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An Application of Tripolar (m, n)-Fuzzy Soft Sets in Ordered Semigroups

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Abstract. In this paper, we introduce the concept of tripolar (m, n)-fuzzy soft sets over ordered semigroups. First, we define several core structures, including tripolar (m, n)-fuzzy soft subsemigroups, tripolar (m, n)-fuzzy soft left and right ideals, tripolar (m, n)-fuzzy soft quasi-ideals, and tripolar (m, n)-fuzzy soft bi-ideals, and we also provide concrete examples to clearly explain these new concepts. Second, we present our main results concerning regular ordered semigroups, where we prove that the concepts of a tripolar (m, n)-fuzzy soft quasi-ideal and a tripolar (m, n)-fuzzy soft bi-ideal are identical (they coincide) under regularity. Finally, we characterize tripolar (m, n)-fuzzy soft quasi-ideals by proving that a tripolar (m, n)-fuzzy soft set is a quasi-ideal if and only if it is the intersection of a tripolar (m, n)-fuzzy soft left ideal and a tripolar (m, n)-fuzzy soft right ideal.

1. Introduction

The study of uncertainty and vagueness is very important in modern mathematics and decision science. Traditional set theory cannot easily handle problems with incomplete or multi-faceted data. To solve this challenge, Zadeh [1] introduced fuzzy sets, which allow elements to have a membership degree in the interval $[0, 1]$. Later, Atanassov [2] extended this idea into intuitionistic fuzzy sets (IFS). An IFS includes both a membership degree and a non-membership degree to better model human hesitation.

However, IFS may not be enough when a problem involves neutral opinions or conflicting negative information. To address this, researchers developed tripolar fuzzy sets. This approach categorizes information into three distinct parts: positive membership, neutral membership, and

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negative membership. This tripolar framework gives a more detailed view for real-world scenarios. For example, in evaluation systems, an expert can have a positive opinion, a neutral stance, or a strictly negative disagreement.

At the same time, Molodtsov [3] introduced soft set theory as a new mathematical tool to handle uncertainty. Unlike fuzzy sets, soft sets use a parameterization function. This feature makes them highly flexible for decision-making. Combining these concepts led to new hybrid structures, such as fuzzy soft sets [4] and intuitionistic fuzzy soft sets [5], which opened new directions in algebra.

Applying fuzzy and soft structures to algebraic systems, especially semigroups, is a very active area of research. Semigroups are fundamental structures used in computer science, control engineering, and theoretical physics. In particular, ordered semigroups add a layer of relational complexity because the algebraic operation interacts directly with a partial order. Within these systems, ideal theory—including left ideals, right ideals, quasi-ideals, and bi-ideals—helps us deeply understand their structural properties.

In 2007, Kehayopulu and Tsingelis [6] showed important results for fuzzy ideals in ordered semigroups. They proved that fuzzy quasi-ideals are fuzzy bi-ideals, and that these two concepts are identical in regular ordered semigroups. They also showed that a fuzzy quasi-ideal is exactly the intersection of a fuzzy right ideal and a fuzzy left ideal. Moreover, Murali Krishna Rao and Venkateswarlu [10] first introduced tripolar fuzzy set theory, which was later applied to ordered semigroups by Wattanasiripong et al. [11–14]. In 2022, Jun and Hur [15] introduced the novel concept of (m,n)-fuzzy sets and successfully investigated its algebraic properties within the framework of BCK-algebras.

Based on these foundational ideas of [6], [10], and [15], this paper introduces the concept of tripolar (m,n)-fuzzy soft sets over ordered semigroups. First, we define several core structures, including tripolar (m,n)-fuzzy soft subsemigroups, tripolar (m,n)-fuzzy soft left and right ideals, tripolar (m,n)-fuzzy soft quasi-ideals, and tripolar (m,n)-fuzzy soft bi-ideals, and we provide concrete examples to explain these new concepts. Second, we present our main results concerning regular ordered semigroups, where we prove that the concepts of a tripolar (m,n)-fuzzy soft quasi-ideal and a tripolar (m,n)-fuzzy soft bi-ideal are identical (they coincide) under regularity. Finally, we characterize tripolar (m,n)-fuzzy soft quasi-ideals by proving that a tripolar (m,n)-fuzzy soft set is a quasi-ideal if and only if it is the intersection of a tripolar (m,n)-fuzzy soft left ideal and a tripolar (m,n)-fuzzy soft right ideal.

2. Preliminaries

In this section, we recall some fundamental definitions and results regarding ordered semigroups and tripolar intuitionistic fuzzy soft set theory that are used throughout this paper.

Let S be a nonempty set together with a binary operation denoted by "$\cdot$". The structure $\langle S;\cdot\rangle$ is referred to as a *groupoid*. If the binary operation "$\cdot$" satisfies the associative property, meaning that

$$x\cdot(y\cdot z)=(x\cdot y)\cdot z$$

holds for all $x, y, z \in S$, then the groupoid $\langle S; \cdot \rangle$ is called a *semigroup*.

A binary relation $\leq$ on a set S is called a *partial order* if it satisfies the following three conditions:

(1) *Reflexivity*: For all $a \in S$, $a \leq a$.
(2) *Antisymmetry*: For all $a, b \in S$, if $a \leq b$ and $b \leq a$, then $a = b$.
(3) *Transitivity*: For all $a, b, c \in S$, if $a \leq b$ and $b \leq c$, then $a \leq c$.

The structure $\langle S; \leq \rangle$ is called a *partially ordered set* (or simply a *poset*).

Definition 2.1. *[6] The structure $\langle S; \cdot, \leq \rangle$ is called an ordered semigroup if it satisfies the following conditions:*

(1) *$\langle S; \cdot \rangle$ is a semigroup.*
(2) *$\langle S; \leq \rangle$ is a partially ordered set.*
(3) *For every $a, b, c \in S$ if $a \leq b$, then $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$.*

For the sake of simplicity, we denote an ordered semigroup $\langle S; \cdot, \leq \rangle$ simply by its carrier set $\mathbf{S}$. For any $a, b \in S$, the product $a \cdot b$ is denoted by ab. Let A and B be nonempty subsets of S. The product AB is defined by

$$AB := \{ab : a \in A \text{ and } b \in B\}.$$

Let $\mathbf{S}$ be an ordered semigroup. A nonempty subset A of S is said to be a *subsemigroup* of $\mathbf{S}$ if $AA \subseteq A$.

Definition 2.2. *[6] Let $\mathbf{S}$ be an ordered semigroup. A nonempty subset A of S is called a left ideal (resp., right ideal) of $\mathbf{S}$ if it satisfies the following conditions:*

(1) *$SA \subseteq A$ (resp., $AS \subseteq A$).*
(2) *For any $x, y \in S$, if $x \leq y$ and $y \in A$, then $x \in A$.*

A nonempty subset of S is called a *two-sided ideal*, or simply an *ideal*, of $\mathbf{S}$ if it is both a left ideal and a right ideal of $\mathbf{S}$.

Let $\langle S; \leq \rangle$ be an ordered set, and let $(\,] : \mathcal{P}(S) \to \mathcal{P}(S)$ be a mapping define as follows,

$$(A] := \{x \in S : x \leq a \text{ for some } a \in A\}.$$

Definition 2.3. *[6] Let $\mathbf{S}$ be an ordered semigroup. A nonempty subset A of S is called a quasi-ideal of $\mathbf{S}$ if it satisfies the following conditions:*

(1) *$(SA] \cap (AS] \subseteq A$.*
(2) *For $x, y \in S$, if $x \leq y$ and $y \in A$, then $x \in A$.*

Definition 2.4. *[6] Let $\mathbf{S}$ be an ordered semigroup. A subsemigroup A of S is called a bi-ideal of $\mathbf{S}$ if it satisfies the following conditions:*

(1) *$ASA \subseteq A$.*
(2) *For $x, y \in S$, if $x \leq y$ and $y \in A$, then $x \in A$.*

In classical set theory, the membership of an element in a set is binary: an element either belongs to the set or it does not. However, many concepts in real-world problems are vague and cannot be defined by precise criteria. To address this uncertainty, Zadeh [1] introduced the concept of fuzzy sets in 1965. Unlike classical sets, a fuzzy set allows elements to have partial membership degrees ranging between 0 and 1.

Definition 2.5. *Let X be a nonempty set. A fuzzy set A in X is defined as an object of the form:*

$$A := \{\langle x, \mu_A(x)\rangle : x \in X\}$$

where $\mu_A\colon X \to [0,1]$ is a mapping called the membership function of A, and $\mu_A(x)$ represents the degree of membership of x in A.

Although fuzzy sets introduced by Zadeh provide a powerful tool for dealing with uncertainty, they assume that the degree of non-membership is simply the complement of the degree of membership (i.e., $1 - \mu_A(x)$). This assumption may not always accurately reflect real-world situations where there is a degree of hesitation or lack of knowledge. To overcome this limitation, Atanassov [2] introduced the concept of intuitionistic fuzzy sets (IFSs) in 1986. An IFS is characterized by two functions: a membership function μ_A and a non-membership function ν_A, with the condition that their sum does not exceed 1. The remaining part, $1 - (\mu_A(x) + \nu_A(x))$, represents the degree of hesitation or uncertainty.

Definition 2.6. *Let X be a nonempty set. An intuitionistic fuzzy set A in X is defined as an object of the form:*

$$A := \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$$

where the mappings $\mu_A\colon X \to [0,1]$ and $\nu_A\colon X \to [0,1]$ denote the degree of membership and the degree of non-membership of the element $x \in X$ to the set A, respectively, satisfying the condition:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1$$

for all $x \in X$.

In 2022, Jun and Hur [15] introduced the novel concept of (m,n)-fuzzy sets and successfully investigated its algebraic properties within the framework of BCK-algebras. This mathematical model provides an elegant extension of intuitionistic, and Pythagorean fuzzy sets by allowing the exponents of membership and non-membership degrees to vary independently as a pair of natural numbers (m,n). Inspired by their pioneering work, we extend the application of (m,n)-fuzzy sets into the class of ordered semigroups.

Definition 2.7. *Let X be a nonempty set, and let $m, n \in \mathbb{N}$. An (m,n)-fuzzy set A in X is an object of the form:*

$$A := \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$$

where the mappings $\mu_A\colon X \to [0,1]$ *and* $\nu_A\colon X \to [0,1]$ *denote the degree of membership and the degree of non-membership of each element* $x \in X$ *to the set* A*, respectively. Furthermore, these mappings satisfy the following condition for all* $x \in X$*:*

$$0 \leq \mu_A^m(x) + \nu_A^n(x) \leq 1.$$

For convenience throughout this paper, we shall write $\mu_A^m(x)$ *and* $\nu_A^n(x)$ *to denote* $(\mu_A(x))^m$ *and* $(\nu_A(x))^n$*, respectively.*

In 1999, Molodtsov [3] introduced soft set theory as a general mathematical tool for dealing with uncertainty. However, the classical soft set theory is based on crisp sets. Recognizing that in many real-world problems, the parameters or the approximations themselves are fuzzy in nature, Maji et al. [4] combined the theories of fuzzy sets and soft sets. They introduced the concept of *fuzzy soft sets*, which allows for a gradation of membership in the approximation of parameters.

Definition 2.8. *Let* U *be an initial universe set and* E *a set of parameters. Let* $\mathcal{F}(U)$ *denote the set of all fuzzy sets in* U*. A pair* (F, E) *is called a fuzzy soft set over* U*, where* F *is a mapping given by:*

$$F\colon E \to \mathcal{F}(U).$$

For any parameter $e \in E$*,* $F(e)$ *is a fuzzy set in* U *defined as an object of the form:*

$$F(e) := \{\langle x, \mu_{F(e)}(x)\rangle : x \in U\}$$

where $\mu_{F(e)} : U \to [0,1]$ *is the membership function.*

Later, Maji et al. [5] generalized the concept of fuzzy soft sets to intuitionistic fuzzy soft sets. This extension is based on the combination of the intuitionistic fuzzy set theory and the soft set theory, which allows for dealing with both the degree of membership and the degree of non-membership of the parameters.

Definition 2.9. *Let* U *be an initial universe set and let* E *be a set of parameters. Let* $\mathcal{IF}(U)$ *denote the set of all intuitionistic fuzzy sets in* U*. A pair* (F, E) *is called an intuitionistic fuzzy soft set over* U*, where* F *is a mapping given by:*

$$F\colon E \to \mathcal{IF}(U).$$

For any parameter $e \in E$*,* $F(e)$ *is an intuitionistic fuzzy set in* U *defined as an object of the form:*

$$F(e) := \{\langle x, \mu_{F(e)}(x), \nu_{F(e)}(x)\rangle : x \in U\}$$

where $\mu_{F(e)}, \nu_{F(e)}\colon U \to [0,1]$ *are the membership and non-membership functions, respectively, satisfying the condition:*

$$0 \leq \mu_{F(e)}(x) + \nu_{F(e)}(x) \leq 1$$

for all $x \in U$*.*

The concept of tripolar fuzzy sets was originally introduced by M. Murali Krishna Rao [10] as a significant generalization of classical fuzzy sets, intuitionistic fuzzy sets, and bipolar fuzzy sets. This framework was specifically developed to capture and model tripolar decision-making information, as formally presented in the following definition.

Definition 2.10. *Let X be a nonempty set. A tripolar fuzzy set A in X is an object of the form:*

$$A := \{\langle x, \mu_A(x), \nu_A(x), \delta_A(x)\rangle : x \in X\}$$

where $\mu_A, \nu_A : X \to [0,1]$ and $\delta_A : X \to [-1,0]$ are mappings such that $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$.

Definition 2.11. *Let U be an initial universe set and E be a set of parameters. Let $\mathcal{TF}(U)$ denote the collection of all tripolar fuzzy sets in U. A pair (F, E) is called a tripolar fuzzy soft set over U, where F is a mapping given by:*

$$F : E \to \mathcal{TF}(U)$$

For each parameter $e \in E$, $F(e)$ is a tripolar fuzzy set in U defined as:

$$F(e) := \{\langle x, \mu_{F(e)}(x), \nu_{F(e)}(x), \delta_{F(e)}(x)\rangle : x \in U\}$$

where $\mu_{F(e)}, \nu_{F(e)} : U \to [0,1]$ and $\delta_{F(e)} : U \to [-1,0]$ are the positive, neutral, and negative membership functions, respectively, satisfying the condition:

$$0 \leq \mu_{F(e)}(x) + \nu_{F(e)}(x) \leq 1$$

for all $x \in U$.

3. Main Results

In this section, we provide the core algebraic results of our investigation into tripolar fuzzy soft sets over ordered semigroups. We first establish the fundamental relationships between various classes of tripolar fuzzy soft ideals—including left, right, quasi, and bi-ideals—and characterize them through the tripolar fuzzy soft product. The section is organized as follows: we begin with preliminary lemmas that describe the behavior of tripolar fuzzy soft products in relation to the ordering of the semigroup. These tools are then employed to prove the equivalence of subsemigroups and ideals with their respective product-based inclusions. Finally, we present a characterization theorem for tripolar fuzzy soft quasi-ideals, showing that they coincide with the intersection of left and right ideals, and discuss these structures within the framework of regular ordered semigroups.

Definition 3.1. *Let U be an initial universe set, E be a set of parameters, and $m, n \in \mathbb{N}$. A pair (F, E) is called a tripolar (m,n)-fuzzy soft set over U, where F is a mapping given by:*

$$F : E \to \mathcal{TF}^{mn}(U)$$

and $\mathcal{TF}^{mn}(U)$ denotes the collection of all tripolar (m,n)-fuzzy sets in U. For each parameter $e \in E$, the image $F(e)$ is characterized as an object of the form:

$$F(e) := \left\{\langle x, \mu_{F(e)}(x), \nu_{F(e)}(x), \delta_{F(e)}(x)\rangle : x \in U\right\}$$

where the mappings $\mu_{F(e)}: U \to [0,1]$, $\nu_{F(e)}: U \to [0,1]$, and $\delta_{F(e)}: U \to [-1,0]$ represent the positive membership, non-membership, and negative membership degrees of the element $x \in U$, respectively. Furthermore, these mappings must satisfy the following structural constraints for all $x \in U$:

$$0 \le (\mu_{F(e)}(x))^m + (\nu_{F(e)}(x))^n \le 1 \quad \text{and} \quad -1 \le \delta_{F(e)}(x) \le 0.$$

For the sake of convenience throughout this paper, we shall utilize the notations $\mu^m_{F(e)}(x)$ and $\nu^n_{F(e)}(x)$ to denote $(\mu_{F(e)}(x))^m$ and $(\nu_{F(e)}(x))^n$, respectively.

To illustrate this concept, we present the following concrete example.

Example 3.1. *Let $U = \{x_1, x_2, x_3\}$ be the initial universe containing three distinct elements, and let $E = \{e_1, e_2\}$ be a set of parameters representing two distinct characteristics. We fix the exponents as natural numbers $m = 2$ and $n = 3$.*

We construct a tripolar $(2,3)$-fuzzy soft set (F,E) over U through the following two systematic steps:
The first step. *For the first parameter $e_1 \in E$, the image $F(e_1)$ is expressed as the tripolar fuzzy set:*

$$F(e_1) = \left\{\langle x, \mu_{F(e_1)}(x), \nu_{F(e_1)}(x), \delta_{F(e_1)}(x)\rangle \mid x \in U\right\}$$

where the component functions $\mu_{F(e_1)}, \nu_{F(e_1)}: U \to [0,1]$ and $\delta_{F(e_1)}: U \to [-1,0]$ are defined piecewise by:

$$\mu_{F(e_1)}(x) := \begin{cases} 0.6 & \text{if } x = x_1 \\ 0.7 & \text{if } x = x_2 \\ 0.5 & \text{if } x = x_3, \end{cases} \quad \nu_{F(e_1)}(x) := \begin{cases} 0.5 & \text{if } x = x_1 \\ 0.4 & \text{if } x = x_2 \\ 0.7 & \text{if } x = x_3, \end{cases} \quad \text{and} \quad \delta_{F(e_1)}(x) := \begin{cases} -0.2 & \text{if } x = x_1 \\ -0.5 & \text{if } x = x_2 \\ 0.0 & \text{if } x = x_3. \end{cases}$$

Similarly, for the second parameter $e_2 \in E$, the image $F(e_2)$ is expressed as the tripolar fuzzy set:

$$F(e_2) = \left\{\langle x, \mu_{F(e_2)}(x), \nu_{F(e_2)}(x), \delta_{F(e_2)}(x)\rangle \mid x \in U\right\}$$

where the component functions $\mu_{F(e_2)}, \nu_{F(e_2)}: U \to [0,1]$ and $\delta_{F(e_2)}: U \to [-1,0]$ are defined piecewise by:

$$\mu_{F(e_2)}(x) := \begin{cases} 0.8 & \text{if } x = x_1 \\ 0.4 & \text{if } x = x_2 \\ 0.3 & \text{if } x = x_3, \end{cases} \quad \nu_{F(e_2)}(x) := \begin{cases} 0.3 & \text{if } x = x_1 \\ 0.6 & \text{if } x = x_2 \\ 0.8 & \text{if } x = x_3, \end{cases} \quad \text{and} \quad \delta_{F(e_2)}(x) := \begin{cases} -0.1 & \text{if } x = x_1 \\ -0.4 & \text{if } x = x_2 \\ -0.7 & \text{if } x = x_3. \end{cases}$$

It is evident that for all $x_i \in U$ and $e_j \in E$ where $j \in \{1,2\}$, the negative membership values satisfy the core boundary constraint $-1 \le \delta_{F(e_j)}(x_i) \le 0$.
The second step. *We verify that the chosen exponents $(m,n) = (2,3)$ are fully compatible with our defined membership functions by evaluating each element in U:*

- *For parameter e_1:*
  - *At x_1: $(\mu_{F(e_1)}(x_1))^2 + (\nu_{F(e_1)}(x_1))^3 = (0.6)^2 + (0.5)^3 = 0.36 + 0.125 = 0.485 \in [0,1]$.*

- *At* x_2: $(\mu_{F(e_1)}(x_2))^2 + (\nu_{F(e_1)}(x_2))^3 = (0.7)^2 + (0.4)^3 = 0.49 + 0.064 = 0.554 \in [0,1]$.
- *At* x_3: $(\mu_{F(e_1)}(x_3))^2 + (\nu_{F(e_1)}(x_3))^3 = (0.5)^2 + (0.7)^3 = 0.25 + 0.343 = 0.593 \in [0,1]$.

- *For parameter* e_2:
  - *At* x_1: $(\mu_{F(e_2)}(x_1))^2 + (\nu_{F(e_2)}(x_1))^3 = (0.8)^2 + (0.3)^3 = 0.64 + 0.027 = 0.667 \in [0,1]$.
  - *At* x_2: $(\mu_{F(e_2)}(x_2))^2 + (\nu_{F(e_2)}(x_2))^3 = (0.4)^2 + (0.6)^3 = 0.16 + 0.216 = 0.376 \in [0,1]$.
  - *At* x_3: $(\mu_{F(e_2)}(x_3))^2 + (\nu_{F(e_2)}(x_3))^3 = (0.3)^2 + (0.8)^3 = 0.09 + 0.512 = 0.602 \in [0,1]$.

Since the constraint holds for all elements, the mathematical notations simplify to the following exponential forms:

$$\mu^2_{F(e_1)}(x_1) = 0.36, \quad \nu^3_{F(e_1)}(x_1) = 0.125,$$
$$\mu^2_{F(e_2)}(x_1) = 0.64, \quad \nu^3_{F(e_2)}(x_1) = 0.027.$$

Hence, (F,E) *is a well-defined tripolar* $(2,3)$*-fuzzy soft set over* U*.*

Example 3.2. *Let* $U = \{x_1, x_2\}$ *be the initial universe and* $E = \{e\}$ *be a set of parameters. We fix the exponents as* $m = 2$ *and* $n = 3$*. We define a pair* (G,E) *over* U*, where the image* $G(e)$ *is given piecewise by the component functions* $\mu_{G(e)}, \nu_{G(e)} : U \to [0,1]$ *and* $\delta_{G(e)} : U \to [-1,0]$ *as follows:*

$$\mu_{G(e)}(x) := \begin{cases} 0.9 & \text{if } x = x_1 \\ 0.5 & \text{if } x = x_2, \end{cases} \quad \nu_{G(e)}(x) := \begin{cases} 0.6 & \text{if } x = x_1 \\ 0.4 & \text{if } x = x_2, \end{cases} \quad \text{and} \quad \delta_{G(e)}(x) := \begin{cases} -0.3 & \text{if } x = x_1 \\ -1.5 & \text{if } x = x_2. \end{cases}$$

We will now demonstrate that (G,E) *fails to be a well-defined tripolar* $(2,3)$*-fuzzy soft set over* U *due to the following structural violations:*

(1) *For the element* $x_1 \in U$*, we evaluate the power sum of its positive and non-membership degrees under the exponents* $(m,n) = (2,3)$*:*

$$(\mu_{G(e)}(x_1))^2 + (\nu_{G(e)}(x_1))^3 = (0.9)^2 + (0.6)^3 = 0.81 + 0.216 = 1.026.$$

Since $1.026 > 1$*, it strictly violates the core constraint that* $0 \le (\mu_{G(e)}(x))^2 + (\nu_{G(e)}(x))^3 \le 1$*.*

(2) *For the element* $x_2 \in U$*, we consider its negative membership value:*

$$\delta_{G(e)}(x_2) = -1.5.$$

Since $-1.5 < -1$*, this value completely falls outside the mandatory interval constraint* $-1 \le \delta_{G(e)}(x) \le 0$*.*

Consequently, because of these mathematical contradictions, (G,E) *is* ***not*** *a tripolar* $(2,3)$*-fuzzy soft set over* U*.*

Note that. By Definition 3.1, a tripolar $(1,1)$-fuzzy soft set is exactly a tripolar fuzzy soft set. This implies that the concept of a tripolar (m,n)-fuzzy soft set is a generalization of the classic tripolar fuzzy soft set.

We denote by $\mathcal{TFSS}^{mn}(U,E)$ the set of all tripolar (m,n)-fuzzy soft sets over U. In the following definition, we introduce the concepts of inclusion relation and intersection operation on $\mathcal{TFSS}^{mn}(U,E)$.

Definition 3.2. *Let (F,E) and (G,E) be two tripolar (m,n)-fuzzy soft sets over U.*

(1) *We say that (F,E) is a tripolar (m,n)-fuzzy soft subset of (G,E), denoted by $(F,E)\,\tilde{\subseteq}\,(G,E)$, if for all $e\in E$ and $x\in U$, the following conditions hold:*

$$\begin{aligned}\mu^m_{F(e)}(x) &\le \mu^m_{G(e)}(x),\\ \nu^n_{F(e)}(x) &\ge \nu^n_{G(e)}(x),\\ \delta_{F(e)}(x) &\ge \delta_{G(e)}(x).\end{aligned}$$

(2) *Two tripolar (m,n)-fuzzy soft sets (F,E) and (G,E) are said to be equal, denoted by $(F,E)=(G,E)$, if $(F,E)\,\tilde{\subseteq}\,(G,E)$ and $(G,E)\,\tilde{\subseteq}\,(F,E)$.*

(3) *The intersection of (F,E) and (G,E), denoted by $(F,E)\,\tilde{\cap}\,(G,E)$, is defined as a tripolar (m,n)-fuzzy soft set (H,E) over U, where for each $e\in E$ and $x\in U$:*

$$\begin{aligned}\mu^m_{H(e)}(x) &:= \min\{\mu^m_{F(e)}(x),\mu^m_{G(e)}(x)\},\\ \nu^n_{H(e)}(x) &:= \max\{\nu^n_{F(e)}(x),\nu^n_{G(e)}(x)\},\\ \delta_{H(e)}(x) &:= \max\{\delta_{F(e)}(x),\delta_{G(e)}(x)\}\end{aligned}$$

for all $x\in U$.

(4) *The union of (F,E) and (G,E), denoted by $(F,E)\,\tilde{\cup}\,(G,E)$, is defined as a tripolar (m,n)-fuzzy soft set (H,E) over U, where for each $e\in E$ and $x\in U$:*

$$\begin{aligned}\mu^m_{H(e)}(x) &:= \max\{\mu^m_{H(e)}(x),\mu^m_{H(e)}(x)\},\\ \nu^n_{H(e)}(x) &:= \min\{\nu^n_{H(e)}(x),\nu^n_{H(e)}(x)\},\\ \delta_{H(e)}(x) &:= \min\{\delta_{H(e)}(x),\delta_{H(e)}(x)\}\end{aligned}$$

for all $x\in U$.

From the above definition, it is easy to verify that the relation $\tilde{\subseteq}$ is a partial order. Consequently, the structure $\langle \mathcal{TIFSS}^{mn}(U,E),\tilde{\subseteq}\rangle$ is a partially ordered set.

Let $\mathbf{S}$ be an ordered semigroup and let $a\in S$. We define the set $\mathbf{S}_a$ by

$$\mathbf{S}_a := \{(x,y)\in S\times S : a\preceq xy\}.$$

We now introduce additional operations on $\mathcal{TFSS}^{mn}(S,E)$ as follows.

Definition 3.3. *Let $\mathbf{S}$ be an ordered semigroup. Let (F,E) and (G,E) be two tripolar (m,n)-fuzzy soft sets over S. The product of (F,E) and (G,E), denoted by $(F,E)\tilde{\diamond}(G,E)$, is defined as a tripolar fuzzy (m,n)-soft set (H,E) over S, where for each $e\in E$ and $x\in S$:*

$$\mu^m_{H(e)}(x) := \begin{cases}\bigvee\limits_{(y,z)\in\mathbf{S}_x}\{\min\{\mu^m_{F(e)}(y),\mu^m_{G(e)}(z)\}\} & \text{if } \mathbf{S}_x\neq\emptyset,\\ 0 & \text{otherwise,}\end{cases}$$

$$\nu^n_{H(e)}(x) := \begin{cases} \bigwedge\limits_{(y,z)\in \mathbf{S}_x} \{\max\{\nu^n_{F(e)}(y), \nu^n_{G(e)}(z)\}\} & \textit{if } \mathbf{S}_x \neq \emptyset, \\ 1 & \textit{otherwise}, \end{cases}$$

$$\delta_{H(e)}(x) := \begin{cases} \bigwedge\limits_{(y,z)\in \mathbf{S}_x} \{\max\{\delta_{F(e)}(y), \delta_{G(e)}(z)\}\} & \textit{if } \mathbf{S}_x \neq \emptyset, \\ 0 & \textit{otherwise}. \end{cases}$$

By Definitions 3.3(1) and 3.4, it is straightforward to verify that the operation $\tilde{\diamond}$ is compatible with the inclusion relation $\tilde{\subseteq}$. That is, for any $(A, E), (B, E), (C, E) \in \mathcal{TFSS}(S, E)$, if $(A, E)\tilde{\subseteq}(B, E)$, then

$$(A, E)\tilde{\diamond}(C, E)\tilde{\subseteq}(B, E)\tilde{\diamond}(C, E) \quad \text{and} \quad (C, E)\tilde{\diamond}(A, E)\tilde{\subseteq}(C, E)\tilde{\diamond}(B, E).$$

Consequently, the structure $\langle \mathcal{TFSS}^{mn}(S, E), \tilde{\diamond}, \tilde{\subseteq}\rangle$ forms an ordered semigroup, which is called a tripolar (m, n)-fuzzy soft ordered semigroup.

Definition 3.4. *Let* $\mathbf{S}$ *be an ordered semigroup and let* (F, E) *be a tripolar* (m, n)*-fuzzy soft set over* S*. Then* (F, E) *is called a* (m, n)*-tripolar fuzzy soft subsemigroup over* $\mathbf{S}$ *if for all* $e \in E$ *and* $x, y \in S$*, the following conditions hold:*

$$\mu^m_{F(e)}(xy) \geq \min\{\mu^m_{F(e)}(x), \mu^m_{F(e)}(y)\},$$
$$\nu^n_{F(e)}(xy) \leq \max\{\nu^n_{F(e)}(x), \nu^n_{F(e)}(y)\},$$
$$\delta_{F(e)}(xy) \leq \max\{\delta_{F(e)}(x), \delta_{F(e)}(y)\}.$$

Example 3.3. *Let* $S = \{a, b, c\}$ *be a set with the multiplication defined by the following Cayley table:*

$*$	a	b	c
a	a	a	a
b	a	a	a
c	a	a	b

We define an order relation $\preceq$ *on* S *as follows:*

$$\preceq := \{(a, a), (b, b), (c, c), (a, b)\}.$$

Then $\mathbf{S} := \langle S, *, \preceq\rangle$ *is an ordered semigroup. Let* $E = \{e\}$ *be a parameter set. We fix the exponents as natural numbers* $m = 2$ *and* $n = 3$*.*

We construct a tripolar $(2, 3)$*-fuzzy soft set* (F, E) *over* $\mathbf{S}$ *by defining the component functions piecewise as follows:*

$$\mu_{F(e)}(x) := \begin{cases} 0.894 & \textit{if } x = a \\ 0.707 & \textit{if } x = b \\ 0.548 & \textit{if } x = c, \end{cases} \quad \nu_{F(e)}(x) := \begin{cases} 0.464 & \textit{if } x = a \\ 0.669 & \textit{if } x = b \\ 0.737 & \textit{if } x = c, \end{cases} \quad \textit{and} \quad \delta_{F(e)}(x) := \begin{cases} -0.6 & \textit{if } x = a \\ -0.4 & \textit{if } x = b \\ -0.2 & \textit{if } x = c. \end{cases}$$

By evaluating these mappings under the exponents $(m,n) = (2,3)$, the simplified degrees $\mu^2_{F(e)}$ and $\nu^3_{F(e)}$ are obtained as:

$$\mu^2_{F(e)}(a) = 0.8, \quad \mu^2_{F(e)}(b) = 0.5, \quad \mu^2_{F(e)}(c) = 0.3,$$
$$\nu^3_{F(e)}(a) = 0.1, \quad \nu^3_{F(e)}(b) = 0.3, \quad \nu^3_{F(e)}(c) = 0.4.$$

It is easy to verify that (F,E) is a well-defined (m,n)-tripolar fuzzy soft subsemigroup over $\mathbf{S}$.

Definition 3.5. *Let $\mathbf{S}$ be an ordered semigroup and let (F,E) be a tripolar (m,n)-fuzzy soft set over S. Then (F,E) is called a tripolar (m,n)-fuzzy soft left ideal over $\mathbf{S}$ if for all $e \in E$ and $x, y \in S$, the following conditions hold:*

(1) *The left ideal condition:*

$$\mu^m_{F(e)}(xy) \geq \mu^m_{F(e)}(y),$$
$$\nu^n_{F(e)}(xy) \leq \nu^n_{F(e)}(y),$$
$$\delta_{F(e)}(xy) \leq \delta_{F(e)}(y).$$

(2) *The order condition: If $x \leq y$, then*

$$\mu^m_{F(e)}(x) \geq \mu^m_{F(e)}(y),$$
$$\nu^n_{F(e)}(x) \leq \nu^n_{F(e)}(y),$$
$$\delta_{F(e)}(x) \leq \delta_{F(e)}(y).$$

Example 3.4. *Let $S = \{a, b, c\}$ be a set with the multiplication defined by the following Cayley table:*

$*$	a	b	c
a	a	a	a
b	a	a	a
c	a	a	b

We define an order relation $\leq$ on S as follows:

$$\leq\ := \{(a,a), (b,b), (c,c), (a,b)\}.$$

Then $\mathbf{S} := \langle S, *, \leq \rangle$ is an ordered semigroup. Let $E = \{e\}$ be a parameter set. We fix the exponents as $m = 2$ and $n = 3$.

We construct a tripolar $(2,3)$-fuzzy soft set (F,E) over S by defining the component functions piecewise as follows:

$$\mu_{F(e)}(x) := \begin{cases} 0.894 & \text{if } x = a \\ 0.707 & \text{if } x = b \\ 0.707 & \text{if } x = c, \end{cases} \quad \nu_{F(e)}(x) := \begin{cases} 0.464 & \text{if } x = a \\ 0.669 & \text{if } x = b \\ 0.669 & \text{if } x = c, \end{cases} \quad \text{and} \quad \delta_{F(e)}(x) := \begin{cases} -0.5 & \text{if } x = a \\ -0.2 & \text{if } x = b \\ -0.2 & \text{if } x = c. \end{cases}$$

By evaluating these mappings under the exponents $(m,n)=(2,3)$, the simplified degrees $\mu^2_{F(e)}$ and $\nu^3_{F(e)}$ are obtained as:

$$\mu^2_{F(e)}(a)=0.8,\quad \mu^2_{F(e)}(b)=0.5,\quad \mu^2_{F(e)}(c)=0.5,$$
$$\nu^3_{F(e)}(a)=0.1,\quad \nu^3_{F(e)}(b)=0.3,\quad \nu^3_{F(e)}(c)=0.3.$$

It is clear that $0\le \mu^2_{F(e)}(x)+\nu^3_{F(e)}(x)\le 1$ and $-1\le \delta_{F(e)}(x)\le 0$ hold for all $x\in S$.

It is easy to verify that (F,E) is a tripolar (m,n)-fuzzy soft left ideal over $\mathbf{S}$.

Definition 3.6. *Let $\mathbf{S}$ be an ordered semigroup and let (F,E) be a tripolar (m,n)-fuzzy soft set over S. Then (F,E) is called a tripolar (m,n)-fuzzy soft right ideal over $\mathbf{S}$ if for all $e\in E$ and $x,y\in S$, the following conditions hold:*

(1) *The right ideal condition:*

$$\mu^m_{F(e)}(xy)\ge \mu^m_{F(e)}(x),$$
$$\nu^n_{F(e)}(xy)\le \nu^n_{F(e)}(x),$$
$$\delta_{F(e)}(xy)\le \delta_{F(e)}(x).$$

(2) *The order condition: If $x\le y$, then*

$$\mu^m_{F(e)}(x)\ge \mu^m_{F(e)}(y),$$
$$\nu^n_{F(e)}(x)\le \nu^n_{F(e)}(y),$$
$$\delta_{F(e)}(x)\le \delta_{F(e)}(y).$$

Example 3.5. *Let $S=\{a,b,c\}$ be a set with the multiplication defined by the following Cayley table:*

$*$	a	b	c
a	a	a	a
b	a	a	a
c	a	a	b

We define an order relation $\preceq$ on S as follows:

$$\preceq\, := \{(a,a),(b,b),(c,c),(a,b)\}.$$

Then $\mathbf{S}:=\langle S,*,\preceq\rangle$ is an ordered semigroup. Let $E=\{e\}$ be a parameter set. We fix the exponents as $m=2$, and $n=3$.

We construct a tripolar $(2,3)$-fuzzy soft set (F,E) over $\mathbf{S}$ by defining the component functions piecewise as follows:

$$\mu_{F(e)}(x):=\begin{cases}0.894 & \text{if } x=a\\ 0.707 & \text{if } x=b\\ 0.548 & \text{if } x=c,\end{cases}\quad \nu_{F(e)}(x):=\begin{cases}0.464 & \text{if } x=a\\ 0.669 & \text{if } x=b\\ 0.737 & \text{if } x=c,\end{cases}\quad \text{and}\quad \delta_{F(e)}(x):=\begin{cases}-0.6 & \text{if } x=a\\ -0.4 & \text{if } x=b\\ -0.2 & \text{if } x=c.\end{cases}$$

By evaluating these mappings under the fixed exponents $(m,n) = (2,3)$, the simplified degrees $\mu^2_{F(e)}$, $\nu^3_{F(e)}$, and $\delta_{F(e)}$ are obtained as:

$$\mu^2_{F(e)}(a) = 0.8, \quad \mu^2_{F(e)}(b) = 0.5, \quad \mu^2_{F(e)}(c) = 0.3,$$
$$\nu^3_{F(e)}(a) = 0.1, \quad \nu^3_{F(e)}(b) = 0.3, \quad \nu^3_{F(e)}(c) = 0.4,$$
$$\delta_{F(e)}(a) = -0.6, \quad \delta_{F(e)}(b) = -0.4, \quad \delta_{F(e)}(c) = -0.2.$$

It is clear that $0 \leq \mu^2_{F(e)}(x) + \nu^3_{F(e)}(x) \leq 1$ and $-1 \leq \delta_{F(e)}(x) \leq 0$ hold for all $x \in S$.

It is easy to calculate that (F,E) is a tripolar (m,n)-fuzzy soft right ideal over **S**.

Definition 3.7. *Let A be a subset of S and let E be a set of parameters. The characteristic tripolar fuzzy soft set over S, denoted by (χ_A, E), is defined by the mapping $\chi_A : E \to \mathcal{TF}(S)$ where for each $e \in E$, the tripolar (m,n)-fuzzy soft set $\chi_A(e)$ is given by:*

$$\chi_A(e) := \{\langle x, \mu_{\chi_A(e)}(x), \nu_{\chi_A(e)}(x), \delta_{\chi_A(e)}(x)\rangle : x \in S\}$$

The positive, neutral, and negative membership functions are defined for each $x \in S$ and $e \in E$ as follows:

(1) ***Positive membership function:*** $\mu_{\chi_A(e)}(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$

(2) ***Neutral membership function:*** $\nu_{\chi_A(e)}(x) := \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{otherwise.} \end{cases}$

(3) ***Negative membership function:*** $\delta_{\chi_A(e)}(x) := \begin{cases} -1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$

If $A = S$, then the characteristic tripolar fuzzy soft set (χ_A, E) is denoted by $(\mathbf{1}, E)$. For each $e \in E$, the tripolar fuzzy set $\mathbf{1}(e)$ is defined by:

$$\mu_{\mathbf{1}(e)}(x) := 1, \quad \nu_{\mathbf{1}(e)}(x) := 0, \quad \text{and} \quad \delta_{\mathbf{1}(e)}(x) := -1$$

for all $x \in S$.

Note that the characteristic tripolar fuzzy soft set (χ_A, E) also constitutes a tripolar (m,n)-fuzzy soft set over **S**.

Definition 3.8. *Let* **S** *be an ordered semigroup and let (F,E) be a tripolar (m,n)-fuzzy soft set over S. Then (F,E) is called tripolar fuzzy soft quasi-ideal over* **S** *if for all $e \in E$ and $x \in S$, the following conditions hold:*

(1) *The quasi-ideal condition:*

$$[(F,E)\tilde{\circ}(\mathbf{1},E)] \,\tilde{\cap}\, [(\mathbf{1},E)\tilde{\circ}(F,E)] \,\tilde{\subseteq}(F,E).$$

(2) *The order condition: If $x \leq y$, then*

$$\mu^m_{F(e)}(x) \geq \mu^m_{F(e)}(y),$$

$$\nu^n_{F(e)}(x) \leq \nu^n_{F(e)}(y),$$
$$\delta_{F(e)}(x) \leq \delta_{F(e)}(y).$$

Note that. The condition (1) of Definition 3.8, means that, for all $e \in E$ and $a \in S$,

$$\min\{\mu^m_{(F\widetilde{\diamond}\mathbf{1})(e)}(a), \mu^m_{(\mathbf{1}\widetilde{\diamond}F)(e)}(a)\} \leq \mu^m_{F(e)}(a),$$
$$\max\{\nu^n_{(F\widetilde{\diamond}\mathbf{1})(e)}(a), \nu^n_{(\mathbf{1}\widetilde{\diamond}F)(e)}(a)\} \geq \nu^n_{F(e)}(a),$$
$$\max\{\delta_{(F\widetilde{\diamond}\mathbf{1})(e)}(a), \delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(a)\} \geq \delta_{F(e)}(a).$$

Example 3.6. *Let $S = \{a, b, c\}$ be a set with the multiplication defined by the following Cayley table:*

$*$	a	b	c
a	a	a	a
b	a	a	a
c	a	a	b

We define an order relation $\preceq$ on S as follows:

$$\preceq := \{(a,a), (b,b), (c,c), (a,b)\}.$$

Then $\mathbf{S} := \langle S, *, \preceq\rangle$ is an ordered semigroup. Let $E = \{e\}$ be a parameter set. We fix the exponents as natural numbers $m = 2$ and $n = 3$.

We construct a tripolar $(2,3)$-fuzzy soft set (F, E) over $\mathbf{S}$ by defining the component functions piecewise as follows:

$$\mu_{F(e)}(x) := \begin{cases} 0.894 & \text{if } x = a \\ 0.707 & \text{if } x = b \\ 0.707 & \text{if } x = c, \end{cases} \quad \nu_{F(e)}(x) := \begin{cases} 0.464 & \text{if } x = a \\ 0.669 & \text{if } x = b \\ 0.669 & \text{if } x = c, \end{cases} \quad \text{and} \quad \delta_{F(e)}(x) := \begin{cases} -0.5 & \text{if } x = a \\ -0.2 & \text{if } x = b \\ -0.2 & \text{if } x = c. \end{cases}$$

By evaluating these mappings under the exponents $(m,n) = (2,3)$, the simplified degrees $\mu^2_{F(e)}$ and $\nu^3_{F(e)}$ are obtained as:

$$\mu^2_{F(e)}(a) = 0.8, \quad \mu^2_{F(e)}(b) = 0.5, \quad \mu^2_{F(e)}(c) = 0.5,$$
$$\nu^3_{F(e)}(a) = 0.1, \quad \nu^3_{F(e)}(b) = 0.3, \quad \nu^3_{F(e)}(c) = 0.3.$$

It is easy to verify that $0 \leq \mu^2_{F(e)}(x) + \nu^3_{F(e)}(x) \leq 1$ and $-1 \leq \delta_{F(e)}(x) \leq 0$ hold for all $x \in S$. It is easy to verify that (F, E) is a tripolar (m,n)-fuzzy soft quasi-ideal over $\mathbf{S}$.

Definition 3.9. *Let $\mathbf{S}$ be an ordered semigroup and let (F, E) be a tripolar (m,n)-fuzzy soft subsemigroup over $\mathbf{S}$. Then (F, E) is called tripolar (m,n)-fuzzy soft bi-ideal over $\mathbf{S}$ if for all $e \in E$ and $x, y, z \in S$, the following conditions hold:*

(1) *The bi-ideal condition:*

$$\mu^m_{F(e)}(xyz) \geq \min\{\mu^m_{F(e)}(x), \mu^m_{F(e)}(z)\},$$
$$\nu^n_{F(e)}(xyz) \leq \max\{\nu^n_{F(e)}(x), \nu^n_{F(e)}(z)\},$$

$$\delta_{F(e)}(xyz) \leq \max\{\delta_{F(e)}(x), \delta_{F(e)}(z)\}.$$

(2) *The order condition: If $x \leq y$, then*

$$\mu^m_{F(e)}(x) \geq \mu^m_{F(e)}(y),$$
$$\nu^n_{F(e)}(x) \leq \nu^n_{F(e)}(y),$$
$$\delta_{F(e)}(x) \leq \delta_{F(e)}(y).$$

Example 3.7. *Let $S = \{a, b, c\}$ be a set with the multiplication defined by the following Cayley table:*

$*$	a	b	c
a	a	a	a
b	a	a	a
c	a	a	b

We define an order relation $\leq$ on S as follows:

$$\leq := \{(a, a), (b, b), (c, c), (a, b)\}.$$

Then $\mathbf{S} := \langle S, *, \leq\rangle$ is an ordered semigroup. Let $E = \{e\}$ be a parameter set. We fix the exponents as natural numbers $m = 2$ and $n = 3$.

We construct a tripolar $(2, 3)$-fuzzy soft set (F, E) over $\mathbf{S}$ by defining the component functions piecewise as follows:

$$\mu_{F(e)}(x) := \begin{cases} 0.894 & \text{if } x = a \\ 0.707 & \text{if } x = b \\ 0.707 & \text{if } x = c, \end{cases} \quad \nu_{F(e)}(x) := \begin{cases} 0.316 & \text{if } x = a \\ 0.669 & \text{if } x = b \\ 0.737 & \text{if } x = c, \end{cases} \quad \text{and} \quad \delta_{F(e)}(x) := \begin{cases} -0.6 & \text{if } x = a \\ -0.4 & \text{if } x = b \\ -0.2 & \text{if } x = c. \end{cases}$$

By evaluating these mappings under the exponents $(m, n) = (2, 3)$, the simplified degrees $\mu^2_{F(e)}$ and $\nu^3_{F(e)}$ are obtained as:

$$\mu^2_{F(e)}(a) = 0.8, \quad \mu^2_{F(e)}(b) = 0.5, \quad \mu^2_{F(e)}(c) = 0.5,$$
$$\nu^3_{F(e)}(a) = 0.1, \quad \nu^3_{F(e)}(b) = 0.3, \quad \nu^3_{F(e)}(c) = 0.4.$$

It is clear that $0 \leq \mu^2_{F(e)}(x) + \nu^3_{F(e)}(x) \leq 1$ and $-1 \leq \delta_{F(e)}(x) \leq 0$ hold for all $x \in S$. Also, the order condition (2) is satisfied since $a \leq b$ implies $\mu^2_{F(e)}(a) \geq \mu^2_{F(e)}(b)$, $\nu^3_{F(e)}(a) \leq \nu^3_{F(e)}(b)$, and $\delta_{F(e)}(a) \leq \delta_{F(e)}(b)$. It is easy to calculate that (F, E) is a well-defined tripolar (m, n)-fuzzy soft bi-ideal over $\mathbf{S}$.

We now investigate the structural relationships among these different types of tripolar (m, n)-fuzzy soft ideals over an ordered semigroup.

Lemma 3.1. *Let $\mathbf{S}$ be an ordered semigroup, and (F, E) a tripolar (m, n)-fuzzy soft set over S. If (F, E) a tripolar (m, n)-fuzzy soft left ideal over $\mathbf{S}$, then (F, E) is a tripolar (m, n)-fuzzy soft quasi-ideal over $\mathbf{S}$.*

Proof. Let (F, E) be a tripolar (m, n)-fuzzy soft left ideal over $\mathbf{S}$. For all $e \in E$, and $x \in S$. We consider two cases as follows: If $\mathbf{S}_x = \emptyset$, then we have

$$\min\{\mu^m_{(F\widetilde{\diamond}\mathbf{1})(e)}(x), \mu^m_{(\mathbf{1}\widetilde{\diamond}F)(e)}(x)\} = 0 \leq \mu^m_{F(e)}(x).$$

Similarly, we have

$$\max\{\nu^n_{(F\widetilde{\diamond}\mathbf{1})(e)}(x), \nu^n_{(\mathbf{1}\widetilde{\diamond}F)(e)}(x)\} = 1 \geq \nu^n_{F(e)}(x),$$

and

$$\max\{\delta_{(F\widetilde{\diamond}\mathbf{1})(e)}(x), \delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(x)\} = 0 \geq \delta_{F(e)}(x).$$

If $\mathbf{S}_x \neq \emptyset$, then for all $(y, z) \in \mathbf{S}_x$, we have

$$\begin{aligned}
\mu^m_{F(e)}(x) &\geq \mu^m_{F(e)}(yz) \\
&\geq \mu^m_{F(e)}(y) \\
&= \min\{\mu^m_{F(e)}(y), \mu^m_{\mathbf{1}(e)}(z)\} \\
&= \bigvee_{(y,z)\in\mathbf{S}_x} \left\{\min\{\mu^m_{F(e)}(y), \mu^m_{\mathbf{1}(e)}(z)\}\right\} \\
&= \mu^m_{(F\widetilde{\diamond}\mathbf{1})(e)}(x) \\
&= \min\{\mu^m_{(F\widetilde{\diamond}\mathbf{1})(e)}(x), \mu^m_{(\mathbf{1}\widetilde{\diamond}F)(e)}(x)\}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\nu^n_{F(e)}(x) &\leq \nu^n_{F(e)}(yz) \\
&\leq \nu^n_{F(e)}(y) \\
&= \max\{\nu^n_{F(e)}(y), \nu^n_{\mathbf{1}(e)}(z)\} \\
&= \bigwedge_{(y,z)\in\mathbf{S}_x} \left\{\max\{\nu^n_{F(e)}(y), \nu^n_{\mathbf{1}(e)}(z)\}\right\} \\
&= \nu^n_{(F\widetilde{\diamond}\mathbf{1})(e)}(x) \\
&= \max\{\nu^n_{(F\widetilde{\diamond}\mathbf{1})(e)}(x), \nu^n_{(\mathbf{1}\widetilde{\diamond}F)(e)}(x)\},
\end{aligned}$$

and

$$\begin{aligned}
\delta_{F(e)}(x) &\leq \delta_{F(e)}(yz) \\
&\leq \delta_{F(e)}(y) \\
&= \max\{\delta_{F(e)}(y), \delta_{\mathbf{1}(e)}(z)\} \\
&= \bigwedge_{(y,z)\in\mathbf{S}_x} \left\{\max\{\delta_{F(e)}(y), \delta_{\mathbf{1}(e)}(z)\}\right\} \\
&= \delta_{(F\widetilde{\diamond}\mathbf{1})(e)}(x) \\
&= \max\{\delta_{(F\widetilde{\diamond}\mathbf{1})(e)}(x), \delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(x)\}.
\end{aligned}$$

For any two cases, we obtain (F, E) is a tripolar (m, n)-fuzzy quasi-ideal over $\mathbf{S}$. $\square$

Similarly to Lemma 3.1, we obtain the following result.

Lemma 3.2. *Let $\mathbf{S}$ be an ordered semigroup, and (F, E) a tripolar (m, n)-fuzzy soft set over S. If (F, E) a tripolar (m, n)-fuzzy soft right ideal over $\mathbf{S}$, then (F, E) is a tripolar (m, n)-fuzzy soft quasi-ideal over $\mathbf{S}$.*

Combining Lemma 3.1 and 3.2, we have the following corollary.

Corollary 3.1. *Let $\mathbf{S}$ be an ordered semigroup and let (F, E) be a tripolar (m, n)-fuzzy soft set over S. If (F, E) is a tripolar (m, n)-fuzzy soft ideal over $\mathbf{S}$, then (F, E) is a tripolar (m, n)-fuzzy soft quasi-ideal over $\mathbf{S}$.*

Proof. By Lemma 3.1 and 3.2. $\square$

The converse of Corollary 3.1, in general, does not hold, as demonstrated by the following counterexample.

Example 3.8. *Let $S = \{a, b, c\}$ be a set with the multiplication defined by the following Cayley table:*

$*$	a	b	c
a	a	a	a
b	a	a	a
c	a	a	b

We define an order relation $\preceq$ on S as follows:

$$\preceq := \{(a, a), (b, b), (c, c), (a, b)\}.$$

Then $\mathbf{S} := \langle S, *, \preceq \rangle$ is an ordered semigroup. Let $E = \{e\}$ be a parameter set. We fix the exponents as natural numbers $m = 2$ and $n = 3$.

We construct a tripolar $(2, 3)$-fuzzy soft set (F, E) over S by defining the component functions piecewise as follows:

$$\mu_{F(e)}(x) := \begin{cases} 0.894 & \text{if } x = a \\ 0.707 & \text{if } x = b \\ 0.707 & \text{if } x = c, \end{cases} \quad \nu_{F(e)}(x) := \begin{cases} 0.464 & \text{if } x = a \\ 0.669 & \text{if } x = b \\ 0.669 & \text{if } x = c, \end{cases} \quad \text{and} \quad \delta_{F(e)}(x) := \begin{cases} -0.5 & \text{if } x = a \\ -0.2 & \text{if } x = b \\ -0.2 & \text{if } x = c. \end{cases}$$

By evaluating these mappings under the exponents $(m, n) = (2, 3)$, the simplified degrees $\mu^2_{F(e)}$ and $\nu^3_{F(e)}$ are obtained as:

$$\mu^2_{F(e)}(a) = 0.8, \quad \mu^2_{F(e)}(b) = 0.5, \quad \mu^2_{F(e)}(c) = 0.5,$$
$$\nu^3_{F(e)}(a) = 0.1, \quad \nu^3_{F(e)}(b) = 0.3, \quad \nu^3_{F(e)}(c) = 0.3.$$

As shown in Example 3.6 previously, (F, E) is a tripolar (m, n)-fuzzy soft quasi-ideal over $\mathbf{S}$. However, (F, E) is not a tripolar (m, n)-fuzzy soft right ideal over $\mathbf{S}$ because

$$\mu^2_{F(e)}(c * c) = \mu^2_{F(e)}(b) = 0.5 \not\geq 0.8 = \mu^2_{F(e)}(c).$$

Therefore, (F, E) is a tripolar (m, n)-fuzzy soft quasi-ideal, but it fails to be a tripolar (m, n)-fuzzy soft ideal over $\mathbf{S}$.

We now consider the connection between tripolar (m, n)-fuzzy soft quasi-ideals and tripolar (m, n)-fuzzy soft bi-ideals.

Lemma 3.3. *Let* $\mathbf{S}$ *be an ordered semigroup and let (F, E) be a tripolar (m, n)-fuzzy soft set over* $\mathbf{S}$. *If (F, E) is a tripolar (m, n)-fuzzy soft quasi-ideal over* $\mathbf{S}$, *then it is a tripolar (m, n)-fuzzy soft bi-ideal over* $\mathbf{S}$.

Proof. Assume that (F, E) is a tripolar (m, n)-fuzzy soft quasi-ideal over $\mathbf{S}$. Let $e \in E$ and $x, y, z \in \mathbf{S}$.

First, we show that (F, E) is a tripolar (m, n)-fuzzy soft subsemigroup over $\mathbf{S}$. Since $xy \in S$, by the definition of a tripolar (m, n)-fuzzy soft quasi-ideal, we observe that:

$$\begin{aligned}
\mu^m_{F(e)}(xy) &\geq \min\Big\{\mu^m_{(F\widetilde{\diamond}\mathbf{1})(e)}(xy), \mu^m_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy)\Big\} \\
&= \min\Bigg\{\bigvee_{(a,b)\in\mathbf{S}_{xy}} \min\Big\{\mu^m_{F(e)}(a), \mu^m_{\mathbf{1}(e)}(b)\Big\}, \bigvee_{(u,v)\in\mathbf{S}_{xy}} \min\Big\{\mu^m_{\mathbf{1}(e)}(u), \mu^m_{F(e)}(v)\Big\}\Bigg\} \\
&\geq \min\Big\{\min\Big\{\mu^m_{F(e)}(x), \mu^m_{\mathbf{1}(e)}(y)\Big\}, \min\Big\{\mu^m_{\mathbf{1}(e)}(x), \mu^m_{F(e)}(y)\Big\}\Big\} \\
&= \min\Big\{\min\Big\{\mu^m_{F(e)}(x), 1\Big\}, \min\Big\{1, \mu^m_{F(e)}(y)\Big\}\Big\} \\
&= \min\Big\{\mu^m_{F(e)}(x), \mu^m_{F(e)}(y)\Big\}.
\end{aligned}$$

Similarly, we have that

$$\begin{aligned}
\nu^n_{F(e)}(xy) &\leq \max\Big\{\nu^n_{(F\widetilde{\diamond}\mathbf{1})(e)}(xy), \nu^n_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy)\Big\} \\
&= \max\Bigg\{\bigwedge_{(a,b)\in\mathbf{S}_{xy}} \max\Big\{\nu^n_{F(e)}(a), \nu^n_{\mathbf{1}(e)}(b)\Big\}, \bigwedge_{(u,v)\in\mathbf{S}_{xy}} \max\Big\{\nu^n_{\mathbf{1}(e)}(u), \nu^n_{F(e)}(v)\Big\}\Bigg\} \\
&\leq \max\Big\{\max\Big\{\nu^n_{F(e)}(x), \nu^n_{\mathbf{1}(e)}(y)\Big\}, \max\Big\{\nu^n_{\mathbf{1}(e)}(x), \nu^n_{F(e)}(y)\Big\}\Big\} \\
&= \max\Big\{\max\Big\{\nu^n_{F(e)}(x), 0\Big\}, \max\Big\{0, \nu^n_{F(e)}(y)\Big\}\Big\} \\
&= \max\Big\{\nu^n_{F(e)}(x), \nu^n_{F(e)}(y)\Big\},
\end{aligned}$$

and

$$\begin{aligned}
\delta_{F(e)}(xy) &\leq \max\Big\{\delta_{(F\widetilde{\diamond}\mathbf{1})(e)}(xy), \delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy)\Big\} \\
&= \max\Bigg\{\bigwedge_{(a,b)\in\mathbf{S}_{xy}} \max\Big\{\delta_{F(e)}(a), \delta_{\mathbf{1}(e)}(b)\Big\}, \bigwedge_{(u,v)\in\mathbf{S}_{xy}} \max\Big\{\delta_{\mathbf{1}(e)}(u), \delta_{F(e)}(v)\Big\}\Bigg\} \\
&\leq \max\Big\{\max\Big\{\delta_{F(e)}(x), \delta_{\mathbf{1}(e)}(y)\Big\}, \max\Big\{\delta_{\mathbf{1}(e)}(x), \delta_{F(e)}(y)\Big\}\Big\} \\
&= \max\Big\{\max\Big\{\delta_{F(e)}(x), -1\Big\}, \max\Big\{-1, \delta_{F(e)}(y)\Big\}\Big\} \\
&= \max\Big\{\delta_{F(e)}(x), \delta_{F(e)}(y)\Big\}.
\end{aligned}$$

Thus, (F, E) satisfies the condition of a tripolar (m, n)-fuzzy soft subsemigroup over $\mathbf{S}$.

Second, we verify that (F,E) satisfies the bi-ideal condition. Since $xyz = x(yz) = (xy)z$, it is clear that $(x,yz) \in \mathbf{S}_{xyz}$ and $(xy,z) \in \mathbf{S}_{xyz}$. By utilizing these pairs in the algebraic condition, we have:

$$\begin{aligned}
\mu^m_{F(e)}(xyz) &\geq \min\Big\{\mu^m_{(F\widetilde{\diamond}\mathbf{1})(e)}(xyz), \mu^m_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xyz)\Big\} \\
&= \min\Bigg\{\bigvee_{(a,b)\in\mathbf{S}_{xyz}} \min\Big\{\mu^m_{F(e)}(a), \mu^m_{\mathbf{1}(e)}(b)\Big\}, \bigvee_{(u,v)\in\mathbf{S}_{xyz}} \min\Big\{\mu^m_{\mathbf{1}(e)}(u), \mu^m_{F(e)}(v)\Big\}\Bigg\} \\
&\geq \min\Big\{\min\Big\{\mu^m_{F(e)}(x), \mu^m_{\mathbf{1}(e)}(yz)\Big\}, \min\Big\{\mu^m_{\mathbf{1}(e)}(xy), \mu^m_{F(e)}(z)\Big\}\Big\} \\
&= \min\Big\{\min\Big\{\mu^m_{F(e)}(x), 1\Big\}, \min\Big\{1, \mu^m_{F(e)}(z)\Big\}\Big\} \\
&= \min\Big\{\mu^m_{F(e)}(x), \mu^m_{F(e)}(z)\Big\}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\nu^n_{F(e)}(xyz) &\leq \max\Big\{\nu^n_{(F\widetilde{\diamond}\mathbf{1})(e)}(xyz), \nu^n_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xyz)\Big\} \\
&= \max\Bigg\{\bigwedge_{(a,b)\in\mathbf{S}_{xyz}} \max\Big\{\nu^n_{F(e)}(a), \nu^n_{\mathbf{1}(e)}(b)\Big\}, \bigwedge_{(u,v)\in\mathbf{S}_{xyz}} \max\Big\{\nu^n_{\mathbf{1}(e)}(u), \nu^n_{F(e)}(v)\Big\}\Bigg\} \\
&\leq \max\Big\{\max\Big\{\nu^n_{F(e)}(x), \nu^n_{\mathbf{1}(e)}(yz)\Big\}, \max\Big\{\nu^n_{\mathbf{1}(e)}(xy), \nu^n_{F(e)}(z)\Big\}\Big\} \\
&= \max\Big\{\max\Big\{\nu^n_{F(e)}(x), 0\Big\}, \max\Big\{0, \nu^n_{F(e)}(z)\Big\}\Big\} \\
&= \max\Big\{\nu^n_{F(e)}(x), \nu^n_{F(e)}(z)\Big\},
\end{aligned}$$

and

$$\begin{aligned}
\delta_{F(e)}(xyz) &\leq \max\Big\{\delta_{(F\widetilde{\diamond}\mathbf{1})(e)}(xyz), \delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xyz)\Big\} \\
&= \max\Bigg\{\bigwedge_{(a,b)\in\mathbf{S}_{xyz}} \max\Big\{\delta_{F(e)}(a), \delta_{\mathbf{1}(e)}(b)\Big\}, \bigwedge_{(u,v)\in\mathbf{S}_{xyz}} \max\Big\{\delta_{\mathbf{1}(e)}(u), \delta_{F(e)}(v)\Big\}\Bigg\} \\
&\leq \max\Big\{\max\Big\{\delta_{F(e)}(x), \delta_{\mathbf{1}(e)}(yz)\Big\}, \max\Big\{\delta_{\mathbf{1}(e)}(xy), \delta_{F(e)}(z)\Big\}\Big\} \\
&= \max\Big\{\max\Big\{\delta_{F(e)}(x), -1\Big\}, \max\Big\{-1, \delta_{F(e)}(z)\Big\}\Big\} \\
&= \max\Big\{\delta_{F(e)}(x), \delta_{F(e)}(z)\Big\}.
\end{aligned}$$

Therefore, (F,E) is a tripolar (m,n)-fuzzy soft bi-ideal over $\mathbf{S}$. □

The converse of Lemma 3.3 does not hold in general, as the following counterexample.

Example 3.9. *Let $S = \{a,b,c\}$ be a set with the multiplication defined by the following Cayley table:*

$*$	a	b	c
a	a	a	a
b	a	a	a
c	a	a	b

We define an order relation $\preceq$ on S as follows:

$$\preceq := \{(a,a),(b,b),(c,c),(a,b)\}.$$

Then $\mathbf{S} := \langle S, *, \preceq\rangle$ is an ordered semigroup. Let $E = \{e\}$ be a parameter set, and fix the exponents as $m = 2$ and $n = 3$.

We construct a tripolar $(2,3)$-fuzzy soft set (F,E) over $\mathbf{S}$ by defining the component functions piecewise as follows:

$$\mu_{F(e)}(x) := \begin{cases} 0.894 & \text{if } x = a \\ 0.707 & \text{if } x = b \\ 0.707 & \text{if } x = c, \end{cases} \quad \nu_{F(e)}(x) := \begin{cases} 0.316 & \text{if } x = a \\ 0.669 & \text{if } x = b \\ 0.737 & \text{if } x = c, \end{cases} \quad \text{and} \quad \delta_{F(e)}(x) := \begin{cases} -0.6 & \text{if } x = a \\ -0.4 & \text{if } x = b \\ -0.2 & \text{if } x = c. \end{cases}$$

Under the exponents $(m,n) = (2,3)$, the corresponding simplified forms are given by:

$$\mu^2_{F(e)}(a) = 0.8, \quad \mu^2_{F(e)}(b) = 0.5, \quad \mu^2_{F(e)}(c) = 0.5,$$
$$\nu^3_{F(e)}(a) = 0.1, \quad \nu^3_{F(e)}(b) = 0.3, \quad \nu^3_{F(e)}(c) = 0.4.$$

As verified previously in Example 3.7, (F,E) is a tripolar (m,n)-fuzzy soft bi-ideal over $\mathbf{S}$. However, (F,E) is not a tripolar (m,n)-fuzzy soft quasi-ideal over $\mathbf{S}$ because of

$$\max\left\{\nu^3_{(F\widetilde{\diamond}\mathbf{1})(e)}(b), \nu^3_{(\mathbf{1}\widetilde{\diamond}F)(e)}(b)\right\} = \max\{0.4, 0.4\} = 0.4 \not\leq 0.3 = \nu^3_{F(e)}(b).$$

Therefore, (F,E) is a tripolar (m,n)-fuzzy soft bi-ideal over $\mathbf{S}$, but it is not a tripolar (m,n)-fuzzy soft quasi-ideal over $\mathbf{S}$.

An ordered semigroup $\mathbf{S}$ is called *regular* if for every $a \in S$ there exist $x \in S$ such that $a \preceq axa$ [10]. The following theorem investigates the relationship between tripolar (m,n)-fuzzy soft bi-ideals and quasi-ideals within the context of regular ordered semigroups. We prove that regularity acts as a bridging condition that allows a bi-ideal to be characterized as a quasi-ideal.

Theorem 3.1. *Let $\mathbf{S}$ be a regular ordered semigroup and let (F,E) be a tripolar (m,n)-fuzzy soft set over $\mathbf{S}$. If (F,E) is a tripolar (m,n)-fuzzy soft bi-ideal over $\mathbf{S}$, then it is a tripolar (m,n)-fuzzy soft quasi-ideal over $\mathbf{S}$.*

Proof. Let (F,E) be a tripolar (m,n)-fuzzy soft bi-ideal over a regular ordered semigroup $\mathbf{S}$. For any $e \in E$ and $a \in S$, since $\mathbf{S}$ is regular, there exists $x \in S$ such that $a \preceq axa$. It follows that $a \preceq axaxa = (ax)a(xa)$. To prove the quasi-ideal condition, let $a \preceq uv$ and $a \preceq yz$ for some $u,v,y,z \in S$. Then

$$a \preceq axa \preceq (uv)x(yz) = u(vxy)z,$$

and we obtain that

$$\begin{aligned} \mu^m_{F(e)}(a) &\geq \mu^m_{F(e)}(u(vxy)z) \\ &\geq \min\{\mu^m_{F(e)}(u), \mu^m_{F(e)}(z)\} \end{aligned}$$

$$= \min\Big\{ \bigvee_{(u,v)\in \mathbf{S}_a} \min\{\mu^m_{F(e)}(u), \mu^m_{\mathbf{1}(e)}(v)\}, \bigvee_{(u,v)\in \mathbf{S}_a} \min\{\mu^m_{\mathbf{1}(e)}(y), \mu^m_{F(e)}(z)\}\Big\}$$
$$= \min\{\mu^m_{(F\widetilde{\diamond}\mathbf{1})(e)}(a), \mu^m_{(\mathbf{1}\widetilde{\diamond}F)(e)}(a)\}.$$

Similarly, we have

$$\begin{aligned}
\nu^n_{F(e)}(a) &\leq \nu^n_{F(e)}(u(vxy)z) \\
&\leq \max\{\nu^n_{F(e)}(u), \nu^n_{F(e)}(z)\} \\
&= \max\Big\{ \bigwedge_{(u,v)\in \mathbf{S}_a} \max\{\nu^n_{F(e)}(u), \nu^n_{\mathbf{1}(e)}(v)\}, \bigwedge_{(u,v)\in \mathbf{S}_a} \max\{\nu^n_{\mathbf{1}(e)}(y), \nu^n_{F(e)}(z)\}\Big\} \\
&= \max\{\nu^n_{(F\widetilde{\diamond}\mathbf{1})(e)}(a), \nu^n_{(\mathbf{1}\widetilde{\diamond}F)(e)}(a)\},
\end{aligned}$$

and

$$\begin{aligned}
\delta_{F(e)}(a) &\leq \delta_{F(e)}(u(vxy)z) \\
&\leq \max\{\delta_{F(e)}(u), \delta_{F(e)}(z)\} \\
&= \max\Big\{ \bigwedge_{(u,v)\in \mathbf{S}_a} \max\{\delta_{F(e)}(u), \delta_{\mathbf{1}(e)}(v)\}, \bigwedge_{(u,v)\in \mathbf{S}_a} \max\{\delta_{\mathbf{1}(e)}(y), \delta_{F(e)}(z)\}\Big\} \\
&= \max\{\delta_{(F\widetilde{\diamond}\mathbf{1})(e)}(a), \delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(a)\}.
\end{aligned}$$

Thus, (F, E) is a tripolar (m, n)-fuzzy soft quasi-ideal over $\mathbf{S}$. □

Combining the results of Lemma 3.3 and Theorem 3.1, we immediately obtain the following corollary, which shows that these two classes of tripolar (m, n)-fuzzy soft ideals coincide in the presence of regularity.

Corollary 3.2. *Let $\mathbf{S}$ be a regular ordered semigroup and let (F, E) be a tripolar (m, n)-fuzzy soft set over S. Then the following statements are equivalent:*

(1) *(F, E) is a tripolar (m, n)-fuzzy soft quasi-ideal over $\mathbf{S}$.*
(2) *(F, E) is a tripolar (m, n)-fuzzy soft bi-ideal over $\mathbf{S}$.*

Proof. (1)⇒(2): By Lemma 3.3, and (2)⇒(1): By Theorem 3.1. □

In the rest of this section, we show that the tripolar (m, n)-fuzzy soft quasi-ideals of an ordered semigroup are just intersections of tripolar (m, n)-fuzzy soft right and tripolar (m, n)-fuzzy soft left ideals.

Lemma 3.4. *Let $\mathbf{S}$ be an ordered semigroup, and let (F, E) be a tripolar (m, n)-fuzzy soft set over S. Then the following holds: For all $e \in E$ and $x, y \in S$.*

(i) $\mu^m_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy) \geq \mu^m_{F(e)}(y)$, $\nu^n_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy) \leq \nu^n_{F(e)}(y)$, *and* $\delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy) \leq \delta_{F(e)}(y)$.
(ii) $\mu^m_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy) \geq \mu^m_{(\mathbf{1}\widetilde{\diamond}F)(e)}(y)$, $\nu^n_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy) \leq \nu^n_{(\mathbf{1}\widetilde{\diamond}F)(e)}(y)$, *and* $\delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy) \leq \delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(y)$.

Proof. (1) By the definition of the soft product, for any $e \in E$ and $x, y \in S$, we have

$$\begin{aligned}
\mu^m_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy) &= \bigvee_{(x,y)\in \mathbf{S}_{xy}} \left\{\min\{\mu^m_{\mathbf{1}(e)}(u), \mu^m_{F(e)}(v)\}\right\} \\
&\geq \min\{\mu^m_{\mathbf{1}(e)}(x), \mu^m_{F(e)}(y)\} \\
&= \min\{1, \mu^m_{F(e)}(y)\} \\
&= \mu^m_{F(e)}(y).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\nu^n_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy) &= \bigwedge_{(x,y)\in \mathbf{S}_{xy}} \{\max\{\nu^n_{\mathbf{1}(e)}(u), \nu^n_{F(e)}(v)\}\} \\
&\leq \max\{\nu^n_{\mathbf{1}(e)}(x), \nu^n_{F(e)}(y)\} \\
&= \max\{0, \nu^n_{F(e)}(y)\} \\
&= \nu^n_{F(e)}(y),
\end{aligned}$$

and

$$\begin{aligned}
\delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy) &= \bigwedge_{(x,y)\in \mathbf{S}_{xy}} \{\max\{\delta_{\mathbf{1}(e)}(u), \delta_{F(e)}(v)\}\} \\
&\leq \max\{\delta_{\mathbf{1}(e)}(x), \delta_{F(e)}(y)\} \\
&= \max\{-1, \delta_{F(e)}(y)\} \\
&= \delta_{F(e)}(y).
\end{aligned}$$

This completes the proof that (1) holds.

(2) For any $u, v \in S$ such that $y \leq uv$, it follows that $xy \leq x(uv) = (xu)v$. Then

$$\begin{aligned}
(\mu^m_{(\mathbf{1}\widetilde{\diamond}F)(e)})(xy) &= \bigvee_{(u,v)\in \mathbf{S}_{y}} \{\min\{\mu^m_{\mathbf{1}(e)}(xu), \mu^m_{F(e)}(v)\}\} \\
&\geq \min\{\mu^m_{\mathbf{1}(e)}(xu), \mu^m_{F(e)}(v)\} \\
&= \mu^m_{F(e)}(v) \\
&= \min\{\mu^m_{\mathbf{1}(e)}(u), \mu^m_{F(e)}(v)\} \\
&= \bigvee_{(u,v)\in \mathbf{S}_{y}} \{\min\{\mu^m_{\mathbf{1}(e)}(u), \mu^m_{F(e)}(v)\}\} \\
&= (\mu^m_{(\mathbf{1}\widetilde{\diamond}F)(e)})(y).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
(\nu^n_{(\mathbf{1}\widetilde{\diamond}F)(e)})(xy) &= \bigwedge_{(u,v)\in \mathbf{S}_{y}} \{\max\{\nu^n_{\mathbf{1}(e)}(xu), \nu^n_{F(e)}(v)\}\} \\
&\leq \max\{\nu^n_{\mathbf{1}(e)}(xu), \nu^n_{F(e)}(v)\}
\end{aligned}$$

$$
\begin{aligned}
&= \nu^n_{F(e)}(v) \\
&= \max\{\nu^n_{\mathbf{1}(e)}(u), \nu^n_{F(e)}(v)\} \\
&= \bigwedge_{(u,v)\in \mathbf{S}_y} \{\max\{\nu^n_{\mathbf{1}(e)}(u), \nu^n_{F(e)}(v)\}\} \\
&= (\nu^n_{(\mathbf{1}\widetilde{\diamond}F)(e)})(y),
\end{aligned}
$$

and

$$
\begin{aligned}
(\delta_{(\mathbf{1}\widetilde{\diamond}F)(e)})(xy) &= \bigwedge_{(u,v)\in \mathbf{S}_y} \{\max\{\delta_{\mathbf{1}(e)}(xu), \delta_{F(e)}(v)\}\} \\
&\le \max\{\delta_{\mathbf{1}(e)}(xu), \delta_{F(e)}(v)\} \\
&= \delta_{F(e)}(v) \\
&= \max\{\delta_{\mathbf{1}(e)}(u), \delta_{F(e)}(v)\} \\
&= \bigwedge_{(u,v)\in \mathbf{S}_y} \{\max\{\delta_{\mathbf{1}(e)}(u), \delta_{F(e)}(v)\}\} \\
&= (\delta_{(\mathbf{1}\widetilde{\diamond}F)(e)})(y).
\end{aligned}
$$

This completed the proof that (2) holds. □

The following lemma can be established analogously to Lemma 3.4.

Lemma 3.5. *Let* $\mathbf{S}$ *be an ordered semigroup and let* (F, E) *be a tripolar* (m, n)*-fuzzy soft set over* S*. Then for any* $e \in E$ *and* $x, y \in S$*, the following properties hold:*

(1) $\mu^m_{(F\widetilde{\diamond}\mathbf{1})(e)}(xy) \ge \mu^m_{F(e)}(x)$, $\nu^n_{(F\widetilde{\diamond}\mathbf{1})(e)}(xy) \le \nu^n_{F(e)}(x)$, *and* $\delta_{(F\widetilde{\diamond}\mathbf{1})(e)}(xy) \le \delta_{F(e)}(x)$.

(2) $\mu^m_{(F\widetilde{\diamond}\mathbf{1})(e)}(xy) \ge \mu^m_{(F\widetilde{\diamond}\mathbf{1})(e)}(x)$, $\nu^n_{(F\widetilde{\diamond}\mathbf{1})(e)}(xy) \le \nu^n_{(F\widetilde{\diamond}\mathbf{1})(e)}(x)$, *and* $\delta_{(F\widetilde{\diamond}\mathbf{1})(e)}(xy) \le \delta_{(F\widetilde{\diamond}\mathbf{1})(e)}(x)$.

In what follows, we establish the relationship between the order relation $\preceq$ and the tripolar (m, n)-fuzzy soft product operation. The next lemma proves that the components of $(\mathbf{1}, E)\widetilde{\diamond}(F, E)$ successfully inherit the order properties of the semigroup.

Lemma 3.6. *Let* $\mathbf{S}$ *be an ordered semigroup and let* (F, E) *be a tripolar* (m, n)*-fuzzy soft set over* S*. For any* $e \in E$ *and* $x, y \in S$*, if* $x \preceq y$*, then the following properties hold:*

(1) $\mu^m_{(\mathbf{1}\widetilde{\diamond}F)(e)}(x) \ge \mu^m_{(\mathbf{1}\widetilde{\diamond}F)(e)}(y)$.

(2) $\nu^n_{(\mathbf{1}\widetilde{\diamond}F)(e)}(x) \le \nu^n_{(\mathbf{1}\widetilde{\diamond}F)(e)}(y)$.

(3) $\delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(x) \le \delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(y)$.

Proof. Let $e \in E$ and $x, y \in S$ such that $x \preceq y$. For the positive membership part, consider the sets $\mathbf{S}_y = \{(u, v) \in S \times S : y \preceq uv\}$ and $\mathbf{S}_x = \{(u, v) \in S \times S : x \preceq uv\}$. Since $x \preceq y$, it follows from the transitivity of $\preceq$ that $y \preceq uv$ implies $x \preceq uv$. Hence, $\mathbf{S}_y \subseteq \mathbf{S}_x$. Thus, we have

$$
\mu^m_{(\mathbf{1}\widetilde{\diamond}F)(e)}(x) = \bigvee_{(u,v)\in \mathbf{S}_x} \{\min\{\mu^m_{\mathbf{1}(e)}(u), \mu^m_{F(e)}(v)\}\}
$$

$$\geq \bigvee_{(u,v)\in \mathbf{S}_y} \{\min\{\mu^m_{\mathbf{1}(e)}(u), \mu^m_{F(e)}(v)\}\}$$
$$= \mu^m_{(\mathbf{1}\widetilde{\diamond}F)(e)}(y).$$

Similarly, we have

$$\begin{aligned}
\nu^n_{(\mathbf{1}\widetilde{\diamond}F)(e)}(x) &= \bigwedge_{(u,v)\in \mathbf{S}_x} \{\max\{\nu^n_{\mathbf{1}(e)}(u), \nu^n_{F(e)}(v)\}\} \\
&\leq \bigwedge_{(u,v)\in \mathbf{S}_y} \{\max\{\nu^n_{\mathbf{1}(e)}(u), \nu^n_{F(e)}(v)\}\} \\
&= \nu^n_{(\mathbf{1}\widetilde{\diamond}F)(e)}(y),
\end{aligned}$$

and

$$\begin{aligned}
\delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(x) &= \bigwedge_{(u,v)\in \mathbf{S}_x} \{\max\{\delta_{\mathbf{1}(e)}(u), \delta_{F(e)}(v)\}\} \\
&\leq \bigwedge_{(u,v)\in \mathbf{S}_y} \{\max\{\delta_{\mathbf{1}(e)}(u), \delta_{F(e)}(v)\}\} \\
&= \delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(y),
\end{aligned}$$

The proof is now complete. □

Similar to Lemma 3.6, we obtain the following result.

Lemma 3.7. *Let* **S** *be an ordered semigroup and let (F, E) be a tripolar (m, n)-fuzzy soft set over S. For any $e \in E$ and $x, y \in S$, if $x \leq y$, then the following properties hold:*

(1) $\mu^m_{(F\widetilde{\diamond}\mathbf{1})(e)}(x) \geq \mu^m_{(F\widetilde{\diamond}\mathbf{1})(e)}(y)$.
(2) $\nu^n_{(F\widetilde{\diamond}\mathbf{1})(e)}(x) \leq \nu^n_{(F\widetilde{\diamond}\mathbf{1})(e)}(y)$.
(3) $\delta_{(F\widetilde{\diamond}\mathbf{1})(e)}(x) \leq \delta_{(F\widetilde{\diamond}\mathbf{1})(e)}(y)$.

Next, we analyze the monotonic inequalities satisfied by the positive, non-membership, and negative membership functions of a tripolar (m, n)-fuzzy soft set when multiplied by the entire algebraic structure from the left. These relations are formally stated below.

Lemma 3.8. *Let* **S** *be an ordered semigroup and let (F, E) be a tripolar (m, n)-fuzzy soft set over S. Then for any $e \in E$ and $x, y \in S$, the following properties hold:*

(1) $\max\{\mu^m_{F(e)}(xy), \mu^m_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy)\} \geq \max\{\mu^m_{F(e)}(y), \mu^m_{(\mathbf{1}\widetilde{\diamond}F)(e)}(y)\}$.
(2) $\min\{\nu^n_{F(e)}(xy), \nu^n_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy)\} \leq \min\{\nu^n_{F(e)}(y), \nu^n_{(\mathbf{1}\widetilde{\diamond}F)(e)}(y)\}$.
(3) $\min\{\delta_{F(e)}(xy), \delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy)\} \leq \min\{\delta_{F(e)}(y), \delta_{(\mathbf{1}\widetilde{\diamond}F)(e)}(y)\}$.

Proof. Let $e \in E$ and $x, y \in S$.

(1) For the positive membership part, by Lemma 3.4, we know that

$$\mu^m_{(\mathbf{1}\widetilde{\diamond}F)(e)}(xy) \geq \mu^m_{F(e)}(y)$$

and by Lemma 3.6, we have that

$$\mu^m_{(\mathbf{1}\tilde{\diamond}F)(e)}(xy) \geq \mu^m_{(\mathbf{1}\tilde{\diamond}F)(e)}(y).$$

It follows that

$$\mu^m_{(\mathbf{1}\tilde{\diamond}F)(e)}(xy) \geq \max\{\mu^m_{F(e)}(y), \mu^m_{(\mathbf{1}\tilde{\diamond}F)(e)}(y)\}.$$

Since $\max\{\mu^m_{F(e)}(y), \mu^m_{(\mathbf{1}\tilde{\diamond}F)(e)}(xy)\}\} \geq \mu^m_{(\mathbf{1}\tilde{\diamond}F)(e)}(xy)$, we obtain

$$\max\{\mu^m_{F(e)}(xy), \mu^m_{(\mathbf{1}\tilde{\diamond}F)(e)}(xy)\} \geq \max\{\mu^p_{F(e)}(y), \mu^p_{(\mathbf{1}\tilde{\diamond}F)(e)}(y)\}.$$

(2) Similarly, for the neutral part, from Lemma 3.4 and 3.6, we have

$$\nu^n_{(\mathbf{1}\tilde{\diamond}F)(e)}(xy) \leq \nu^n_{F(e)}(y)$$

and

$$\nu^n_{(\mathbf{1}\tilde{\diamond}F)(e)}(xy) \leq \nu^n_{(\mathbf{1}\tilde{\diamond}F)(e)}(y).$$

This implies

$$\nu^n_{(\mathbf{1}\tilde{\diamond}F)(e)}(xy) \leq \max\{\nu^n_{F(e)}(y), \nu^n_{(\mathbf{1}\tilde{\diamond}F)(e)}(y)\}.$$

Consequently,

$$\min\{\nu^n_{F(e)}(xy), \nu^n_{(\mathbf{1}\tilde{\diamond}F)(e)}(xy)\} \leq \min\{\nu^n_{F(e)}(y), \nu^n_{(\mathbf{1}\tilde{\diamond}F)(e)}(y)\}.$$

The proof for the negative part (3) follows an identical argument as in (2). □

In a similar manner to Lemma 3.8, we obtain the following results for the right-hand composition.

Lemma 3.9. *Let* $\mathbf{S}$ *be an ordered semigroup and let* (F, E) *be a tripolar* (m, n)*-fuzzy soft set over* S*. Then for any* $e \in E$ *and* $x, y \in S$*, the following properties hold:*

(1) $\max\{\mu^m_{F(e)}(xy), \mu^m_{(\mathbf{1}\tilde{\diamond}F)(e)}(xy)\} \geq \max\{\mu^m_{F(e)}(x), \mu^m_{(\mathbf{1}\tilde{\diamond}F)(e)}(x)\}$.
(2) $\min\{\nu^n_{F(e)}(xy), \nu^n_{(\mathbf{1}\tilde{\diamond}F)(e)}(xy)\} \leq \min\{\nu^n_{F(e)}(x), \nu^n_{(\mathbf{1}\tilde{\diamond}F)(e)}(x)\}$.
(3) $\min\{\delta_{F(e)}(xy), \delta_{(\mathbf{1}\tilde{\diamond}F)(e)}(xy)\} \leq \min\{\delta_{F(e)}(x), \delta_{(\mathbf{1}\tilde{\diamond}F)(e)}(x)\}$.

A fundamental problem in algebraic fuzzy structures is the construction of the smallest fuzzy ideal containing a given fuzzy set, a process often referred to as ideal generation. The following lemma provides a concrete operational mechanism to construct a tripolar (m, n)-fuzzy soft left ideal from an arbitrary tripolar (m, n)-fuzzy soft set satisfying the order conditions by utilizing the soft union and product operations.

Lemma 3.10. *Let* $\mathbf{S}$ *be an ordered semigroup and let* (F, E) *be a tripolar* (m, n)*-fuzzy soft set over* S *satisfying the order-reversing property: for any* $e \in E$ *and* $x, y \in S$*,* $x \leq y$ *implies* $\mu^m_{F(e)}(x) \geq \mu^m_{F(e)}(y)$*,* $\nu^n_{F(e)}(x) \leq \nu^n_{F(e)}(y)$*, and* $\delta_{F(e)}(x) \leq \delta_{F(e)}(y)$*. Then the tripolar* (m, n)*-fuzzy soft set* $(G, E) = (F, E)\tilde{\cup}[(\mathbf{1}, E)\tilde{\diamond}(F, E)]$ *is a tripolar* (m, n)*-fuzzy soft left ideal over* $\mathbf{S}$*.*

Proof. Let $(G,E) = (F,E)\tilde{\cup}[(\mathbf{1},E)\tilde{\diamond}(F,E)]$. For any $e \in E$ and $x,y \in S$, by applying Lemma 3.7, we have

$$\begin{aligned}\mu^m_{G(e)}(xy) &= \max\{\mu^m_{F(e)}(xy), \mu^m_{(\mathbf{1}\tilde{\diamond}F)(e)}(xy)\}\\ &\geq \max\{\mu^m_{F(e)}(y), \mu^m_{(\mathbf{1}\tilde{\diamond}F)(e)}(y)\}\\ &= \mu^m_{G(e)}(y).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\nu^n_{G(e)}(xy) &= \min\{\nu^n_{F(e)}(xy), \nu^n_{(\mathbf{1}\tilde{\diamond}F)(e)}(xy)\}\\ &\leq \min\{\nu^n_{F(e)}(y), \nu^n_{(\mathbf{1}\tilde{\diamond}F)(e)}(y)\}\\ &= \nu^n_{G(e)}(y),\end{aligned}$$

and

$$\begin{aligned}\delta_{G(e)}(xy) &= \min\{\delta_{F(e)}(xy), \delta_{(\mathbf{1}\tilde{\diamond}F)(e)}(xy)\}\\ &\leq \min\{\delta_{F(e)}(y), \delta_{(\mathbf{1}\tilde{\diamond}F)(e)}(y)\}\\ &= \delta_{G(e)}(y).\end{aligned}$$

This shows that (G,E) satisfies the left ideal property. Next, suppose $x,y \in S$ with $x \leq y$. By Lemma 3.6, we have that

$$\begin{aligned}\mu^m_{G(e)}(x) &= \max\{\mu^m_{F(e)}(x), \mu^m_{(\mathbf{1}\tilde{\diamond}F)(e)}(x)\}\\ &\geq \max\{\mu^m_{F(e)}(y), (\mu^m_{(\mathbf{1}\tilde{\diamond}F)(e)})(y)\}\\ &= \mu^m_{G(e)}(y).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\nu^n_{G(e)}(x) &= \min\{\nu^n_{F(e)}(x), \nu^n_{(\mathbf{1}\tilde{\diamond}F)(e)}(x)\}\\ &\leq \min\{\nu^n_{F(e)}(y), \nu^n_{(\mathbf{1}\tilde{\diamond}F)(e)}(y)\}\\ &= \nu^n_{G(e)}(y),\end{aligned}$$

and

$$\begin{aligned}\delta_{G(e)}(x) &= \min\{\delta_{F(e)}(x), \delta_{(\mathbf{1}\tilde{\diamond}F)(e)}(x)\}\\ &\leq \min\{\delta_{F(e)}(y), \delta_{(\mathbf{1}\tilde{\diamond}F)(e)}(y)\}\\ &= \delta_{G(e)}(y).\end{aligned}$$

Hence, (G,E) is a tripolar (m,n)-fuzzy soft left ideal over $\mathbf{S}$. □

Analogously to Lemma 3.10, we obtain the following result.

Lemma 3.11. *Let* $\mathbf{S}$ *be an ordered semigroup and let* (F, E) *be a tripolar* (m, n)*-fuzzy soft set over* S *satisfying the order-reversing property. Then the tripolar* (m, n)*-fuzzy soft set* $(H, E) = (F, E)\tilde{\cup}[(F, E)\tilde{\diamond}(\mathbf{1}, E)]$ *is a tripolar* (m, n)*-fuzzy soft right ideal over* $\mathbf{S}$*.*

Next, we present a fundamental algebraic property concerning the operations of soft intersection and soft union. The following lemma demonstrates that the distributive law holds perfectly for any tripolar (m, n)-fuzzy soft sets over an ordered semigroup.

Lemma 3.12. *Let* $\mathbf{S}$ *be an ordered semigroup and let* (F, E)*,* (G, E)*, and* (H, E) *be tripolar* (m, n)*-fuzzy soft sets over* S*. Then the following distributive law holds:*

$$(F, E)\tilde{\cap}\Big((G, E)\tilde{\cup}(H, E)\Big) = \Big((F, E)\tilde{\cap}(G, E)\Big)\tilde{\cup}\Big(F, E)\tilde{\cap}(H, E)\Big).$$

Proof. Let (F, E), (G, E), and (H, E) be tripolar (m, n)-fuzzy soft sets over S, $e \in E$, and $x \in S$.

For the positive membership part, we have

$$\begin{aligned}
\mu^m_{(F\tilde{\cap}(G\tilde{\cup}H))(e)}(x) &= \min\{\mu^m_{F(e)}(x), \mu^m_{(G\tilde{\cup}H)(e)}(x)\} \\
&= \min\{\mu^m_{F(e)}(x), \max\{\mu^m_{G(e)}(x), \mu^m_{H(e)}(x)\}\} \\
&= \max\{\min\{\mu^m_{F(e)}(x), \mu^m_{G(e)}(x)\}, \min\{\mu^m_{F(e)}(x), \mu^m_{H(e)}(x)\}\} \\
&= \max\{\mu^m_{(F\tilde{\cap}G)(e)}(x), \mu^m_{(F\tilde{\cap}H)(e)}(x)\} \\
&= \mu^m_{((F\tilde{\cap}G)\tilde{\cup}(F\tilde{\cap}H))(e)}(x).
\end{aligned}$$

For the neutral membership part, we have

$$\begin{aligned}
\nu^n_{(F\tilde{\cap}(G\tilde{\cup}H))(e)}(x) &= \max\{\nu^n_{F(e)}(x), \nu^n_{(G\tilde{\cup}H)(e)}(x)\} \\
&= \max\{\nu^n_{F(e)}(x), \min\{\nu^n_{G(e)}(x), \nu^q_{H(e)}(x)\}\} \\
&= \min\{\max\{\nu^n_{F(e)}(x), \nu^n_{G(e)}(x)\}, \max\{\nu^n_{F(e)}(x), \nu^n_{H(e)}(x)\}\} \\
&= \min\{\nu^n_{(F\tilde{\cup}G)(e)}(x), \nu^n_{(F\tilde{\cup}H)(e)}(x)\} \\
&= \nu^n_{((F\tilde{\cup}G)\tilde{\cap}(F\tilde{\cup}H))(e)}(x).
\end{aligned}$$

For the negative membership part, we have

$$\begin{aligned}
\delta_{(F\tilde{\cap}(G\tilde{\cup}H))(e)}(x) &= \max\{\delta_{F(e)}(x), \delta_{(G\tilde{\cup}H)(e)}(x)\} \\
&= \max\{\delta_{F(e)}(x), \min\{\delta_{G(e)}(x), \delta_{H(e)}(x)\}\} \\
&= \min\{\max\{\delta_{F(e)}(x), \delta_{G(e)}(x)\}, \max\{\delta_{F(e)}(x), \delta_{H(e)}(x)\}\} \\
&= \min\{\delta_{(F\tilde{\cup}G)(e)}(x), \delta_{(F\tilde{\cup}H)(e)}(x)\} \\
&= \delta_{((F\tilde{\cup}G)\tilde{\cap}(F\tilde{\cup}H))(e)}(x).
\end{aligned}$$

Thus, $(F, E)\tilde{\cap}\Big((G, E)\tilde{\cup}(H, E)\Big) = \Big((F, E)\tilde{\cap}(G, E)\Big)\tilde{\cup}\Big((F, E)\tilde{\cap}(H, E)\Big)$, which means the distributive law holds. $\square$

Lemma 3.13. *Let* $\mathbf{S}$ *be an ordered semigroup. If* (F,E) *and* (G,E) *are tripolar* (m,n)*-fuzzy soft quasi-ideals over* $\mathbf{S}$*, then their intersection* $(H,E) = (F,E)\widetilde{\cap}(G,E)$ *is also a tripolar* (m,n)*-fuzzy soft quasi-ideal over* $\mathbf{S}$.

Proof. Let (F,E) and (G,E) be two tripolar (m,n)-fuzzy soft quasi-ideals over $\mathbf{S}$. Let $(H,E) = (F,E)\tilde{\cap}(G,E)$, where $H(e) = F(e) \cap G(e)$ for all $e \in E$. For the sake of simplicity, for each parameter $e \in E$, we denote the components of $F(e)$, $G(e)$, and $H(e)$ at any element $x \in S$ by $(\mu^m_{F(e)}(x), \nu^n_{F(e)}(x), \delta_{F(e)}(x))$, $(\mu^m_{G(e)}(x), \nu^n_{G(e)}(x), \delta_{G(e)}(x))$, and $(\mu^m_{H(e)}(x), \nu^n_{H(e)}(x), \delta_{H(e)}(x))$, respectively. By the definition of soft intersection, we have:

$$\mu^m_{H(e)}(x) = \min\{\mu^m_{F(e)}(x), \mu^m_{G(e)}(x)\},$$

$$\nu^n_{H(e)}(x) = \max\{\nu^n_{F(e)}(x), \nu^n_{G(e)}(x)\},$$

$$\delta_{H(e)}(x) = \max\{\delta_{F(e)}(x), \delta_{G(e)}(x)\}.$$

We need to verify that (H,E) satisfies both the algebraic condition and the order condition of a tripolar (m,n)-fuzzy soft quasi-ideal over $\mathbf{S}$. Since (F,E) and (G,E) are tripolar (m,n)-fuzzy soft quasi-ideals of $\mathbf{S}$, for any $x \in S$, we have:

$$\begin{aligned}
\mu^m_{H(e)}(x) &= \min\{\mu^m_{F(e)}(x), \mu^m_{G(e)}(x)\} \\
&\geq \min\Big\{ \bigvee_{(a,b)\in \mathbf{S}_x} \{\min\{\mu^m_{F(e)}(a), \mu^m_{\mathbf{1}(e)}(b)\}\}, \bigvee_{(c,d)\in \mathbf{S}_x} \{\min\{\mu^m_{\mathbf{1}(e)}(c), \mu^m_{F(e)}(d)\}\}, \\
&\quad \bigvee_{(a,b)\in \mathbf{S}_x} \{\min\{\mu^m_{G(e)}(a), \mu^m_{\mathbf{1}(e)}(b)\}\}, \bigvee_{(c,d)\in \mathbf{S}_x} \{\min\{\mu^m_{\mathbf{1}(e)}(c), \mu^m_{G(e)}(d)\}\}\Big\} \\
&= \min\{ \bigvee_{(a,b)\in \mathbf{S}_x} \{\min\{\mu^m_{F(e)}(a), \mu^m_{\mathbf{1}(e)}(b)\}\}, \bigvee_{(c,d)\in \mathbf{S}_x} \{\min\{\mu^m_{G(e)}(a), \mu^m_{\mathbf{1}(e)}(b)\}\}, \\
&\quad \bigvee_{(a,b)\in \mathbf{S}_x} \{\min\{\mu^m_{\mathbf{1}(e)}(c), \mu^m_{F(e)}(d)\}\}, \bigvee_{(c,d)\in \mathbf{S}_x} \{\min\{\mu^m_{\mathbf{1}(e)}(c), \mu^m_{G(e)}(d)\}\}\} \\
&= \min\{ \bigvee_{(a,b)\in \mathbf{S}_x} \{\min\{\min\{\mu^m_{F(e)}(a), \mu^m_{G(e)}(a)\}, \mu^m_{\mathbf{1}(e)}(b)\}\}, \\
&\quad \bigvee_{(c,d)\in \mathbf{S}_x} \{\min\{\mu^m_{\mathbf{1}(e)}(c), \min\{\mu^m_{F(e)}(d), \mu^m_{G(e)}(d)\}\}\}\} \\
&= \min\{ \bigvee_{(a,b)\in \mathbf{S}_x} \{\min\{\mu^m_{H(e)}(a), \mu^m_{\mathbf{1}(e)}(b)\}\}, \bigvee_{(c,d)\in \mathbf{S}_x} \{\min\{\mu^m_{\mathbf{1}(e)}(c), \mu^m_{H(e)}(d)\}\} \\
&= \min\{\mu^m_{(H\widetilde{\diamond}\mathbf{1})(e)}(x), \mu^m_{(\mathbf{1}\widetilde{\diamond}H)(e)}(x)\}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\nu^n_{H(e)}(x) &= \max\{\nu^n_{F(e)}(x), \nu^n_{G(e)}(x)\} \\
&\leq \max\{ \bigwedge_{(a,b)\in \mathbf{S}_x} \{\max\{\nu^n_{F(e)}(a), \nu^n_{\mathbf{1}(e)}(b)\}\}, \bigwedge_{(c,d)\in \mathbf{S}_x} \{\max\{\nu^q_{\mathbf{1}(e)}(c), \nu^n_{F(e)}(d)\}\},
\end{aligned}$$

$$\begin{aligned}
&\bigwedge_{(a,b)\in\mathbf{S}_x}\{\max\{\nu^n_{G(e)}(a),\nu^n_{\mathbf{1}(e)}(b)\}\},\bigwedge_{(c,d)\in\mathbf{S}_x}\{\max\{\nu^n_{\mathbf{1}(e)}(c),\nu^n_{G(e)}(d)\}\}\}\\
&=\max\{\bigwedge_{(a,b)\in\mathbf{S}_x}\{\max\{\nu^n_{F(e)}(a),\nu^n_{\mathbf{1}(e)}(b)\}\},\bigwedge_{(c,d)\in\mathbf{S}_x}\{\max\{\nu^n_{G(e)}(a),\nu^n_{\mathbf{1}(e)}(b)\}\},\\
&\bigwedge_{(a,b)\in\mathbf{S}_x}\{\max\{\nu^n_{\mathbf{1}(e)}(c),\nu^n_{F(e)}(d)\}\},\bigwedge_{(c,d)\in\mathbf{S}_x}\{\max\{\nu^n_{\mathbf{1}(e)}(c),\nu^n_{G(e)}(d)\}\}\}\\
&=\max\{\bigwedge_{(a,b)\in\mathbf{S}_x}\{\max\{\max\{\nu^n_{F(e)}(a),\nu^n_{G(e)}(a)\},\nu^n_{\mathbf{1}(e)}(b)\}\},\\
&\bigwedge_{(c,d)\in\mathbf{S}_x}\{\max\{\nu^n_{\mathbf{1}(e)}(c),\max\{\nu^n_{F(e)}(d),\nu^n_{G(e)}(d)\}\}\}\\
&=\max\{\bigwedge_{(a,b)\in\mathbf{S}_x}\{\max\{\nu^n_{H(e)}(a),\nu^n_{\mathbf{1}(e)}(b)\}\},\bigwedge_{(c,d)\in\mathbf{S}_x}\{\max\{\nu^n_{\mathbf{1}(e)}(c),\nu^n_{H(e)}(d)\}\}\}\\
&=\max\{\nu^n_{(H\widetilde{\diamond}\mathbf{1})(e)}(x),\nu^n_{(\mathbf{1}\widetilde{\diamond}H)(e)}(x)\},
\end{aligned}$$

and

$$\begin{aligned}
\delta_{H(e)}(x)&=\max\{\delta_{F(e)}(x),\delta_{G(e)}(x)\}\\
&\leq\max\{\bigwedge_{(a,b)\in\mathbf{S}_x}\{\max\{\delta_{F(e)}(a),\delta_{\mathbf{1}(e)}(b)\}\},\bigwedge_{(c,d)\in\mathbf{S}_x}\{\max\{\delta_{\mathbf{1}(e)}(c),\delta_{F(e)}(d)\}\},\\
&\bigwedge_{(a,b)\in\mathbf{S}_x}\{\max\{\delta_{G(e)}(a),\delta_{\mathbf{1}(e)}(b)\}\},\bigwedge_{(c,d)\in\mathbf{S}_x}\{\max\{\delta_{\mathbf{1}(e)}(c),\delta_{G(e)}(d)\}\}\}\\
&=\max\{\bigwedge_{(a,b)\in\mathbf{S}_x}\{\max\{\delta_{F(e)}(a),\delta_{\mathbf{1}(e)}(b)\}\},\bigwedge_{(c,d)\in\mathbf{S}_x}\{\max\{\delta_{G(e)}(a),\delta_{\mathbf{1}(e)}(b)\}\},\\
&\bigwedge_{(a,b)\in\mathbf{S}_x}\{\max\{\delta_{\mathbf{1}(e)}(c),\delta_{F(e)}(d)\}\},\bigwedge_{(c,d)\in\mathbf{S}_x}\{\max\{\delta_{\mathbf{1}(e)}(c),\delta_{G(e)}(d)\}\}\}\\
&=\max\{\bigwedge_{(a,b)\in\mathbf{S}_x}\{\max\{\max\{\delta_{F(e)}(a),\delta_{G(e)}(a)\},\delta_{\mathbf{1}(e)}(b)\}\},\\
&\bigwedge_{(c,d)\in\mathbf{S}_x}\{\max\{\delta_{\mathbf{1}(e)}(c),\max\{\delta_{F(e)}(d),\delta_{G(e)}(d)\}\}\}\\
&=\max\{\bigwedge_{(a,b)\in\mathbf{S}_x}\{\max\{\delta_{H(e)}(a),\delta_{\mathbf{1}(e)}(b)\}\},\bigwedge_{(c,d)\in\mathbf{S}_x}\{\max\{\delta_{\mathbf{1}(e)}(c),\delta_{H(e)}(d)\}\}\}\\
&=\max\{\delta_{(H\widetilde{\diamond}\mathbf{1})(e)}(x),\delta_{(\mathbf{1}\widetilde{\diamond}H)(e)}(x)\}.
\end{aligned}$$

Thus, the algebraic conditions for a tripolar (m,n)-fuzzy soft quasi-ideal are completely satisfied. Let $x,y\in S$ such that $x\leq y$. Since (F,E) and (G,E) are tripolar (m,n)-fuzzy soft quasi-ideals, they both satisfy the order condition. Thus, we have:

$$\begin{aligned}
\mu^m_{H(e)}(x)&=\min\{\mu^m_{F(e)}(x),\mu^m_{G(e)}(x)\}\\
&\geq\min\{\mu^m_{F(e)}(y),\mu^m_{G(e)}(y)\}\\
&=\mu^m_{H(e)}(y).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}\nu^n_{H(e)}(x) &= \max\{\nu^n_{F(e)}(x), \nu^n G(e)(x)\}\\ &\leq \max\{\nu^n_{F(e)}(y), \nu^n_{G(e)}(y)\}\\ &= \nu^n_{H(e)}(y),\end{aligned}$$

and

$$\begin{aligned}\delta_{H(e)}(x) &= \max\{\delta_{F(e)}(x), \delta_{G(e)}(x)\}\\ &\leq \max\{\delta_{F(e)}(y), \delta_{G(e)}(y)\}\\ &= \delta_{H(e)}(y).\end{aligned}$$

This confirms that (H,E) satisfies the order condition. Consequently, $(H,E) = (F,E)\tilde{\cap}(G,E)$ is a tripolar (m,n)-fuzzy soft quasi-ideal over $\mathbf{S}$. □

One of the most significant results in the theory of ideals is the ability to represent a complex structure through simpler, well-defined components. In the following theorem, we provide a complete characterization of tripolar (m,n)-fuzzy soft quasi-ideals by showing that they are exactly the intersections of tripolar (m,n)-fuzzy soft left and right ideals. This result simplifies the study of quasi-ideals by linking them directly to the properties of left and right ideals established in the previous lemmas.

Theorem 3.2. *Let $\mathbf{S}$ be an ordered semigroup and let (F,E) be a tripolar (m,n)-fuzzy soft set over S. Then the following conditions are equivalent.*

1. *(F,E) is a tripolar (m,n)-fuzzy soft quasi-ideal over $\mathbf{S}$.*
2. *There exist a tripolar (m,n)-fuzzy soft left ideal (L,E) and a tripolar (m,n)-fuzzy soft right ideal (R,E) over $\mathbf{S}$ such that $(F,E) = (L,E)\tilde{\cap}(R,E)$.*

Proof. (1)⇒(2): Assume that (F,E) is a tripolar (m,n)-fuzzy soft quasi-ideal over $\mathbf{S}$. Define

$$(L,E) = (F,E)\tilde{\cup}\Big((\mathbf{1},E)\tilde{\diamond}(F,E)\Big) \quad \text{and} \quad (R,E) = (F,E)\tilde{\cup}\Big((F,E)\tilde{\diamond}(\mathbf{1},E)\Big).$$

According to Lemma 3.10 and Lemma 3.11, (L,E) and (R,E) are a tripolar (m,n)-fuzzy soft left ideal and a tripolar (m,n)-fuzzy soft right ideal over $\mathbf{S}$, respectively. By Lemma 3.12, we obtain

$$\begin{aligned}(L,E)\tilde{\cap}(R,E) &= \Big((F,E)\tilde{\cup}\Big((\mathbf{1},E)\tilde{\diamond}(F,E)\Big)\Big)\tilde{\cap}\Big((F,E)\tilde{\cup}\Big((F,E)\tilde{\diamond}(\mathbf{1},E)\Big)\Big)\\ &= (F,E)\tilde{\cup}\Big(\Big((\mathbf{1},E)\tilde{\diamond}(F,E)\Big)\tilde{\cap}\Big((F,E)\tilde{\diamond}(\mathbf{1},E)\Big)\Big)\\ &\tilde{\subseteq}(F,E)\tilde{\cup}(F,E)\\ &= (F,E).\end{aligned}$$

Therefore, $(L,E)\tilde{\cap}(R,E) = (F,E)$.

(2)⇒(1): Conversely, let $(F,E) = (L,E)\tilde{\cap}(R,E)$ for some tripolar (m,n)-fuzzy soft left ideal (L,E) and tripolar (m,n)-fuzzy soft right ideal (R,E). Since any tripolar (m,n)-fuzzy soft left (right) ideal is a tripolar (m,n)-fuzzy soft quasi-ideal, and by Lemma 3.13, the intersection of tripolar (m,n)-fuzzy soft quasi-ideals is a tripolar (m,n)-fuzzy soft quasi-ideal, we conclude that (F,E) is a tripolar (m,n)-fuzzy soft quasi-ideal over $\mathbf{S}$. □

4. Conclusion

In this paper, we studied a new algebraic structure called the tripolar (m,n)-fuzzy soft set over ordered semigroups. We introduced several core definitions, including:

- Tripolar (m,n)-fuzzy soft subsemigroups,
- Tripolar (m,n)-fuzzy soft left and right ideals,
- Tripolar (m,n)-fuzzy soft quasi-ideals, and
- Tripolar (m,n)-fuzzy soft bi-ideals.

We also provided concrete counterexamples to illustrate the basic relationships and differences among these structures.

The main results of this research are summarized as follows:

(1) We proved that every tripolar (m,n)-fuzzy soft quasi-ideal is the intersection of a tripolar (m,n)-fuzzy soft left ideal and a tripolar (m,n)-fuzzy soft right ideal.
(2) We showed that tripolar (m,n)-fuzzy soft quasi-ideals and bi-ideals are identical (they coincide) when the underlying ordered semigroup is regular.

These new findings successfully generalize previous results in fuzzy soft set theory. By adding neutral and negative dimensions, our framework can handle complex information more effectively.

For future research, we plan to extend this work in two main directions:

- First, we will study the properties of tripolar (m,n)-fuzzy soft prime ideals.
- Second, we will apply these mathematical structures to practical decision-making problems, especially when dealing with neutral or conflicting data.

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