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Balance Spherical Interval Valued Fuzzy Graphs

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Abstract. Spherical interval-valued fuzzy graphs (SIvFGs) represent a further advancement of the original concept of spherical fuzzy graphs (SFGs). Within this framework, the balanced spherical fuzzy graph forms a distinct subclass that highlights equilibrium in structural relationships. In this study, we introduce the concept of balanced spherical interval-valued fuzzy graphs based on density functions and investigate their fundamental properties and examine their key properties. We also establish a set of comprehensive criteria for determining whether a given SIvFG is balanced. Furthermore, an analytical approach is developed to determine whether an SIvFG is balanced. Overall, this work illustrates how balanced spherical interval-valued fuzzy graphs can effectively model and interpret the interconnections among neighboring nations.

1. Introduction

In 1965, Zadeh [44] introduced the concept of fuzzy subsets as a means to represent and manage uncertainty. Since its introduction, fuzzy set theory has become a vital research area, inspiring extensive studies across numerous disciplines such as life and medical sciences, management, social sciences, engineering, statistics, graph theory, artificial intelligence, signal processing, multi-agent systems, pattern recognition, robotics, computer networks, expert systems, decision-making, and automata theory. Rosenfeld [34] later defined the notion of fuzzy graphs (FGs) in 1975, establishing a structure analogous to classical graph theory. Since then, various extensions of fuzzy graphs have been proposed using different types of fuzzy sets and relations. For example, Santhimaheswari et al. [32] investigated strongly edge-regular and totally regular FGs, while Al-Hawary [6–10] explored related concepts. Parvathi and Karunambigai [31] extended FGs to intuitionistic fuzzy graphs (IFGs), which were later studied in depth by Akram and Davvaz [2]

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through the notion of strong IFGs. Naz et al. [30] introduced Pythagorean fuzzy graphs (PYFGs) as a generalization of IFGs and demonstrated their applications in decision-making. Karunambigai et al. [25] developed ideas related to the density and balance of IFGs and defined their direct products. Samanta and Pal [38] presented fuzzy planar graphs, along with classifications of edges as effective or considerable, while Pal et al. [36] explored fuzzy k-competition graphs. Akram et al. [3] further examined specific types of PYFGs and their decision-making applications. Sahoo and Pal [35] discussed intuitionistic fuzzy competition graphs. The concept of neutrality was introduced by Cuong and Kreinovich [13, 14], leading to the development of picture fuzzy sets (PFSs) and their graphical counterpart—picture fuzzy graphs (PFGs). Several researchers, such as Zuo et al. [47] and Xiao et al. [42], expanded this framework through operations like cartesian and direct products, and also introduced picture fuzzy multigraphs (PFMGs). The ideas of PFG dominance and edge domination were later examined by Ismayil and colleagues [21,22]. Further advancements include the introduction of q-rung picture fuzzy line graphs by Akram and Habib [4], and the development of picture Dombi fuzzy graphs by Mohanta et al. [29]. Jan et al. [23] extended this concept to interval-valued q-rung orthopair fuzzy graphs (IVQROPFGs), while Koczy et al. [26] applied PFGs to social and Wi-Fi networks. Moreover, Jan et al. proposed a model combining complex intuitionistic fuzzy sets and soft sets to evaluate operating systems. Spherical fuzzy sets (SFSs), proposed by Gündoğdu and Kahraman [20], further extend picture fuzzy sets by allowing degrees of truth (η), indeterminacy (ϑ), and falsity (ω) to satisfy the condition $0 \leq \eta^2 + \vartheta^2 + \omega^2 \leq 1$. In 2024, P. Khamrot and T. Gaketem [18] studied spherical interval-valued fuzzy ideals in semigroups. This framework offers a more flexible representation of uncertainty. Akram et al. [5] applied spherical fuzzy graphs (SFGs) to decision-making, and Zeadam et al. [46] introduced T-spherical fuzzy information for solving shortest path problems. Other studies include works by Jan et al. [24] on double domination, Guleria et al. [19] on operations and selection processes, Shoaib et al. [39] on complex spherical fuzzy graphs, and Mohamed et al. [28] on their energy properties. Balanced fuzzy graphs have proven significant across many fields. Rashmanlou and Pal [33] introduced balanced interval-valued fuzzy graphs, while Ghorai and Pal [17], Sankar and Ezhilmaran [1], and others explored various balanced forms, including m-polar, bipolar, picture, and neutrosophic fuzzy graphs [11, 27, 40]. In 2025, B. Some et al. [41] presented the research to spherical fuzzy graphs.

In this paper, we introduce the concept of balanced spherical interval-valued fuzzy graphs based on density functions and investigate their fundamental properties. We also establish comprehensive criteria for determining whether a spherical interval-valued fuzzy graph is balanced. Finally, we demonstrate how balanced spherical interval-valued fuzzy graphs can effectively model the interrelationships among neighboring nations.

2. Preliminaries

In this section, we recall some basic definitions and concepts that will be used throughout the paper.

For any $\eta_i \in [0,1]$ where $i \in \mathcal{J}$ define

$$\bigvee_{i\in\mathcal{J}} \eta_i := \sup_{i\in\mathcal{J}}\{\eta_i\} \quad \text{and} \quad \bigwedge_{i\in\mathcal{J}} \eta_i := \inf_{i\in\mathcal{J}}\{\eta_i\}.$$

We see that for any $\eta_1, \eta_2 \in [0,1]$, we have

$$\eta_1 \vee \eta_2 = \max\{\eta_1, \eta_2\} \quad \text{and} \quad \eta_1 \wedge \eta_2 = \min\{\eta_1, \eta_2\}.$$

Definition 2.1. *[44] A fuzzy subset (fuzzy set) $\acute{\eta}$ of a non-empty set $\mathfrak{T}$ is a function from $\mathfrak{T}$ into the closed interval $[0,1]$, i.e, $\acute{\eta} : \mathfrak{T} \to [0,1]$.*

Definition 2.2. *[18] Let $\mathfrak{T}$ be a non-empty set. A* Pythagorean fuzzy set *(PFS) $P := \{u, \eta(u), \vartheta(u) \mid u \in \mathfrak{G}\}$ where $\eta : \mathfrak{T} \to [0,1]$ and $\vartheta : \mathfrak{T} \to [0,1]$ represent the degree of membership and non-membership of the object $z \in \mathfrak{T}$ to the set P subset to the condition $0 \leq (\eta(u))^2 + (\vartheta(u))^2 \leq 1$ for all $u \in \mathfrak{T}$. For the sake of simplicity, a PFS is denoted as $P = (\eta(u); \vartheta(u))$.*

Definition 2.3. *[18] Let $\mathfrak{T}$ be a non-empty set. A* spherical fuzzy set *(SF-set) $\mathrm{SP} := \{u, \eta(u), \vartheta(u), \omega(u) \mid u \in \mathfrak{G}\}$ where $\eta : \mathfrak{T} \to [0,1]$, $\vartheta : \mathfrak{T} \to [0,1]$ and $\omega : \mathfrak{T} \to [0,1]$ represent the degree of membership, non-membership and hesitancy of the object $u \in \mathfrak{T}$ to the set $\mathrm{SP} = (\eta; \vartheta, \omega)$ subset to the condition $0 \leq (\eta(u))^2 + (\vartheta(u))^2 + (\omega(u))^2 \leq 1$ for all $u \in \mathfrak{T}$. For the sake of simplicity, an $\mathrm{SP} = (\eta(u); \vartheta(u), \omega(u))$ is denoted as $\mathrm{SP} = (\eta; \vartheta, \omega)$.*

Definition 2.4. *[33] An* interval valued fuzzy set *$\acute{\eta}$ in a set $\mathfrak{T}$ is a mapping $\acute{\eta} : \mathfrak{T} \to \mathcal{I}([0,1])$, where $\mathcal{I}([0,1]) = \{[u,v] \mid 0 \leq u \leq v \leq 1\}$. Denote $[0,0]$ by $\tilde{0}$ and $[1,1]$ by $\tilde{1}$.*

Definition 2.5. *For $\acute{\eta} = [\eta^-, \eta^+]$, $\acute{\rho} = [\rho^-, \rho^+] \in \mathcal{I}([0,1])$, define:*

(1) $\acute{\eta} \leq \acute{\rho} \iff \eta^- \leq \rho^-, \eta^+ \leq \rho^+$,
(2) $\acute{\eta} = \acute{\rho} \iff \eta^- = \rho^-, \eta^+ = \rho^+$,
(3) $\acute{\eta} \curlywedge \acute{\rho} = [\eta^- \wedge \rho^-, \eta^+ \wedge \rho^+]$,
(4) $\acute{\eta} \curlyvee \acute{\rho} = [\eta^- \vee \rho^-, \eta^+ \vee \rho^+]$.

Definition 2.6. *[18] Let $\mathfrak{T}$ be a non-empty set. A* spherical interval valued fuzzy set *(SIVF-set) $\overline{SP} := \{u, \acute{\eta}(u), \acute{\vartheta}(u), \acute{\omega}(u) \mid u \in \mathfrak{G}\}$ where $\acute{\eta} : \mathfrak{T} \to \mathcal{I}([0,1])$, $\acute{\vartheta} : \mathfrak{T} \to \mathcal{I}([0,1])$ and $\acute{\omega} : \mathfrak{T} \to \mathcal{I}([0,1])$ represent the degree of membership, non-membership and hesitancy of the object $u \in \mathfrak{T}$ to the set $\overline{SP}$ subset to the condition $\overline{0} \leq (\acute{\eta}(u))^2 + (\acute{\vartheta}(u))^2 + (\acute{\omega}(u))^2 \leq \overline{1}$ for all $u \in \mathfrak{T}$. For the sake of simplicity a PFS is denoted as $\overline{SP} := (\acute{\eta}, \acute{\vartheta}, \acute{\omega})$.*

Definition 2.7. *[41] A spherical fuzzy graph (SPFG) $G = (V, E)$ is of the form:*

(1) $V = \{v = v_1, v_2, v_3, \ldots, v_n\}$ *such that* $\eta_V : V \to [0,1]$, $\vartheta_V : V \to [0,1]$, *and* $\omega_V : V \to [0,1]$ *denote the degree of membership, neutral, and non-membership of the element* $v_i \in V$ *respectively, and*

$$0 \leq \eta_V^2(v_i) + \vartheta_V^2(v_i) + \omega_V^2(v_i) \leq 1, \quad \forall v_i \in V.$$

(2) $E \subseteq V \times V$ *where* $\eta_E : V \times V \to [0,1]$, $\vartheta_E : V \times V \to [0,1]$, *and* $\omega_E : V \times V \to [0,1]$ *are such that*

$$\eta_E(v_i, v_j) \leq \eta_V(v_i) \wedge \eta_V(v_j),$$
$$\vartheta_E(v_i, v_j) \leq \vartheta_V(v_i) \wedge \vartheta_V(v_j),$$
$$\omega_E(v_i, v_j) \leq \omega_V(v_i) \vee \omega_V(v_j), \quad \forall v_i, v_j \in V,$$

and

$$0 \leq \eta_E^2(v_i, v_j) + \vartheta_E^2(v_i, v_j) + \omega_E^2(v_i, v_j) \leq 1, \quad \forall (v_i, v_j) \in V \times V.$$

Where $\eta_E : V \times V \to [0,1]$, $\vartheta_E : V \times V \to [0,1]$, *and* $\omega_E : V \times V \to [0,1]$ *represent the membership, neutral, and non-membership grades of E respectively.*

Definition 2.8. *[41] An SPFG* $G = (V, E)$ *is defined as a* complete SPFG *if*

$$\eta_E(v_i, v_j) = \eta_V(v_i) \wedge \eta_V(v_j),$$
$$\vartheta_E(v_i, v_j) = \vartheta_V(v_i) \wedge \vartheta_V(v_j),$$
$$\omega_E(v_i, v_j) = \omega_V(v_i) \vee \omega_V(v_j), \quad \forall v_i, v_j \in V,$$

and

$$0 \leq \eta_E^2(v_i, v_j) + \vartheta_E^2(v_i, v_j) + \omega_E^2(v_i, v_j) \leq 1, \quad \forall (v_i, v_j) \in V \times V.$$

Definition 2.9. *[41] An SFG on a non-empty set X is a pair* $G = (V, E)$ *defined as a* strong SPFG *if*

$$\eta_E(v_i, v_j) = \eta_V(v_i) \wedge \eta_V(v_j),$$
$$\vartheta_E(v_i, v_j) = \vartheta_V(v_i) \wedge \vartheta_V(v_j),$$
$$\omega_E(v_i, v_j) = \omega_V(v_i) \vee \omega_V(v_j), \quad \forall v_i, v_j \in V$$

and

$$0 \leq \eta_E^2(v_i v_j) + \vartheta_E^2(v_i v_j) + \omega_E^2(v_i v_j) \leq 1, \quad \forall (v_i, v_j) \in V \times V.$$

Definition 2.10. *[41] An SFG* $G = (V, E)$ *is called a* single valued average SPFG *if*

$$\eta_E(v_i, v_j) = \frac{1}{2}(\eta_V(v_i) \wedge \eta_V(v_j)),$$
$$\vartheta_E(v_i, v_j) = \frac{1}{2}(\vartheta_V(v_i) \wedge \vartheta_V(v_j)),$$
$$\omega_E(v_i, v_j) = \frac{1}{2}(\omega_V(v_i) \vee \omega_V(v_j)), \quad \forall v_i, v_j \in V$$

and

$$0 \leq \eta_E^2(v_i, v_j) + \vartheta_E^2(v_i, v_j) + \omega_E^2(v_i, v_j) \leq 1, \quad \forall (v_i, v_j) \in V \times V.$$

Definition 2.11. *[41] The complement of an SPFG $G = (V, E)$ is a spherical fuzzy graph $G^C = (V^C, E^C)$ if and only if*

(1) $V = V^C$
(2) $\eta_V^C(v_i) = \eta_V(v_i),\ \vartheta_V^C(v_i) = \vartheta_V(v_i),\ \omega_V^C(v_i) = \omega_V(v_i),\ \forall v_i \in V.$
(3)

$$\eta_E^C(v_i, v_j) = \eta_V(v_i) \wedge \eta_V(v_j) - \eta_E(v_i, v_j),$$
$$\vartheta_E^C(v_i, v_j) = \vartheta_V(v_i) \wedge \vartheta_V(v_j) - \vartheta_V(v_i, v_j),$$
$$\omega_E^C(v_i, v_j) = \omega_V(v_i) \vee \omega_V(v_j) - \omega_V(v_i, v_j), \quad \forall (v_i, v_j) \in V \times V.$$

Theorem 2.1. *[41] Let $G = (V, E)$ be SPFG. Then G is a self-complementary if and only G is an average SPFG.*

Definition 2.12. *[41] Let $G = (V, E)$ be an SPFG, then the size of G is denoted by Λ_G and is defined by*

$$\Lambda_G = (\Lambda_\eta(G), \Lambda_\vartheta(G), \Lambda_\omega(G)),$$

where

$$\Lambda_\eta(G) = \sum_{v_i \neq v_j} \eta_E(v_i, v_j),$$
$$\Lambda_\vartheta(G) = \sum_{v_i \neq v_j} \vartheta_E(v_i, v_j),$$
$$\Lambda_\omega(G) = \sum_{v_i \neq v_j} \omega_E(v_i, v_j) \quad \forall v_i, v_j \in V.$$

Definition 2.13. *[41] Let $G = (V, E)$ be an SPFG. Then the weight of G is denoted by*

$$M_G = (M_\eta(G), M_\vartheta(G), M_\omega(G)),$$

where

$$M_\eta(G) = \sum_{(v_i, v_j) \in E} \eta_V(v_i) \wedge \eta_V(v_j),$$
$$M_\vartheta(G) = \sum_{(v_i, v_j) \in E} \vartheta_V(v_i) \wedge \vartheta_V(v_j),$$
$$M_\omega(G) = \sum_{(v_i, v_j) \in E} \omega_V(v_i) \vee \omega_V(v_j), \quad \forall v_i, v_j \in V.$$

Definition 2.14. *[41] Let $G = (V, E)$ be an SPFG. Then the density of G is denoted by*

$$\kappa(G) = (\kappa_\eta(G), \kappa_\vartheta(G), \kappa_\omega(G)),$$

where

$$\kappa_\eta(G) = \frac{\Lambda_\eta(G)}{M_\eta(G)}, \quad \kappa_\vartheta(G) = \frac{\Lambda_\vartheta(G)}{M_\vartheta(G)}, \quad \kappa_\omega(G) = \frac{\Lambda_\omega(G)}{M_\omega(G)}.$$

for all $v_i, v_j \in V$.

Definition 2.15. *[41] An SPFG $G = (V, E)$ is said to be* balanced *if all of its subgraphs are in G, i.e.,*

$$\kappa(S) \leq \kappa(G)$$

for any subgraph S of G. Now $\kappa(S) \leq \kappa(G)$ holds if

$$\kappa_{\eta}(S) \leq \kappa_{\eta}(G), \quad \kappa_{\vartheta}(S) \leq \kappa_{\vartheta}(G), \quad \kappa_{\omega}(S) \leq \kappa_{\omega}(G).$$

Theorem 2.2. *[41] Let $G = (V, E)$ be an average SPFG, and $\kappa(G)$ is the density of G. Then $\kappa(G) = (1, 1, 1)$.*

3. Main Result

Definition 3.1. *A spherical interval valued fuzzy graph (SPIvFG) $G = (V, E)$ is of the form:*

(1) *$V = \{v = v_1, v_2, v_3, \ldots, v_n\}$ such that $\acute{\eta}_V : V \to \mathcal{I}([0, 1])$, $\acute{\vartheta}_V : V \to \mathcal{I}([0, 1])$, and $\acute{\omega}_V : V \to \mathcal{I}([0, 1])$ denote the degree of membership, neutral, and non-membership of the element $v_i \in V$ respectively, and*

$$\acute{0} \leq \acute{\eta}_V^2(v_i) + \acute{\vartheta}_V^2(v_i) + \acute{\omega}_V^2(v_i) \leq \acute{1}, \quad \forall v_i \in V.$$

(2) *$E \subseteq V \times V$ where $\acute{\eta}_E : V \times V \to \mathcal{I}([0, 1])$, $\acute{\vartheta}_E : V \times V \to \mathcal{I}([0, 1])$, and $\acute{\omega}_E : V \times V \to \mathcal{I}([0, 1])$ are such that*

$$\begin{aligned} \acute{\eta}_E(v_i, v_j) &\leq \acute{\eta}_V(v_i) \wedge \eta_V(v_j), \\ \acute{\vartheta}_E(v_i, v_j) &\leq \acute{\vartheta}_V(v_i) \wedge \acute{\vartheta}_V(v_j), \\ \acute{\omega}_E(v_i, v_j) &\leq \acute{\omega}_V(v_i) \vee \acute{\omega}_V(v_j), \quad \forall v_i, v_j \in V, \end{aligned}$$

and

$$\acute{0} \leq \acute{\eta}_E^2(v_i, v_j) + \acute{\vartheta}_E^2(v_i, v_j) + \acute{\omega}_E^2(v_i, v_j) \leq \acute{1}, \quad \forall (v_i, v_j) \in V \times V.$$

Where $\acute{\eta}_E : V \times V \to \mathcal{I}([0, 1])$, $\acute{\vartheta}_E : V \times V \to \mathcal{I}([0, 1])$, and $\acute{\omega}_E : V \times V \to \mathcal{I}([0, 1])$ represent the membership, neutral, and non-membership grades of E respectively.

Definition 3.2. *An SPIvFG $G = (V, E)$ is defined as a* complete SPIvFG *if*

$$\begin{aligned} \acute{\eta}_E(v_i, v_j) &= \acute{\eta}_V(v_i) \wedge \acute{\eta}_V(v_j), \\ \acute{\vartheta}_E(v_i, v_j) &= \acute{\vartheta}_V(v_i) \wedge \acute{\vartheta}_V(v_j), \\ \acute{\omega}_E(v_i, v_j) &= \acute{\omega}_V(v_i) \vee \acute{\omega}_V(v_j), \quad \forall v_i, v_j \in V, \end{aligned}$$

and

$$\acute{0} \leq \acute{\eta}_E^2(v_i, v_j) + \acute{\vartheta}_E^2(v_i, v_j) + \acute{\omega}_E^2(v_i, v_j) \leq \acute{1}, \quad \forall (v_i, v_j) \in V \times V.$$

Definition 3.3. *An SIvFG on a non-empty set X is a pair $G = (V, E)$ defined as a* strong SPIvFG *if*

$$\acute{\eta}_E(v_i, v_J) = \acute{\eta}_V(v_i) \wedge \acute{\eta}_V(v_J),$$

$$\acute{\vartheta}_E(v_i, v_J) = \acute{\vartheta}_V(v_i) \wedge \acute{\vartheta}_V(v_J),$$

$$\acute{\omega}_E(v_i, v_J) = \acute{\omega}_V(v_i) \vee \acute{\omega}_V(v_J), \quad \forall v_i, v_J \in V$$

and

$$\acute{0} \leq \acute{\eta}_E^2(v_i, v_J) + \acute{\vartheta}_E^2(v_i, v_J) + \acute{\omega}_E^2(v_i, v_J) \leq \acute{1}, \quad \forall (v_i, v_J) \in V \times V.$$

Definition 3.4. *An SIvFG $G = (V, E)$ is called a* single valued average SPIvFG *if*

$$\acute{\eta}_E(v_i, v_j) = \frac{1}{2}(\acute{\eta}_V(v_i) \wedge \acute{\eta}_V(v_j),$$

$$\acute{\vartheta}_E(v_i, v_j) = \frac{1}{2}(\acute{\vartheta}_V(v_i) \wedge \acute{\vartheta}_V(v_j),$$

$$\acute{\omega}_E(v_i, v_j) = \frac{1}{2}(\acute{\omega}_V(v_i) \vee \acute{\omega}_V(v_j), \quad \forall v_i, v_j \in V$$

and

$$\acute{0} \leq \acute{\eta}_E^2(v_i, v_j) + \acute{\vartheta}_E^2(v_i, v_j) + \acute{\omega}_E^2(v_i, v_j) \leq \acute{1}, \quad \forall (v_i, v_j) \in V \times V.$$

Definition 3.5. *The complement of an SPIvFG $G = (V, E)$ is an SPIvFG $G^C = (V^C, E^C)$ if and only if*

(1) $V = V^C$

(2) $\acute{\eta}_V^C(v_i) = \acute{\eta}_V(v_i)$, $\acute{\vartheta}_V^C(v_i) = \acute{\vartheta}_V(v_i)$, $\acute{\omega}_V^C(v_i) = \acute{\omega}_V(v_i)$, $\forall v_i \in V$.

(3)

$$\acute{\eta}_E^C(v_i, v_j) = \acute{\eta}_V(v_i) \wedge \acute{\eta}_V(v_j) - \acute{\eta}_E(v_i, v_j),$$

$$\acute{\vartheta}_E^C(v_i, v_j) = \acute{\vartheta}_V(v_i) \wedge \acute{\vartheta}_V(v_j) - \acute{\vartheta}_V(v_i, v_j),$$

$$\acute{\omega}_E^C(v_i, v_j) = \acute{\omega}_V(v_i) \vee \acute{\omega}_V(v_j) - \acute{\omega}_V(v_i, v_j), \quad \forall (v_i, v_j) \in V \times V.$$

Theorem 3.1. *Let $G = (V, E)$ be SPIvFG. Then G is a self-complementary if and only G is an average SPIvFG.*

Proof. Assume that $G = (V, E)$ be an SPIvFG and $G^C = (V^C, E^C)$ be its complement of G. Then $\acute{\eta}_V^C = \acute{\eta}_V$, $\acute{\vartheta}_V^C = \acute{\vartheta}_V$, $\acute{\omega}_V^C = \acute{\omega}_V$ and

$$\acute{\eta}_E(u, v) = \acute{\eta}_V(u) \wedge \acute{\eta}_V(v) - \acute{\eta}_E(u, v),$$

$$\acute{\vartheta}_E(u, v) = \acute{\vartheta}_V(u) \wedge \acute{\vartheta}_V(v) - \acute{\vartheta}_E(u, v),$$

$$\acute{\omega}_E(u, v) = \acute{\omega}_V(u) \vee \acute{\omega}_V(v) - \acute{\omega}_E(u, v), \quad \forall u, v \in V.$$

Let G be a single-valued average SIvFG. Then

$$\acute{\eta}_E(u, v) = \frac{1}{2}(\acute{\eta}_V(u) \wedge \acute{\eta}_V(v)),$$

$$\acute{\vartheta}_E(u, v) = \frac{1}{2}(\acute{\vartheta}_V(u) \wedge \acute{\vartheta}_V(v)),$$

$$\acute{\omega}_E(u, v) = \frac{1}{2}(\acute{\omega}_V(u) \vee \acute{\omega}_V(v)), \quad \forall u, v \in V.$$

Consider,

$$\begin{aligned} \acute{\eta}_E^C(u, v) &= \acute{\eta}_V(u) \wedge \acute{\eta}_V(v) - \acute{\eta}_V(u, v) \\ &= \acute{\eta}_V(u) \wedge \acute{\eta}_V(v) - \frac{1}{2}(\acute{\eta}_V(u) \wedge \acute{\eta}_V(v)) \end{aligned}$$

$$= \frac{1}{2}(\acute{\eta}_V(u) \wedge \acute{\eta}_V(v)), \quad \forall u, v \in V.$$

Similarly,

$$\acute{\vartheta}_E^C(u,v) = \acute{\vartheta}_E(u,v), \quad \acute{\omega}_E^C(u,v) = \acute{\omega}_E(u,v).$$

Thus, G^C is similar to G. Hence, G is a self-complementary SPIvFG.

Conversely, let G be a self-complementary SPIvFG. Then

$$\acute{\eta}_E^C(u,v) = \acute{\eta}_E(u,v), \quad \acute{\vartheta}_E^C(u,v) = \acute{\vartheta}_E(u,v), \quad \acute{\omega}_E^C(u,v) = \acute{\omega}_E(u,v), \quad \forall u, v \in V.$$

Consider,

$$\begin{aligned} \acute{\eta}_E(u,v) = \acute{\eta}_E^C(u,v) &= \acute{\eta}_V(u) \wedge \acute{\eta}_V(v) - \acute{\eta}_E(u,v) \\ &\Rightarrow 2\acute{\eta}_E(u,v) = \acute{\eta}_V(u) \wedge \acute{\eta}_V(v) \\ &\Rightarrow \acute{\eta}_E(u,v) = \frac{1}{2}(\acute{\eta}_V(u) \wedge \acute{\eta}_V(v)). \end{aligned}$$

Similarly, it can be shown that

$$\acute{\vartheta}_E(u,v) = \frac{1}{2}(\acute{\vartheta}_V(u) \wedge \acute{\vartheta}_V(v))$$

and

$$\acute{\omega}_E(u,v) = \frac{1}{2}(\acute{\omega}_V(u) \vee \acute{\omega}_V(v)), \quad \forall u, v \in V.$$

Hence, G is a single-valued average SPIvFG. □ □

Definition 3.6. *Let $G = (V, E)$ be an SPIvFG, then the size of G is denoted by Λ_G and is defined by*

$$\Lambda_G = (\Lambda_{\acute{\eta}}(G), \Lambda_{\acute{\vartheta}}(G), \Lambda_{\acute{\omega}}(G)),$$

where

$$\begin{aligned} \Lambda_{\acute{\eta}}(u) &= \sum_{\{v_i,v_j\} \in E} \acute{\eta}_E(v_i, v_j), \\ \Lambda_{\acute{\vartheta}}(u) &= \sum_{\{v_i,v_j\} \in E} \acute{\vartheta}_E(v_i, v_j), \\ \Lambda_{\acute{\omega}}(u) &= \sum_{\{v_i,v_j\} \in E} \acute{\omega}_E(v_i, v_j) \quad \forall v_i, v_j \in V. \end{aligned}$$

Definition 3.7. *Let $G = (V, E)$ be an SPIvFG. Then the weight of G is denoted by*

$$M_G = (M_{\acute{\eta}}(G), M_{\acute{\vartheta}}(G), M_{\acute{\omega}}(G)),$$

where

$$\begin{aligned} M_{\acute{\eta}}(G) &= \sum_{(v_i,v_j) \in E} \acute{\eta}_V(v_i) \wedge \acute{\eta}_V(v_j), \\ M_{\acute{\vartheta}}(G) &= \sum_{(v_i,v_j) \in E} \acute{\vartheta}_V(v_i) \wedge \acute{\vartheta}_V(v_j), \\ M_{\acute{\omega}}(G) &= \sum_{(v_i,v_j) \in E} \acute{\omega}_V(v_i) \vee \acute{\omega}_V(v_j), \quad \forall v_i, v_j \in V. \end{aligned}$$

Definition 3.8. *Let $G = (V, E)$ be an SPIvFG. Then the density of G is denoted by*

$$\kappa(G) = (\kappa_{\acute{\eta}}(G), \kappa_{\acute{\vartheta}}(G), \kappa_{\acute{\omega}}(G)),$$

where

$$\kappa_{\acute{\eta}}(G) = \frac{\Lambda_{\acute{\eta}}(G)}{M_{\acute{\eta}}(G)}, \quad \kappa_{\acute{\vartheta}}(G) = \frac{\Lambda_{\acute{\vartheta}}(G)}{M_{\acute{\vartheta}}(G)}, \quad \kappa_{\acute{\omega}}(G) = \frac{\Lambda_{\acute{\omega}}(G)}{M_{\acute{\omega}}(G)}.$$

for all $v_i, v_j \in V$.

Definition 3.9. *An SPIvFG $G = (V, E)$ is said to be* balanced *if all of its subgraphs are in G, i.e.,*

$$\kappa(S) \preceq \kappa(G)$$

for any subgraph S of G. Now $\kappa(S) \preceq \kappa(G)$ holds if

$$\kappa_{\acute{\eta}}(S) \leq \kappa_{\acute{\eta}}(G), \quad \kappa_{\acute{\vartheta}}(S) \leq \kappa_{\acute{\vartheta}}(G), \quad \kappa_{\acute{\omega}}(S) \geq \kappa_{\acute{\omega}}(G).$$

Definition 3.10. *An SIvFG $G = (V, E)$ is said to be strictly balanced if $\kappa(S) = \kappa(G)$ for all subgraph S of G holds if $\kappa_{\acute{\eta}}(S) = \kappa_{\acute{\eta}}(G)$, $\kappa_{\acute{\vartheta}}(S) = \kappa_{\acute{\vartheta}}(G)$, $\kappa_{\acute{\omega}}(S) = \kappa_{\acute{\omega}}(G)$.*

Theorem 3.2. *Let $G = (V, E)$ be an average SPIvFG, and $\kappa(G)$ is the density of G. Then $\kappa(G) = (\acute{1}, \acute{1}, \acute{1})$.*

Proof. Since $\kappa(G)$ is the density of PIvFG G, the density of G is given by

$$\kappa(G) = (\kappa_{\acute{\eta}}(G), \kappa_{\acute{\vartheta}}(G), \kappa_{\acute{\omega}}(G)).$$

Thus,

$$\kappa_{\acute{\eta}}(G) = \frac{\Lambda_{\acute{\eta}}(G)}{M_{\acute{\eta}}(G)} = \acute{1}.$$

Similarly $\kappa_{\acute{\vartheta}}(G) = \acute{1}$, $\kappa_{\acute{\omega}}(G) = \acute{1}$. Therefore, $\kappa(G) = (\acute{1}, \acute{1}, \acute{1})$. □

Theorem 3.3. *A SIvFG $G = (V, E)$ is a strictly balanced if and only if*

$$\acute{\eta}_E(v_i, v_j) = \xi_1(\acute{\eta}_V(v_i) \wedge \acute{\eta}_V(v_j)), \quad \acute{\vartheta}_E(v_i, v_j) = \xi_2(\acute{\vartheta}_V(v_i) \wedge \acute{\vartheta}_V(v_j)),$$

$$\acute{\omega}_E(v_i, v_j) = \xi_3(\acute{\omega}_V(v_i) \vee \acute{\omega}_V(v_j)), \qquad \forall v_i, v_j \in V \text{ where } \kappa(G) = (\xi_1, \xi_2, \xi_3).$$

Proof. Assume that the SFG $G = (V, E)$ has n nodes and is strictly balanced, and that $V = \{v_1, v_2, \ldots, v_n\}$. Then by Definition 3.10 $\kappa(S) = \kappa(G)$ for all subgraph S of G, since $\kappa(G) = (\xi_1, \xi_2, \xi_3)$.

Since G contains n nodes and $V = \{v_1, v_2, v_3, \ldots, v_n\}$ then G has $2^n - (n+1)$ subgraphs. Among the n subgraphs of G, $\binom{n}{2} = N_1$ subgraphs are the subgraphs, each containing 2 nodes. In N_1 subgraph, let us consider any arbitrary subgraph, say S_λ of G. Let $S_\lambda = (V_\lambda, E_\lambda)$ where $V_\lambda = \{v_{\lambda_i}, v_{\lambda_j}\}$. Now, the density of S_λ is given by

$$\kappa(S_\lambda) = (\kappa_{\acute{\eta}}(S_\lambda), \kappa_{\acute{\vartheta}}(S_\lambda), \kappa_{\acute{\omega}}(S_\lambda)).$$

Therefore, we have

$$\kappa_{\acute{\eta}}(S_\lambda) = \frac{\acute{\eta}_{E_\lambda}(v_{\lambda_i}, v_{\lambda_j})}{\min\{\acute{\eta}_V(v_{\lambda_i}), \acute{\eta}_V(v_{\lambda_j})\}},$$

$$\kappa_{\acute{\vartheta}}(S_\lambda) = \frac{\acute{\vartheta}_{E_\lambda}(v_{\lambda_i}, v_{\lambda_j})}{\min\{\acute{\vartheta}_V(v_{\lambda_i}), \acute{\vartheta}_V(v_{\lambda_j})\}},$$

$$\kappa_{\acute{\omega}}(S_\lambda) = \frac{\acute{\omega}_{E_\lambda}(v_{\lambda_i}, v_{\lambda_j})}{\max\{\acute{\omega}_V(v_{\lambda_i}), \acute{\omega}_V(v_{\lambda_j})\}} \qquad \forall v_{\lambda_i}, v_{\lambda_j} \in V_\lambda.$$

Since S_λ is an arbitrary subgraph G with two nodes and G is strictly balanced, so $\kappa(S_\lambda) = \kappa(G)$. Set $\kappa_{\acute{\eta}}(S_\lambda) = \xi_1, \quad \kappa_{\acute{\vartheta}}(S_\lambda) = \xi_2, \quad \kappa_{\acute{\omega}}(S_\lambda) = \xi_3$. Then

$$\acute{\eta}_{E_\lambda}(v_{\lambda_i}, v_{\lambda_j}) = \xi_1 \times \min\{\acute{\eta}_V(v_{\lambda_i}), \acute{\eta}_V(v_{\lambda_j})\},$$

$$\acute{\vartheta}_{E_\lambda}(v_{\lambda_i}, v_{\lambda_j}) = \xi_2 \times \min\{\acute{\vartheta}_V(v_{\lambda_i}), \acute{\vartheta}_V(v_{\lambda_j})\},$$

$$\acute{\omega}_{E_\lambda}(v_{\lambda_i}, v_{\lambda_j}) = \xi_3 \times \max\{\acute{\omega}_V(v_{\lambda_i}), \acute{\omega}_V(v_{\lambda_j})\}.$$

As S_λ represents any arbitrary subgraph containing two nodes from the set of nodes of G, the relation mentioned above holds for all $v_i, v_j \in V$ not belonging to the vertex set V_λ.

Conversely, let $G = (V, E)$ be a SFG where $V = \{v_1, v_2, v_3, \ldots, v_n\}$. $\kappa(G)$ be the density of G where $\kappa(G) = (\xi_1, \xi_2, \xi_3)$. The membership function of all edges satisfies the relation

$$\acute{\eta}_E(v_i, v_j) = \xi_1 \times \min\{\acute{\eta}_V(v_i), \acute{\eta}_V(v_j)\},$$

$$\acute{\vartheta}_E(v_i, v_j) = \xi_2 \times \min\{\acute{\vartheta}_V(v_i), \acute{\vartheta}_V(v_j)\},$$

$$\acute{\omega}_E(v_i, v_j) = \xi_3 \times \max\{\acute{\omega}_V(v_i), \acute{\omega}_V(v_j)\}, \qquad \forall v_i, v_j \in V.$$

Let G is strictly balanced and $S_m = (V_m, E_m)$ be any subgraph of G, where $V_m = \{v_{m_1}, v_{m_2}, v_{m_3}, \ldots, v_{m_r}\}$, $m_1, m_2, m_3, \ldots, m_r \in \{1, 2, 3, \ldots, n\}$ such that $m_i \neq m_j$ for all i, j with $\kappa(S_m) = (\kappa_{\acute{\eta}}(S_m), \kappa_{\acute{\vartheta}}(S_m), \kappa_{\omega\eta}(S_m))$ be the density of the subgraph S_m. Then

$$\kappa_{\acute{\eta}}(S_m) = \frac{\sum\limits_{v_{m_i} \neq v_{m_j}} \acute{\eta}_{E_m}(v_{m_i}, v_{m_j})}{\sum\limits_{(v_{m_i}, v_{m_j}) \in E} \min\{\acute{\eta}_V(v_{m_i}), \acute{\eta}_V(v_{m_j})\}} \quad = \xi_1 \times \frac{\sum\limits_{(v_{m_i}, v_{m_j}) \in E} \min\{\acute{\eta}_{V_m}(v_{m_i}), \acute{\eta}_{V_m}(v_{m_j})\}}{\sum\limits_{(v_{m_i}, v_{m_j}) \in E} \min\{\acute{\eta}_{V_m}(v_{m_i}), \acute{\eta}_{V_m}(v_{m_j})\}} = \xi_1.$$

Similarly, we can show that

$$\kappa_{\acute{\vartheta}}(S_\xi) = \xi_2 \qquad \text{and} \qquad \kappa_{\acute{\omega}}(S_\xi) = \xi_3.$$

Therefore, $\kappa(S_m) = \big(\kappa_{\acute{\eta}}(S_m),\ \kappa_{\acute{\vartheta}}(S_m),\ \kappa_{\acute{\omega}}(S_m)\big) = (\xi_1, \xi_2, \xi_3)$. Since S_m is an arbitrary subgraph of the SIvFG of G, $\kappa(S) = \kappa(G)$ for all subgraphs S of G. Hence G is strictly balanced. □

Corollary 3.1. *A SIvFGs $G = (V, E)$ is balanced iff*

$$\acute{\eta}_E(v_i, v_j) = \{\xi_1 \times \acute{\eta}_V(v_i) \wedge \acute{\eta}_V(v_j)\},$$
$$\acute{\vartheta}_E(v_i, v_j) = \{\xi_2 \times \acute{\vartheta}_V(v_i) \vee \acute{\vartheta}_V(v_j)\},$$
$$\acute{\omega}_E(v_i, v_j) = \{\xi_3 \times \acute{\omega}_V(v_i) \vee \acute{\omega}_V(v_j)\}, \qquad \forall (v_i, v_j) \in E,$$

where $\kappa(G) = (\xi_1, \xi_2, \xi_3)$.

Corollary 3.2. *Let $G = (V, E)$ be a balanced SIvFGs and S be any subgraph of G. Then*

$$\kappa(S) = \kappa(G) \quad \text{or} \quad \kappa(S) = (\acute{0}, \acute{0}, \acute{0}).$$

Theorem 3.4. *Let $G = (V, E)$ be a complete SIvFG. Then $\kappa(G) = (\acute{1}, \acute{1}, \acute{1})$.*

Proof. Let $G = (V, E)$ be a complete spherical fuzzy graph, where

$$\acute{\eta}_E(v_i, v_j) = \acute{\eta}_V(v_i) \wedge \acute{\eta}_V(v_j),$$
$$\acute{\vartheta}_E(v_i, v_j) = \acute{\vartheta}_V(v_i) \wedge \acute{\vartheta}_V(v_j),$$
$$\acute{\omega}_E(v_i, v_j) = \acute{\omega}_V(v_i) \vee \acute{\omega}_V(v_j)\}, \forall v_i, v_j \in E.$$

Since, $\kappa(G)$ is the density of G,

$$\kappa(G) = (\kappa_{\acute{\eta}}(G), \kappa_{\acute{\vartheta}}(G), \kappa_{\acute{\omega}}(G)).$$

Thus,

$$\kappa_{\acute{\eta}}(G) = \frac{S_{\acute{\eta}}(G)}{W_{\acute{\eta}}(G)} = \acute{1}.$$

Similarly, $\kappa_{\acute{\vartheta}}(G) = 1$, $\kappa_{\acute{\omega}}(G) = \acute{1}$. □

Theorem 3.5. *Any single valued complete SPIvFG $G = (V, E)$ is strictly balanced.*

Proof. Let $G = (V, E)$ be a complete SPIvFG and let $\kappa(G)$ be the density of G. Since G is a complete SPIvFG we have

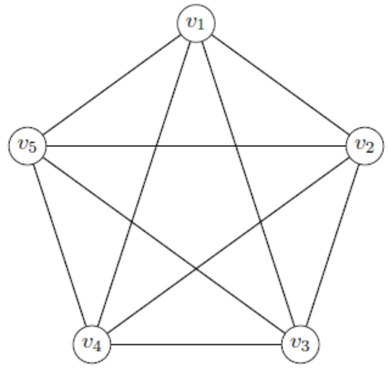
$$\acute{\eta}_E(v_i, v_j) = \acute{\eta}_V(v_i) \wedge \acute{\eta}_V(v_j)\},$$
$$\acute{\vartheta}_E(v_i, v_j) = \acute{\vartheta}_V(v_i) \wedge \vartheta_V(v_j),$$
$$\acute{\omega}_E(v_i, v_j) = \acute{\omega}_V(v_i) \vee \acute{\omega}_V(v_j), \quad \forall v_i, v_j \in V.$$

Now, the above relations can be written as

$$\acute{\eta}_E(v_i, v_j) = \xi_1 \acute{\eta}_V(v_i) \wedge \acute{\eta}_V(v_j),$$
$$\acute{\vartheta}_E(v_i, v_j) = \xi_2 \acute{\vartheta}_V(v_i) \wedge \acute{\vartheta}_V(v_j),$$
$$\acute{\omega}_E(v_i, v_j) = \xi_3 \acute{\omega}_V(v_i) \vee \acute{\omega}_V(v_j)\}, \quad \forall v_i, v_j \in V$$

where $\xi_1 = \xi_2 = \xi_3 = \acute{1}$. □

An example of a Spherical Interval-valued Fuzzy Graph is Balanced.



Vertex and Edge Values

$$\begin{aligned}
v_1 &= ([0.6, 0.7], [0.2, 0.4], [0.3, 0.5]) \\
v_2 &= ([0.2, 0.4], [0.4, 0.6], [0.3, 0.5]) \\
v_3 &= ([0.3, 0.5], [0.4, 0.6], [0.5, 0.6]) \\
v_4 &= ([0.1, 0.3], [0.2, 0.4], [0.5, 0.7]) \\
v_5 &= ([0.4, 0.6], [0.3, 0.5], [0.1, 0.3]) \\
v_1v_2 &= ([0.2, 0.4], [0.2, 0.4], [0.3, 0.5]) \\
v_1v_3 &= ([0.3, 0.5], [0.2, 0.4], [0.5, 0.6]) \\
v_1v_4 &= ([0.1, 0.3], [0.2, 0.4], [0.5, 0.7]) \\
v_1v_5 &= ([0.4, 0.6], [0.2, 0.4], [0.3, 0.5]) \\
v_2v_3 &= ([0.2, 0.4], [0.4, 0.6], [0.5, 0.6]) \\
v_2v_4 &= ([0.1, 0.3], [0.2, 0.4], [0.5, 0.7]) \\
v_2v_5 &= ([0.2, 0.4], [0.3, 0.5], [0.3, 0.5]) \\
v_3v_4 &= ([0.1, 0.3], [0.2, 0.4], [0.5, 0.7]) \\
v_3v_5 &= ([0.3, 0.5], [0.3, 0.5], [0.5, 0.6]) \\
v_4v_5 &= ([0.1, 0.3], [0.2, 0.4], [0.5, 0.7])
\end{aligned}$$

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