

A Computational Matrix Method For Multi-Dimensional Stochastic Volterra Integral Equations

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Abstract. Euler wavelets are constructed using Euler polynomials, which generally contain fewer terms than those used in alternative wavelet methods. As a result, the corresponding operational matrices are sparser, enabling more efficient numerical computation. This paper provides a comprehensive comparison of existing approximation methods for stochastic Volterra integral equations with multi-dimensional Brownian motion. The proposed approach employs Euler wavelet approximation in combination with block-pulse functions (BPFs). The operational matrices associated with block-pulse functions enable the transformation of stochastic Volterra equations into algebraic equations. This methodology offers a direct and efficient computational strategy by converting the single-integral problem into a linear algebraic system that is straightforward to solve. Several numerical examples are included to illustrate the effectiveness and accuracy of the proposed method.

1. INTRODUCTION

Given that Gaussian white noise is a formal derivative of Brownian motion in many cases, all such phenomena are mathematically modeled and essentially represented by complex stochastic differential or integro-differential equations of the Itô type. Many population models, from deterministic to stochastic, have been considered in the mathematical literature, from simple discrete random variables to very complex continuous stochastic models ([1], [2], [3], [4], [5]). However, some authors, for instance [6], [7], [8], [9], [11], [12], and ([13]–[24]), used an iterative approach or the Euler-Maruyama approximation to arrive at the numerical solution of the stochastic Volterra integral equation.

A general form of such an equation is typically expressed as:

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$$X(t) = f(t) + \int_0^t A_0(t,s)X(s)ds + \sum_{i=1}^n \int_0^t A_i(t,s)X(s)dB_i(s), \quad t \in [0, T]. \quad (1.1)$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a probability space. Consider $X(t)$ as an unknown stochastic process, where $t \in [0, T]$. The functions $X(t)$, $f(t)$, $A_0(t, s)$, and $A_i(t, s)$, for $i = 1, 2, \dots, n$ and $t, s \in [0, T]$, denote additional stochastic processes defined on this interval. The processes $B_1(t), B_2(t), \dots, B_n(t)$ represent independent Brownian motions. The integrals $\int_0^t A_i(t, s)X(s)dB_i(s)$, for $i = 1, 2, \dots, n$, are interpreted as Itô integrals. The present study is motivated by the analysis of general multidimensional stochastic terms. Based on recent analytical developments in [13], this work investigates forward stochastic Volterra integral equations (SVIEs) driven by multidimensional Brownian motion. This study develops a reliable and accurate numerical approximation for the class of SVIEs under consideration. The proposed approach utilizes Euler wavelets, which are constructed from Euler polynomials and generally require fewer terms than polynomials employed in alternative wavelet-based methods. Consequently, the resulting operational matrices are sparser, facilitating efficient numerical evaluation and reduced computational cost. This sparsity is particularly advantageous when many collocation points are used to improve solution accuracy.

The structure of the paper is as follows. Section 2 provides definitions related to stochastic calculus. Section 3 describes the block-pulse function and the operational integration matrix. Section 4 introduces stochastic integration as the operational matrix. Section 5 discusses key properties of Euler wavelets. Section 6 presents a stochastic approach for solving stochastic Volterra integral equations. Section 7 details the implementation of the operational matrices. Section 8 addresses the convergence analysis. Section 9 presents numerical simulations. Section 10 concludes with brief remarks.

2. STOCHASTIC CONCEPTS AND ITÔ INTEGRAL

Several definitions of stochastic calculus are presented in this section. A stochastic process is defined as a collection of random variables $\{X(t), t \in [0, T]\}$ specified on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If the following criteria are satisfied, the stochastic process $B(t)$ is classified as a Brownian motion:

- $B(0) = 0$ (with the probability 1),
- for $0 \leq t_1 \leq t_2 \leq T$ the random variable given by the increment $B(t_2) - B(t_1)$ is normally distributed with mean zero and variance $t_2 - t_1$,
- for $0 \leq t_1 < t_2 < t_3 < t_4 \leq T$ the increments $B(t_2) - B(t_1)$ and $B(t_4) - B(t_3)$ are independent,
- for $t \geq 0$, $B(t)$ is a continuous function of t .

Definition 1.1. A process $f(t, \omega) : [0, \infty) \times \Omega \rightarrow \mathcal{R}$ is called $\mathcal{F}_t$ -adapted if for each $t \geq 0$, the function $\omega \rightarrow f(t, \omega)$ is $\mathcal{F}_t$ measurable, where $\mathcal{F}_t$ is an increasing sequence of σ -algebras of $\mathcal{F}$.

Let $W = W(s, T)$ be the class of functions $f(t, \omega) : [0, \infty) \times \Omega \rightarrow \mathcal{R}$, which satisfy the conditions:

- the function $(t, \omega) \rightarrow f(t, \omega)$ is $\mathcal{B} \times \mathcal{F}$ measurable, where $\mathcal{B}$ is the Borel algebra,

- f is $\mathcal{F}_t$ -adapted for all $t \geq 0$,
- $E \left[\int_S^T f(t, \omega)^2 dt \right] < \infty$.

Theorem 1. (The Itô isometry) : Let $f \in W(S, T)$, Then

$$E \left[\left(\int_S^T f(t, \omega) dB_t \right)^2 \right] = E \left[\int_S^T f^2(t, \omega) dt \right]. \quad (2.1)$$

Refer to [25] for additional properties of stochastic integrals.

3. BLOCK-PULSE FUNCTIONS (BPFs)

This section outlines the notations, definitions, known results, and formulas concerning BPFs, which are pertinent to this paper, as previously discussed in [17]- [18].

The block-pulse functions Φ_i is defined over the unit interval $[0, 1)$ as: $0 \leq i < m$, and $m \in \{1, 2, \dots\}$,

$$\Phi_i(t) = \begin{cases} 1 & (i-1)h \leq t < ih, \\ 0 & \text{otherwise,} \end{cases} \quad (3.1)$$

where $t \in [0, T)$, $i = 1, 2, \dots, m$, and $h = \frac{T}{m}$.

The block-pulse functions have the following properties:

(1) Disjointness: The BPFs are disjointed with each other in the interval $t \in [0, T)$.

$$\Phi_i(t)\Phi_j(t) = \delta_{ij}\Phi_i(t), \quad (3.2)$$

where $i, j = 1, 2, \dots, m$, and δ_{ij} denotes the Kronecker delta.

(2) Orthogonality: The BPFs are orthogonal with each other in the interval $t \in [0, T)$.

$$\int_0^T \Phi_i(t)\Phi_j(t)dt = h\delta_{ij}, \quad i, j = 1, 2, \dots, m. \quad (3.3)$$

(3) The third property is completeness: For every $f \in L^2[0, T)$, when $m \rightarrow \infty$, Parseval's identity holds, that is

$$\int_0^T f^2(t)dt = \sum_{i=1}^{\infty} f_i^2 \|\Phi_i(t)\|^2,$$

where $f_i = \frac{1}{h} \int_0^T f(t)\Phi_i(t)dt$. The set of function can be described by an m vector

$$\Phi(t) = (\Phi_1(t), \Phi_2(t), \dots, \Phi_m(t))^T, \quad t \in [0, T).$$

Therefore, the relationship between BPFs and their integrals can be expressed in the following matrix form.

The derivation of the matrix representation and the disjointness property is presented below:

$$\Phi(t)\Phi^T(t) = \begin{pmatrix} \phi_1(t) & 0 & \cdots & 0 \\ 0 & \phi_2(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi_m(t) \end{pmatrix}_{m \times m}, \quad (3.4)$$

additionally, yield

$$\Phi^T(t)\Phi(t) = 1,$$

and,

$$\Phi(t)\Phi^T(t)F^T = \tilde{F}\Phi(t), \quad (3.5)$$

where $\tilde{F}$ usually denotes a diagonal matrix whose diagonal entries are related to a constant vector $F = (f_1, f_2, \dots, f_m)^T$. It is also straightforward to show that for an $m \times m$ matrix B

$$\Phi^T(t)B\Phi(t) = \tilde{B}\Phi(t), \quad (3.6)$$

where $\tilde{B} = \text{diag}(B)$ is a m -vector.

3.1. Functions approximation. A real bounded function $f(t)$, can be expanded into a block pulse series, which is square integrable in the interval $t \in ([0, T])$. Then, it minimizes the mean square error $f(t)$ and its approximation

$$f(t) \simeq \hat{f}(t) = \sum_{i=1}^m f_i \Phi_i(t) = \Phi^T(t)f, \quad (3.7)$$

where f_i is the block pulse coefficient with respect to the i th BPF $\Phi_i(t)$. The vector representation is given as follows:

$$f = (f_1, f_2, \dots, f_m)^T,$$

where $f_m(t)$ is a linear combination of BPFs.

Let $A(t, s) \in L^2([0, T_1] \times [0, T_2])$. It can be similarly expanded with respect to BPFs such as

$$A(t, s) \simeq \hat{A}_m(t, s) = \Psi^T(s)A\Phi(t) = \Phi^T(t)A^T\Psi(s), \quad (3.8)$$

where $\Psi(s)$ and $\Phi(t)$ are m_1 and m_2 dimensional BPFs vectors respectively, and $A = (a_{ij}), i = 1, 2, \dots, m_1, j = 1, 2, \dots, m_2$ is the $m_1 \times m_2$ block pulse coefficient matrix with

$$a_{ij} = \frac{1}{h_1 h_2} \int_0^{T_1} \int_0^{T_2} A(t, s) \Psi_i(s) \Phi_j(t) ds dt,$$

where $h_1 = \frac{T_1}{m_1}, h_2 = \frac{T_2}{m_2}$. For convenience, let $m_1 = m_2 = m$.

3.2. Integration operational matrix. Computing $\int_0^t \Phi_i(s) ds$ follows

$$\int_0^t \phi_i(s) ds = \begin{cases} 0 & 0 \leq t < (i-1)h, \\ t - (i-1)h & (i-1)h \leq t < ih, \\ h & ih \leq t < T. \end{cases} \quad (3.9)$$

At midpoint of $[(i-1)h, ih)$, the term $t - (i-1)h$, equals to $\frac{h}{2}$. Hence, for $(i-1)h \leq t < ih$, $t - (i-1)h$ can be approximated by $\frac{h}{2}$.

According to [17], it follows that:

$$\int_0^t \Phi(s) ds = P\Phi(t), \quad (3.10)$$

where operational matrix of integration is given by

$$P = \frac{h}{2} \begin{pmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{m \times m}. \quad (3.11)$$

Thus, the integral of every function $f(t)$ can be approximated as follows

$$\int_0^t f(s)ds \simeq \int_0^t F^T \Phi(s)ds \simeq F^T P \Phi(t). \quad (3.12)$$

4. STOCHASTIC INTEGRATION OPERATIONAL MATRIX

The integral of Itô of a single BPF $\phi_i(t)$ may be computed as follows,

$$\int_0^t \phi_i(s)dB(s) = \begin{cases} 0 & 0 \leq t < (i-1)h, \\ B(t) - B((i-1)h) & (i-1)h \leq t < ih, \\ B(ih) - B((i-1)h) & ih \leq t < T. \end{cases} \quad (4.1)$$

Now expressing $\int_0^t \phi_i(s)dB(s)$, in terms of the BPFs follows

$$\int_0^t \phi_i(s)dB(s) \simeq (B(ih/2) - B((i-1)h/2))\phi_i(t) + (B(ih) - B((i-1)h)) \sum_{j=i+1}^m \phi_j(t), \quad (4.2)$$

Therefore

$$\int_0^t \Phi(s)dB(s) \simeq P_S \Phi(t), \quad (4.3)$$

In this case, the stochastic operational matrix of integration can be expressed as follows:

$$P_S = \begin{pmatrix} \gamma_1 & \rho_1 & \rho_1 & \cdots & \rho_1 \\ 0 & \gamma_2 & \rho_2 & \cdots & \rho_2 \\ 0 & 0 & \gamma_3 & \cdots & \rho_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \gamma_m \end{pmatrix}_{m \times m}, \quad (4.4)$$

where $\rho_i = B(ih) - B((i-1)h)$, $i = 1, 2, \dots, m-1$; $\gamma_j = B(ih/2) - B((i-1)h/2)$, $j = 1, 2, \dots, m$.

This can be approximated by computing the Itô integral for every function $f(t)$ as follows:

$$\int_0^t f(s)dB(s) \simeq F^T \Phi(s)dB(s) \simeq F^T P_S \Phi(t). \quad (4.5)$$

5. DEFINITION AND FEATURES OF EULER WAVELETS

The Euler wavelets presented in this section exhibit unique characteristics and are applicable to several mathematical disciplines. In particular, they share significant similarities with the theory of Bernoulli polynomials and numbers.

The Euler Wavelets $\mathbf{E}_n(t)$ of degree n can be expressed as follows:

$$\sum_{k=0}^n \binom{n}{k} \mathbf{E}_k(t) + \mathbf{E}_n(t) = 2t^n. \quad (5.1)$$

Therefore, the Euler vector $\mathbf{E}(t)$ is specified:

$$\mathbf{E}(t) = [\mathbf{E}_0(t), \mathbf{E}_1(t), \dots, \mathbf{E}_N(t)]^T. \quad (5.2)$$

The Euler Wavelets given by (5.1). can be defined in the matrix form

$$\mathbf{E}(t) = A T_n(t), \quad (5.3)$$

where $T_n(t) = [1, t, t^2, \dots, t^N]^T$ and since A is a lower triangular matrix with nonzero diagonal elements, it is nonsingular, and hence A^{-1} exists.

Definition 2. Euler polynomials have several distinctive features:

- (i) $\mathbf{E}'_n(t) = n\mathbf{E}_{n-1}(t)$, $n = 1, 2, \dots$,
- (ii) $\mathbf{E}_n(t+1) + \mathbf{E}_n(t) = 2t^n$, $n = 1, 2, \dots$,
- (iii) $\mathbf{E}_n(t) = \frac{1}{n+1} \sum_{k=1}^{n+1} (2 - 2^{k+1}) \binom{n+1}{k} B_k(0) t^{n+1-k}$, $n = 1, 2, \dots$,

where $B_k(t)$, $k = 0, 1, \dots$, are the Bernoulli polynomials of order k which has been defined:

$$\sum_{k=0}^n \binom{n+1}{k} B_k(t) = (n+1)t^n.$$

5.1. Functions approximation. Assume an arbitrary function $f(t) \in L^2[0, 1)$, which can be approximated by a truncated Euler series as follows:

$$f(t) \simeq \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} \mathbf{E}_{nm}(t) = C^T \mathbf{E}(t) = \mathbf{E}^T(t) C, \quad (5.4)$$

in which C and $\mathbf{E}(t)$ have dimension $m = NM$ for

$$C = (c_{10}, c_{11}, \dots, c_{1(M-1)}, c_{20}, c_{21}, \dots, c_{2(M-1)}, \dots, c_{2^{k-1}0}, c_{2^{k-1}1}, \dots, c_{2^{k-1}(M-1)})^T.$$

$$\mathbf{E} = (\mathbf{E}_{10}, \mathbf{E}_{11}, \dots, \mathbf{E}_{1(M-1)}, \mathbf{E}_{20}, \mathbf{E}_{21}, \dots, \mathbf{E}_{2(M-1)}, \dots, \mathbf{E}_{2^{k-1}0}, \mathbf{E}_{2^{k-1}1}, \dots, \mathbf{E}_{2^{k-1}(M-1)})^T. \quad (5.5)$$

Similarly, let $A(t, s) \in L^2([0, T_1] \times [0, T_2])$. It can be similarly expanded with respect to Euler such as

$$A(t, s) \simeq \hat{A}(t, s) = \mathbf{E}^T(s) A \mathbf{E}(t) = \mathbf{E}^T(t) A^T \mathbf{E}(s), \quad (5.6)$$

where $\mathbf{E}(s)$ and $\mathbf{E}(t)$ are m_1 and m_2 dimensional Euler vectors respectively, and $A = (a_{ij})$, $i = 1, 2, \dots, m_1$, $j = 1, 2, \dots, m_2$ is the $m_1 \times m_2$ Euler coefficient matrix with

$$a_{ij} = \frac{1}{h_1 h_2} \int_0^{T_1} \int_0^{T_2} A(t, s) \mathbf{E}_i(s) \mathbf{E}_j(t) ds dt,$$

where $h_1 = \frac{T_1}{m_1}, h_2 = \frac{T_2}{m_2}$. For convenience, we put $m_1 = m_2 = m$.

5.2. Euler Wavelets and Block Pulse Functions. Now the relation between the Euler wavelet and block pulse functions is discussed.

Theorem 2. Let $\mathbf{E}(t)$ and $\Phi(t)$ be the m -dimensional Euler wavelet and block pulse function vectors. Then $\mathbf{E}(t)$ is expressed by block pulse functions $\Phi(t)$ as

$$\mathbf{E}(t) = \Theta \Phi(t), \quad (5.7)$$

where Θ is an $m \times m$ matrix and

$$\Theta_{ij} = \mathbf{E}_i \left(\frac{2j-1}{2m} \right), \quad i, j = 1, 2, \dots, m.$$

Proof: Let $\mathbf{E}_i(t), i = 1, 2, \dots, m$, be the i -th element of Euler wavelet vectors. Expanding $\mathbf{E}_i(t)$ into an m -term of BPFs, then

$$\mathbf{E}_i(t) = \sum_{j=1}^m \Theta_{ij} \phi_j(t) = \Theta_i^T \Phi(t), \quad i = 1, 2, \dots, m.$$

where Θ_i is the i -th row and Θ_{ij} is the (i, j) -th element of matrix Θ . By using orthogonality of BPFs, then

$$\Theta_{ij} = \frac{1}{h} \int_0^1 \mathbf{E}_i(t) \phi_j(t) dt = \frac{1}{h} \int_0^1 \int_{\frac{j-1}{m}}^{\frac{j}{m}} \mathbf{E}_i(t) dt = 2^2 m \int_0^1 \int_{\frac{j-1}{m}}^{\frac{j}{m}} h_{i-1}(t) dt.$$

Thus, the final equation can be formulated using the mean value theorem for integrals.

$$\Theta_{ij} = 2^2 m \left(\frac{j}{m} - \frac{j-1}{m} \right) h_{i-1}(\chi_j), \quad \chi_j \in \left(\frac{j-1}{m}, \frac{j}{m} \right).$$

As $h_{i-1}(t)$ is constant on the interval $\left(\frac{j-1}{m}, \frac{j}{m} \right)$, choose $\chi_j = \frac{2j-1}{2m}$. Consequently

$$\Theta_{ij} = 2^2 h_{i-1} \left(\frac{2j-1}{2m} \right), \quad i, j = 1, 2, \dots, m.$$

Therefore, proof is established.

Remark 1. It is clear from the definition of matrix Θ in (5.7), that

$$\Theta^{-1} = \frac{1}{m} \Theta^T.$$

The following remarks are drawn from (3.5), (3.6), and Theorem 2:

Remark 2. An m -vector F ,

$$\mathbf{E}(t) \mathbf{E}^T(t) F = \tilde{F} \mathbf{E}(t),$$

in which $\tilde{F}$ is an $m \times m$ matrix as $\tilde{F} = \Theta \bar{F} \Theta^{-1}$ where $\bar{F} = \text{diag}(\Theta^T F)$.

Remark 3. Let B be any $m \times m$ matrix. Subsequently, the Euler vector $\mathbf{E}(t)$ is obtained

$$\mathbf{E}^T(t) B \mathbf{E}(t) = \hat{B}^T \mathbf{E}(t),$$

where $\hat{B}^T = \Gamma \Theta^{-1}$ and $\Gamma = \text{diag}(\Theta^T B \Theta)$ is a m -vector.

6. STOCHASTIC OPERATIONAL MATRIX OF EULER WAVELETS

The operational matrix of stochastic integration for the Euler wavelet basis is derived in this section. To achieve this derivation, the BPFs matrix Θ , shown in (5.7), is utilized.

Theorem 3. Let $\mathbf{E}(t)$ be the Euler wavelet vector defined in (5.5). The integral of this vector is derived as

$$\int_0^t \mathbf{E}(s) ds \simeq \frac{1}{m} \Theta P \Theta^T \mathbf{E}(t) = \Lambda \mathbf{E}(t), \quad (6.1)$$

where Θ is introduced in (5.7) and P is the operational matrix of integration for BPFs derived in (3.11).

Proof: Using Theorem 2, let $\mathbf{E}(t)$ be an Euler wavelet vector,

$$\int_0^t \mathbf{E}(s) ds \simeq \int_0^t \Theta \Phi(s) ds = \Theta \int_0^t \Phi(s) ds = \Theta P \Phi(t),$$

Theorem 2 and Remark 1 now provide

$$\int_0^t \mathbf{E}(s) ds \simeq \Theta P \Phi(t) = \frac{1}{m} \Theta P \Theta^T \mathbf{E}(t) = \Lambda \mathbf{E}(t),$$

therefore the proof is established.

Theorem 4. Assume $\mathbf{E}(t)$ is the Euler wavelet vector defined in (5.5). The Itô integral of this vector is derived as

$$\int_0^t \mathbf{E}(s) dB(s) \simeq \frac{1}{m} \Theta P_s \Theta^T \mathbf{E}(t) = \Lambda_s \mathbf{E}(t), \quad (6.2)$$

where Λ_s is stochastic operational matrix for Euler wavelet, Θ is introduced in (5.7) and P_s is the stochastic operational matrix of integration for BPFs derived in (4.4).

Proof: Given an Euler wavelet vector $\mathbf{E}(t)$, we may use Theorem 2 to obtain

$$\int_0^t \mathbf{E}(s) dB(s) \simeq \int_0^t \Theta \Phi(s) dB(s) = \Theta \int_0^t \Phi(s) dB(s) = \Theta P_s \Phi(t),$$

Theorem 2 and Remark 1 yield

$$\int_0^t \mathbf{E}(s) dB(s) \simeq \Theta P_s \Phi(t) = \frac{1}{m} \Theta P_s \Theta^T \mathbf{E}(t) = \Lambda_s \mathbf{E}(t),$$

therefore the proof is established.

7. MULTIDIMENSIONAL STOCHASTIC ITÔ-VOLTERRA INTEGRAL EQUATIONS

This section transforms multidimensional stochastic Itô-Volterra integral equations into linear algebraic equations using the stochastic operational matrix of Euler wavelet. The function $X(t)$ obtained from (1.1) enables the approximation of $X(t)$, $f(t)$, $A_0(t, s)$, and $A_i(t, s)$ in the following forms:

$$\begin{cases} X(t) \simeq X^T \mathbf{E}(t) = \mathbf{E}^T(t)X, \\ f(t) \simeq F^T \mathbf{E}(t) = \mathbf{E}^T(t)F, \\ A_0(t, s) \simeq \mathbf{E}^T(s)A_0 \mathbf{E}(t) = \mathbf{E}^T(t)A_0^T \mathbf{E}(s), \\ A_i(t, s) \simeq \mathbf{E}^T(s)A_i \mathbf{E}(t) = \mathbf{E}^T(t)A_i^T \mathbf{E}(s), \quad i = 1, 2, \dots, n, \end{cases} \quad (7.1)$$

X and F are Euler coefficient vectors; A_0 and A_i are Euler coefficient matrices defined in equations (5.5) and (5.6). Substituting the above approximation into (1.1) yields

$$\begin{aligned} X^T \mathbf{E}(t) &= F^T \mathbf{E}(t) + \mathbf{E}^T(t)A_0^T \left(\int_0^t \mathbf{E}(s) \mathbf{E}^T(s) X ds \right) + \\ &\sum_{i=1}^n \mathbf{E}^T(t)A_i^T \left(\int_0^t \mathbf{E}(s) \mathbf{E}^T(s) X dB_i(s) \right), \quad t \in [0, T]. \end{aligned} \quad (7.2)$$

By applying Remark 2,

$$X^T \mathbf{E}(t) = F^T \mathbf{E}(t) + \mathbf{E}^T(t)A_0^T \left(\int_0^t \tilde{X} \mathbf{E}(s) ds \right) + \sum_{i=1}^n \mathbf{E}^T(t)A_i^T \left(\int_0^t \tilde{X} \mathbf{E}(s) dB_i(s) \right), \quad t \in [0, T] \quad (7.3)$$

where $\tilde{X}$ is an $m \times m$ matrix. By applying Λ and Λ_s , the following expression is obtained:

$$X^T \mathbf{E}(t) = F^T \mathbf{E}(t) + \mathbf{E}^T(t)A_0^T \tilde{X} \Lambda \mathbf{E}(t) + \sum_{i=1}^n \mathbf{E}^T(t)A_i^T \tilde{X} \Lambda_s \mathbf{E}(t). \quad (7.4)$$

Let $Z_0 = A_0^T \tilde{X} \Lambda$ and $Z_i = A_i^T \tilde{X} \Lambda_{si}$, be m -dimensional vectors. Utilizing Remark 2, the following result is obtained:

$$X^T \mathbf{E}(t) - \hat{Z}_0^T \mathbf{E}(t) - \sum_{i=1}^n \hat{Z}_i^T \mathbf{E}(t) = F^T \mathbf{E}(t).$$

This equation holds for $t \in [0, 1)$. Consequently,

$$X^T - \hat{Z}_0^T - \sum_{i=1}^n \hat{Z}_i^T = F^T. \quad (7.5)$$

For the unknown vector X , equation (7.5) represents a linear system of equations. To find X , solve this linear system. Next, substitute the obtained X into equation (7.1) to approximate the solution of equation (1.1).

8. ERROR ESTIMATE AND RATE OF CONVERGENCE

This section presents a detailed analysis of the exact solution corresponding to the Euler approximation and evaluates its convergence rate. The error characteristics of the proposed method, denoted by $X_N(t)$ and representing the Euler approximation of $X(t)$, are systematically evaluated. The approximate and exact solution values are compared at selected points based on the absolute error metric.

$$e_N(t) = |X(t) - X_N(t)|, \quad t \in [0, 1],$$

where $X(t)$ and $X_N(t)$ represent exact and approximate solutions, respectively.

Theorem 6. Assume $f(t)$ and $A(t, s)$ are smooth functions, and $f_N(t)$ and $A_N(t, s)$ are their Euler approximations. Therefore, the error bound can obtained thus

$$(i) \|f(t) - f_N(t)\|_\infty \leq C\tilde{F}(2\pi)^{-N}, \quad t \in [0, 1],$$

$$(ii) \|A(t, s) - A_N(t, s)\|_\infty \leq \tilde{C}N(2\pi)^{-N},$$

where $\tilde{F} = \sup\{\|f^{(i)}(X)\|_\infty, i = 0, 1, 2, \dots\}$ and $\tilde{C}$ is a positive constant.

Theorem 7. Let $X(t)$ and $X_N(t)$ be solutions of the exact and approximate of equation (1.1), respectively of Euler approximation. Furthermore, assume that

$$(i) \|X(t)\|_\infty \leq \rho_i \quad t \in [0, 1], i = 1, 2,$$

$$(ii) \|A_i(t, s)\|_\infty M_i \quad t, s \in [0, 1] \times [0, 1], i = 0, 1, 2,$$

$$(iii) (M_0 + \mu(N)) + \|B_i(t)\|_\infty (M_i + \lambda(N)) < 1.$$

Consequently,

$$\|X(t) - X_N(t)\|_\infty \leq \frac{\eta(N) + (M_0 + \mu(N) + \mu(N)\rho_1) + \sum_{i=1}^n \|B(t)\|_\infty ((M_i + \lambda(N)) + \lambda(N)\rho_2)}{1 - (M_0 + \mu(N)) - \sum_{i=1}^n \|B(t)\|_\infty (M_i + \lambda(N))}, \quad (8.1)$$

where

$$\eta(N) = C\tilde{F}(2\pi)^{-N},$$

$$\mu(N) = \tilde{C}_1 N(2\pi)^{-N},$$

$$\lambda(N) = \tilde{C}_2 N(2\pi)^{-N},$$

Proof. Equation (1.1) yields the following result.

$$\begin{aligned} X(t) - X_N(t) &= f(t) - f_N(t) + \int_0^t A_0(t, s)X(s)ds - \int_0^t A_{0N}(t, s)X_N(s)ds \\ &\quad + \sum_{i=1}^n \int_0^t A_i(t, s)X(s)dB_i(s) - \sum_{i=1}^n \int_0^t A_{iN}(t, s)X_N(s)dB_i(s). \end{aligned}$$

Consequently,

$$\begin{aligned} \|X(t) - X_N(t)\|_\infty &\leq \|f(t) - f_N(t)\|_\infty + \|A_0(t, s)X(s) - A_{0N}(t, s)X_N(s)\|_\infty \\ &\quad + \left\| \sum_{i=1}^n B_i(t) \right\|_\infty \|A_i(t, s)X(s) - A_{iN}(t, s)X_N(s)\|_\infty. \end{aligned} \quad (8.2)$$

By combining assumptions (i) and (ii) with Theorems 6 and 7, the following result is obtained.

$$\begin{aligned} \|A_0(t, s)X(s) - A_{0N}(t, s)X_N(s)\|_\infty &\leq \|A_0(t, s)\|_\infty \|X(s) - X_N(s)\|_\infty \\ &\quad + \|A_i(t, s) - A_{iN}(t, s)\|_\infty \left(\|X(s) - X_N(s)\|_\infty + \|X(s)\|_\infty \right) \\ &\leq (M_0 + \mu(N))\|X(t) - X_N(t)\|_\infty + (M_0 + \mu(N)) + \mu(N)\rho_i, \end{aligned} \quad (8.3)$$

and

$$\begin{aligned} \|A_i(t,s)X(s) - A_{iN}(t,s)X_N(s)\|_\infty &\leq \|A_i(t,s)\|_\infty \|X(s) - X_N(s)\|_\infty \\ &\quad + \|A_i(t,s) - A_{iN}(t,s)\|_\infty \left(\|X(s) - X_N(s)\|_\infty + \|X(s)\|_\infty \right) \\ &\leq (M_i + \lambda(N)) \|X(t) - X_N(t)\|_\infty + (M_i + \lambda(N)) + \lambda(N)\rho_i, \end{aligned} \quad (8.4)$$

Equations (8.2)-(8.4) and assumption (iii) yield

$$\begin{aligned} \|X(t) - X_N(t)\|_\infty &\leq \\ &\frac{\eta(N) + (M_0 + \mu(N) + \mu(N)\rho_1) + \sum_{i=1}^n \|B(t)\|_\infty ((M_i + \lambda(N)) + \lambda(N)\rho_2)}{1 - (M_0 + \mu(N)) - \sum_{i=1}^n \|B(t)\|_\infty (M_i + \lambda(N))}. \end{aligned} \quad (8.5)$$

9. NUMERICAL RESULTS

A comprehensive comparative analysis was performed to quantify the discrepancies between the Euler approximation and the BPFs method. The simulation results, as depicted by the pink (BPFs method), red (Euler method), and blue (exact solution) lines, reveal distinct error profiles for each approach. Simulations were conducted for $m = 16, 32, 64$ and $n = 50, 100, 150, 200$ to systematically evaluate the deviation from the exact solution. The absolute error, defined as $E = |X_i - Y_i|$ where X_i denotes the approximate solution from either the BPF or Euler method and Y_i represents the exact solution, is summarized in Tables 1 and 2. The absolute error within the interval $([0, 1])$ is presented in Figures 2 and 5. The BPF method exhibits a maximum error at approximately 0.1s, which decreases to near zero after 0.4s. In contrast, the Euler method reaches its maximum error at approximately 0.2s, then gradually decreases. All computations were performed using Python.

Example 1 ([10]). (The basic Black-Scholes model): The Black-Scholes model is a stochastic framework for modeling financial markets with derivative instruments. The model is defined by a two-dimensional continuous-time process $\{(X_0(t), X(t)) : 0 \leq t \leq T\}$, where $X_0(t)$ denotes the risk-free asset and $X(t)$ denotes the risky asset. The risk-free asset evolves according to the differential equation $dX_0(t) = rX_0(t)dt$, where $r \geq 0$ denotes the instantaneous interest rate. Setting $X_0(0) = 1$ yields the solution $X_0(t) = e^{rt}$. The evolution of the stock price is governed by

$$dX(t) = \lambda X(t)dt + \mu X(t)dB(t), \quad (9.1)$$

where $\{B(t) : 0 \leq t \leq T\}$ is a standard Brownian motion with $B(0) = 0$, λ is a constant and $\mu(t)$ as a function. The model is accurate for the range $[0, T]$, where T is the option's maturity. $X(t) = \exp\left(\left(\lambda - \frac{1}{2}\mu^2\right)t + \mu B(t)\right)$ provides the exact solution. In Table 1, the results obtained for this example's $\lambda = -8, \mu = 1$, and $n = 500$ are presented. The comparison between the exact solutions and absolute error are depicted in Figures 1–3, respectively.

TABLE 1. Comparison Table of Absolute Error of Multiple Experiments of Example 1 at $m = 16, 32$, and 64 .

n	m=16		m=32		m=64	
	Absolute Error		Absolute Error		Absolute Error	
	BPF	Euler	BPF	Euler	BPF	Euler
50	0.04331	0.030051	0.030889	0.012457	0.027406	0.004436
100	0.039773	0.029888	0.027541	0.01246	0.027491	0.004895
150	0.042872	0.030309	0.02946	0.013007	0.024479	0.004735
200	0.03026	0.03026	0.029278	0.012506	0.026244	0.004642

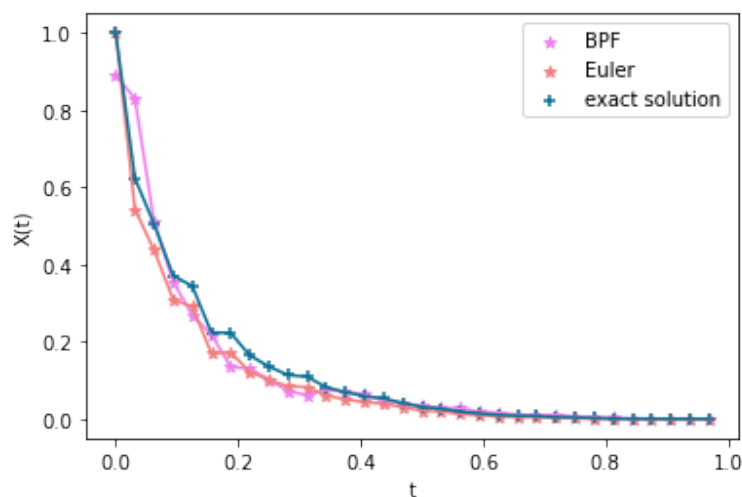


FIGURE 1. The comparison between the exact solution and the approximation of both BPF and Euler solutions of Example 1.

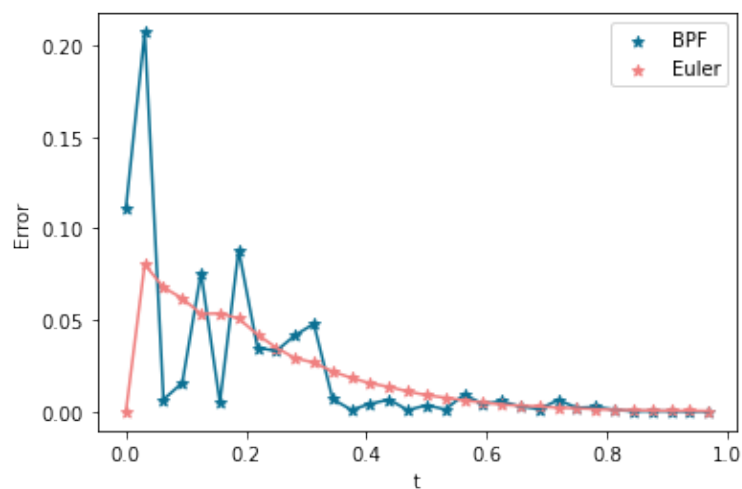


FIGURE 2. Comparison of the absolute errors of the BPF and Euler expansions for Example 1.

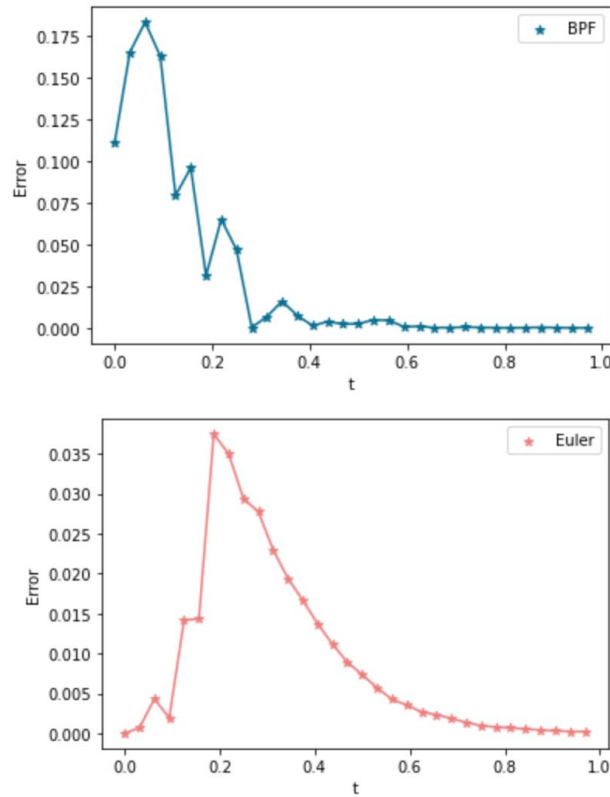


FIGURE 3. The absolute error of BPF and Euler expansions comparison at random points of Example 1.

Example 2 ([10]). Consider the linear SVIDE

$$y(t) = y_0 + \int_0^t u(s)y(s)ds + \sum_{i=1}^m \int_0^t \beta_i(s)y(s)dB_i(s), \quad s, t \in [0, 1), \quad (9.2)$$

with the exact solution is $y(t) = y_0 e^{(u(s) - \frac{1}{2} \sum_{i=1}^m \beta_i^2(s))dt + \sum_{i=1}^m \beta_i(s)dB_i(t)}$, for $0 \leq t < 1$ where $y(t)$ is an unknown stochastic process defined on a complete probability space, and $B_t = (B_1(t), B_2(t), \dots, B_m(t))$ is an m -dimension Brownian motion process. The numerical results for $y_0 = \frac{1}{200}, u(s) = \frac{1}{20}, \beta_1 = \frac{1}{50}, \beta_2 = \frac{2}{50}, \beta_3 = \frac{4}{50}, \beta_4 = \frac{9}{50}$. Table 2 presented numerical results. The comparison between the exact solutions and absolute error are depicted in Figures 4–6, respectively.

TABLE 2. Comparison Table of Absolute Error of Multiple Experiments of Example 2 at $m = 16, 32$ and 64 .

n	m=16		m=32		m=64	
	Absolute Error		Absolute Error		Absolute Error	
	BPF	Euler	BPF	Euler	BPF	Euler
50	0.000466	0.000022	0.000369	0.000024	0.027406	0.004436
100	0.000391	0.000023	0.000457	0.000022	0.027491	0.004895
150	0.000415	0.000022	0.000417	0.000022	0.024479	0.004735
200	0.00046	0.000022	0.000434	0.000022	0.026244	0.004642

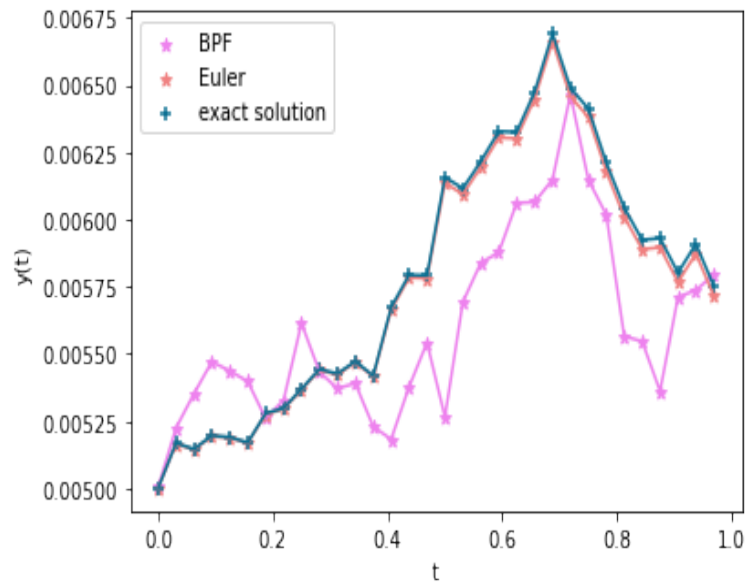


FIGURE 4. The comparison between the exact solution and the approximation of both BPF and Euler solutions of Example 2.

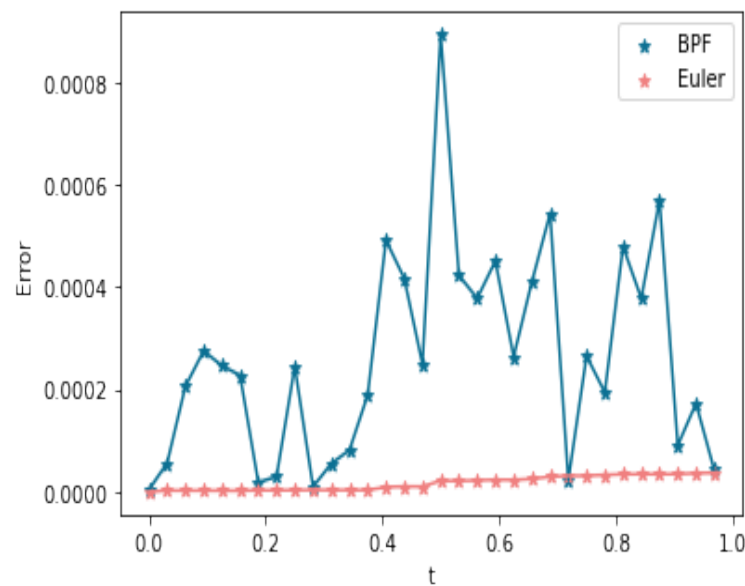


FIGURE 5. Comparison of the absolute errors of the BPF and Euler expansions for Example 2.

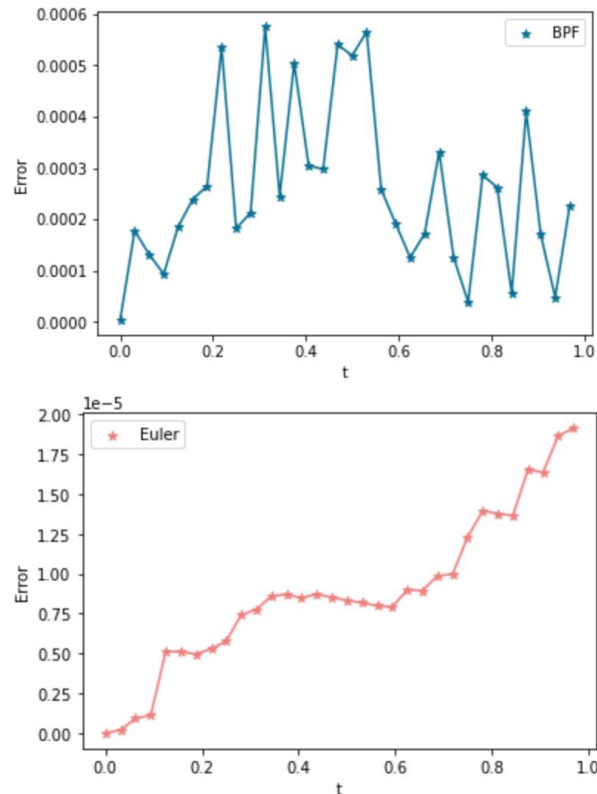


FIGURE 6. The absolute error of BPF and Euler expansions comparison at random points of Example 2.

10. CONCLUSION

This study presents a method for solving stochastic Volterra integral equations by comparing the Euler approximation with a computational technique based on the properties of BPFs. Numerical examples demonstrate the effectiveness of function-based approximations for integral equations and facilitate detailed error analysis of these methods. The Euler method exhibits a trend closely aligned with the exact solution obtained by the BPFs method, although it generally produces larger errors. In certain cases, the Euler method yields greater errors than the BPFs method. Overall, the results indicate that the proposed procedure achieves high accuracy.

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Data Availability: Data sharing does not apply to this paper, as no data sets were generated or analyzed during the current study.

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REFERENCES

- [1] K. Maleknejad, M. Khodabin, M. Rostami, Numerical Solution of Stochastic Volterra Integral Equations by a Stochastic Operational Matrix Based on Block Pulse Functions, *Math. Comput. Model.* 55 (2012), 791–800. <https://doi.org/10.1016/j.mcm.2011.08.053>.
- [2] F. Mohammadi, An Efficient Computational Method for Solving Stochastic Itô-Volterra Integral Equations, *TWMS J. Appl. Eng. Math.* 5 (2015), 286–297.
- [3] J.J. Levin, J.A. Nohel, On a System of Integrodifferential Equations Occurring in Reactor Dynamics. II, *Arch. Ration. Mech. Anal.* 11 (1962), 210–243. <https://doi.org/10.1007/BF00253938>.
- [4] M. Ahmadinia, H. Afshari A., M. Heydari, Numerical Solution of Itô-Volterra Integral Equation by Least Squares Method, *Numer. Algorithms* 84 (2020), 591–602. <https://doi.org/10.1007/s11075-019-00770-2>.
- [5] X. Wen, J. Huang, A Numerical Method for Linear Stochastic Itô-Volterra Integral Equation Driven by Fractional Brownian Motion, in: 2019 IEEE International Conference on Artificial Intelligence and Computer Applications (ICAICA), IEEE, 2019, pp. 121–125. <https://doi.org/10.1109/ICAICA.2019.8873448>.
- [6] X. Dai, A. Xiao, Lévy-Driven Stochastic Volterra Integral Equations with Doubly Singular Kernels: Existence, Uniqueness, and a Fast EM Method, *Adv. Comput. Math.* 46 (2020), 29. <https://doi.org/10.1007/s10444-020-09780-4>.
- [7] X. Dai, W. Bu, A. Xiao, Well-Posedness and EM Approximations for Non-Lipschitz Stochastic Fractional Integro-Differential Equations, *J. Comput. Appl. Math.* 356 (2019), 377–390. <https://doi.org/10.1016/j.cam.2019.02.002>.
- [8] S. Alipour, F. Mirzaee, An Iterative Algorithm for Solving Two Dimensional Nonlinear Stochastic Integral Equations: A Combined Successive Approximations Method with Bilinear Spline Interpolation, *Appl. Math. Comput.* 371 (2020), 124947. <https://doi.org/10.1016/j.amc.2019.124947>.
- [9] B. Hashemi, M. Khodabin, K. Maleknejad, Numerical Solution Based on Hat Functions for Solving Nonlinear Stochastic Itô Volterra Integral Equations Driven by Fractional Brownian Motion, *Mediterr. J. Math.* 14 (2017), 24. <https://doi.org/10.1007/s00009-016-0820-7>.
- [10] M. Saffarzadeh, G.B. Loghmani, M. Heydari, An Iterative Technique for the Numerical Solution of Nonlinear Stochastic Itô-Volterra Integral Equations, *J. Comput. Appl. Math.* 333 (2018), 74–86. <https://doi.org/10.1016/j.cam.2017.09.035>.
- [11] M. Saffarzadeh, M. Heydari, G.B. Loghmani, Convergence Analysis of an Iterative Algorithm to Solve System of Nonlinear Stochastic Itô-Volterra Integral Equations, *Math. Methods Appl. Sci.* 43 (2020), 5212–5233. <https://doi.org/10.1002/mma.6261>.
- [12] M. Saffarzadeh, M. Heydari, G.B. Loghmani, Convergence Analysis of an Iterative Numerical Algorithm for Solving Nonlinear Stochastic Itô-Volterra Integral Equations with m -Dimensional Brownian Motion, *Appl. Numer. Math.* 146 (2019), 182–198. <https://doi.org/10.1016/j.apnum.2019.07.010>.
- [13] N. Momenzade, A.R. Vahidi, E. Babolian, A Computational Method for Solving Stochastic Itô-Volterra Integral Equation with Multi-Stochastic Terms, *Math. Sci.* 12 (2018), 295–303. <https://doi.org/10.1007/s40096-018-0269-x>.
- [14] K.E. Yao, Y. Zhang, Y. Shi, Numerical Solution for Stochastic Volterra-Fredholm Integral Equations with Delay Arguments, *Acta Polytech.* 64 (2024), 128–141. <https://doi.org/10.14311/ap.2024.64.0128>.
- [15] K.E. Yao, M. Samar, Y. Shi, Approximation Approach for Backward Stochastic Volterra Integral Equations, *Math. Model. Control* 4 (2024), 390–399. <https://doi.org/10.3934/mmc.2024031>.
- [16] M.H. Heydari, M.R. Hooshmandasl, F.M. Maalek Ghaini, C. Cattani, A Computational Method for Solving Stochastic Itô-Volterra Integral Equations Based on Stochastic Operational Matrix for Generalized Hat Basis Functions, *J. Comput. Phys.* 270 (2014), 402–415. <https://doi.org/10.1016/j.jcp.2014.03.064>.
- [17] Z.H. Jiang, W. Schaufelberger, Block Pulse Functions and Their Applications in Control Systems, Springer Berlin Heidelberg, 1992. <https://doi.org/10.1007/BFb0009162>.
- [18] G. Prasada Rao, Piecewise Constant Orthogonal Functions and Their Application to Systems and Control, Springer-Verlag, 1983. <https://doi.org/10.1007/BFb0041228>.

- [19] M. Khodabin, K. Maleknejad, M. Rostami, M. Nouri, Numerical Approach for Solving Stochastic Volterra–Fredholm Integral Equations by Stochastic Operational Matrix, *Comput. Math. Appl.* 64 (2012), 1903–1913. <https://doi.org/10.1016/j.camwa.2012.03.042>.
- [20] S.U. Khan, M. Ali, I. Ali, A Spectral Collocation Method for Stochastic Volterra Integro-Differential Equations and Its Error Analysis, *Adv. Differ. Equ.* 2019 (2019), 161. <https://doi.org/10.1186/s13662-019-2096-2>.
- [21] E.Y. Kutorzi, Y. Zhang, Y. Shi, Y. Gao, Approximate Solution for Stochastic Volterra Integral Equations with Constant Delay, in: *Proceedings of the 2024 9th International Conference on Mathematics and Artificial Intelligence*, ACM, New York, NY, USA, 2024, pp. 53–60. <https://doi.org/10.1145/3670085.3670099>.
- [22] B. Øksendal, *Stochastic Differential Equations: An Introduction with Applications*, 5th ed., Springer Berlin Heidelberg, 1998. <https://doi.org/10.1007/978-3-662-03620-4>.
- [23] F.C. Klebaner, *Introduction to Stochastic Calculus with Applications*, 2nd ed., Imperial College Press, 2005. <https://doi.org/10.1142/p386>.
- [24] G.N. Milstein, *Numerical Integration of Stochastic Differential Equations*, Springer Netherlands, 1995. <https://doi.org/10.1007/978-94-015-8455-5>.
- [25] P.E. Kloeden, E. Platen, *Numerical Solution of Stochastic Differential Equations*, Springer Berlin Heidelberg, 1992. <https://doi.org/10.1007/978-3-662-12616-5>.