

Tensor Product of 2-K-Frames in 2-Hilbert Spaces

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Abstract. The tensor product of 2-K-frames in 2-Hilbert spaces are studied and some results on it are presented. It is shown that the tensor product of two 2-K-frames is a 2-K-frame for the tensor product of Hilbert spaces. Some results on tensor product of 2-K-frames are established.

1. INTRODUCTION

The concept of frames in Hilbert spaces have been introduced by Duffin and Schaeffer in their work on nonharmonic Fourier series in 1952 [5], reintroduced by Daubechies et al. [4], in 1986, and since then work on frames is going on. Now frames play an important role not only in the theoretics but also in many kinds of applications.

The concept of linear 2-normed spaces was first introduced by S. Gahler in 1965 [6] and has been developed extensively in different subjects by many authors. A concept which is related to a 2-normed space is 2-inner product space which has been studied by many mathematicians in the last three decades.

The notion of K-frames was proposed by Găvruta to study the atomic systems with respect to a bounded linear operator K in Hilbert spaces in [7], and its properties have been exposed in [12].

The 2-frames in 2-Hilbert space were introduced by Ali Akbar Arefijammaal and Ghadir Sadeghi [1], and some results on it are presented [2].

The tensor product of 2-frames in 2-Hilbert spaces is introduced by Reddy, G.U. in 2016 [9] and some results on tensor frame operator are presented.

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The 2-K-frames in 2-Hilbert space, where K is a θ -bounded linear operator on 2-Hilbert space H , for a fixed element θ in H were introduced by S. Tatouti and N. Bounader, and some results on it are presented in 2026 [11].

In the purpose of this article, 2-K-frames in 2-Hilbert spaces are studied and some results on it are presented. The tensor product of 2-K-frames in 2-Hilbert spaces is introduced, where K is a θ -bounded linear operator on H , for a fixed element θ in H . It is shown that the tensor product of two 2-K-frames is a 2-K-frame for the tensor product of Hilbert spaces. Some results on tensor product of 2-K-frames are established.

2. NOTATION AND PRELIMINARIES

Throughout this paper, we will adopt such notations:

- $\mathbb{K}$ denotes the field $\mathbb{R}$ or $\mathbb{C}$.
- X is the notation used for linear space of dimension greater than 1 over the field $\mathbb{K}$.
- $L(X)$ is the collection for all bounded linear operators from X to X .
- $\langle \cdot, \cdot \rangle$ is the notation used for inner product on X .
- $\langle \cdot, \cdot | \cdot \rangle$ is the notation used for 2-inner product on X .
- $(H, \langle \cdot, \cdot | \cdot \rangle)$ is the notation used for 2-Hilbert space.
- $B_\theta(H)$ is the set of all θ -bounded linear operators on H , for an element θ fixed in H .
with $\|K\|_\theta = \sup_{\|f, \theta\|=1} \{\|Kf, \theta\|\}, \quad K \in B_\theta(H)$.
- $l_2(\mathbb{N})$ refers to the Hilbert space $\{(a_n)_{n \in \mathbb{N}}; a_n \in \mathbb{C}, \sum_{n \in \mathbb{N}} |a_n|^2 < \infty\}$.

The following definitions from [12], [8] are useful in our discussion.

Definition 2.1. [12] Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space, K is a bounded linear operator on X as an element in $L(H)$, and K^* its adjoint operator. A sequence $F = \{x_n\}_{n \in \mathbb{N}} \subset X$ is a K -frame if there exist two constants $0 < E \leq F < \infty$ satisfying:

$$E\|K^*x\|^2 \leq \sum_{n \in \mathbb{N}} |\langle x, x_n \rangle|^2 \leq F\|x\|^2, \quad \forall x \in X. \quad (2.1)$$

The above inequality is called the k -frame inequality.

The numbers E, F are called lower and upper K -frame bounds respectively.

$F = \{x_n\}_{n \in \mathbb{N}} \subset X$ is called a Bessel sequence if only the right inequality of (2.1) holds.

$F = \{x_n\}_{n \in \mathbb{N}} \subset X$ is just an ordinary frames for X if $K = I_X$, where I_X is the identity operator on X .

Definition 2.2. [12] Given a K -frame $F = \{x_n\}_{n \in \mathbb{N}}$ for a Hilbert space X where $K \in L(X)$. The following operators are defined.

The K -synthesis operator is well defined and bounded:

$$T : l_2(\mathbb{N}) \rightarrow X, \quad \text{where,}$$

$$T(\{a_n\}_{n \in \mathbb{N}}) = \sum_{n \in \mathbb{N}} a_n x_n$$

The K -analysis operator is defined and bounded as the adjoint operator of T :

$$T^* : X \rightarrow l_2(\mathbb{N}), \quad \text{where,}$$

$$T^*x = \{\langle x, x_n \rangle\}_{n \in \mathbb{N}}, \quad \forall x \in X.$$

The K -frame operator is defined by $S = TT^*$:

$$S : X \rightarrow X, \quad \text{where,}$$

$$Sx = \sum_{n \in \mathbb{N}} \langle x, x_n \rangle x_n \quad \forall x \in X.$$

By a simple calculation we have

$$\langle EKK^*x, x \rangle \leq \langle Sx, x \rangle \leq \langle Fx, x \rangle, \quad \forall x \in X.$$

We review here the basic definitions of 2-inner product spaces, 2-normed spaces from [1], [8].

Definition 2.3. [8] Let X be a linear space of dimension greater than 1 over the field $\mathbb{K}$.

A $\mathbb{K}$ valued function $\langle \cdot, \cdot | \cdot \rangle$ defined on $X \times X \times X$ is called a 2-inner product on X if the following properties are verified

- a) $\langle f, f | g \rangle \geq 0$ and $\langle f, f | g \rangle = 0$ if and only if f, g are linearly dependent vectors.
- b) $\langle f, f | g \rangle = \overline{\langle g, g | f \rangle}$ for all $f, g \in X$,
- c) $\langle h, f | g \rangle = \overline{\langle f, h | g \rangle}$ for all $f, h, g \in X$,
- d) $\langle \alpha f, h | g \rangle = \alpha \langle f, h | g \rangle \quad \forall \alpha \in K$ and for all $f, h, g \in X$,
- e) $\langle f + f', h | g \rangle = \langle f, h | g \rangle + \langle f', h | g \rangle$ for all $f, f', h, g \in X$.

$(X, \langle \cdot, \cdot | \cdot \rangle)$ is called a 2-inner product space (or 2-pre-Hilbert space).

Definition 2.4. [8] Let X be a linear space of dimension greater than 1 over the field $\mathbb{K}$. A real-valued function $\| \cdot, \cdot \|$ is defined on $X \times X$ by

$$\|f, g\|^2 = \langle f, f | g \rangle, \quad \forall f, g \in X. \quad (2.2)$$

This function is called a 2-norm and satisfies the following conditions:

- a) $\|f, g\| \geq 0$ and $\|f, g\| = 0$ if and only if f, g are linearly dependent vectors,
- b) $\|f, g\| = \|g, f\|$ for all $f, g \in X$,
- c) $\|\alpha f, g\| = |\alpha| \|f, g\| \quad \forall \alpha \in \mathbb{K}$ and for all $f, g \in X$,
- d) $\|f + f', g\| \leq \|f, g\| + \|f', g\|$ for all $f, f', g \in X$.

The pair $(X, \|\cdot, \cdot\|)$ is called a linear 2-normed space.

Using the above properties, we can prove the Cauchy-Schwarz inequality

$$|\langle f, h | g \rangle|^2 \leq \langle f, f | g \rangle \langle h, h | g \rangle; \quad (2.3)$$

Whenever a 2-inner product space $(X, \langle \cdot, \cdot | \cdot \rangle)$ is given, we consider it as a linear 2-normed space $(X, \|\cdot, \cdot\|)$ with the 2-norm defined by (2.2).

Lemma 2.1. [1] Let $(X, \langle \cdot, \cdot | \cdot \rangle)$ be a 2-inner product space and $f, g \in X$. Then,

$$\|f, g\| = \sup \{ |\langle f, h | g \rangle| \mid h \in X, \|h, g\| = 1 \}. \quad (2.4)$$

Example 2.1. [1] If $(X, \langle \cdot, \cdot | \cdot \rangle)$ is a 2-inner product space, the $\mathbb{K}$ valued function defined on $X \times X \times X$ by; for all $f, h, g \in X$

$$\langle f, h | g \rangle = \begin{vmatrix} \langle f, h \rangle & \langle f, g \rangle \\ \langle g, h \rangle & \langle g, g \rangle \end{vmatrix} = \langle f, h \rangle \langle g, g \rangle - \langle g, h \rangle \langle f, g \rangle. \quad (2.5)$$

is called the standard 2-inner product defined on X .

We introduce the definition of a 2-Hilbert space from [1] as follows.

Definition 2.5. [1] Let $(X, \langle \cdot, \cdot | \cdot \rangle)$ be a 2-inner product space, θ is a fixed element in X .

A sequence $\{f_n\}_{n \in \mathbb{N}} \subset X$ is said to be convergent if there exists an element $f \in X$ such that $\lim_{n \rightarrow +\infty} \|f_n - f, \theta\| = 0$, for all $\theta \in X$.

A sequence $\{f_n\}_{n \in \mathbb{N}} \subset X$ is said to be a Cauchy sequence if for every $\varepsilon > 0$ there exists $N > 0$ such that for every $m, n \geq N$, $0 < \|f_n - f_m, \theta\| < \varepsilon$, for all $\theta \in X$.

A 2-inner product space X is said a 2-Hilbert space if it is complete. That is, every Cauchy sequence in X is convergent in this space.

For the rest of this paper,

- L_θ denotes the subspace of H generated by a fixed element $\theta \in H$,
- M_θ denotes the algebraic complement of L_θ in H . So we have $L_\theta \oplus M_\theta = H$,
- $(M_\theta, \|\cdot\|_\theta)$ is a norm space where, $\|f\|_\theta = \langle f, f | \theta \rangle^{\frac{1}{2}}$, $f \in M_\theta$.
- H_θ be the completion of the inner product space M_θ See more Lemma 5.1.2 in [3].

Definition 2.6. [1] Let $(H, \langle \cdot, \cdot | \cdot \rangle)$ be a 2-Hilbert space. Then, A sequence $\{f_n\}_{n \in \mathbb{N}} \subset H$ is a 2-frame associated to θ if there exist two constants $0 < E \leq F < \infty$ such that

$$E\|f, \theta\|^2 \leq \sum_{n \in \mathbb{N}} |\langle f, f_n | \theta \rangle|^2 \leq F\|f, \theta\|^2 \quad \forall f \in H. \quad (2.6)$$

Then the definition of 2-frame can be written as

$$E\|f\|_{\theta}^2 \leq \sum_{n \in \mathbb{N}} |\langle f, f_n | \theta \rangle|^2 \leq F\|f\|_{\theta}^2 \quad \forall f \in H_{\theta}. \quad (2.7)$$

Given a 2-frame $F = \{f_n\}_{n \in \mathbb{N}}$ for H associated to θ . The 2-frame operator is defined by

$$S_{\theta} : H_{\theta} \rightarrow H_{\theta}, \quad \text{where,}$$

$$S_{\theta} f = \sum_{n \in \mathbb{N}} \langle f, f_n | \theta \rangle f_n, \quad \forall f \in H_{\theta}.$$

This operator S_{θ} is a bounded, invertible, and self-adjoint operator. The result leads to the following decomposition:

$$f = S_{\theta} S_{\theta}^{-1} f = \sum_{n \in \mathbb{N}} \langle S_{\theta}^{-1} f, f_n | \theta \rangle f_n, \quad \forall f \in H_{\theta}. \quad (2.8)$$

3. 2-K-FRAME

First we need to define $B_{\theta}(H)$ the collection of all θ -bounded linear operators on H , where θ is a fixed element in H .

A linear operator K on a 2-Hilbert space H is said a θ -bounded operator if there exists $M > 0$ such that

$$\|K\|_{\theta} < M, \text{ where } \|K\|_{\theta} = \sup_{\|f, \theta\|=1} \{\|Kf, \theta\|, f \in H\}.$$

We denote by $B_{\theta}(H)$ the set of all θ -bounded linear operators on H .

The definition of 2-K-frame and some properties on it from [10] as follows.

Definition 3.1. [11] Let $(H, \langle \cdot, \cdot \rangle, |\cdot|)$ be a 2-Hilbert space, θ is a fixed element in H , and $K \in B_{\theta}(H)$. A sequence $F = \{k_n\}_{n \in \mathbb{N}}$ of elements in H is called a 2-K-frame for H associated to θ , if there exist two constants $0 < E \leq F < \infty$ satisfying :

$$E\|K^* f, \theta\|^2 \leq \sum_{n \in \mathbb{N}} |\langle f, k_n | \theta \rangle|^2 \leq F\|f, \theta\|^2, \quad \forall f \in H. \quad (3.1)$$

E and F are the lower and upper 2-K-frame bounds respectively.

$F = \{k_n\}_{n \in \mathbb{N}}$ is called a 2-frame for H associated to θ if $K = I_H$, where I_H is the identity operator on H .

$F = \{k_n\}_{n \in \mathbb{N}}$ is called a 2-Bessel sequence for H associated to θ if only the right inequality of (3.1) holds.

Definition 3.2. [11] Let $\{k_n\}_{n \in \mathbb{N}} \subset H$ be a 2-K-frame associated to θ in a 2-Hilbert space H , where $K \in B_{\theta}(H)$. Then the 2-pre-K-frame operator or the 2-K-synthesis operator T_{θ} is defined as follow

$$T_{\theta} : l_2(\mathbb{N}) \rightarrow H_{\theta}, \quad \text{where;}$$

$$T_{\theta}(\{a_n\}_{n \in \mathbb{N}}) = \sum_{n \in \mathbb{N}} a_n k_n$$

Definition 3.3. [11] Let $\{k_n\}_{n \in \mathbb{N}} \subset H$ be a 2-K-frame associated to θ in a 2-Hilbert space H , where $K \in B_{\theta}(H)$. Then, the 2-K-analysis operator T_{θ}^* is defined by

$$T_{\theta}^*: H_{\theta} \rightarrow l_2(\mathbb{N}), \quad \text{where;} \\ T_{\theta}^*(f) = \{\langle f, k_n | \theta \rangle\}_{n \in \mathbb{N}}, \quad \forall f \in H_{\theta}.$$

Definition 3.4. [11] Let $\{k_n\}_{n \in \mathbb{N}} \subset H$ be a 2-K-frame associated to θ in a 2-Hilbert space H , where $K \in B_{\theta}(H)$. Then the 2-K-frame operator $S_{\theta} = T_{\theta} T_{\theta}^*$ for F is defined as:

$$S_{\theta}: H_{\theta} \rightarrow H_{\theta} \\ S_{\theta} f = \sum_{n \in \mathbb{N}} \langle f, k_n | \theta \rangle k_n \quad \forall f \in H_{\theta}.$$

Remark 3.1. Let θ be a fixed element in H and let $\{k_n\}_{n \in \mathbb{N}}$ be a 2-K-frame for a 2-Hilbert space H associated to θ , where $K \in B_{\theta}(H)$, and the range $R(K)$ of K is closed. There exists $\{f_n\}_{n \in \mathbb{N}}$ a 2-Bessel sequence for H associated to θ such that

$$\forall f \in H, \quad Kf = \alpha K\theta + \sum_{n \in \mathbb{N}} \langle f, f_n | \theta \rangle k_n, \quad (3.2)$$

for some $\alpha \in \mathbb{K}$.

Proposition 3.1. [11] Suppose that $\{k_n\}_{n \in \mathbb{N}}$ is a 2-K-frame for a 2-Hilbert space H associated to θ , a fixed element in H , and $T \in B_{\theta}(H)$. Then $\{Tk_n\}_{n \in \mathbb{N}}$ is a 2-TK-frame for H associated to θ .

As a consequence of above proposition we have.

Consequence 1. For every 2-frame $\{f_n\}_{n \in \mathbb{N}}$ for 2-Hilbert space H associated to θ , $\{Kf_n\}_{n \in \mathbb{N}} \subset H$ is a 2-K-frame for H associated to θ for all $K \in B_{\theta}(H)$.

4. MAIN RESULTS

4.1. Tensor Product of 2-K-Frames. First of all of this section we need to recall the notion of the tensor product from [9].

Vocabulary and notations

$(H_1, \langle \cdot, \cdot \rangle_1)$, $(H_2, \langle \cdot, \cdot \rangle_2)$ are two 2-Hilbert spaces, $\eta \in H_1$, $\theta \in H_2$.
 $\| \cdot \|_1, \| \cdot \|_2$ are norms generated by $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ respectively.
 $K_1 \in B_{\eta}(H_1)$, and $K_2 \in B_{\theta}(H_2)$.

Definition 4.1. [9] The tensor product of H_1 and H_2 is denoted by $H_1 \otimes H_2$ and is an inner product space with respect to the inner product given by $\forall f_1, g_1 \in H_1, \forall f_2, g_2 \in H_2$

$$\langle f_1 \otimes f_2, g_1 \otimes g_2 | h_1 \otimes h_2 \rangle = \langle f_1, g_1 | h_1 \rangle_1 \langle f_2, g_2 | h_2 \rangle_2. \quad (4.1)$$

$$\| f_1 \otimes f_2, g_1 \otimes g_2 \| = \| f_1, g_1 \|_1 \| f_2, g_2 \|_2. \quad (4.2)$$

The space $H_1 \otimes H_2$ is completion with the above inner product. Therefore, the space $H_1 \otimes H_2$ is a 2-Hilbert space.

In order to define a tensor product of 2-K-frame for the tensor product of Hilbert spaces $H_1 \otimes H_2$ associated to $\eta \otimes \theta$, where $K = K_1 \otimes K_2$, $K_1 \in B_\eta(H_1)$, $K_2 \in B_\theta(H_2)$. We will first show the existence of the adjoint K^* of K as follows:

Lemma 4.1. *Let $(H_1, \langle \cdot, \cdot \rangle_1)$, $(H_2, \langle \cdot, \cdot \rangle_2)$ are two 2-Hilbert spaces, $\eta \otimes \theta$ is a fixed element in $H_1 \otimes H_2$, $K_1 \in B_\eta(H_1)$, and $K_2 \in B_\theta(H_2)$. There exist K^* the adjoint operator of $K = K_1 \otimes K_2$ such that*

$$\|K\|_{\eta \otimes \theta} = \|K^*\|_{\eta \otimes \theta} = \|K_1^*\|_\eta \|K_2^*\|_\theta. \quad (4.3)$$

and

$$\begin{cases} \langle f_1 \otimes f_2, K^*(g_1 \otimes g_2) | \eta \otimes \theta \rangle = \langle K(f_1 \otimes f_2), g_1 \otimes g_2 | \eta \otimes \theta \rangle, \\ \forall f_1, g_1 \in H_1, \forall f_2, g_2 \in H_2. \end{cases} \quad (4.4)$$

Proof. $K_1 \in B_\eta(H_1)$, $K_2 \in B_\theta(H_2)$ from Corollary 3.1.4 in [11] there exists K_1^* and K_2^* the adjoints operators of K_1 and K_2 respectively such that

$$\begin{aligned} \langle K_1 f_1, g_1 | \eta \rangle_1 &= \langle f_1, K_1^* g_1 | \eta \rangle_1, \quad \forall f_1, g_1 \in H_1 \quad \text{and} \quad \|K_1\|_\eta = \|K_1^*\|_\eta. \\ \langle K_2 f_2, g_2 | \theta \rangle_2 &= \langle f_2, K_2^* g_2 | \theta \rangle_2, \quad \forall f_2, g_2 \in H_2 \quad \text{and} \quad \|K_2\|_\theta = \|K_2^*\|_\theta. \end{aligned}$$

Then for each $f_1 \otimes f_2, g_1 \otimes g_2 \in H_1 \otimes H_2$ we have

$$\begin{aligned} \langle K(f_1 \otimes f_2), g_1 \otimes g_2 | \eta \otimes \theta \rangle &= \langle K_1 \otimes K_2(f_1 \otimes f_2), g_1 \otimes g_2 | \eta \otimes \theta \rangle \\ &= \langle K_1 f_1, g_1 | \eta \rangle_1 \langle K_2 f_2, g_2 | \theta \rangle_2 \\ &= \langle f_1, K_1^* g_1 | \eta \rangle_1 \langle f_2, K_2^* g_2 | \theta \rangle_2 \\ &= \langle f_1 \otimes f_2, K_1^* g_1 \otimes K_2^* g_2 | \eta \otimes \theta \rangle \\ &= \langle f_1 \otimes f_2, K_1^* \otimes K_2^*(g_1 \otimes g_2) | \eta \otimes \theta \rangle \end{aligned}$$

which proves that there exists an operator denoted by $K^* = K_1^* \otimes K_2^*$ as an adjoint operator of the product operator $K = K_1 \otimes K_2$.

Furthermore, we have

$$\begin{aligned} \|K^*\|_{\eta \otimes \theta} &= \sup\{\|K^*(f \otimes g), \eta \otimes \theta\|, f \otimes g \in H_1 \otimes H_2; \|f \otimes g, \eta \otimes \theta\| = 1\} \\ &= \sup\{\|K_1^* \otimes K_2^*(f \otimes g), \eta \otimes \theta\|, f \otimes g \in H_1 \otimes H_2; \|f \otimes g, \eta \otimes \theta\| = 1\} \\ &= \sup\{\|K_1^* f, \eta\|_1 \|K_2^* g, \theta\|_2, f \in H_1, g \in H_2; \|f, \eta\|_1 = 1, \|g, \theta\|_2 = 1\} \\ &= \sup\{\|K_1^* f, \eta\|_1, f \in H_1; \|f, \eta\|_1 = 1\} \sup\{\|K_2^* g, \theta\|_2, g \in H_2; \|g, \theta\|_2 = 1\} \\ &= \|K_1^*\|_\eta \|K_2^*\|_\theta, \end{aligned}$$

On the other hand,

$$\begin{aligned}
 \|K\|_{\eta \otimes \theta} &= \sup\{\|K(f \otimes g), \eta \otimes \theta\|, f \otimes g \in H_1 \otimes H_2; \|f \otimes g, \eta \otimes \theta\| = 1\} \\
 &= \sup\{\|K_1 \otimes K_2(f \otimes g), \eta \otimes \theta\|, f \otimes g \in H_1 \otimes H_2; \|f \otimes g, \eta \otimes \theta\| = 1\} \\
 &= \sup\{\|K_1 f, \eta\|_1 \|K_2 g, \theta\|_2, f \in H_1, g \in H_2; \|f, \eta\|_1 = 1, \|g, \theta\|_2 = 1\} \\
 &= \sup\{\|K_1 f, \eta\|_1, f \in H_1; \|f, \eta\|_1 = 1\} \sup\{\|K_2 g, \theta\|_2, g \in H_2; \|g, \theta\|_2 = 1\} \\
 &= \|K_1\|_{\eta} \|K_2\|_{\theta}.
 \end{aligned}$$

then the adjoint for $K = K_1 \otimes K_2$ is the operator $K^* = K_1^* \otimes K_2^*$ satisfying

$$\|K^*\|_{\eta \otimes \theta} = \|K_1^*\|_{\eta} \|K_2^*\|_{\theta} = \|K_1\|_{\eta} \|K_2\|_{\theta} = \|K\|_{\eta \otimes \theta}.$$

Hence, the result □

Definition 4.2. Let $\{k_n\}_{n \in \mathbb{N}}, \{l_m\}_{m \in \mathbb{N}}$ be two sequences of vectors in 2-Hilbert spaces H_1, H_2 respectively, $\eta \otimes \theta$ is a fixed element in $H_1 \otimes H_2$, and $K = K_1 \otimes K_2$ where $K_1 \in B_{\eta}(H_1), K_2 \in B_{\theta}(H_2)$. Then, the sequence of vectors $\{k_n \otimes l_m\}_{n,m \in \mathbb{N}}$ is said to be a tensor product of 2- K_1 -frame, 2- K_2 -frame for the tensor product of Hilbert spaces $H_1 \otimes H_2$ associated to $\eta \otimes \theta$ if there exist two constants E, F with $0 < E \leq F < \infty$ such that: for all $f \otimes g \in H_1 \otimes H_2$,

$$E\|K^*(f \otimes g), \eta \otimes \theta\|^2 \leq \sum_{n,m \in \mathbb{N}} |\langle f \otimes g, k_n \otimes l_m, \eta \otimes \theta \rangle|^2 \leq F\|f \otimes g, \eta \otimes \theta\|^2. \quad (4.5)$$

The numbers E and F are called lower and upper frame bounds of the tensor product of 2- K -frame, respectively.

$\{k_n \otimes l_m\}_{n,m \in \mathbb{N}}$ is called a tensor product of 2-frame for H associated to $\eta \otimes \theta$ if $K_1 = I_{H_1}$, and $K_2 = I_{H_2}$.

The family $\{k_n \otimes l_m\}_{n,m \in \mathbb{N}}$ is a 2- $(\mathcal{K}_1 \otimes \mathcal{K}_2)$ -frame for $\mathcal{H}_1 \otimes \mathcal{H}_2$ associated with $\eta \otimes \theta$

Theorem 4.1. Let $\{k_n\}_{n \in \mathbb{N}}$ and $\{l_m\}_{m \in \mathbb{N}}$ be two sequences in 2-Hilbert spaces H_1 and H_2 respectively. Then, the sequence $\{k_n \otimes l_m\}_{n,m \in \mathbb{N}}$ is a 2- $K_1 \otimes K_2$ -frame for $H_1 \otimes H_2$ associated to $\eta \otimes \theta$ if and only if $\{k_n\}_{n \in \mathbb{N}}$ and $\{l_m\}_{m \in \mathbb{N}}$ are the 2- K_1 -frame for H_1 associated to η and 2- K_2 -frame for H_2 associated to θ respectively.

Proof. Suppose that $\{k_n \otimes l_m\}_{n,m \in \mathbb{N}}$ is a 2- $K_1 \otimes K_2$ -frame for $H_1 \otimes H_2$ associated to $\eta \otimes \theta$. Then, for each $f \otimes g \in H_1 \otimes H_2 - \{0 \otimes 0\}$

$$E\|K^*(f \otimes g), \eta \otimes \theta\|^2 \leq \sum_{n,m \in \mathbb{N}} |\langle f \otimes g, k_n \otimes l_m, \eta \otimes \theta \rangle|^2 \leq F\|f \otimes g, \eta \otimes \theta\|^2.$$

On using (4.1) and (4.2) the above equation becomes

$$E\langle K_1^* f, K_1^* f | \eta \rangle_1 \langle K_2^* g, K_2^* g | \theta \rangle_2 \leq \sum_{i \in \mathbb{N}} |\langle f, k_n | \eta \rangle_1|^2 \sum_{m \in \mathbb{N}} |\langle g, l_m | \theta \rangle_2|^2 \leq F\langle f, f | \eta \rangle_1 \langle g, g | \theta \rangle_2.$$

This gives

$$E \frac{\langle K_2^* g, K_2^* g | \theta \rangle_2}{\sum_{m \in \mathbb{N}} |\langle g, l_m | \theta \rangle_2|^2} \langle K_1^* f, K_1^* f | \eta \rangle_1 \leq \sum_{n \in \mathbb{N}} |\langle f, k_n | \eta \rangle_1|^2 \leq F \frac{\langle g, g | \theta \rangle_2}{\sum_{m \in \mathbb{N}} |\langle g, l_m | \theta \rangle_2|^2} \langle f, f | \eta \rangle_1.$$

That is

$$E_1 \langle K_1^* f, K_1^* f | \eta \rangle_1 \leq \sum_{n \in \mathbb{N}} |\langle f, k_n | \eta \rangle_1|^2 \leq F_1 \langle f, f | \eta \rangle_1.$$

Therefore

$$E_1 \|K_1^* f, \eta\|_1^2 \leq \sum_{n \in \mathbb{N}} |\langle f, k_n | \eta \rangle_1|^2 \leq F_1 \|f, \eta\|_1^2.$$

where,

$$E_1 = E \frac{\langle K_2^* g, K_2^* g | \theta \rangle_2}{\sum_{m \in \mathbb{N}} |\langle g, l_m | \theta \rangle_2|^2}, \quad F_1 = F \frac{\langle g, g | \theta \rangle_2}{\sum_{m \in \mathbb{N}} |\langle g, l_m | \theta \rangle_2|^2}.$$

Which shows that $\{k_n\}_{n \in \mathbb{N}}$ is a $2-K_1$ -frame for H_1 associated to η .

Similarly we can prove that $\{l_m\}_{m \in \mathbb{N}}$ is a $2-K_2$ -frame for H_2 associated to θ .

Conversely, assume that $\{k_n\}_{n \in \mathbb{N}}$ is a $2-K_1$ -frame for H_1 associated to η with frame bounds E_1, F_1 , and $\{l_m\}_{m \in \mathbb{N}}$ is a $2-K_2$ -frame for H_2 associated to θ with frame bounds E_2, F_2 . Then

$$E_1 \|K_1^* f, \eta\|_1^2 \leq \sum_{n \in \mathbb{N}} |\langle f, k_n | \eta \rangle_1|^2 \leq F_1 \|f, \eta\|_1^2, \quad \forall g \in H_1. \quad (4.6)$$

And

$$E_2 \|K_2^* g, \theta\|_2 \leq \sum_{m \in \mathbb{N}} |\langle g, l_m | \eta \rangle_2|^2 \leq F_2 \|g, \theta\|_2^2, \quad \forall g \in H_2. \quad (4.7)$$

multiplying the relations (4.6) and (4.7) we get

$$E_1 E_2 \|K^*(f \otimes g), \eta \otimes \theta\|^2 \leq \sum_{n, m \in \mathbb{N}} |\langle f \otimes g, k_n \otimes l_m, \eta \otimes \theta \rangle|^2 \leq F_1 F_2 \|f \otimes g, \eta \otimes \theta\|^2.$$

Which shows that $\{k_n \otimes l_m\}_{n, m \in \mathbb{N}}$ is a tensor product of $2-K_1$ -frame, $2-K_2$ -frame for $H_1 \otimes H_2$ associated to $\eta \otimes \theta$ with lower and upper boundes $E = E_1 E_2, F = F_1 F_2$ respectively. $\square$

Example 4.1. Suppose that the Hilbert space $H = \mathbb{C}^3$, and suppose that $\{e_i\}_{i=1}^3$ be the orthonormal basis for $\mathbb{C}^3$, $\eta_0 = \langle 1, 0, 0 \rangle \in H$ $\theta_0 = \langle 0, 0, 1 \rangle \in H$ where

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Let $F = \{e_i\}_{i=1}^3 = \{e_1, e_2, e_3\}$.

$\forall f \in \mathbb{C}^3$, since $\{e_i\}_{i=1}^3$ is an orthonormal basis for H , there exist x, y, z such that $f = xe_1 + ye_2 + ze_3$.

Obviously the sequence F is 2-frame for the standad 2-inner product space H associated to η_0 . indeed by a simple calculation we have

$$\|f, \eta_0\|^2 = y^2 + z^2, \quad \sum_{i=1}^3 |\langle f, e_i | \eta_0 \rangle|^2 = y^2 + z^2$$

Define $K_1 \in B_{\eta_0}(H)$ as follows:

$$K_1 : H \rightarrow H, \quad K_1 e_1 = e_1, K_1 e_2 = e_1, K_1 e_3 = e_2.$$

So, we have for each $f \in H$

$$f = xe_1 + ye_2 + ze_3, \quad K_1 f = (x + y)e_1 + ze_2.$$

$$\|K_1 f, \eta_0\|^2 = z^2, \quad \|f, \eta_0\|^2 = y^2 + z^2, \quad \|K_1\|_{\eta_0} \leq 1.$$

And

$$\langle K_1^* e_1, f, \eta_0 \rangle = \langle e_1, K_1 f, \eta_0 \rangle = \langle e_1, (x + y)e_1 + ze_3, \eta_0 \rangle = \langle e_1, f, \eta_0 \rangle \text{ i.e } K_1^* e_1 = e_1.$$

$$\langle K_1^* e_2, f, \eta_0 \rangle = \langle e_2, K_1 f, \eta_0 \rangle = \langle e_2, (x + y)e_1 + ze_3, \eta_0 \rangle = \langle e_3, f, \eta_0 \rangle \text{ i.e } K_1^* e_2 = e_3.$$

$$\langle K_1^* e_3, f, \eta_0 \rangle = \langle e_3, K_1 f, \eta_0 \rangle = \langle e_3, (x + y)e_1 + ze_3, \eta_0 \rangle = \langle e_1, f, \eta_0 \rangle \text{ i.e } K_1^* e_3 = e_1.$$

Since

$$\sum_{i=1}^3 |\langle f, k_i | \eta_0 \rangle|^2 = |\langle f, e_1 | \eta_0 \rangle|^2 + |\langle f, e_1 | \theta_0 \rangle|^2 + |\langle f, e_2 | \eta_0 \rangle|^2 = y^2, \quad \|K_1^* f, \eta_0\| = y^2,$$

then

$$\|K_1^* f, \eta_0\|^2 \leq \sum_{i=1}^3 |\langle f, k_i | \eta_0 \rangle|^2 \leq \|f, \eta_0\|^2$$

which shows that $\{k_i\}_{i=1}^3 = \{K_1 e_i\}_{i=1}^3$ is a 2- K_1 -frame for the standad 2-inner product space $\mathbb{C}^3$ associated to η_0 .

Similarly, we have the sequence F is 2-frame for the standad 2-inner product space $H = \mathbb{C}^3$ associated to θ_0 .

$\forall f \in \mathbb{C}^3$, since $\{e_i\}_{i=1}^3$ is an orthonormal basis for H , there exist x', y', z' such that $g = x'e_1 + y'e_3 + z'e_3$.

$$\|g, \theta_0\|^2 = x'^2 + y'^2, \quad \sum_{i=1}^3 |\langle g, e_i | \theta_0 \rangle|^2 = x'^2 + y'^2$$

Define $K_2 \in B_{\theta_0}(H)$ as follows:

$$K_2 : H \rightarrow H, \quad K_2 e_1 = e_2, K_2 e_2 = e_3, K_2 e_3 = e_3.$$

So, we have for each $g \in H$

$$g = x'e_1 + y'e_2 + z'e_3, \quad K_2 g = x'e_2 + (y' + z')e_3.$$

$$\|K_2 g, \theta_0\|^2 = x'^2, \quad \|g, \theta_0\|^2 = x'^2 + y'^2, \quad \|K_2\|_{\theta_0} \leq 1.$$

$$\langle K_2^* e_1, g, \theta_0 \rangle = \langle e_1, K_2 g, \theta_0 \rangle = \langle e_1, x'e_2 + (y' + z')e_3, \theta_0 \rangle = \langle e_3, g, \theta_0 \rangle \text{ i.e } K_2^* e_1 = e_3.$$

$$\langle K_2^* e_2, g, \theta_0 \rangle = \langle e_2, K_2 g, \theta_0 \rangle = \langle e_1, x'e_2 + (y' + z')e_3, \theta_0 \rangle = \langle e_1, g, \theta_0 \rangle \text{ i.e } K_2^* e_2 = e_1.$$

$$\langle K_2^* e_3, g, \theta_0 \rangle = \langle e_3, K_2 g, \theta_0 \rangle = \langle e_3, x'e_2 + (y' + z')e_3, \theta_0 \rangle = \langle e_3, g, \theta_0 \rangle \text{ i.e } K_2^* e_3 = e_3.$$

Since

$$\sum_{j=1}^3 |\langle g, l_j | \theta_0 \rangle|^2 = |\langle g, e_2 | \theta_0 \rangle|^2 + |\langle g, e_3 | \theta_0 \rangle|^2 + |\langle g, e_3 | \theta_0 \rangle|^2 = x'^2, \quad \|K_2^* f, \theta_0\| = x'^2,$$

then

$$\|K_2^* g, \theta_0\|^2 \leq \sum_{i=1}^3 |\langle g, l_i | \theta_0 \rangle|^2 \leq \|g, \theta_0\|^2$$

which shows that $\{l_j\}_{j=1}^3 = \{K_2 e_j\}_{j=1}^3$ is a 2- K_2 -frame for $\mathbb{C}^3$ associated to θ_0 .

So from Theorem 4.1 $\{k_i \otimes l_j\}_{i,j=1}^3$ is a tensor product of 2- K_1 -frame, 2- K_2 -frame for $\mathbb{C}^3 \otimes \mathbb{C}^3$ associated to $\eta_0 \otimes \theta_0$.

Theorem 4.2. Suppose that $\{k_n \otimes l_m\}_{n,m \in \mathbb{N}}$ is a tensor product of 2-frame for $H_1 \otimes H_2$ associated to $\eta \otimes \theta$, where η and θ are two fixed elements in H_1 and H_2 respectively. For all $K_1 \in B_\eta(H_1)$, $K_2 \in B_\theta(H_2)$, $\{(K_1 \otimes K_2)(k_n \otimes l_m)\}_{n,m \in \mathbb{N}}$ is a tensor product of 2- K_1 -frame, 2- K_2 -frame for $H_1 \otimes H_2$ associated to $\eta \otimes \theta$.

Proof. Suppose that $\{k_n \otimes l_m\}_{n,m \in \mathbb{N}}$ is a tensor product of 2-frame for $H_1 \otimes H_2$ associated to $\eta \otimes \theta$. According to Theorem 4.1 $\{k_n\}_{n \in \mathbb{N}}$ is a 2-frames for H_1 associated to η and $\{l_m\}_{m \in \mathbb{N}}$ is a 2-frame for H_2 associated to θ .

by applying the (consequence 1) ,

for all $K_1 \in B_\eta(H_1)$, $K_2 \in B_\theta(H_2)$ we have

$$\begin{cases} \{K_1 k_n\}_{n \in \mathbb{N}} \text{ is a 2-}K_1\text{-frame for } H_1 \text{ associated with } \eta, \\ \{K_2 l_m\}_{m \in \mathbb{N}} \text{ is a 2-}K_2\text{-frame for } H_2 \text{ associated with } \theta. \end{cases}$$

So we conclude

$\{K_1 \otimes K_2(k_n \otimes l_m)\}_{n,m \in \mathbb{N}}$ is a tensor product of 2- K_1 -frame and 2- K_2 -frame for $H_1 \otimes H_2$ associated to $\eta \otimes \theta$. $\square$

Theorem 4.3. let $K = K_1 \otimes K_2$, where $K_1 \in B_\eta(H_1)$, and $K_2 \in B_\theta(H_2)$. Suppose that $\{k_n \otimes l_m\}_{n,m \in \mathbb{N}}$ is a tensor product of 2- K_1 -frame, 2- K_2 -frame for $H_1 \otimes H_2$ associated to $\eta \otimes \theta$. Then $\{T_1 \otimes T_2(k_n \otimes l_m)\}_{n,m \in \mathbb{N}}$ is a tensor product of 2- $T_1 K_1$ -frame, 2- $T_2 K_2$ -frame for $H_1 \otimes H_2$ associated to $\eta \otimes \theta$ for all $T_1 \in B_\eta(H_1)$ and $T_2 \in B_\theta(H_2)$.

Proof. $\{k_n \otimes l_m\}_{n,m \in \mathbb{N}}$ is a tensor product of 2- K_1 -frame, 2- K_2 -frame for $H_1 \otimes H_2$ associated to $\eta \otimes \theta$. According to Theorem 4.1 $\{k_n\}_{n \in \mathbb{N}}$ and $\{l_m\}_{m \in \mathbb{N}}$ are the 2- K_1 -frame for H_1 associated to η and 2- K_2 -frame for H_2 associated to θ respectively. By using Proposition 3.1 we have,

$$\begin{cases} \{T_1 k_n\}_{n \in \mathbb{N}} \text{ is a 2-}T_1 K_1\text{-frame for } H_1 \text{ associated with } \eta, \\ \{T_2 l_m\}_{m \in \mathbb{N}} \text{ is a 2-}T_2 K_2\text{-frame de } H_2 \text{ associated with } \theta. \end{cases}$$

Which gives,

$$E_1 \| (K_1^* T_1^* f), \eta \|_1^2 \leq \sum_{n \in \mathbb{N}} |\langle f, T_1 k_n / \eta \rangle|^2 \leq F_1 \| f, \eta \|_1^2, \quad \forall f \in H_1. \quad (4.8)$$

$$E_2 \| (K_2^* T_2^* g), \theta \|_1^2 \leq \sum_{m \in \mathbb{N}} |\langle g, T_2 l_m / \theta \rangle|^2 \leq F_2 \| g, \theta \|_1^2, \quad \forall g \in H_2. \quad (4.9)$$

By multiplying the two above inequalities (4.8) and (4.9), there exist two constants $E = E_1 E_2$, $F = F_1 F_2$ such that for each $f \otimes g \in H_1 \otimes H_2$,

$$E \| K^* T_1^* \otimes T_2^* (f \otimes g), \eta \otimes \theta \|^2 \leq \sum_{n,m \in \mathbb{N}} |\langle f \otimes g, T_1 \otimes T_2 (k_n \otimes l_m), \eta \otimes \theta \rangle|^2 \leq F \| f \otimes g, \eta \otimes \theta \|^2.$$

Which shows that $\{T_1 \otimes T_2(k_n \otimes l_m)\}_{n,m \in \mathbb{N}}$ is a $2\text{-}T_1K_1 \otimes T_2K_2$ -frame for $H_1 \otimes H_2$ associated to $\eta \otimes \theta$. $\square$

4.2. Frame operator of tensor Product of 2-K-Frames. In this paragraph, we define the frame operator of any tensor product of two 2-K-frames.

Remark 4.1. If the $\{k_n\}_{n \in \mathbb{N}}$, $\{l_m\}_{m \in \mathbb{N}}$, and $\{k_n \otimes l_m\}_{n,m \in \mathbb{N}}$ are the $2\text{-}K_1$ -frame for H_1 associated to η , $2\text{-}K_2$ -frame for H_2 associated to θ , and $2\text{-}K_1 \otimes K_2$ -frame for $H_1 \otimes H_2$ associated to $\eta \otimes \theta$ respectively. From (3.2) in [10] S_η , S_θ , and $S_{\eta \otimes \theta}$ are the frame operators respectively of above frames, we have the following.

$$S_\eta f = \sum_{n \in \mathbb{N}} \langle f, k_n | \eta \rangle k_n, \quad \forall f \in H_1. \quad S_\theta g = \sum_{m \in \mathbb{N}} \langle g, l_m | \theta \rangle l_m, \quad \forall g \in H_2. \quad (4.10)$$

$$S_{\eta \otimes \theta}(f \otimes g) = \sum_{n,m \in \mathbb{N}} \langle f \otimes g, k_n \otimes l_m | \eta \otimes \theta \rangle (k_n \otimes l_m), \quad \forall f \otimes g \in H_1 \otimes H_2. \quad (4.11)$$

Theorem 4.4. If $\{k_n\}_{n \in \mathbb{N}}$, $\{l_m\}_{m \in \mathbb{N}}$ and $\{k_n \otimes l_m\}_{n,m \in \mathbb{N}}$ are the $2\text{-}K_1$ -frame for H_1 associated to η , $2\text{-}K_2$ -frame for H_2 associated to θ , and $2\text{-}K_1 \otimes K_2$ -frame for $H_1 \otimes H_2$ associated to $\eta \otimes \theta$, with the frame operators S_η , S_θ , and $S_{\eta \otimes \theta}$ respectively, then $S_{\eta \otimes \theta} = S_\eta \otimes S_\theta$

Proof. $\forall f \in H_1, \forall g \in H_1, \forall f \otimes g \in H_1 \otimes H_2$, we have

$$\begin{aligned} S_{\eta \otimes \theta}(f \otimes g) &= \sum_{n,m \in \mathbb{N}} \langle f \otimes g, k_n \otimes l_m | \eta \otimes \theta \rangle (k_n \otimes l_m) \\ &= \sum_{n,m \in \mathbb{N}} \langle f, k_n | \eta \rangle_1 \langle g, l_m | \theta \rangle_2 (k_n \otimes l_m) \\ &= \left(\sum_{n \in \mathbb{N}} \langle f, k_n | \eta \rangle_1 k_n \right) \otimes \left(\sum_{m \in \mathbb{N}} \langle g, l_m | \theta \rangle_2 l_m \right) \\ &= S_\eta f \otimes S_\theta g = (S_\eta \otimes S_\theta)(f \otimes g). \end{aligned}$$

Hence $S_{\eta \otimes \theta} = S_\eta \otimes S_\theta$. $\square$

We call $S_{\eta \otimes \theta}$ the frame operator for $2\text{-}K_1 \otimes K_2$ -frame for $H_1 \otimes H_2$ associated to $\eta \otimes \theta$. By a simple calculation we have

$$\langle S_\eta \otimes S_\theta(f \otimes g), f \otimes g | \eta \otimes \theta \rangle = \sum_{n,m \in \mathbb{N}} |\langle f \otimes g, k_n \otimes l_m | \eta \otimes \theta \rangle|^2, \quad \forall f \otimes g \in H_1 \otimes H_2.$$

Consequence 2. Let $\{k_n\}_{n \in \mathbb{N}}$ be a $2\text{-}K_1$ -frame for H_1 associated to η and $\{l_m\}_{m \in \mathbb{N}}$ be a $2\text{-}K_2$ -frame for H_2 associated to θ , with the frame operators S_η and S_θ respectively. The frame operator $S_{\eta \otimes \theta}$ for a $2\text{-}K_1 \otimes K_2$ -frame for $H_1 \otimes H_2$ associated to $\eta \otimes \theta$; $\{k_n \otimes l_m\}_{n,m \in \mathbb{N}}$ Checks the following inequality:

$$E_1 K_1 K_1^* \leq S_\eta \leq F_1 I_{H_1}, \quad E_2 K_2 K_2^* \leq S_\theta \leq F_2 I_{H_2}, \quad (4.12)$$

$$E_1 E_2 (K_1 \otimes K_2) (K_1 \otimes K_2)^* \leq S_{\eta \otimes \theta} \leq F_1 F_2 I_{H_1 \otimes H_2}. \quad (4.13)$$

Example 4.2. Let $\{e_i\}_{i=1}^3$ be an orthonormal basis for a three-dimensional vector space $\mathbb{C}^3$ with the inner product. $\eta_o = (1, 0, 0) \in \mathbb{C}^3$, $\theta_o = (0, 0, 1) \in \mathbb{C}^3$

Consider the sequences $\{k_i\}_{i=1}^3$, $\{l_j\}_{j=1}^3$, and $\{k_i \otimes l_j\}_{i,j=1}^3$ seen in Example 4.1 are $2\text{-}K_1$ -frame for $H_1 = \mathbb{C}^3$

associated to η_0 , 2- K_2 -frame for $H_2 = \mathbb{C}^3$ associated to θ_0 , 2- $K_1 \otimes K_2$ -frame for $H_1 \otimes H_2$ associated to $\eta_0 \otimes \theta_0$ respectively.

We define their frame operator S_η , S_θ , and $S_\eta \otimes S_\theta$ as follows:

For each $f = xe_1 + ye_2 + ze_3 \in H_1$, $g = x'e_1 + y'e_2 + z'e_3 \in H_2$,

$$S_{\eta_0}f = \sum_{i=1}^3 \langle f, k_i | \eta_0 \rangle k_i = ye_2,$$

$$S_{\theta_0}g = \sum_{j=1}^3 \langle g, l_j | \theta_0 \rangle l_j = y'e_2,$$

$$S_{\eta_0 \otimes \theta_0}(f \otimes g) = yy'e_2 \otimes e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Next we use the frame operator of 2- $K_1 \otimes K_2$ -frame to give another equivalent characterization for 2- $K_1 \otimes K_2$ -frame.

Theorem 4.5. Let $\{k_n\}_{n \in \mathbb{N}}$, and $\{l_m\}_{m \in \mathbb{N}}$ are two 2-Bessel sequences in H_1, H_2 respectively, $K_1 \in B_\eta(H_1)$, $K_2 \in B_\theta(H_2)$. Then $\{k_i \otimes l_m\}_{n,m \in \mathbb{N}}$ is a tensor product of 2- K_1 -frame, 2- K_2 -frame for $H_1 \otimes H_2$ associated to $\eta \otimes \theta$, if and only if there exists $E > 0$ such that

$$S_\eta \otimes S_\theta \geq E(K_1 K_1^* \otimes K_2 K_2^*),$$

where S_η , and S_θ are the frame operators for $\{k_n\}_{n \in \mathbb{N}}$ and $\{l_m\}_{m \in \mathbb{N}}$ respectively.

Proof. $\{k_n \otimes l_m\}_{n,m \in \mathbb{N}}$ is a tensor product of 2- K_1 -frame, 2- K_2 -frame for $H_1 \otimes H_2$ associated to $\eta \otimes \theta$, with frame bounds E, F and frame operator $S_\eta \otimes S_\theta$, if and only if

for all $f \in H_1, g \in H_2$,

$$E \|K_1^* \otimes K_2^*(f \otimes g), \eta \otimes \theta\|^2 \leq \sum_{n,m \in \mathbb{N}} |(f \otimes g, k_n \otimes l_m, \eta \otimes \theta)|^2 \leq F \|f \otimes g, \eta \otimes \theta\|^2.$$

that is, $\forall f \in H_1, g \in H_2$,

$$\langle K_1 K_1^* \otimes K_2 K_2^*(f \otimes g), f \otimes g | \eta \otimes \theta \rangle \leq \langle S_\eta \otimes S_\theta(f \otimes g), f \otimes g | \eta \otimes \theta \rangle \leq \langle f \otimes g, f \otimes g | \eta \otimes \theta \rangle,$$

So the conclusion holds

$$S_\eta \otimes S_\theta \geq E(K_1 K_1^* \otimes K_2 K_2^*).$$

□

In general the frame operator of 2- $K_1 \otimes K_2$ -frame is not invertible on $H_1 \otimes H_2$, but we can show that it is invertible on a subspace $R(K_1 \otimes K_2)$, when $R(K_1)$, and $R(K_2)$ are closed. What we will prove in the following.

Proposition 4.1. Let $K = K_1 \otimes K_2$, where $K_1 \in B_\eta(H_1)$, and $K_2 \in B_\theta(H_2)$. Suppose that $\{k_n \otimes l_m\}_{n,m \in \mathbb{N}}$ is a tensor product of 2- K_1 -frame, 2- K_2 -frame for $H_1 \otimes H_2$ associated to $\eta \otimes \theta$ as element fixed in $H_1 \otimes H_2$. If the range $R(K_1)$ of K_1 and the range $R(K_2)$ of K_2 are closed, then the 2-K-frame operator,

$$S_{\eta \otimes \theta} : R(K_1 \otimes K_2) \rightarrow S_{\eta \otimes \theta}(R(K_1 \otimes K_2))$$

is a homeomorphism and $S_{\eta \otimes \theta}^{-1} = S_\eta^{-1} \otimes S_\theta^{-1}$, which verifies: $\forall f \otimes g \in S_{\eta \otimes \theta}((R(K_1 \otimes K_2)))$

$$\frac{1}{F} \|f \otimes g, \eta \otimes \theta\| \leq \|S_{\eta \otimes \theta}^{-1}(f \otimes g), \eta \otimes \theta\| \leq \frac{1}{E} \|K_1^\dagger\|_\eta^2 \|K_2^\dagger\|_\theta^2 \|f \otimes g, \eta \otimes \theta\|, \quad (4.14)$$

where $K_1^\dagger$ is the pseudo-inverse of K_1 , $K_2^\dagger$ is the pseudo-inverse of K_2

Proof. $\{k_n \otimes l_m\}_{n,m \in \mathbb{N}}$ is a tensor product of 2- K_1 -frame, 2- K_2 -frame for $H_1 \otimes H_2$ associated to $\eta \otimes \theta$ as element fixed in $H_1 \otimes H_2$. Then,

$\{k_n\}_{n \in \mathbb{N}}$ and $\{l_m\}_{m \in \mathbb{N}}$ are 2- K_1 -frame for H_1 associated to η and 2- K_2 -frame for H_2 associated to θ respectively.

Since, K_1 and K_2 had a closed ranges, then the frame operators

S_η is a homeomorphism from $R(K_1)$ onto $S_\eta(R(K_1))$,

S_θ is a homeomorphism from $R(K_2)$ onto $S_\theta(R(K_2))$,

and verifying the following inequalities :

$$E_1 \|K_1^\dagger\|_\eta^{-2} \|f, \eta\|_1 \leq \|S_\eta(f), \eta\|_1 \leq F_1 \|f, \eta\|_1, \quad \forall f \in R(K_1), \quad (4.15)$$

$$E_2 \|K_2^\dagger\|_\theta^{-2} \|g, \theta\|_2 \leq \|S_\theta(g), \theta\|_2 \leq F_2 \|g, \theta\|_2 \quad \forall g \in R(K_2). \quad (4.16)$$

$$\frac{1}{F_1} \|f, \eta\|_1 \leq \|S_\eta^{-1}(f), \eta\|_1 \leq \frac{1}{E_1} \|K_1^\dagger\|_\eta^2 \|f, \eta\|_1, \quad \forall f \in S_\eta(R(K_1)), \quad (4.17)$$

$$\frac{1}{F_2} \|g, \theta\|_2 \leq \|S_\theta^{-1}(g), \theta\|_2 \leq \frac{1}{E_2} \|K_2^\dagger\|_\theta^2 \|g, \theta\|_2, \quad \forall g \in S_\theta(R(K_2)). \quad (4.18)$$

multiplying the Equations (4.17) and (4.18) we get,

$$\forall f \otimes g \in S_{\eta \otimes \theta}(R(K_1) \otimes R(K_2)) = S_{\eta \otimes \theta}(R(K_1 \otimes K_2))$$

$$\frac{1}{F_1 F_2} \|f, \eta\|_1 \|g, \theta\|_2 \leq \|S_\eta^{-1}(f) \otimes S_\theta^{-1}(g), \eta \otimes \theta\| \leq \frac{1}{E_1 E_2} \|K_1^\dagger\|_\eta^2 \|K_2^\dagger\|_\theta^2 \|f, \eta\|_1 \|g, \theta\|_2$$

which gives:

$$\forall f \otimes g \in S_{\eta \otimes \theta}(R(K_1 \otimes K_2))$$

$$\frac{1}{F} \|f \otimes g, \eta \otimes \theta\| \leq \|S_{\eta \otimes \theta}^{-1}(f \otimes g), \eta \otimes \theta\| \leq \frac{1}{E} \|K_1^\dagger\|_\eta^2 \|K_2^\dagger\|_\theta^2 \|f \otimes g, \eta \otimes \theta\|$$

So, we have $S_{\eta \otimes \theta}$ is a homeomorphism.

Moreover by the lemma 3.6 in [10], $S_{\eta \otimes \theta}^{-1} = S_{\eta}^{-1} \otimes S_{\theta}^{-1}$. $\square$

Based on equation (3.2), we can therefore make the following remark.

Remark 4.2. *If the sequences $\{k_n \otimes l_m\}_{n,m \in \mathbb{N}}$ is a tensor product of 2- K_1 -frame, 2- K_2 -frame for the 2-Hilbert spaces $H_1 \otimes H_2$. If K_1, K_2 had a closed ranges, there exist two 2-Bessel sequence $\{f_n\}_{n \in \mathbb{N}}$ for H_1 associated to η , and $\{g_n\}_{n \in \mathbb{N}}$ for H_2 associated to θ such that:*

$$\forall f \otimes g \in H_1 \otimes H_2 \quad K_1 \otimes K_2(f \otimes g) = \alpha K_1 \otimes K_2(\eta \otimes \theta) + \sum_{n,m \in \mathbb{N}} \langle f \otimes g, k_n \otimes l_m | \eta \otimes \theta \rangle k_n \otimes l_m. \quad (4.19)$$

for some $\alpha \in \mathbb{K}$.

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REFERENCES

- [1] A.A. Arefijamaal, G. Sadeghi, Frames in 2-Inner Product Spaces, Iran. J. Math. Sci. Inform. 8 (2013), 123–130. <https://doi.org/10.7508/ijmsi.2013.02.011>.
- [2] Y.J. Cho, P.C.S. Lin, S.S. Kim, A. Misiak, Theory of 2-Inner Product Spaces, Nova Science Publishers, 2001.
- [3] O. Christensen, Frames and Bases: An Introductory Course, Birkhäuser Boston, 2008. <https://doi.org/10.1007/978-0-8176-4678-3>.
- [4] I. Daubechies, A. Grossmann, Y. Meyer, Painless Nonorthogonal Expansions, J. Math. Phys. 27 (1986), 1271–1283. <https://doi.org/10.1063/1.527388>.
- [5] R.J. Duffin, A.C. Schaeffer, A Class of Nonharmonic Fourier Series, Trans. Am. Math. Soc. 72 (1952), 341–366. <https://doi.org/10.1090/S0002-9947-1952-0047179-6>.
- [6] S. Gähler, Lineare 2-normierte Räume, Math. Nachr. 28 (1964), 1–43. <https://doi.org/10.1002/mana.19640280102>.
- [7] L. Găvruta, Frames for Operators, Appl. Comput. Harmon. Anal. 32 (2012), 139–144. <https://doi.org/10.1016/j.acha.2011.07.006>.
- [8] H. Mazaheri, R. Kazemi, Some Results on 2-Inner Product Spaces, Novi Sad J. Math. 37 (2007), 35–40. <https://eudml.org/doc/229469>
- [9] G.U. Reddy, Tensor Product of 2-Frames in 2-Hilbert Spaces, Adv. Pure Math. 6 (2016), 517–522. <https://doi.org/10.4236/apm.2016.68039>.
- [10] G.U. Reddy, N.G. Reddy, B.K. Reddy, Frame Operator and Hilbert-Schmidt Operator in Tensor Product of Hilbert Spaces, J. Dyn. Syst. Geom. Theor. 7 (2009), 61–70. <https://doi.org/10.1080/1726037X.2009.10698563>.
- [11] S. Tatouti, N. Bounader, 2-K-Frames in 2-Hilbert Spaces, Int. J. Anal. Appl. 24 (2026), 101. <https://doi.org/10.28924/2291-8639-24-2026-101>.
- [12] X. Xiao, Y. Zhu, L. Găvruta, Some Properties of K-Frames in Hilbert Spaces, Results Math. 63 (2013), 1243–1255. <https://doi.org/10.1007/s00025-012-0266-6>.