

Arctan–Weibull Log Logistic Mixture Distribution: Theory and Applications to Lifetime Data

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Abstract. This paper introduces the ArcTangent Weibull Log Logistic Mixture (AT-WLLM) distribution, a novel and flexible lifetime model constructed by applying an arctangent transformation to a mixture of Weibull and log-logistic distribution functions. The proposed model exhibits several desirable properties, including the ability to capture a wide range of hazard rate shapes commonly encountered in reliability engineering and survival analysis. The theoretical properties of the AT-WLLM distribution are thoroughly investigated. Parameter estimation is carried out using the maximum likelihood method, and a comprehensive simulation study is conducted to assess the performance of the estimators. Across all parameter configurations considered, the MLE procedure for the AT-WLLM distribution exhibits numerical stability and satisfactory finite-sample performance. The practical applicability of the model is demonstrated through three real-data analyses, in which the AT-WLLM distribution is compared with seven well-known competing models. The empirical results consistently affirm the superior flexibility and strength of the AT-WLLM distribution in modeling lifetime data.

1. INTRODUCTION

The pursuit of probability distributions that can accurately capture the complexity of modern real-world data remains a fundamental challenge in the evolving landscape of statistical modeling. Most traditional distributions such as the Weibull distribution [1] and log-logistic distribution [2] have served as workhorses for numerous disciplines, but have proven inadequate when confronted with data sets exhibiting multiple failure regimes in reliability engineering, complex hazard patterns in survival analysis, heavy-tailed behavior in financial and environmental applications, and

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mixed population in biological and medical studies. This gap between model flexibility and practical applicability has driven the development of more complicated distributional families, among which mixture models have emerged as particularly promising due to their enhanced capacity to capture multimodal properties and complex patterns. Moreover, when the heterogeneous nature of a process is taken into account, it is more appealing and appropriate to employ mixture models that capture the underlying behavior of the process. A substantial body of literature documents numerous contributions that employ two-fold mixture approaches for reliability and failure-time modeling. Examples include mixtures of two Weibull models [3], the use of the method of moments for mixtures of two exponential distributions [4], mixtures of two inverse Weibull distributions [5], Weibull and inverse Weibull mixture models [6], mixtures of two different distributions [7], two-fold Weibull mixtures [8], a new mixture of Pareto and length-biased Pareto distributions [9], the two-component unit Weibull mixture distribution [10], and two-component Weibull-log-logistic mixture models [11].

Other mixture-based lifetime distributions include the three-component mixture of Rayleigh distributions [12, 13], the three-component mixture of exponential distributions [14], the three-component mixture of Pareto distributions [15, 16], and the three-component mixture of Kumaraswamy distributions [17].

The novelty of the AT-WLLM distribution lies in its construction, rather than in the introduction of additional parameters. First, although the AT-WLLM model involves a mixing weight, it does not belong to the class of conventional finite mixture distributions. Classical mixtures combine baseline distributions directly, whereas the present model aggregates nonlinear transformations of the baseline cumulative distribution functions. Consequently, the resulting density and hazard functions are not convex combinations of the corresponding Weibull and log-logistic quantities. This structural distinction leads to hazard-rate behavior that cannot be reproduced by standard mixture or competing-risk models [18]. Recent works in mixture models include [35], [36], [37], [38], [39], [40], [41], and [42].

Second, the use of a trigonometric transformation as a CDF generator distinguishes the proposed model from most existing generalized lifetime distributions. The literature on distribution generators has focused primarily on polynomial, exponential and beta-type mechanisms [19], [20]. In contrast, trigonometric mappings such as the arctangent function remain largely unexplored in this context, despite their smoothness and boundedness. This choice introduces a different mode of shape control that affects skewness, tail behavior, and hazard dynamics in a unified manner.

Thirdly, AT-WLLM distribution achieves considerable flexibility without increasing model complexity. With only five parameters, it is capable of exhibiting increasing, decreasing, unimodal, and bathtub-shaped hazard functions, a feature that is often associated with heavily parameterized models [21]. This flexibility arises from the nonlinear interaction between the baseline distributions and the trigonometric transformation rather than from additional shape parameters.

Finally, by combining Weibull and log-logistic baselines, the AT-WLLM distribution accommodates both light-tailed and heavy-tailed behavior within a single coherent framework. This transformation moderates the contribution of each baseline distribution, producing a smooth transition between different distributional regimes rather than a simple dominance of one component over the other [22].

The remainder of this paper is organized as follows. Section 2 reviews the general framework of M -fold mixture models and introduces the proposed ArcTan–Weibull–Log–Logistic Mixture (AT–WLLM) distribution, including its cumulative distribution, probability density, survival, and hazard rate functions. In Section 3, the mathematical properties of the AT–WLLM model are derived, including crude moments, moment generating function, limiting behavior, entropy measures, order statistics, and special cases. Section 4 develops the parameter estimation procedure using the maximum likelihood method and presents the associated score equations. Section 5 reports an extensive Monte Carlo simulation study to evaluate the finite-sample performance of the maximum likelihood estimators under different parameter configurations and sample sizes. Section 6 assesses the practical applicability of the proposed model through three real data applications drawn from economic and epidemiological contexts, with comprehensive comparisons against several competing lifetime distributions using information criteria and goodness-of-fit tests. Finally, Section 7 concludes the paper with a summary of the main findings and directions for future research.

2. MATERIALS AND METHODS

2.1. M-fold Mixture Model. The cumulative distribution function (CDF), probability density function (PDF), and hazard function (HF) of a general M -fold mixture model involving M subpopulations have been extensively discussed in [23,24]. For such a model, the cumulative distribution function is defined as

$$F(x) = \sum_{k=1}^M p_k G_k(x), \quad (2.1)$$

where $G_k(x)$ denotes the CDF of the k th subpopulation, p_k is the mixing proportion associated with the k th component, $p_k > 0$, and $\sum_{k=1}^M p_k = 1$.

The corresponding probability density function is given by

$$f(x) = \sum_{k=1}^M p_k g_k(x), \quad (2.2)$$

where $g_k(x)$ is the PDF of $G_k(x)$.

The survival function takes the form

$$S(x) = 1 - \sum_{k=1}^M p_k G_k(x), \quad (2.3)$$

and the hazard rate function is expressed as

$$h(x) = \frac{f(x)}{S(x)} = \sum_{k=1}^M q_k(x) h_k(x), \quad (2.4)$$

where $h_k(x)$ is the hazard rate of the k th subpopulation and

$$q_k(x) = \frac{p_k S_k(x)}{\sum_{k=1}^M p_k S_k(x)}, \quad k = 1, 2, \dots, M. \quad (2.5)$$

with $S_k(x) = 1 - F_k(x)$. It follows that

$$\sum_{k=1}^M q_k(x) = 1, \quad (2.6)$$

so that the overall hazard function is a mixture of the component hazards weighted by the conditional probabilities $q_k(x)$.

2.2. Theoretical Motivation of the Model. The AT-WLLM distribution adopts the arctangent transformation of the components of the cumulative distribution function (CDFs) to surmount the inherent shape limitations in conventional linear mixtures $F(x) = pG_1(x) + (1-p)G_2(x)$. This approach is theoretically rooted in the well-known T-X family framework studied by [25], where a new distribution is constructed by applying a continuous CDF $T(u)$ to the CDF of a classical distribution $G(x)$. Given that $T(u)$ is a valid CDF on $[0, 1]$, then $F(x) = T(G(x))$ is also a valid CDF. By choosing $T(u) = \frac{4}{\pi} \arctan(u)$ (where $u \mapsto \frac{4}{\pi} \arctan(u)$ maps the interval $[0, 1]$ unto $[0, 1]$) produces the arctangent transformation of equation (2.1) into a two-component structure of (CDF) in a smooth, monotonic manner with the resulting mixture. Consider a mixture model consisting of two distinct subpopulations ($m = 2$), where each subpopulation follows a different lifetime distribution. For such a two-component structure, the cumulative distribution function (CDF) of the resulting mixture can be written as

$$F_{y,z}(x; \psi) = \frac{4}{\pi} \left[p \cdot \arctan(G_y(x; \theta_y)) + (1-p) \cdot \arctan(G_z(x; \omega_z)) \right], \quad (2.7)$$

where $\psi = (p, \theta_y, \omega_z)$ denotes the full parameter vector, $p \in (0, 1)$ is the mixing proportion, and $G_y(\cdot)$ and $G_z(\cdot)$ represent the baseline CDFs of the two subpopulations. This theoretical motivation for the AT-WLLM distribution rests on several pillars: (a) Mathematical foundation through the well-researched T-X family, (b) Enhanced shape flexibility beyond the linear mixture, (c) Improved tail behavior suitable for extreme value applications, (e) Numerical stability, (f) Interpretability of component parameters and mixing weight, and (g) Unified handling of boundary conditions. The listed pillars positions the arctan transformation as a powerful instrument for constructing a flexible and interpretable distribution, while strategically placing the AT-WLLM as a valuable contribution to the family of mixture models.

In this work, we construct a new lifetime model by combining the Weibull distribution [1] and the Log-Logistic distribution [2]. The resulting distribution is referred to as the *Arc-Tan-Weibull-Log-Logistic Mixture* (AT-WLLM) distribution.

Definition 2.1. Let X be a continuous random variable following the AT-WLLM distribution. Then its cumulative distribution function is given by

$$F(x) = \frac{4}{\pi} \left[p \cdot \arctan(G_{\text{Weibull}}(x)) + (1-p) \cdot \arctan(G_{\text{Log-logistic}}(x)) \right], \quad (2.8)$$

where $p \in [0, 1]$ is the mixture weight (proportion) which determines the contribution of each component to the mixture. It does not directly affect the shape of the component distributions., $G_{\text{Weibull}}(x) = 1 - \exp\left[-\left(\frac{x}{\delta}\right)^k\right]$, $\delta > 0$ is the scale parameter which affects the spread of the Weibull component, $k > 0$ is the shape parameter which controls the shape of the Weibull Component, $g_{\text{Weibull}}(x) = \frac{k}{\delta} \left(\frac{x}{\delta}\right)^{k-1} \exp\left[-\left(\frac{x}{\delta}\right)^k\right]$ are the CDF and PDF of Weibull distribution respectively, and $G_{\text{Log-logistic}}(x) = \frac{(x/\beta)^\alpha}{1 + (x/\beta)^\alpha}$, where, $\alpha > 0, \beta > 0$ are the shape and scale parameter of Log-Logistic distribution respectively, $g_{\text{Log-logistic}}(x) = \frac{\alpha}{\beta} \frac{(x/\beta)^{\alpha-1}}{[1+(x/\beta)^\alpha]^2}$ are the log-logistic CDF and PDF respectively.

2.3. Probability Density Function (PDF).

Theorem 2.1. The probability density function (PDF) of the AT-WLLM distribution is given by

$$f(x) = \frac{4p}{\pi} \frac{kx^{k-1}e^{-(\frac{x}{\delta})^k}}{\delta^k \left[1 + \left(1 - e^{-(\frac{x}{\delta})^k}\right)^2\right]} + \frac{4(1-p)}{\pi} \frac{\alpha(\frac{x}{\beta})^{\alpha-1}}{\beta \left(1 + (\frac{x}{\beta})^\alpha\right)^2 \left[1 + \left(\frac{(\frac{x}{\beta})^\alpha}{1+(\frac{x}{\beta})^\alpha}\right)^2\right]} \quad (2.9)$$

Proof. Starting from the CDF of the AT-WLLM distribution in (2.8), we differentiate with respect to x and apply the chain rule:

Let the cumulative distribution function of the proposed distribution be

$$F(x) = \frac{4}{\pi} \left[p \arctan(G_W(x)) + (1-p) \arctan(G_L(x)) \right], \quad x > 0,$$

where $0 \leq p \leq 1$, $G_W(x) = 1 - \exp\left[-\left(\frac{x}{\delta}\right)^k\right]$, $\delta > 0$, $k > 0$, $G_L(x) = \frac{(x/\beta)^\alpha}{1+(x/\beta)^\alpha}$, $\alpha > 0$, $\beta > 0$. Differentiating $F(x)$ with respect to x , we obtain the corresponding probability density function as

$$f(x) = \frac{4}{\pi} \left[p \frac{d}{dx} \arctan(G_W(x)) + (1-p) \frac{d}{dx} \arctan(G_L(x)) \right].$$

Using the chain rule, $\frac{d}{dx} \arctan(u) = \frac{u'}{1+u^2}$, it follows that

$$f(x) = \frac{4}{\pi} \left[p \frac{G'_W(x)}{1 + G_W^2(x)} + (1-p) \frac{G'_L(x)}{1 + G_L^2(x)} \right].$$

For the Weibull baseline distribution, $G_W(x) = 1 - \exp\left[-\left(\frac{x}{\delta}\right)^k\right]$, and differentiating gives $G'_W(x) = g_W(x) = \frac{k}{\delta^k} x^{k-1} \exp\left[-\left(\frac{x}{\delta}\right)^k\right]$. For the log-logistic baseline distribution, $G_L(x) = \frac{(x/\beta)^\alpha}{1+(x/\beta)^\alpha}$, and differentiating gives $G'_L(x) = g_L(x) = \frac{\alpha}{\beta} \frac{(x/\beta)^{\alpha-1}}{(1+(x/\beta)^\alpha)^2}$. Substituting into the density yields

$$f(x) = \frac{4}{\pi} \left[p \frac{\frac{k}{\delta^k} x^{k-1} \exp\left[-\left(\frac{x}{\delta}\right)^k\right]}{1 + \left(1 - \exp\left[-\left(\frac{x}{\delta}\right)^k\right]\right)^2} + (1-p) \frac{\frac{\alpha}{\beta} (x/\beta)^{\alpha-1}}{(1 + (x/\beta)^\alpha)^2 \left[1 + \left(\frac{(x/\beta)^\alpha}{1+(x/\beta)^\alpha}\right)^2\right]} \right].$$

Simplifying further yields the stated expression for equation (2.9).

□

2.4. Survival Function.

Definition 2.2. The corresponding survival function of the AT-WLLM distribution defined by $S(x) = 1 - F(x)$ is given as

$$S(x) = 1 - \frac{4}{\pi} \left[p \cdot \arctan\left(1 - e^{-\left(\frac{x}{\delta}\right)^k}\right) + (1-p) \cdot \arctan\left(\frac{\left(\frac{x}{\beta}\right)^\beta}{1 + \left(\frac{x}{\beta}\right)^\alpha}\right) \right] \quad (2.10)$$

2.5. Hazard Rate Function (HRF).

Definition 2.3. The hazard rate function of the AT-WLLM distribution defined as $h(x) = \frac{f(x)}{1-F(x)}$ is given by

$$h(x) = \frac{\frac{4}{\pi} \left[\frac{p g_{\text{Weibull}}(x)}{1 + G_{\text{Weibull}}^2(x)} + \frac{(1-p) g_{\text{Log-logistic}}(x)}{1 + G_{\text{Log-logistic}}^2(x)} \right]}{1 - \frac{4}{\pi} \left[p \cdot \arctan(G_{\text{Weibull}}(x)) + (1-p) \cdot \arctan(G_{\text{Log-logistic}}(x)) \right]}, \quad x > 0,$$

where G_{Weibull} , g_{Weibull} and $G_{\text{Log-logistic}}$, $g_{\text{Log-logistic}}$ denote the CDF and PDF respectively.

$$= \frac{\frac{4p}{\pi} \frac{kx^{k-1} e^{-\left(\frac{x}{\delta}\right)^k}}{\delta^k \left[1 + \left(1 - e^{-\left(\frac{x}{\delta}\right)^k}\right)^2\right]} + \frac{4(1-p)}{\pi} \frac{\alpha \left(\frac{x}{\beta}\right)^{\alpha-1}}{\beta \left(1 + \left(\frac{x}{\beta}\right)^\alpha\right)^2 \left[1 + \left(\frac{\left(\frac{x}{\beta}\right)^\alpha}{1 + \left(\frac{x}{\beta}\right)^\alpha}\right)^2\right]}}{1 - \frac{4}{\pi} \left[p \arctan\left(1 - e^{-\left(\frac{x}{\delta}\right)^k}\right) + (1-p) \arctan\left(\frac{\left(\frac{x}{\beta}\right)^\alpha}{1 + \left(\frac{x}{\beta}\right)^\alpha}\right) \right]} \quad (2.11)$$

To illustrate the flexibility of the proposed AT-WLLM distribution, several parameter configurations are considered to represent different mixing structures and hazard rate behaviors. These parameter sets are chosen to reflect balanced mixtures and commonly observed hazard patterns in lifetime data analysis. The specific parameter values used to generate the PDF and HRF plots are summarized in **Table 1**. To illustrate the flexibility of the proposed AT-WLLM distribution, several parameter configurations are considered to represent different mixing structures and hazard rate behaviors. These parameter sets are chosen to reflect balanced mixtures and commonly observed hazard patterns in lifetime data analysis. The specific parameter values used to generate the PDF and HRF plots are summarized in **Table 1**. **Figure 1** presents the corresponding plots of the PDF

TABLE 1. Parameter Sets Used in the Plots of the PDF and HRF

Set	p	k_w	δ_w	α_l	β_l
Balanced mixture	0.50	2.00	1.50	2.00	1.50
Weibull-dominated	0.97	1.50	2.00	2.5	1.00
Log-logistic-dominated	0.20	2.50	1.00	1.50	2.05
Early high hazard	0.60	0.80	1.00	3.00	2.00
Bathtub hazard	0.50	3.00	2.50	2.00	0.05
Increasing hazard	0.70	3.00	2.50	2.00	0.90

and HRF for the parameter sets listed in **Table 1**, highlighting the wide range of distributional shapes captured by the AT-WLLM model.

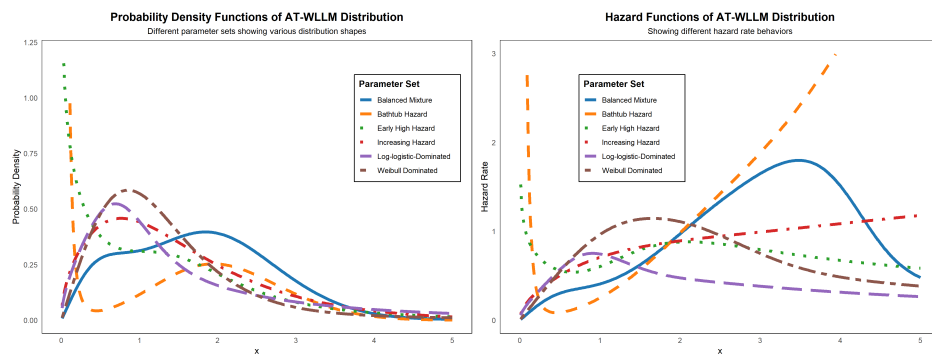


FIGURE 1. Plots of the PDF and HRF illustrating the flexibility of the AT-WLLM model.

The plots reveal that the AT-WLLM distribution is capable of modeling diverse density shapes, including reversed-J, right-skewed, approximately symmetric, unimodal, and multimodal forms. In addition, the hazard rate function exhibits a variety of patterns such as bathtub, upside-down bathtub, and decreasing hazard rates.

These shapes are commonly encountered in real-life applications across several fields, including reliability engineering, insurance and actuarial science, epidemiology, forensic science, and biometrics. For instance, the reversed-J density shape is often associated with the early surge observed during infectious disease outbreaks such as COVID-19, while insurance claim sizes typically follow a right-skewed distribution. In engineering applications, the lifetimes of mechanical systems may exhibit approximately symmetric behavior, whereas electronic components are frequently characterized by a bathtub-shaped hazard rate. Similarly, economic and business cycles, marked by periods of expansion and contraction, are well represented by an upside-down bathtub hazard structure.

2.6. Series Expansion of the CDF and PDF of AT-WLLM Distribution. To study the Mathematical properties of the distribution it is essential to derive the linear representation of the CDF and PDF

into forms that are mathematically tractable and analytically useful. Here, consider the cumulative distribution function of $AT - WLLM$ given by

$$F(x) = \frac{4}{\pi} [p \arctan(G_W(x)) + (1-p) \arctan(G_L(x))], \quad x > 0,$$

where $G_W(x) = 1 - \exp\left[-\left(\frac{x}{\delta}\right)^k\right]$, $\delta, k > 0$, and $G_L(x) = \frac{\left(\frac{x}{\beta}\right)^\alpha}{1 + \left(\frac{x}{\beta}\right)^\alpha}$, $\alpha, \beta > 0$. The Maclaurin expansion of the inverse tangent function is $\arctan(z) = \sum_{r=0}^{\infty} \frac{(-1)^r}{2r+1} z^{2r+1}$, $|z| \leq 1$. Substituting into the CDF yields

$$F(x) = \frac{4}{\pi} \sum_{r=0}^{\infty} \frac{(-1)^r}{2r+1} [p G_W^{2r+1}(x) + (1-p) G_L^{2r+1}(x)]. \quad (2.12)$$

Hence,

$$F(x) = \frac{4}{\pi} \sum_{r=0}^{\infty} \frac{(-1)^r}{2r+1} \left[p \left(1 - e^{-(x/\delta)^k}\right)^{2r+1} + (1-p) \left(\frac{(x/\beta)^\alpha}{1 + (x/\beta)^\alpha}\right)^{2r+1} \right].$$

Weibull component expansion

$$G_W^{2r+1}(x) = (1 - e^{-(x/\delta)^k})^{2r+1}$$

Using the binomial expansion $(1-z)^m = \sum_{j=0}^m (-1)^j \binom{m}{j} z^j$, with $z = e^{-(x/\delta)^k}$ and $m = 2r+1$, we obtain $(1 - e^{-(x/\delta)^k})^{2r+1} = \sum_{j=0}^{2r+1} (-1)^j \binom{2r+1}{j} e^{-j(x/\delta)^k}$.

$$G_W^{2r+1}(x) = \sum_{j=0}^{2r+1} (-1)^j \binom{2r+1}{j} e^{-j(x/\delta)^k}. \quad (2.13)$$

To expand the Log-Logistic Component, Write

$$G_L^{2r+1}(x) = \left(\frac{(x/\beta)^\alpha}{1 + (x/\beta)^\alpha}\right)^{2r+1} = (x/\beta)^{\alpha(2r+1)} (1 + (x/\beta)^\alpha)^{-(2r+1)}.$$

Applying the generalized binomial expansion $(1+z)^{-m} = \sum_{j=0}^{\infty} (-1)^j \binom{m+j-1}{j} z^j$, with $z = (x/\beta)^\alpha$ and $m = 2r+1$, gives $(1 + (x/\beta)^\alpha)^{-(2r+1)} = \sum_{j=0}^{\infty} (-1)^j \binom{2r+j}{j} (x/\beta)^{\alpha j}$. Therefore,

$$G_L^{2r+1}(x) = \sum_{j=0}^{\infty} (-1)^j \binom{2r+j}{j} (x/\beta)^{\alpha(2r+1+j)}. \quad (2.14)$$

Substituting equations (2.6) and (2.14) expansions into (2.12) yields

$$F(x) = \frac{4}{\pi} \sum_{r=0}^{\infty} \frac{(-1)^r}{2r+1} \left[p \sum_{j=0}^{2r+1} (-1)^j \binom{2r+1}{j} e^{-j(x/\delta)^k} + (1-p) \sum_{j=0}^{\infty} (-1)^j \binom{2r+j}{j} (x/\beta)^{\alpha(2r+1+j)} \right]. \quad (2.15)$$

Differentiating equation (2.15) term-by-term gives, the PDF becomes

$$f(x) = \frac{4}{\pi} \sum_{r=0}^{\infty} \frac{(-1)^r}{2r+1} \left[p \sum_{j=0}^{2r+1} (-1)^{j+1} \binom{2r+1}{j} j \frac{k}{\delta^k} x^{k-1} e^{-j(x/\delta)^k} \right. \\ \left. + (1-p) \sum_{j=0}^{\infty} (-1)^j \binom{2r+j}{j} \frac{\alpha(2r+1+j)}{\beta} (x/\beta)^{\alpha(2r+1+j)-1} \right]. \quad (2.16)$$

3. MATHEMATICAL PROPERTIES OF THE MODEL

3.1. Crude Moments.

Definition 3.1. The m^{th} crude moment of a continuous random variable X with density $f(x)$ is defined as

$$\mu'_m = E[X^m] = \int_0^\infty x^m f(x) dx. \quad (3.1)$$

Theorem 3.1. For the AT-WLLM distribution, the m th crude moment is given by

$$\mu'_m = \frac{4}{\pi} \left[p I_{\text{Weibull}}(m) + (1-p) I_{\text{Log-logistic}}(m) \right],$$

where $I_{\text{Weibull}}(m)$ and $I_{\text{Log-logistic}}(m)$ denote the contributions of the Weibull and log-logistic components, respectively.

$$\begin{aligned} \mu'_m = & \frac{4p}{\pi} \sum_{r=0}^{\infty} \frac{(-1)^r}{2r+1} \sum_{j=1}^{2r+1} (-1)^{j+1} \binom{2r+1}{j} \delta^m j^{-m/k} \Gamma\left(1 + \frac{m}{k}\right) \\ & + \frac{4(1-p)}{\pi} \sum_{r=0}^{\infty} \frac{(-1)^r}{2r+1} \sum_{j=0}^{\infty} (-1)^j \binom{2r+j}{j} (2r+1+j) \beta^m B\left(2r+j+1 + \frac{m}{\alpha}, 1 - \frac{m}{\alpha}\right) \end{aligned} \quad (3.2)$$

Proof. Substituting the series expansion of the density given in equation (3.2) into equation (3.1) and simplifying gives

$$\mu'_s = \frac{4}{\pi} \sum_{r=0}^{\infty} \frac{(-1)^r}{2r+1} \left[p \sum_{j=0}^{2r+1} (-1)^{j+1} \binom{2r+1}{j} j^{\frac{k}{\delta}} I_1 + (1-p) \sum_{j=0}^{\infty} (-1)^j \binom{2r+j}{j} \frac{\alpha(2r+1+j)}{\beta} I_2 \right]$$

where $I_1 = \int_0^\infty x^{s+k-1} e^{-j(x/\delta)^k} dx$, and $I_2 = \int_0^\infty x^s (x/\beta)^{\alpha(2r+1+j)-1} dx$. To evaluate the Weibull Integral, use the transformation $u = j\left(\frac{x}{\delta}\right)^k$. $x = \delta\left(\frac{u}{j}\right)^{1/k}$, $dx = \frac{\delta}{k} j^{-1/k} u^{1/k-1} du$.

Hence, $I_1 = \int_0^\infty \delta^{s+k} j^{-(s+k)/k} u^{(s+k)/k-1} e^{-u} \frac{1}{k} du$. Therefore, $I_1 = \frac{\delta^{s+k}}{k} j^{-(s+k)/k} \Gamma\left(\frac{s+k}{k}\right)$. Thus, the Weibull contribution becomes

$$I_1(m) = \frac{4p}{\pi} \sum_{r=0}^{\infty} \frac{(-1)^r}{2r+1} \sum_{j=0}^{2r+1} (-1)^{j+1} \binom{2r+1}{j} \delta^s j^{-s/k} \Gamma\left(1 + \frac{s}{k}\right).$$

To evaluate of Log-Logistic Integral, the log-logistic term, is rewritten as

$$I_2 = \beta^{-\alpha(2r+1+j)+1} \int_0^\infty x^{s+\alpha(2r+1+j)-1} dx$$

To obtain convergence, use the beta transformation $u = \left(\frac{x}{\beta}\right)^\alpha$. Then $x = \beta u^{1/\alpha}$, $dx = \frac{\beta}{\alpha} u^{1/\alpha-1} du$. Hence,

$$I_2 = \frac{\beta^{s+1}}{\alpha} \int_0^\infty u^{2r+j+s/\alpha} (1+u)^{-2r-2} du.$$

Recognizing the beta integral, $\int_0^\infty u^{a-1} (1+u)^{-(a+b)} du = B(a, b)$, we obtain

$$I_2 = \frac{\beta^{s+1}}{\alpha} B\left(2r+j+1 + \frac{s}{\alpha}, 1 - \frac{s}{\alpha}\right).$$

Thus, the log-logistic contribution becomes

$$I_2(m) = \frac{4(1-p)}{\pi} \sum_{r=0}^{\infty} \frac{(-1)^r}{2r+1} \sum_{j=0}^{\infty} (-1)^j \binom{2r+j}{j} (2r+1+j) \beta^s B\left(2r+j+1 + \frac{s}{\alpha}, 1 - \frac{s}{\alpha}\right).$$

The expression for the Crude Moment is therefore,

$$\mu'_s = I_1(m) + I_2(m)$$

□

3.2. Moment Generating Function.

Series Expansion Approach.

Theorem 3.2. The moment generating function (MGF) of the AT-WLLM distribution is given by

$$\begin{aligned} M_X(t) &= \frac{4p}{\pi} \sum_{r=0}^{\infty} \frac{(-1)^r}{2r+1} \sum_{j=0}^{2r+1} (-1)^{j+1} \binom{2r+1}{j} \sum_{n=0}^{\infty} \frac{t^n}{n!} \delta^n j^{-nk} \Gamma\left(1 + \frac{n}{k}\right) \\ &+ \frac{4(1-p)}{\pi} \sum_{r=0}^{\infty} \frac{(-1)^r}{2r+1} \sum_{j=0}^{\infty} (-1)^j \binom{2r+j}{j} (2r+1+j) \sum_{n=0}^{\infty} \frac{t^n}{n!} \beta^n B\left(2r+j+1 + \frac{n}{\alpha}, 1 - \frac{n}{\alpha}\right) \end{aligned} \quad (3.3)$$

Proof. Starting from the definition of the MGF,

$$M_X(t) = E[e^{tX}] = \int_0^{\infty} e^{tx} f(x) dx,$$

Using the series expansion of the AWLLM density, we obtain

$$M_X(t) = \frac{4}{\pi} \sum_{r=0}^{\infty} \frac{(-1)^r}{2r+1} \left[p \sum_{j=0}^{2r+1} (-1)^{j+1} \binom{2r+1}{j} j \frac{k}{\delta^k} I_1 + (1-p) \sum_{j=0}^{\infty} (-1)^j \binom{2r+j}{j} \frac{\alpha(2r+1+j)}{\beta} I_2 \right],$$

where $I_1 = \int_0^{\infty} x^{k-1} e^{tx} e^{-j(x/\delta)^k} dx$, and $I_2 = \int_0^{\infty} e^{tx} (x/\beta)^{\alpha(2r+1+j)-1} dx$. Expanding the exponential term and simplifying, we obtain, $I_1 = \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_0^{\infty} x^{n+k-1} e^{-j(x/\delta)^k} dx$. Using the generalized gamma integral $\int_0^{\infty} x^{a-1} e^{-cx^k} dx = \frac{1}{k} c^{-a/k} \Gamma\left(\frac{a}{k}\right)$, with $a = n+k$, $c = \frac{j}{\delta^k}$, we obtain

$$I_1 = \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{\delta^{n+k}}{k} j^{-(n+k)/k} \Gamma\left(\frac{n+k}{k}\right).$$

Thus, the Weibull contribution becomes

$$M_{X1}(t) = \frac{4p}{\pi} \sum_{r=0}^{\infty} \frac{(-1)^r}{2r+1} \sum_{j=0}^{2r+1} (-1)^{j+1} \binom{2r+1}{j} \sum_{n=0}^{\infty} \frac{t^n}{n!} \delta^n j^{-n/k} \Gamma\left(1 + \frac{n}{k}\right).$$

Similarly, the Log-Logistic contribution is obtained by using the generalized beta integral. To obtain convergence, the beta transformation $u = \left(\frac{x}{\beta}\right)^{\alpha}$. Then $x = \beta u^{1/\alpha}$, $dx = \frac{\beta}{\alpha} u^{1/\alpha-1} du$. $I_2 =$

$\frac{\beta^{s+1}}{\alpha} \int_0^\infty u^{2r+j+s/\alpha} (1+u)^{-2r-2} du$. Recognizing the beta integral, $\int_0^\infty u^{a-1} (1+u)^{-(a+b)} du = B(a, b)$, is used. Thus, the log-logistic contribution becomes

$$M_{X2}(t) = \frac{4(1-p)}{\pi} \sum_{r=0}^{\infty} \frac{(-1)^r}{2r+1} \sum_{j=0}^{\infty} (-1)^j \binom{2r+j}{j} (2r+1+j) \sum_{n=0}^{\infty} \frac{t^n}{n!} \beta^n B\left(2r+j+1+\frac{n}{\alpha}, 1-\frac{n}{\alpha}\right).$$

Therefore,

$$M_X(t) = M_{X1}(t) + M_{X2}(t)$$

Proved. $\square$

SPECIAL CASES AND PROPERTIES

From equation (2.7), some well-known special distributions can be derived as sub-models of the AT-WLLM distribution under certain specific parameter settings:

- (i) When $p = 1$: the model simplifies to the arctangent-transformed Weibull distribution [26];
- (ii) $p = 0$: the model simplifies to the arctangent-transformed log-logistic distribution [26];
- (iii) $k_w = 1$: the Weibull component reduces to the exponential distribution;
- (iv) $\alpha_l = 1$: the log-logistic component becomes the logistic distribution.

Limiting Behavior.

Proposition 3.1. *The limiting behavior of the AT-WLLM distribution is characterized by:*

$$\begin{aligned} \lim_{x \rightarrow 0^+} F(x) &\sim \frac{4}{\pi} \left[\frac{p}{\delta^k} x^k + \frac{1-p}{\beta^\alpha} x^\alpha \right], \\ \lim_{x \rightarrow \infty} S(x) &\sim \frac{2}{\pi} \left[p e^{-(x/\delta)^k} + (1-p) \left(\frac{\beta}{x} \right)^\alpha \right]. \end{aligned} \quad (3.4)$$

These expressions highlight the interplay between the Weibull and log-logistic tail behaviors in determining the overall distributional form.

3.3. Renyi Entropy.

Definition 3.2. *The Renyi entropy of order ρ ($\rho > 0$, $\rho \neq 1$) for a continuous random variable X with pdf $f(x)$ is defined as:*

$$H_\rho(X) = \frac{1}{1-\rho} \log \left(\int_{-\infty}^{\infty} [f(x)]^\rho dx \right) \quad (3.5)$$

Theorem 3.3. *The Renyi entropy for the AT-WLLM distribution is given by:*

$$I_R(\rho) = \frac{1}{1-\rho} \log \left[\left(\frac{4}{\pi} \right)^\rho \sum_{j=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{m=0}^{2r} D_{jrsm} I_{jrsm} \right]. \quad (3.6)$$

where:

$$I_{jrsm} = \int_0^\infty x^M \exp \left[-N \left(\frac{x}{\delta} \right)^k \right] \left(1 + \left(\frac{x}{\beta} \right)^\alpha \right)^{-Q} dx.$$

Proof. The Rényi entropy $I_R(\rho)$ of $AT - WLLM$ random variable X with probability density function $f(x)$ raised to power ρ is given as

$$f(x)^\rho = \left(\frac{4}{\pi}\right)^\rho \left[p \frac{g_W(x)}{1 + G_W(x)^2} + (1-p) \frac{g_L(x)}{1 + G_L(x)^2} \right]^\rho.$$

Using the generalized binomial expansion, $(A + B)^\rho = \sum_{j=0}^{\infty} \binom{\rho}{j} A^{\rho-j} B^j$, we obtain

$$f(x)^\rho = \left(\frac{4}{\pi}\right)^\rho \sum_{j=0}^{\infty} \binom{\rho}{j} p^{\rho-j} (1-p)^j \left(\frac{g_W(x)}{1 + G_W(x)^2} \right)^{\rho-j} \left(\frac{g_L(x)}{1 + G_L(x)^2} \right)^j.$$

Next, expand $(1 + G_W(x)^2)^{-(\rho-j)} = \sum_{r=0}^{\infty} (-1)^r \binom{\rho-j+r-1}{r} G_W(x)^{2r}$, and $(1 + G_L(x)^2)^{-j} = \sum_{s=0}^{\infty} (-1)^s \binom{j+s-1}{s} G_L(x)^{2s}$. Substituting these expansions gives

$$f(x)^\rho = \left(\frac{4}{\pi}\right)^\rho \sum_{j=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} D_{jrs} g_W(x)^{\rho-j} g_L(x)^j G_W(x)^{2r} G_L(x)^{2s},$$

where

$$D_{jrs} = \binom{\rho}{j} (-1)^{r+s} \binom{\rho-j+r-1}{r} \binom{j+s-1}{s} p^{\rho-j} (1-p)^j.$$

Now, expand $G_W(x)^{2r} = \left(1 - \exp\left[-\left(\frac{x}{\delta}\right)^k\right]\right)^{2r}$. Applying the binomial expansion again, $G_W(x)^{2r} = \sum_{m=0}^{2r} (-1)^m \binom{2r}{m} \exp\left[-m\left(\frac{x}{\delta}\right)^k\right]$.

Hence $f(x)^\rho = \left(\frac{4}{\pi}\right)^\rho \sum_{j=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{m=0}^{2r} D_{jrsm} x^M \exp\left[-N\left(\frac{x}{\delta}\right)^k\right] \left(1 + \left(\frac{x}{\beta}\right)^\alpha\right)^{-Q}$, where $M = (k-1)(\rho-j) + j(\alpha-1) + 2s\alpha$, $N = (\rho-j) + m$, $Q = 2j + 2s$, and the full coefficient is

$$D_{jrsm} = \binom{\rho}{j} (-1)^{r+s+m} \binom{\rho-j+r-1}{r} \binom{j+s-1}{s} \binom{2r}{m} p^{\rho-j} (1-p)^j \left(\frac{k}{\delta^k}\right)^{\rho-j} \left(\frac{\alpha}{\beta}\right)^j \beta^{-j(\alpha-1)-2s\alpha}.$$

Therefore,

$$\int_0^\infty f(x)^\rho dx = \left(\frac{4}{\pi}\right)^\rho \sum_{j=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{m=0}^{2r} D_{jrsm} I_{jrsm},$$

where

$$I_{jrsm} = \int_0^\infty x^M \exp\left[-N\left(\frac{x}{\delta}\right)^k\right] \left(1 + \left(\frac{x}{\beta}\right)^\alpha\right)^{-Q} dx.$$

Finally, substituting into the Rényi entropy definition yields

$$I_R(\rho) = \frac{1}{1-\rho} \log \left[\left(\frac{4}{\pi}\right)^\rho \sum_{j=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{m=0}^{2r} D_{jrsm} I_{jrsm} \right].$$

□

It is noteworthy that I_{jrsm} can be obtain using the generalised beta-gamma integral.

3.4. Order Statistics. Let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ denote the order statistics from a random sample of size n .

Theorem 3.4. *The probability density function of the r -th order statistic $X_{(r)}$ from a sample of size n is:*

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} [F(x)]^{r-1} [1-F(x)]^{n-r} f(x) \quad (3.7)$$

where $F(x)$ and $f(x)$ are the CDF and PDF of the AT-WLLM distribution.

Proof.

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} [F(x)]^{r-1} [1-F(x)]^{n-r} f(x).$$

Substituting the proposed CDF gives

$$\begin{aligned} f_{X_{(r)}}(x) &= \frac{n!}{(r-1)!(n-r)!} \left[\frac{4}{\pi} (p \arctan(G_W(x)) + (1-p) \arctan(G_L(x))) \right]^{r-1} \\ &\times \left[1 - \frac{4}{\pi} (p \arctan(G_W(x)) + (1-p) \arctan(G_L(x))) \right]^{n-r} f(x). \end{aligned}$$

Using the binomial expansion, $[1-F(x)]^{n-r} = \sum_{m=0}^{n-r} (-1)^m \binom{n-r}{m} [F(x)]^m$. Hence,

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} \sum_{m=0}^{n-r} (-1)^m \binom{n-r}{m} [F(x)]^{m+r-1} f(x).$$

Substituting the series form of $F(x)$ yields

$$\begin{aligned} f_{X_{(r)}}(x) &= \frac{n!}{(r-1)!(n-r)!} \sum_{m=0}^{n-r} (-1)^m \binom{n-r}{m} \\ &\times \left(\frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} [pG_W^{2k+1}(x) + (1-p)G_L^{2k+1}(x)] \right)^{m+r-1} f(x). \end{aligned}$$

□

Thus, the order statistics admit an infinite series representation suitable for numerical evaluation.

Theorem 3.5. *The m -th moment of the r -th order statistic is:*

$$E[X_{(r)}^m] = \frac{n!}{(r-1)!(n-r)!} \int_0^{\infty} x^m [F(x)]^{r-1} [1-F(x)]^{n-r} f(x) dx \quad (3.8)$$

Theorem 3.6. *The cumulative distribution function of the r -th order statistic is:*

$$F_{X_{(r)}}(x) = \sum_{j=r}^n \binom{n}{j} [F(x)]^j [1-F(x)]^{n-j} \quad (3.9)$$

4. PARAMETER ESTIMATION

Maximum Likelihood Estimation. Here, the maximum likelihood method is employed to estimate the parameters of the AT-WLLM distribution following the procedure given by [43] and [44]. Let $\mathbf{X} = (X_1, X_2, \dots, X_n)$ be a random sample of size n drawn from the AT-WLLM distribution. Denote the vector of unknown parameters by $\boldsymbol{\theta} = (p, k_w, \delta_w, \alpha_l, \beta_l)^\top$, where $p \in (0, 1)$ is the mixing proportion, (k_w, δ_w) are the shape and scale parameters of the Weibull component, and (α_l, β_l) are the shape and scale parameters of the log-logistic component.

The log-likelihood function corresponding to the observed sample $\mathbf{X}$ is given by

$$\ell(\boldsymbol{\theta}) = n \log\left(\frac{4}{\pi}\right) + \sum_{i=1}^n \log[p A(x_i) + (1-p) B(x_i)], \quad (4.1)$$

where

$$A(x) = \frac{g_{\text{Weibull}}(x)}{1 + (G_{\text{Weibull}}(x))^2}, \quad B(x) = \frac{g_{\text{Log-logistic}}(x)}{1 + (G_{\text{Log-logistic}}(x))^2},$$

and $g_{\text{Weibull}}(\cdot)$, $G_{\text{Weibull}}(\cdot)$, $g_{\text{Log-logistic}}(\cdot)$, and $G_{\text{Log-logistic}}(\cdot)$ denote the corresponding probability density and distribution functions of the Weibull and log-logistic components, respectively.

Score Equations. The maximum likelihood estimator $\hat{\boldsymbol{\theta}}$ is obtained by solving the system of score equations defined by the first derivatives of the log-likelihood function with respect to the model parameters. In particular, the score function with respect to the mixing proportion p is

$$\frac{\partial \ell}{\partial p} = \sum_{i=1}^n \frac{A(x_i) - B(x_i)}{p A(x_i) + (1-p) B(x_i)} = 0. \quad (4.2)$$

Let $\boldsymbol{\phi}_w = (k_w, \delta_w)$ and $\boldsymbol{\phi}_l = (\alpha_l, \beta_l)$ denote the parameter vectors of the Weibull and log-logistic components, respectively. The corresponding score equations are given by

$$\begin{aligned} \frac{\partial \ell}{\partial \phi_{w,j}} &= \sum_{i=1}^n \frac{p \frac{\partial A(x_i)}{\partial \phi_{w,j}}}{p A(x_i) + (1-p) B(x_i)} = 0, & j = 1, 2, \\ \frac{\partial \ell}{\partial \phi_{l,j}} &= \sum_{i=1}^n \frac{(1-p) \frac{\partial B(x_i)}{\partial \phi_{l,j}}}{p A(x_i) + (1-p) B(x_i)} = 0, & j = 1, 2. \end{aligned} \quad (4.3)$$

Because the resulting system of equations is nonlinear and does not admit closed-form solutions, numerical optimization techniques are required. Accordingly, the maximum likelihood estimates $\hat{\boldsymbol{\theta}} = (\hat{p}, \hat{k}_w, \hat{\delta}_w, \hat{\alpha}_l, \hat{\beta}_l)^\top$ are obtained using iterative numerical methods such as the Newton-Raphson algorithm or quasi-Newton optimization procedures.

5. SIMULATION STUDY

This section investigates the finite-sample performance of the maximum likelihood estimation (MLE) procedure for the AT-WLLM distribution using a Monte Carlo simulation study. For each selected parameter configuration, 1,000 independent random samples are generated from

the AT-WLLM distribution according to equation (2.7). The sample sizes considered are $n \in \{25, 60, 100, \dots, 500\}$, allowing the behavior of the estimators to be examined over a wide range of small to large samples. For each sample size and parameter setting, the simulation experiment is replicated 1,000 times. In every replication, parameter estimation is carried out by numerically maximizing the log-likelihood function using the `optim` routine in R (version 4.5.1), with the L-BFGS-B algorithm employed to enforce the parameter constraints of the AT-WLLM model. The performance of the MLEs is evaluated using three standard accuracy measures: the average estimate (AV), the bias (Bias), and the root mean square error (RMSE). These measures are computed across replications for each parameter and sample size and are summarized through graphical displays to illustrate convergence behavior and estimation precision. The interpretation of the simulation results follows commonly accepted guidelines in the statistical literature, with emphasis on closeness of the estimators to the true parameter values which facilitates a clearer comparison of the estimator performance across different sample sizes, decreasing bias and RMSE as indicators of consistency and efficiency (see, e.g., [27]). The numerical Average estimate, RMSE, and bias results for the first parameter configuration, $\theta = (0.3, 1.5, 2.0, 2.0, 1.5)$, are summarized in Table 2.

TABLE 2. Monte Carlo simulation results for $\theta = (0.3, 1.5, 2.0, 2.0, 1.5)$

Parameter	Sample Size																							
	$n = 25$			$n = 60$			$n = 100$			$n = 200$			$n = 300$			$n = 400$			$n = 500$					
	Avg	RMSE	Bias	Avg	RMSE	Bias	Avg	RMSE	Bias	Avg	RMSE	Bias	Avg	RMSE	Bias	Avg	RMSE	Bias	Avg	RMSE	Bias	Avg	RMSE	Bias
p	0.558	0.390	0.258	0.474	0.363	0.174	0.457	0.367	0.157	0.383	0.321	0.083	0.339	0.308	0.039	0.316	0.286	0.016	0.316	0.291	0.016			
δ_w	1.680	0.714	0.180	1.675	0.683	0.175	1.645	0.678	0.145	1.609	0.615	0.109	1.565	0.622	0.065	1.557	0.710	0.057	1.566	0.610	0.066			
k_w	3.780	3.285	1.780	3.809	3.361	1.809	3.515	3.173	1.514	3.386	2.856	1.386	3.518	3.178	1.518	3.103	2.606	1.103	3.226	2.788	1.226			
α_l	4.146	3.748	2.146	3.119	2.610	1.119	2.716	1.912	0.716	2.326	1.087	0.326	2.181	0.605	0.181	2.145	0.471	0.145	2.156	0.398	0.156			
β_l	2.224	2.423	0.724	2.430	2.599	0.930	2.513	2.548	1.013	2.234	1.939	0.734	2.029	1.509	0.529	1.909	1.297	0.409	1.323	1.323	0.447			

The RMSE and bias results for the second parameter configuration, $\theta = (0.2, 2.0, 1.0, 0.5, 0.5)$, are reported in Table 3.

TABLE 3. Monte Carlo simulation results for $\theta = (0.2, 0.2, 1.0, 0.5, 0.5)$

Parameter	Sample Size																							
	$n = 25$			$n = 60$			$n = 100$			$n = 200$			$n = 300$			$n = 400$			$n = 500$					
	Avg	RMSE	Bias	Avg	RMSE	Bias	Avg	RMSE	Bias	Avg	RMSE	Bias	Avg	RMSE	Bias	Avg	RMSE	Bias	Avg	RMSE	Bias	Avg	RMSE	Bias
p	0.479	0.329	0.279	0.428	0.279	0.228	0.383	0.230	0.183	0.322	0.165	0.122	0.311	0.149	0.111	0.301	0.133	0.101	0.286	0.112	0.086			
δ_w	0.139	0.172	-0.061	0.110	0.113	-0.090	0.095	0.115	-0.105	0.081	0.124	-0.119	0.078	0.125	-0.122	0.075	0.127	-0.125	0.071	0.130	-0.129			
k_w	3.077	3.113	2.077	3.277	3.243	2.277	3.452	3.324	2.452	3.925	3.559	2.925	3.906	3.457	2.906	3.939	3.320	2.939	4.094	3.427	3.094			
α_l	0.942	0.898	0.442	0.783	0.318	0.283	0.758	0.272	0.258	0.734	0.239	0.234	0.735	0.239	0.235	0.734	0.236	0.234	0.732	0.234	0.232			
β_l	3.476	4.462	2.976	2.688	3.610	2.188	1.903	2.446	1.403	1.297	1.410	0.797	1.178	1.192	0.678	1.062	0.829	0.562	0.960	0.596	0.460			

The simulation results for the third parameter configuration, $\theta = (0.5, 2.5, 1.2, 0.5, 3.5)$, are summarized in Table 4,

TABLE 4. Monte Carlo simulation results for $\theta = (0.5, 2.5, 1.2, 0.5, 3.5)$

Parameter	Sample Size																				
	$n = 25$			$n = 60$			$n = 100$			$n = 200$			$n = 300$			$n = 400$			$n = 500$		
	Avg	RMSE	Bias	Avg	RMSE	Bias	Avg	RMSE	Bias	Avg	RMSE	Bias	Avg	RMSE	Bias	Avg	RMSE	Bias	Avg	RMSE	Bias
p	0.440	0.253	-0.059	0.377	0.236	-0.123	0.355	0.225	-0.145	0.318	0.220	-0.182	0.301	0.230	-0.198	0.309	0.202	-0.190	0.298	0.213	-0.202
δ_w	2.512	1.524	0.013	2.522	1.351	0.023	2.536	1.096	0.037	2.499	0.577	-0.001	2.478	0.387	-0.022	2.482	0.321	-0.018	2.454	0.313	-0.045
k_w	3.023	3.419	1.823	2.427	2.523	1.228	2.375	2.399	1.176	1.924	1.575	0.724	1.902	1.369	0.702	1.617	0.541	0.417	1.677	0.564	0.478
α_l	1.023	1.554	0.524	0.797	0.873	0.297	0.715	0.283	0.216	0.672	0.183	0.173	0.675	0.187	0.175	0.661	0.166	0.161	0.670	0.173	0.171
β_l	4.220	3.299	0.720	3.697	2.748	0.198	3.522	2.552	0.023	2.829	1.496	-0.671	2.826	1.762	-0.674	2.530	1.101	-0.969	2.457	1.107	-1.042

Overall, the Monte Carlo results presented in Tables 2–4 show that the estimated values for different parameter values get closer to the true values as the sample sizes increase indicating consistence property of the estimator. The tables also demonstrate that the average bias of the MLEs decreases steadily toward zero as the sample size increases, indicating asymptotic unbiasedness. In addition, RMSE values consistently decline with increasing sample size, reflecting improved estimation precision for all parameters. Across all parameter configurations considered, the MLE procedure for the AT-WLLM distribution exhibits numerical stability and satisfactory finite-sample performance.

6. RESULTS AND DISCUSSION

6.1. Evaluation of the Model and Applications. In this section, the practical relevance and modeling capability of the proposed ArcTan-Weibull-Log-Logistic Mixture (AT-WLLM) distribution are demonstrated through the analysis of three real lifetime datasets drawn from multidisciplinary application areas. The performance of the AT-WLLM model is evaluated by comparing it with several well-established lifetime distributions commonly used in reliability analysis and survival studies. The competing models considered in this study include the Weibull-Log-Logistic Mixture (WLLM) model [11], the ArcTan-Weibull (AT-Weibull) model [26], the Weibull distribution [1], the ArcTan-Log-Logistic (AT-LLog) model [26], the Log-Logistic (LLog) distribution [2, 22, 28], the Gamma distribution [29], and the Lognormal distribution [30]. These models provide a comprehensive benchmark for assessing the flexibility and goodness-of-fit of the proposed AT-WLLM distribution. All analyses are conducted using the R statistical computing environment. For each dataset, maximum likelihood estimation is employed to fit the competing models, and their relative performance is assessed using both information-based model selection criteria and formal goodness-of-fit tests. Specifically, model comparison is performed using the negative twice logarithmic likelihood (-2LogLik), the Akaike Information Criterion (AIC), and the Bayesian Information Criterion (BIC). In addition, goodness-of-fit is evaluated using the Anderson-Darling (AD) statistic, the Cramér-von-Mises (CvM) statistic, and the Kolmogorov-Smirnov (KS) statistic, along with their associated p -values.

For completeness, the cumulative distribution functions and parameter sets of all competing models considered in the analysis are summarized in Table 5.

TABLE 5. Distribution Models with CDFs and Parameters

Distribution	Cumulative Distribution Function (CDF)	Parameters
WLLM	$F(x) = p \cdot \left(1 - e^{-(x/\delta_w)^{k_w}}\right) + (1-p) \cdot \left(\frac{(x/\beta_l)^{\alpha_l}}{1 + (x/\beta_l)^{\alpha_l}}\right)$	$p, k_w, \delta_w, \alpha_l, \beta_l$
AT-Weibull	$F(x) = \frac{4}{\pi} \cdot \arctan\left(1 - e^{-(x/\delta_w)^{k_w}}\right)$	k_w, δ_w
AT-LLog	$F(x) = \frac{4}{\pi} \cdot \arctan\left(\frac{(x/\beta_l)^{\alpha_l}}{1 + (x/\beta_l)^{\alpha_l}}\right)$	α_l, β_l
Weibull	$F(x) = 1 - \exp\left[-\left(\frac{x}{\delta_w}\right)^{k_w}\right]$	k_w, δ_w
LLog	$F(x) = \frac{1}{1 + \left(\frac{x}{\beta_l}\right)^{-\alpha_l}} = \frac{\left(\frac{x}{\beta_l}\right)^{\alpha_l}}{1 + \left(\frac{x}{\beta_l}\right)^{\alpha_l}}$	α_l, β_l
Gamma	$F(x) = \frac{\gamma\left(k, \frac{x}{\zeta}\right)}{\Gamma(k)}$	k, ζ
Lognormal	$F(x) = \Phi\left(\frac{\ln x - \mu}{\sigma}\right)$	μ, σ

6.2. Data Description.

Economic Data: Set 1. This dataset consists of the total milk production from the first birth of 107 cows of the SINDI breed. The data were originally reported by [31] and subsequently analyzed by [32]. The observed values are presented in Table 6 and are commonly used in the literature to illustrate the performance of lifetime and production models.

TABLE 6. Observed values: total milk production from the first birth of 107 cows of the SINDI race

0.5902	0.4823	0.7804	0.3406	0.5744	0.5481	0.7290	0.1131	0.6012	0.1479
0.2356	0.5394	0.4990	0.6058	0.6891	0.5770	0.0671	0.3323	0.4800	0.5483
0.7261	0.2361	0.1525	0.6927	0.3188	0.4612	0.3906	0.4438	0.6220	0.2160
0.6707	0.3627	0.5350	0.5707	0.7131	0.5853	0.6768	0.6789	0.4151	0.4576
0.6750	0.5447	0.5627	0.5113	0.5150	0.4143	0.3945	0.0776	0.2605	0.7471
0.6196	0.4260	0.4365	0.8781	0.5140	0.6907	0.4143	0.3413	0.4694	0.0854
0.4332	0.6821	0.5629	0.4675	0.7629	0.5941	0.6860	0.6174	0.4741	0.5553
0.5878	0.3598	0.5232	0.8492	0.8147	0.4470	0.5285	0.6465	0.4553	0.0650
0.3751	0.1546	0.3635	0.4111	0.5349	0.4517	0.4049	0.2681	0.3175	0.4752
0.5529	0.3134	0.3891	0.4530	0.1167	0.0168	0.4564	0.3480	0.2303	0.3259
0.4371	0.6114	0.7687	0.3383	0.2747	0.0609	0.6488			

Maximum likelihood estimation is employed to fit the AT-WLLM model and all competing distributions to the data. The estimated parameters for each model are reported in Table 7.

TABLE 7. Maximum Likelihood Estimates for All Distributions

Model	p	k_w	δ_w	α_l	β_l	k	ζ	μ	σ
AT-WLLM	0.9148	3.8588	0.5979	2.4702	0.0990	–	–	–	–
WLLM	0.8997	3.6210	0.5638	2.5717	0.0937	–	–	–	–
AT-Weibull	–	2.6895	0.5720	–	–	–	–	–	–
AT-LLog	–	–	–	3.2128	0.5085	–	–	–	–
Weibull	–	2.6012	0.5236	–	–	–	–	–	–
LLog	–	–	–	3.3158	0.4533	–	–	–	–
Gamma	–	–	–	–	–	3.6911	7.8727	–	–
Lognormal	–	–	–	–	–	–	–	-0.8990	0.6444

Model comparison based on information criteria and goodness-of-fit statistics for all fitted models are summarized in Table 8.

TABLE 8. Model Comparison Results and Goodness-of-Fit Statistics

Model	–2LogLik	AIC	BIC	KS	p_{value}	CvM	p_{value}	AD	p_{value}
AT-WLLM	-61.9612	-51.9613	-38.5972	0.0421	0.9913	0.0174	0.9989	0.1101	0.9998
WLLM	-60.9162	-51.9163	-38.5522	0.0439	0.9862	0.0176	0.9988	0.1191	0.9998
Weibull	-42.6950	-38.6950	-33.3494	0.0832	0.4486	0.1895	0.2890	1.4842	0.1803
AT-Weibull	-39.2532	-35.2532	-29.9076	0.0855	0.4148	0.2028	0.2623	1.6829	0.1383
Gamma	-18.8490	-14.8491	-9.5035	0.1364	0.0373	0.6398	0.0177	3.8037	0.0109
AT-LLog	-10.2546	-6.2547	-0.9091	0.1142	0.1226	0.3191	0.1194	3.3313	0.0187
LLog	-8.4654	-4.4655	0.8801	0.1136	0.1262	0.3483	0.0992	3.4648	0.0161
Lognormal	17.2100	21.2100	26.5557	0.1743	0.0030	1.1532	0.0010	6.6029	0.0005

Figures 2 and 3 provide a graphical assessment of model performance, including the fitted probability density functions, cumulative distribution functions, survival functions, and AIC-based PDF comparisons, as well as Quantile–Quantile (Q–Q) plots.

Overall, the results in Tables 7, 8, and Figures 2–3 indicate that the AT-WLLM distribution provides the best fit among all competing models for this dataset. Specifically, the AT-WLLM model achieves the smallest –2LogLik, AIC, and BIC values, and consistently yields the lowest goodness-of-fit statistics with the highest corresponding p -values. The graphical diagnostics further confirm the superior performance of the AT-WLLM distribution, as evidenced by its close agreement with the empirical distribution and the near-linear behavior observed in the Q–Q plots.

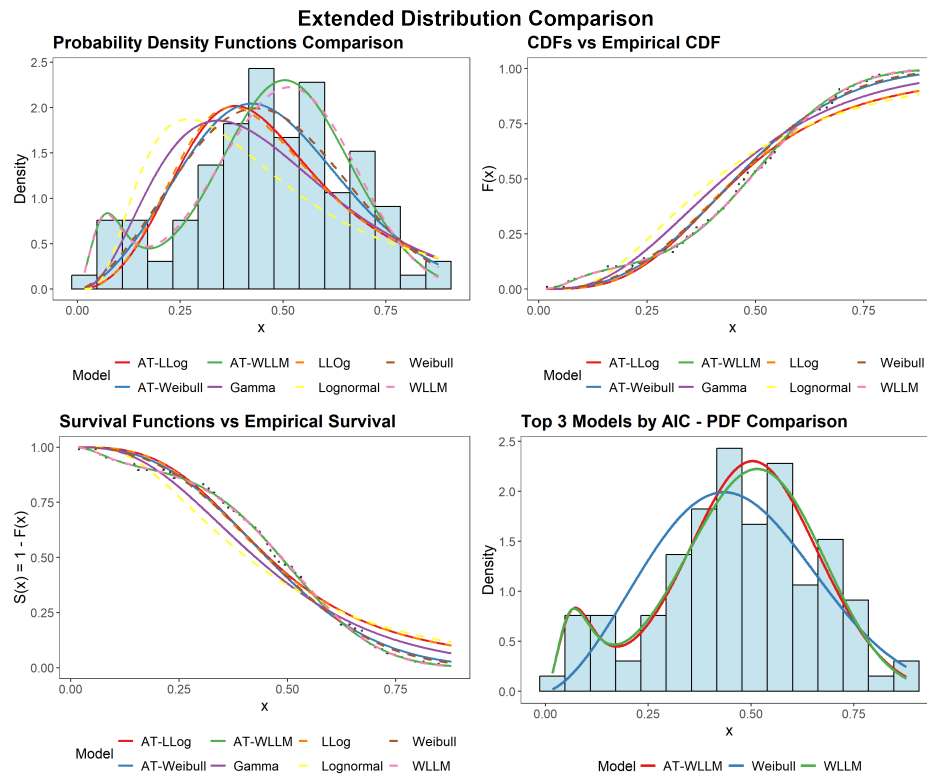


FIGURE 2. Fitted PDF, CDF, survival, and AIC–PDF comparison plots for Data Set 1.

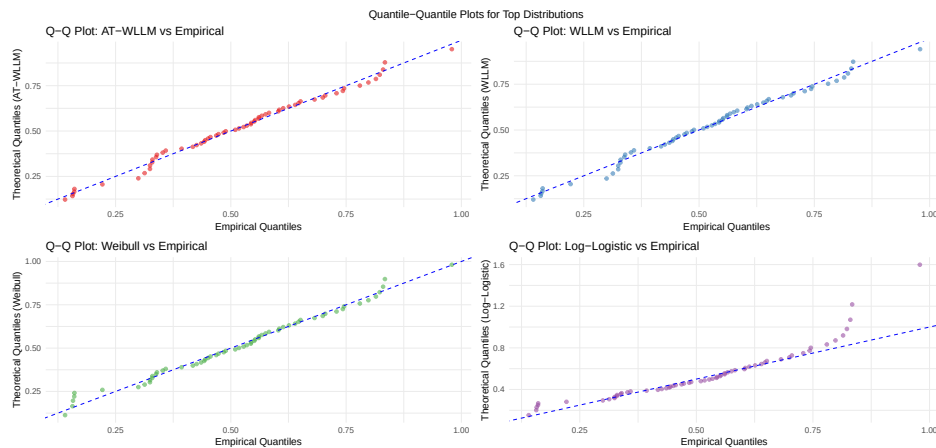


FIGURE 3. Q–Q plots for Data Set 1.

EPIDEMIOLOGICAL DATA: SET 2

The second dataset analyzed in this study consists of COVID–19 records for Italy, covering a total of 172 consecutive days from March 1 to August 20, 2020. This dataset was recently reported by [33] and is widely used to assess the performance of lifetime and survival models in epidemiological applications. The observed values are presented in Table 9.

TABLE 9. Observed Values: COVID-19 data from Italy covering a period of 172 days (March 1 to August 20, 2020)

0.0107	0.0490	0.0601	0.0460	0.0533	0.0630	0.0297	0.0885	0.0540	0.1720	0.0847
0.0713	0.0989	0.0495	0.1025	0.1079	0.0984	0.1124	0.0807	0.1044	0.1212	0.1167
0.1255	0.1416	0.1315	0.1073	0.1629	0.1485	0.1453	0.2000	0.2070	0.1520	0.1628
0.1666	0.1417	0.1221	0.1767	0.1987	0.1408	0.1456	0.1443	0.1319	0.1053	0.1789
0.2032	0.2167	0.1387	0.1646	0.1375	0.1421	0.2012	0.1957	0.1297	0.1754	0.1390
0.1761	0.1119	0.1915	0.1827	0.1548	0.1522	0.1369	0.2495	0.1253	0.1597	0.2195
0.2555	0.1956	0.1831	0.1791	0.2057	0.2406	0.1227	0.2196	0.2641	0.3067	0.1749
0.2148	0.2195	0.1993	0.2421	0.2430	0.1994	0.1779	0.0942	0.3067	0.1965	0.2003
0.1180	0.1686	0.2668	0.2113	0.3371	0.1730	0.2212	0.4972	0.1641	0.2667	0.2690
0.2321	0.2792	0.3515	0.1398	0.3436	0.2254	0.1302	0.0864	0.1619	0.1311	0.1994
0.3176	0.1856	0.1071	0.1041	0.1593	0.0537	0.1149	0.1176	0.0457	0.1264	0.0476
0.1620	0.1154	0.1493	0.0673	0.0894	0.0365	0.0385	0.2190	0.0777	0.0561	0.0435
0.0372	0.0385	0.0769	0.1491	0.0802	0.0870	0.0476	0.0562	0.0138	0.0684	0.1172
0.0321	0.0327	0.0198	0.0182	0.0197	0.0298	0.0545	0.0208	0.0079	0.0237	0.0169
0.0336	0.0755	0.0263	0.0260	0.0150	0.0054	0.0375	0.0043	0.0154	0.0146	0.0210
0.0115	0.0052	0.2512	0.0084	0.0125	0.0125	0.0109	0.0071			

The AT-WLLM model and all competing distributions are fitted to the data using maximum likelihood estimation. The estimated parameters for each model are reported in Table 10. The results

TABLE 10. Maximum Likelihood Estimates for All Distributions

Model	p	k_w	δ_w	α_l	β_l	k	ζ	μ	σ
AT-WLLM	0.6963	2.7491	0.1984	1.6664	0.0379	–	–	–	–
WLLM	0.7111	2.5378	0.1809	1.7191	0.0298	–	–	–	–
AT-Weibull	–	1.4824	0.1668	–	–	–	–	–	–
AT-LLog	–	–	–	1.7964	0.1310	–	–	–	–
Weibull	–	1.4358	0.1424	–	–	–	–	–	–
LLog	–	–	–	1.8598	0.1061	–	–	–	–
Gamma	–	–	–	–	–	1.6102	12.3772	–	–
Lognormal	–	–	–	–	–	–	–	-2.3811	0.9850

of the model comparison based on the log-likelihood and information criteria, and goodness-of-fit statistics for all fitted models are summarized in Table 11.

TABLE 11. Model Comparison Results and Goodness-of-Fit Statistics

Model	-2LogLik	AIC	BIC	KS	p_{value}	CvM	p_{value}	AD	p_{value}
AT-WLLM	-409.8136	-399.8136	-384.0471	0.0269	0.9996	0.0160	0.9994	0.1309	0.9996
WLLM	-409.1562	-399.1562	-383.3897	0.0323	0.9936	0.0222	0.9946	0.1668	0.9969
Weibull	-388.3548	-384.3547	-378.0481	0.0926	0.1032	0.3814	0.0808	2.2468	0.0676
AT-Weibull	-393.1874	-379.1874	-372.8808	0.0973	0.0755	0.4406	0.0565	2.7104	0.0386
Gamma	-380.2918	-376.2918	-369.9853	0.1174	0.0170	0.5845	0.0243	3.1103	0.0241
AT-LLog	-343.8498	-339.8499	-333.5433	0.1067	0.0390	0.6574	0.0161	5.1981	0.0023
LLog	-343.3618	-339.3619	-333.0553	0.1097	0.0312	0.6941	0.0131	5.2405	0.0022
Lognormal	-338.1268	-334.1269	-327.8203	0.1664	0.0001	1.2192	0.0007	6.7415	0.0004

Figures 4 and 5 provide a visual assessment of the fitted models, including density, distribution, survival, and AIC–PDF comparison plots, as well as Quantile–Quantile (Q–Q) plots.

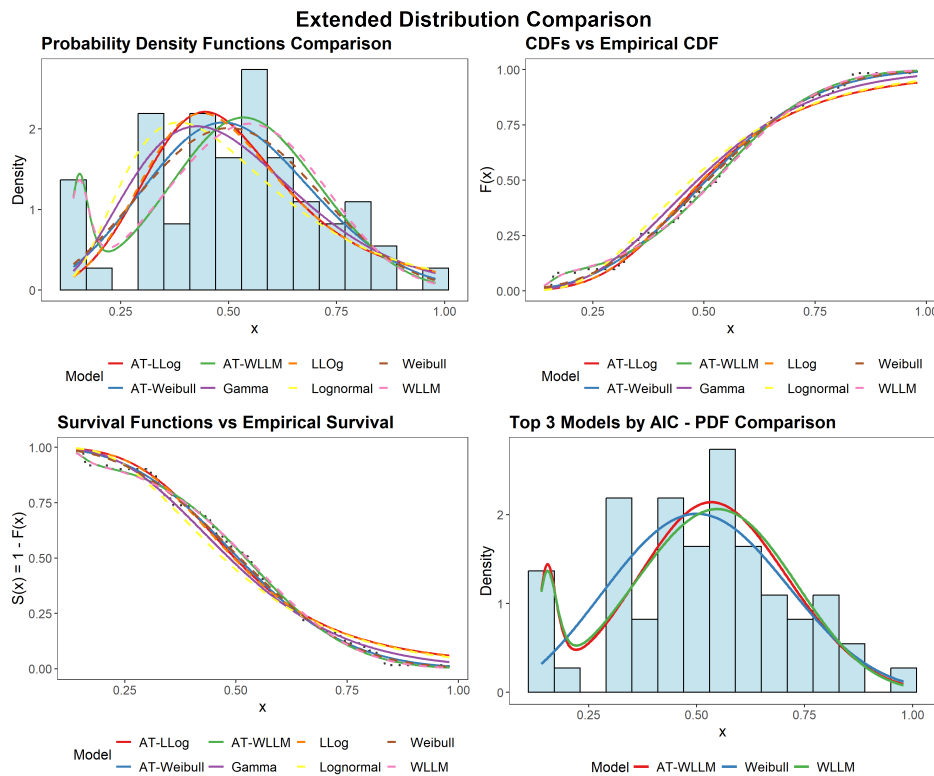


FIGURE 4. Fitted PDF, CDF, survival, and AIC–PDF comparison plots for Data Set 2.

From the model comparison results in Table 11, it is evident that the AT-WLLM distribution outperforms the competing models in terms of the -2LogLik , AIC, and BIC criteria and goodness-of-fit statistics. The results show that the AT-WLLM model consistently achieves the smallest Anderson–Darling, Cramér–von–Mises, and Kolmogorov–Smirnov statistics, along with the largest corresponding p -values.

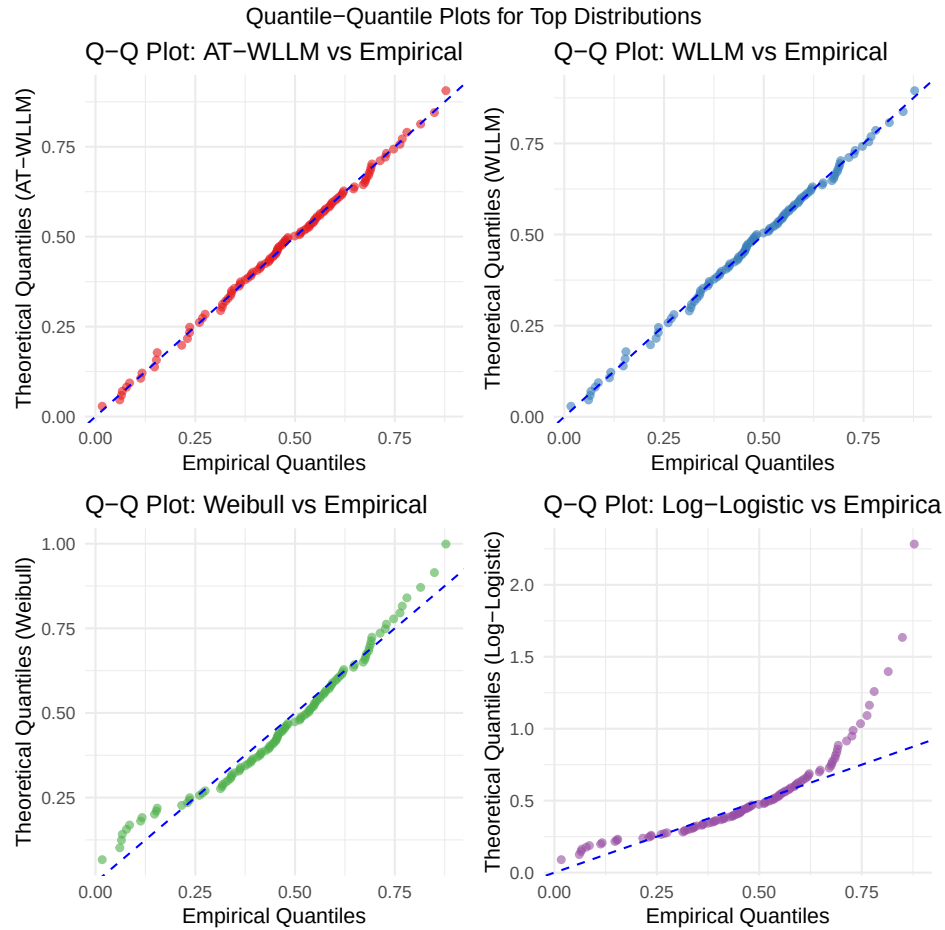


FIGURE 5. Q–Q plots for Data Set 2.

The graphical diagnostics in Figures 4 and 5 further support these findings. In particular, the fitted curves produced by the AT-WLLM distribution closely track the empirical distribution of the data, and the Q–Q plot exhibits minimal deviation from the reference line. These results confirm the superior flexibility and goodness-of-fit of the AT-WLLM model for modeling COVID–19 data from Italy.

ECONOMIC DATA: SET 3

The third dataset analyzed in this study consists of trade share values drawn from the well-known *Determinants of Economic Growth* dataset. This dataset contains observations for up to 61 countries and includes economic growth indicators as well as variables that influence trade openness and economic performance. The dataset has been previously studied by [34] and is commonly used to evaluate the adequacy of statistical models in economic applications. The observed trade share values are presented in Table 12.

TABLE 12. Observed Values: Determinants of Economic Growth Data

0.1405	0.1566	0.1577	0.1604	0.1608	0.2215	0.2994	0.3131	0.3246
0.3247	0.3295	0.3300	0.3379	0.3397	0.3523	0.3589	0.3933	0.4176
0.4258	0.4356	0.4421	0.4444	0.4505	0.4558	0.4683	0.4733	0.4846
0.4890	0.5096	0.5177	0.5278	0.5347	0.5433	0.5442	0.5508	0.5527
0.5606	0.5608	0.5671	0.5753	0.5828	0.6030	0.6050	0.6136	0.6261
0.6395	0.6469	0.6512	0.6816	0.6994	0.7048	0.7292	0.7430	0.7455
0.7798	0.7984	0.8147	0.8230	0.8302	0.8342	0.9794		

The AT-WLLM distribution and all competing models are fitted to the data using the maximum likelihood method. The corresponding parameter estimates for each model are reported in Table 13.

TABLE 13. Maximum Likelihood Estimates for All Distributions

Model	p	k_w	δ_w	α_l	β_l	k	ζ	μ	σ
AT-WLLM	0.9401	3.6854	0.6398	10.0000	0.1568	–	–	–	–
WLLM	0.9261	3.4905	0.6033	10.0000	0.1566	–	–	–	–
AT-Weibull	–	3.0727	0.6197	–	–	–	–	–	–
AT-LLog	–	–	–	3.9499	0.5434	–	–	–	–
Weibull	–	2.9499	0.5764	–	–	–	–	–	–
LLog	–	–	–	4.0949	0.4958	–	–	–	–
Gamma	–	–	–	–	–	5.8887	11.4531	–	–
Lognormal	–	–	–	–	–	–	–	-0.7525	0.4508

Model comparison results based on the $-2\log$ -likelihood, Akaike Information Criterion (AIC), and Bayesian Information Criterion (BIC) are summarized in Table 14.

Figures 6 and 7 provide a visual comparison of the fitted models through density, distribution, survival, and AIC–PDF plots, as well as Quantile–Quantile (Q–Q) diagnostics.

Table 13 presents the maximum likelihood estimates for all fitted models, while Table 14 summarizes the model comparison results based on the $-2\log$ -likelihood, AIC, and BIC criteria. Among the competing models, the AT-WLLM distribution achieves the smallest $-2\log$ -likelihood (-37.5783), as well as the lowest AIC (-27.5785) and BIC (-17.0242) values, indicating the best overall fit to the data. Although the Weibull and AT-Weibull models exhibit slightly smaller BIC

TABLE 14. Model Comparison Results and Goodness-of-Fit Statistics

Model	-2LogLik	AIC	BIC	KS	p_{value}	CvM	p_{value}	AD	p_{value}
AT-WLLM	-37.5783	-27.5785	-17.0242	0.0544	0.9893	0.0265	0.9868	0.2191	0.9843
WLLM	-36.8110	-26.8110	-16.2567	0.0538	0.9904	0.0326	0.9680	0.2477	0.9715
Weibull	-29.3736	-25.2736	-21.0518	0.0591	0.9750	0.0305	0.9755	0.3122	0.9285
AT-Weibull	-28.6472	-24.6472	-20.4254	0.0620	0.9619	0.0362	0.9527	0.3704	0.8771
Gamma	-23.4094	-19.4095	-15.1877	0.0891	0.6852	0.1175	0.5068	0.8866	0.4221
AT-LLog	-18.6478	-14.6479	-10.4262	0.0906	0.6645	0.0815	0.6851	0.9728	0.3715
LLog	-18.4610	-14.4611	-10.2393	0.0896	0.6780	0.0865	0.6570	0.9968	0.3586
Lognormal	-15.9042	-11.9042	-7.6824	0.1193	0.3238	0.2237	0.2261	1.6017	0.1541

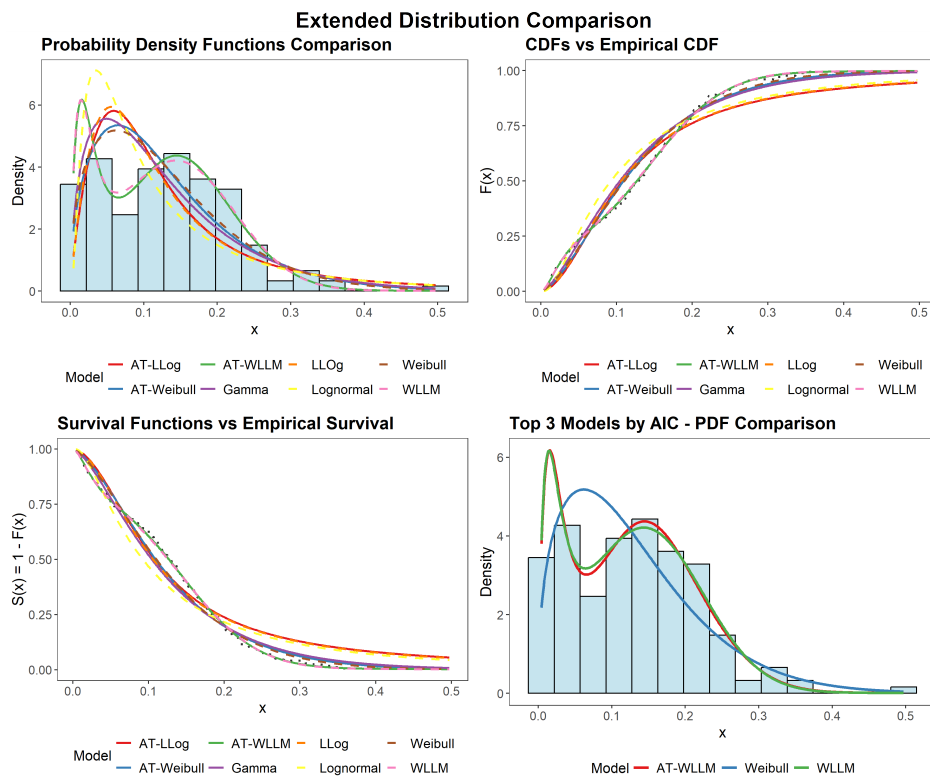


FIGURE 6. Fitted PDF, CDF, survival, and AIC–PDF comparison plots for Data Set 3.

values, they perform less favorably in terms of AIC and log-likelihood. The goodness-of-fit statistics reported further reinforced the superior performance of the AT-WLLM distribution. While the KS statistic and its associated p -value indicate a marginal advantage for the WLLM model, the Anderson–Darling and Cramér–von–Mises statistics, together with their corresponding p -values, consistently favor the AT-WLLM model. These results, combined with the graphical diagnostics, clearly demonstrate the superior flexibility and goodness-of-fit of the proposed AT-WLLM distribution for modeling economic trade share data.

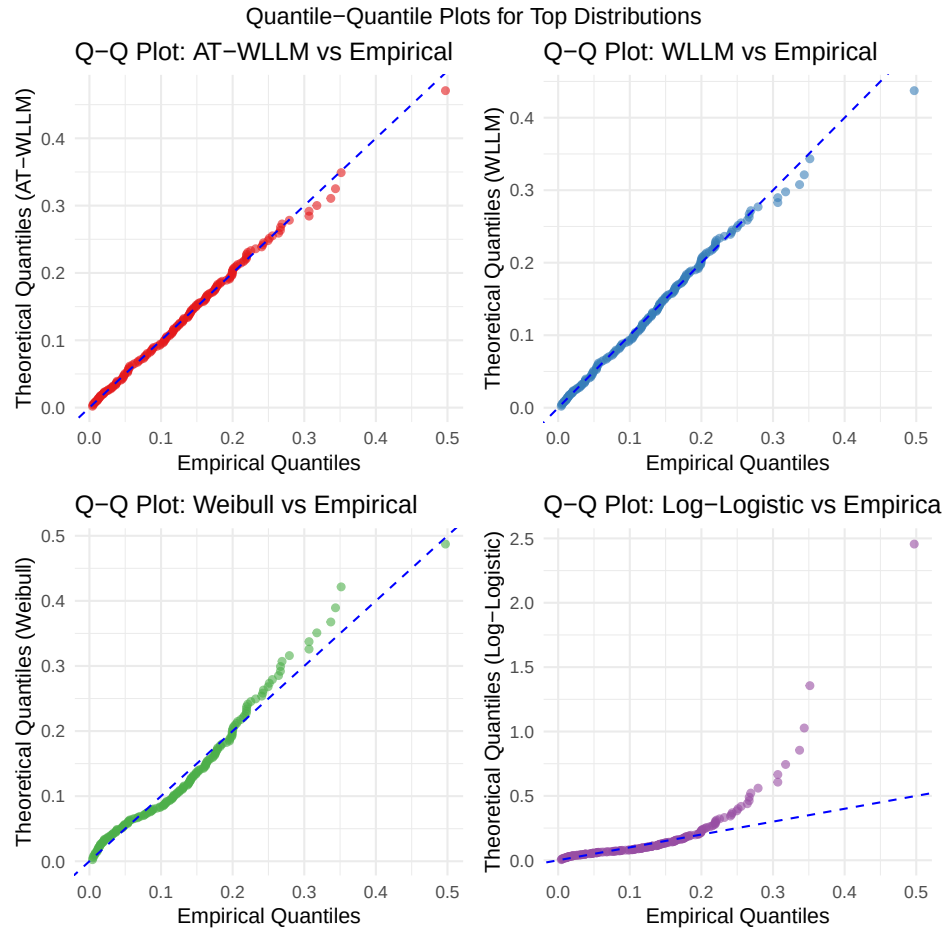


FIGURE 7. Q-Q plot for Data Set 3.

6.3. Performance Assessment for the Datasets. The performance of the proposed AT-WLLM distribution is evaluated by comparing it with several competing models, including the Weibull-Log-Logistic Mixture (WLLM), Arctan-Weibull, ArcTan-Log-Logistic, Weibull, Log-Logistic, Gamma, and Lognormal distributions. The inclusion of the classical Weibull-Log-Logistic mixture model provides an important benchmark, enabling a clear assessment of the advantages offered by the trigonometric transformation employed in the AT-WLLM framework. Across all datasets considered, the AT-WLLM distribution consistently provides the best overall fit, demonstrating strong robustness and superior modeling capability. The results indicate that the arctangent transformation introduces substantial flexibility beyond that achievable with linear mixture models, leading to improved goodness-of-fit and more accurate representation of complex data structures. In general, mixture models particularly the AT-WLLM and WLLM; outperform single component models such as the Arctan-Weibull, ArcTan-Log-Logistic, Weibull, Log-Logistic, Gamma, and Lognormal distributions. This observation highlights the importance of accounting for population heterogeneity and reinforces the applicability of mixture-based approaches for modeling lifetime data with diverse characteristics. Overall, the empirical findings

confirm that the Arctan–Weibull–Log–Logistic mixture model represents a meaningful extension of traditional mixture modeling methods reported in the literature. Moreover, the results provide strong empirical support for the theoretical advantages of trigonometric transformations in enhancing model flexibility and fit.

7. CONCLUSION

This paper introduces a novel trigonometric mixture distribution, the Arctan—Weibull—Log—Logistic Mixture (AT–WLLM) model, designed to capture complex hazard rate behaviors while accommodating diverse tail characteristics. The proposed distribution retains mathematical tractability and offers enhanced flexibility compared to conventional mixture and single-component lifetime models. Theoretical properties of the AT–WLLM distribution are derived in detail, and parameter estimation is carried out using the maximum likelihood method. A comprehensive simulation study demonstrates that the estimators exhibit desirable finite-sample and asymptotic properties, particularly as the sample size increases. The mixing parameter $\hat{p}$ provides a useful mechanism for guiding the selection and interpretation of sub-model contributions. Empirical analyses based on real economic and epidemiological datasets further illustrate the practical utility of the proposed model. In all applications, the AT–WLLM distribution consistently outperforms competing models, emerging as the most flexible and best-fitting alternative. In summary, the Arctan–Weibull–Log–Logistic Mixture distribution provides a powerful and flexible framework for modeling lifetime data with complex hazard structures. The results presented in this study establish a solid foundation for future research and applications in economics and finance, reliability engineering, survival analysis, biological research, environmental sciences, biometrics and biomedical engineering, and related fields.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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