in \Gamma_k$ lie, and to what extent they determine p.

Preliminary Information

Circles which are internally tangent to the unit circle at 1 will play an important role in what follows. Given $\alpha > 0$, denote by T_α the circle of diameter α passing through 1 and $1-\alpha$ in the complex plane. That is,

$$T_\alpha = \left\{ z \in \mathbb{C} \, : \, \left| z - \left(1 - \frac{\alpha}{2}\right) \right| = \frac{\alpha}{2} \right\}.$$

For example, T_2 is the unit circle (a circle of diameter 2 centered at the origin). A key result of [1] will be used to establish a geometric relationship between the critical points of a polynomial in Γ_k.

Theorem 1 ([1])**.** *Let $f(z) = (z-1)(z-r_1)\cdots(z-r_n)$, where $|z_k| = 1$ for each k. Let $c_1, c_2, \ldots, c_n$ denote the critical points of $f(z)$, and suppose that $1 \neq c_k \in T_{\alpha_k}$ for each k. Then*

$$\sum_{k=1}^{n} \frac{1}{\alpha_k} = n. \tag{2}$$

A general result related to the geometry of complex polynomials is the Gauss-Lucas Theorem.

Theorem 2 (Gauss-Lucas Theorem)**.** *Let p be a complex-valued polynomial. The critical points of p are located in the convex hull of its roots.*

An additional fact of interest is related to fractional linear transformations.

Theorem 3 ([4])**.** *A fractional linear transformation T sends the unit circle to the unit circle if and only if $T(z) = \frac{\bar{\alpha}z+\bar{\beta}}{\beta z+\alpha}$ for some $\alpha, \beta \in \mathbb{C}$.*

Critical Points

We begin by analyzing a few special cases for future reference.

Example 1. Suppose $p \in \Gamma_k$ has nontrivial critical point $c = 1$. This occurs if and only if $z = 1$ is a repeated root of p. That is, r_1 and/or r_2 must be 1. Hence, $p(z) = (z-1)^{k+1}(z-r)^k$ for some $r \in T_2$. Conversely, given $p(z) = (z-1)^{k+1}(z-r)^k$ for some $r \in T_2$, differentiation yields

$$p'(z) = (2k+1)(z-1)^{k-1}(z-r)^{k-1}\left[(z-1)\left(z - \frac{k}{2k+1} - \frac{(k+1)}{2k+1}r\right)\right].$$

Therefore, $p \in \Gamma_k$ has a nontrivial critical point at $z = 1$ if and only if $p(z) = (z-1)^{k+1}(z-r)^k$ with $r \in T_2$. In this case, the other nontrivial critical point is $\frac{k}{2k+1} + \frac{(k+1)}{2k+1}r \in T_{\frac{2k+2}{2k+1}}$.

Now that we know which polynomials in Γ_k have nontrivial critical point $c = 1$, we may assume that $c \neq 1$ throughout the remainder of the paper.

Example 2. Suppose $p \in \Gamma_k$ has nontrivial critical point $1 \neq c \in T_2$. This occurs if and only if $z = c$ is a repeated root of p with multiplicity greater than k. That is, $r_1 = r_2 = c$ so that $p(z) = (z-1)(z-c)^{2k}$. Conversely, given $p(z) = (z-1)(z-c)^{2k}$, differentiation yields

$$p'(z) = (2k+1)(z-c)^{2k-2}\left[(z-c)\left(z - \frac{2k}{2k+1} - \frac{1}{2k+1}c\right)\right].$$

Therefore, $p \in \Gamma_k$ has nontrivial critical point $c \neq 1$ on T_2 if and only if $p(z) = (z-1)(z-c)^{2k}$. In this case, the other nontrivial critical point is $\frac{2k}{2k+1} + \frac{1}{2k+1}c \in T_{\frac{2}{2k+1}}$.

Let's now determine where the nontrivial critical points of $p \in \Gamma_k$ lie. The Gauss-Lucas Theorem guarantees that the nontrivial critical points will lie within the unit disk. But we can say more; there is a *desert* in the unit disk, the open disk $\{z \mid z \in T_\alpha \text{ with } 0 < \alpha < \frac{2}{2k+1}\}$, in which nontrivial critical points of p cannot occur.

Theorem 4. *No polynomial $p \in \Gamma_k$ has a nontrivial critical point strictly inside $T_{\frac{2}{2k+1}}$.*

Proof. Let $c_1 \neq 1$ and $c_2 \neq 1$ be nontrivial critical points of $p(z) = (z-1)(z-r_1)^k(z-r_2)^k$ with $c_1 \in T_\alpha$ and $c_2 \in T_\beta$. As the $2k-2$ trivial critical points lie on T_2, Theorem 1 gives

$$(2k-2)\left(\frac{1}{2}\right) + \frac{1}{\alpha} + \frac{1}{\beta} = 2k$$

which simplifies to

$$\frac{1}{\alpha} + \frac{1}{\beta} = k+1. \tag{3}$$

Suppose to the contrary that $\alpha < \frac{2}{2k+1}$. Then

$$\begin{aligned}\frac{1}{\beta} &= k+1-\frac{1}{\alpha} \\ &< k+1-\frac{2k+1}{2} \\ &= \frac{1}{2}.\end{aligned}$$

But then $\beta > 2$ which violates Theorem 2. □

Theorem 5. *Let $c_1 \neq 1$ and $c_2 \neq 1$ be nontrivial critical points of $p \in \Gamma_k$ with $c_1 \in T_\alpha$ and $c_2 \in T_\beta$. If c_1 lies on $T_{\frac{2}{k+1}}$ so does c_2. Otherwise, c_1 and c_2 lie on opposite sides of $T_{\frac{2}{k+1}}$.*

Proof. Let $c_1 \neq 1$ and $c_2 \neq 1$ be nontrivial critical points of $p \in \Gamma_k$ with $c_1 \in T_\alpha$ and $c_2 \in T_\beta$. Then, from equation (3), $\frac{1}{\alpha} + \frac{1}{\beta} = k+1$. Therefore, $\alpha = \frac{2}{k+1}$ if and only if $\beta = \frac{2}{k+1}$. Additionally, if $\alpha < \frac{2}{k+1}$, then

$$\begin{aligned}\frac{1}{\beta} &= k+1-\frac{1}{\alpha}\\ &< k+1-\frac{k+1}{2}\\ &= \frac{k+1}{2}\end{aligned}$$

and $\beta > \frac{2}{k+1}$. □

Now that we know where the nontrivial critical points lie, let's investigate to what extent they determine the polynomial. Given $p \in \Gamma_k$ with roots at 1, r_1 and r_2, and a nontrivial critical point c, we have

$$0 = q'(c) = (2k+1)c^2 - (2k + (k+1)(r_1+r_2))c + k(r_1+r_2) + r_1 r_2.$$

Direct calculations give

$$r_2 = \frac{(k - c(k+1))r_1 + (2k+1)c^2 - 2k}{-r_1 + c(k+1) - k}.$$

Definition 1. Given $c \in \mathbb{C}$, define

$$f_c(z) = \frac{(k - c(k+1))z + (2k+1)c^2 - 2k}{-z + c(k+1) - k}$$

and let S_c denote the image of the unit circle under f_c.

That is, $f_c(T_2) = S_c$ and $f_c(r_1) = r_2$.

Theorem 6. *Polynomial* $p(z) = (z-1)(z-r_1)^k(z-r_2)^k \in \Gamma_k$ *has nontrivial critical* $c \neq 1$ *if and only if* $f_c(r_1) = r_2$.

As fractional linear transformations send circles and lines to circles and lines, S_c will be a circle when $c \notin T_{\frac{2}{k+1}}$. To see this, note that S_c is a line when

$$|c(k+1) - k| = 1 \longleftrightarrow \left|c - \left(1 - \frac{1}{k+1}\right)\right| = \frac{1}{k+1}$$

which is equivalent to $c \in T_{\frac{2}{k+1}}$. We have established the following theorem (See Theorem 8). Let's investigate a special case.

Example 3. Suppose $1 \neq c \in T_2$. Using the fact that

$$f_c(c) = c, \quad f_c(1) = \frac{(2k+1)c - k}{k+1}, \text{ and } f_c(-1) = \frac{c^2(2k+1) + c(1-k) - k}{c(k+1) + (1-k)}$$

direct calculations give

$$\left| f_c(z) - \left(\frac{2k+1}{k+1}\right) c \right| = \frac{k}{k+1}$$

for $z \in \{c, \pm 1\}$. Therefore, for $1 \neq c \in T_2$, S_c is a circle with radius $\frac{k}{k+1}$ and center $\left(\frac{2k+1}{k+1}\right) c$, which is externally tangent to T_2 at c.

When $1 \neq c \in T_2$, it follows from Example 2 that the other critical point of p lies on the boundary of the desert at $c_2 = \frac{2k}{2k+1} + \frac{1}{2k+1}c$. Similar calculations show that S_{c_2} is a circle with radius $\frac{k}{k+1}$ and center $\left(\frac{1}{k+1}\right)c$, which is internally tangent to T_2 at c.

When $c = 1$, $f_c(z) = \frac{-z+1}{-z+1} = 1$ and $(f_c)^{-1}$ does not exist. If $c \neq 1$, then $(f_c)^{-1} = f_c$ so that $f_c(r_2) = r_1$. Hence, f_c restricts to a one-to-one correspondence from $S_c \cap T_2$ to itself, and if c is a nontrivial critical point of p, then $\{r_1, r_2\} \subseteq S_c \cap T_2$. This observation allows us to classify the polynomials in Γ_k which have a critical point at $1 \neq c$ in the unit disk! We simply need to study the intersection of circles T_2 and S_c.

Theorem 7. *If $c \notin \{1, -\frac{1}{2k+1}\}$ lies on T_α for some $\alpha \in \left[\frac{2}{2k+1}, 2\right]$, then there is a unique $p \in \Gamma_k$ with nontrivial critical point at c.*

Proof. Let $c \in \mathbb{C}$. In order to determine if there is a polynomial $p \in \Gamma_k$ with critical point at c we must study the intersection of S_c and T_2. As S_c and T_2 are circles, their intersection is disjoint, contains one point, contains two points, or is all of T_2.

If $S_c \cap T_2 = \emptyset$, then there is no polynomial in Γ_k with a nontrivial critical point at c. At a minimum, this occurs when $c \in T_\alpha$ with $\alpha > 2$ (Theorem 2) and $\alpha < \frac{2}{2k+1}$ (Theorem 4).

If $S_c \cap T_2 = \{r\}$, then $f_c(r) = r$ and by Theorem 6, r is a nontrivial critical point of $p(z) = (z-1)(z-r)^{2k}$. Conversely, as illustrated in Example 3, if $p(z) = (z-1)(z-r)^{2k}$, then $S_c \cap T_2 = \{r\}$.

If $S_c \cap T_2 = \{r, s\}$ with $r \neq s$, there are two possibilities: $f_c(r) = r$ and $f_c(s) = s$, or $f_c(r) = s$ and $f_c(s) = r$. We will rule out the first possibility. If $f_c(r) = r$ and $f_c(s) = s$, then by Theorem 6, c is a nontrivial critical point of $p(z) = (z-1)(z-r)^{2k}$ and $p(z) = (z-1)(z-s)^{2k}$. By the Gauss-Lucas Theorem, c lies on line segments $\overline{1r}$ and $\overline{1s}$. A contradiction. Therefore, $f_c(r) = s$ and $f_c(s) = r$, and it follows by Theorem 6 that $p(z) = (z-1)(z-r)^k(z-s)^k$ is the only polynomial in Γ_k with a nontrivial critical point at c.

If $S_c \cap T_2 = T_2$, then $f_c(T_2) = T_2$. As

$$f_c(z) = \frac{(k - c(k+1))z + (2k+1)c^2 - 2k}{-z + c(k+1) - k} = -\frac{(k - c(k+1))z + (2k+1)c^2 - 2k}{z + k - c(k+1)},$$

according to Theorem 3, $f_c(T_2) = T_2$ exactly when $k - c(k+1) = \overline{k - c(k+1)}$ and $(2k+1)c^2 - 2k = 1$. The first equation implies $c \in \mathbb{R}$, and the second equation simplifies to

$$((2k+1)c + 1)(c-1) = 0.$$

Since $c \neq 1$, $S_c \cap T_2 = T_2$ precisely when $c = -\frac{1}{2k+1}$. Therefore, $c = -\frac{1}{2k+1}$ is the nontrivial critical point of $p \in \Gamma_k$ if and only if $p(z) = (z-1)(z-r)^k \left(z - f_{-\frac{1}{2k+1}}(r)\right)^k$ for $r \in T_2$.

In order to establish uniqueness, we need to show that if $c \neq -\frac{1}{2k+1}$ lies on T_α with $\alpha \in (\frac{2}{2k+1}, 2)$, then $|S_c \cap T_2| = 2$. This claim follows from a simple ‘root dragging'-type argument. Without loss of generality, suppose that $S_c \cap T_2 = \emptyset$ and S_c lies inside T_2. As we 'drag' c to T_2 along a line segment going away from the origin, S_c is continuously transformed into a circle externally tangent to T_2. The Intermediate Value Theorem implies that there exists a c_0 on the line segment with S_{c_0} internally tangent to T_2. As c never crosses $T_{\frac{2}{2k+1}}$, this is a contradiction. □

Now that we have proven uniqueness, let's revisit Theorem 5.

Theorem 8. *Suppose c_1 and c_2 are nontrivial critical points of $p \in \Gamma_k$. If $1 \neq c_1 \in T_{\frac{2}{k+1}}$, then $c_2 = \bar{c_1}$.*

Stated differently, if $c \in T_{\frac{2}{k+1}}$, then S_c is a vertical line passing through $f_c(1) = \frac{(2k+1)c-k}{k+1}$. We use this fact, along with uniqueness, to provide a proof.

Proof. Let $c = x + iy \in T_{\frac{2}{k+1}}$. Suppose $r = e^{i\theta}$ with $\cos(\theta) = \left(\frac{2k+1}{k+1}x - \frac{k}{k+1}\right)$ and

$$q(z) = (z-1)(z-r)^k(z-\bar{r})^k \in \Gamma_k.$$

Then

$$q'(z) = (z-r)^{k-1}(z-\bar{r})^{k-1}\left[(2k+1)z^2 - ((k+1)(r+\bar{r})+2k)\,z + k(r+\bar{r}) + r\bar{r}\right]$$

and q has nontrivial critical points when

$$(2k+1)z^2 - ((k+1)2\cos(\theta)+2k)\,z + 2k\cos(\theta) + 1 = 0.$$

Using $\cos(\theta) = \left(\frac{2k+1}{k+1}x - \frac{k}{k+1}\right)$ yields

$$\begin{aligned}
(2k+1)z^2 - (2(2k+1)x)z + \frac{2k(2k+1)}{k+1}x - \frac{(2k+1)(k-1)}{k+1} &= 0\\
(2k+1)\left[z^2 - 2xz + \frac{2k}{k+1}x - \frac{k-1}{k+1}\right] &= 0\\
z^2 - (c+\bar{c})z + c\bar{c} &= 0\\
(z-c)(z-\bar{c}) &= 0
\end{aligned}$$

and $q \in \Gamma_k$ has nontrivial critical points at c and $\bar{c}$. Therefore, by uniqueness, if $p \in \Gamma_k$ has nontrivial critical point at $1 \neq c_1 \in T_{\frac{2}{k+1}}$, then $c_2 = \bar{c_1}$. □

Centers

Given $p(z) = (z-1)(z-r_1)^k(z-r_2)^k \in \Gamma_k$, we saw in Equation (1) that the nontrivial critical points are the solutions of

$$q(z) = (2k+1)z^2 - (2k + (k+1)(r_1+r_2))z + k(r_1+r_2) + r_1r_2.$$

We define $g \in \mathbb{C}$ to be the *center* of $p(z)$ if $q'(g) = 0$. Since q has degree 2, every $p \in \Gamma_k$ has the unique center

$$g = \frac{k}{2k+1} + \frac{k+1}{2k+1}\left(\frac{r_1+r_2}{2}\right).$$

As in [1] we will use a geometric construction to show exactly where the center can lie.

Theorem 9. *Let* $g \in \mathbb{C}$

- $p \in \Gamma_k$ *has center* $\frac{k}{2k+1}$ *if and only if* $p(z) = (z-1)(z-r_1)^k(z-r_2)^k$ *with* $r_2 = -r_1$.
- *If* $0 < |g - \frac{k}{2k+1}| \leq \frac{k+1}{2k+1}$, *then there is a unique polynomial in* Γ_k *with center* g.
- *If* $|g - \frac{k}{2k+1}| > \frac{k+1}{2k+1}$, *then there is no polynomial in* Γ_k *with center* g.

Proof. Suppose g is the center of $p \in \Gamma_k$. By the Gasuss-Lucas Theorem, g is contained in $\triangle r_1 r_2 1$, where r_1 and r_2 are points to be constructed on T_2. Even though we do not know r_1 and r_2, their midpoint, w, lies in the unit disk with $g = \frac{k}{2k+1} + \frac{k+1}{2k+1}w$. Therefore $|g - \frac{k}{2k+1}| \leq \frac{k+1}{2k+1}$.

If $0 < |g - \frac{k}{2k+1}| \leq \frac{k+1}{2k+1}$, then $g \neq \frac{k}{2k+1}$ and $w \neq 0$. As $\overline{r_1 r_2}$ is a chord of T_2, its perpendicular bisector passes through w and the origin O. Since w lies in the unit disk, the line through w perpendicular to $\overline{Ow}$ intersects T_2 in two places, r_1 and r_2.

If $g = \frac{k}{2k+1}$, then $w = 0$ is the midpoint of $\overline{r_1 r_2}$ and it follows that $r_2 = -r_1$. □

This proof completes the extension of [1] to the class of generalized cubics Γ_k. This paper completely characterizes where the critical points and centers of a $p \in \Gamma_k$ can lie and to what extent they determine a polynomial in Γ_k.

References

[1] Christopher Frayer, Myeon Kwon, Christopher Schafahuser, and James A. Swenson, *The Geometry of Cubic Polynomials*, Math. Magazine **87** (2014), no. 2, 113–124.

[2] Dan Kalman, *An elementary proof of Marden's theorem*, Amer. Math. Monthly **115** (2008), no. 4, 330–338.

[3] Sam Northshield, *Geometry of Cubic Polynomials*, Math. Magazine **86** (April 2013), 136–143.

[4] E.B Saff and A.D Snider, *Fundamentals of Com,plex Analysis for Mathematics, Science, and Engineering*, Prentice-Hall, Anglewood Cliffs, New Jersey, 1993.

University of Wisconsin-Platteville, United States