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A REAL PALEY-WIENER THEOREM FOR THE GENERALIZED DUNKL TRANSFORM

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Abstract. In this article, we prove a real Paley-Wiener theorem for the generalized Dunkl transform on $\mathbb{R}$.

1. Introduction

In [3] N.B Andersen proved a real Paley-Wiener theorem for the dunkl transform. In this paper, we first prove a real Paley-Wiener theorem for the generalized dunkl transform.
Let Λ_α denote the Dunkl operator and $\mathcal{F}_{\alpha,n}$ the Dunkl transform, Chettaoui, C., Trimèche proved in [4] the following theorem:

Theorem 1.1. *Let $1 \leq p \leq \infty$. Let $f \in \mathcal{S}(\mathbb{R})$ (the Schwartz space on $\mathbb{R}$). Then*

$$\lim_{m\to\infty} \|\Lambda_\alpha^m f\|_p^{\frac{1}{m}} = \sup\{|\lambda|, \lambda \in Supp\mathcal{F}_\alpha(f)\}.$$

N.B Andersen in [3] gave a simple proof of the above theorem by using the real Paley-Wiener theorem for the Dunkl transform. Our second result is to prove the above theorem for the generalized Dunkl transform.
The structure of the paper is as follows: In section 2 we set some notations and collect some basic results about the Dunkl operator and the Dunkl transform, and we give also some facts about harmonic analysis related to the first-order singular differential-difference operator $\Lambda_{\alpha,n}$, and the generalized Dunkl transform. In section 3 we state and prove a real Paley-Wiener theorem for the generalized Dunkl transform. In section 4 we give a characterization of the support of the generalized Dunkl transform on $\mathbb{R}$

2. Preliminaries

Throughout this paper we assume that $\alpha > \frac{-1}{2}$, and we denote by

- $E(\mathbb{R})$ the space of functions $\mathbb{C}^\infty$ on $\mathbb{R}$, provided with the topology of compact convergence for all derivatives. That is the topology defined by semi-norms

$$P_{a,m}(f) = sup_{x\in[-a,a]} \sum_{k=0}^{m} | \frac{d^k}{dx^k} f(x) |, \quad a > 0, \quad m = 0, 1, ...$$

- $D_a(\mathbb{R})$, the space of $\mathbb{C}^\infty$ function on $\mathbb{R}$, which are supported in $[-a, a]$, equipped with the topology induced by $E(\mathbb{R})$.
- $D(\mathbb{R}) = \bigcup_{a>0} D_a(\mathbb{R})$, endowed with inductive limit topology.
- $E_n(\mathbb{R})$ (resp $D_n(\mathbb{R})$) stand for the subspace of $E(\mathbb{R})$ (resp $D(\mathbb{R})$) consisting of functions f such that

$$f(0) = ... = f^{(2n-1)}(0).$$

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- L^p_α the class of measurable functions f on $\mathbb{R}$ for which $\|f\|_{p,\alpha} < \infty$, where

$$\|f\|_{p,\alpha} = \left(\int_{\mathbb{R}} |f(x)|^p |x|^{2\alpha+1} dx \right)^{\frac{1}{p}}, \quad if p < \infty,$$

and $\|f\|_{\infty,\alpha} = \|f\|_\infty = esssup_{x\geq 0}|f(x)|$.
- $L^p_{\alpha,n}$ the class of measurable functions f on $\mathbb{R}$ for which

$$\|f\|_{p,\alpha,n} = \|\mathcal{M}^{-1} f\|_{p,\alpha+2n} < \infty.$$

- $D^p_{\alpha,n}(\mathbb{R}) = D_n(\mathbb{R}) \bigcap L^p_{\alpha,n}(\mathbb{R})$.
- $\mathbf{H}_a,\ a > 0$, the space of entire rapidly decreasing functions of exponential type a ; that is, $f \in \mathbf{H}_a,\ a > 0$ if and only if, f is entire on $\mathbb{C}$ and for all j=0,1......

$$q_j(f) = \sup_{\lambda\in\mathbb{C}} |(1+\lambda)^m f(\lambda) e^{-a|Im\lambda|} < \infty$$

$\mathbf{H}_a,\ a > 0$ is equipped with the topology defined by the semi-norms $q_j,\ j = 0, 1.....$
- $\mathbf{H} = \bigcup_{a>0} \mathbf{H}_a$, equipped with the inductive limit topology.

2.1. **Dunkl transform.** In this subsection we recall some facts about harmonic analysis related to Dunkl operator Λ_α associated with reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. We cite here, as briefly as possible, only some properties. For more details we refer to [2, 4, 5].
The Dunkl operator Λ_α is defined as follow:

$$\Lambda_\alpha f(x) = f'(x) + (\alpha + \frac{1}{2})\frac{f(x) - f(-x)}{x}. \tag{1}$$

The Dunkl kernel e_α is defined by

$$e_\alpha(z) = j_\alpha(iz) + \frac{z}{2(\alpha+1)} j_{\alpha+1}(z) \quad (z \in \mathbb{C}) \tag{2}$$

where

$$j_\alpha(z) = \Gamma(\alpha+1)\Sigma_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{2})^{2n}}{n!\ \Gamma(n+\alpha+1)} \quad (z \in \mathbb{C}).$$

is the normalized spherical Bessel function of index α. The functions $e_\alpha(\lambda)\quad \lambda \in \mathbb{C}$, are solutions of the differential-difference equation

$$\Lambda_\alpha u = \lambda u, \quad u(0) = 1.$$

Furthermore, Dunkl kernel e_α possesses the Laplace type integral representation

$$e_\alpha(z) = a_\alpha \int_{-1}^{1} (1-t^2)^{\alpha-\frac{1}{2}}(1+t)e^{zt} dt,$$

where

$$a_\alpha = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2}}). \tag{3}$$

The Dunkl transform of a function $f \in D(\mathbb{R})$ is defined by

$$\mathcal{F}_\alpha(f)(\lambda) = \int_{\mathbb{R}} f(x) e_\alpha(-i\lambda x)|x|^{2\alpha+1} dx, \quad \lambda \in \mathbb{C}. \tag{4}$$

Theorem 2.1. **(i):** *The Dunkl transform $\mathcal{F}_\alpha$ is a topological automorphism from $D(\mathbb{R})$ onto $\mathbb{H}$. More precisely $f \in D_a(\mathbb{R})$ if, and only if, $\mathcal{F}_\alpha(f) \in \mathbb{H}_a$*

(ii): *For every* $f \in D(\mathbb{R})$,

$$f(x) = \int_{\mathbb{R}} \mathcal{F}_\alpha(f)(\lambda) e_\alpha(i\lambda x)|\lambda|^{2\alpha+1} d\lambda,$$

$$\int_{\mathbb{R}} |f(x)|^2 |x|^{2\alpha+1} dx = m_\alpha \int_{\mathbb{R}} |\mathcal{F}_\alpha(f)(\lambda)|^2 |\lambda|^{2\alpha+1} d\lambda,$$

where

$$m_\alpha = \frac{1}{2^{2(\alpha+1)}(\Gamma(\alpha+1))^2}. \tag{5}$$

2.2. **Generalized Dunkl transform.** In this section, we recall some properties about Generalized Dunkl transform. We refer to [1] for more details and references.
The first-order singular differential-difference operator on $\mathbb{R}$ is defined as follow

$$\Lambda_{\alpha,n} f(x) = f'(x) + (\alpha + \frac{1}{2}) \frac{f(x) - f(-x)}{x} - 2n \frac{f(-x)}{x}, \tag{6}$$

Lemma 2.2. **(i):** *The map*

$$M_n(f)(x) = x^{2n} f(x)$$

is a topological isomorphism

- *from* $E(\mathbb{R})$ *onto* $E_n(\mathbb{R})$*;*
- *from* $D(\mathbb{R})$ *onto* $D_n(\mathbb{R})$*.*

(ii): *For all* $f \in E(\mathbb{R})$,

$$\Lambda_{\alpha,n} \circ M_n(f) = M_n \circ \Lambda_{\alpha+2n}(f),$$

where $\Lambda_{\alpha+2n}$ *is the Dunkl operator of order* $\alpha + 2n$ *given by (1)*

(iii): *Let* $f \in E_n(\mathbb{R})$ *and* $g \in D_n(\mathbb{R})$*. Then*

$$\int_{\mathbb{R}} \Lambda_{\alpha,n} f(x) g(x) |x|^{2\alpha+1} dx = -\int_{\mathbb{R}} f(x) \Lambda_{\alpha,n} g(x) |x|^{2\alpha+1} dx. \tag{7}$$

2.3. **Generalized Dunkl Transform.** For $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$ put

$$\Psi_{\lambda,\alpha,n}(x) = x^{2n} e_{\alpha+2n}(i\lambda x), \tag{8}$$

where $e_{\alpha+2n}$ is the Dunkl kernel of index $\alpha + 2n$ given by (2).

Proposition 2.3. **(i):** $\Psi_{\lambda,\alpha,n}$ *satisfies the differential-difference equation*

$$\Lambda_{\alpha,n} \Psi_{\lambda,\alpha,n} = i\lambda \Psi_{\lambda,\alpha,n}. \tag{9}$$

Definition 2.4. *The generalized Dunkl transform of a function* $f \in D_n(\mathbb{R})$ *is defined by*

$$\mathcal{F}_{\alpha,n}(f)(\lambda) = \int_{\mathbb{R}} f(x) \Psi_{-\lambda,\alpha,n}(x) |x|^{2\alpha+1} dx, \quad \lambda \in \mathbb{C}. \tag{10}$$

Proposition 2.5. *For every* $f \in D_n(\mathbb{R})$,

$$\mathcal{F}_{\alpha,n}(\Lambda_{\alpha,n} f)(\lambda) = i\lambda \mathcal{F}_{\alpha,n}(f)(\lambda), \tag{11}$$

Theorem 2.6. **(i):** *For all* $f \in D_n(\mathbb{R})$*, we have the inversion formula*

$$f(x) = m_{\alpha+2n} \int_{\mathbb{R}} \mathcal{F}_{\alpha,n}(f)(\lambda) \Psi_{\lambda,\alpha,n}(x) |\lambda|^{2\alpha+4n+1} d\lambda,$$

where $m_{\alpha+2n}$ *is given by (5).*

(ii): *For every* $f \in D_n(\mathbb{R})$*, we have the Plancherel formula*

$$\int_{\mathbb{R}} |f(x)|^2 |x|^{2\alpha+1} dx = m_{\alpha+2n} \int_{\mathbb{R}} |\mathcal{F}_{\alpha,n}(f)(\lambda)|^2 |\lambda|^{2\alpha+4n+1} d\lambda. \tag{12}$$

3. A Real Paley-Wiener Theorem

In this section, we give a short and simple proof of a real Paley-Wiener theorem for the Dunkl transform.
We define the real Paley-Wiener space $PW_R(\mathbb{R})$ as the space of all $f \in S(\mathbb{R})$ such that, for $N \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$

$$\sup_{x\in\mathbb{R}, m\in\mathbb{N}_0} R^{-m} m^{-N} (1+|x|)^N |\Lambda_{\alpha,n}^m f(x)| < \infty. \tag{13}$$

Our real Paley-Wiener Theorem is the following:

Theorem 3.1. *Let $R > 0$. The Generalized Dunkl transform $\mathcal{F}_{\alpha,n}$ is a bijection from $PW_R(\mathbb{R})$ onto $\mathcal{C}_R^\infty(\mathbb{R})$, and by symmetry a bijection from $\mathcal{C}_R^\infty(\mathbb{R})$ onto $PW_R(\mathbb{R})$.*

Proof. Let $f \in PW_R(\mathbb{R})$, and λ outside $[-R, R]$. Then (7) and (9) yield

$$\begin{aligned}
\mathcal{F}_{\alpha,n} f(\lambda) &= \int_{\mathbb{R}} f(x)\Psi_{-\lambda,\alpha,n}(x)|x|^{2\alpha+1}dx, \\
&= (-i\lambda)^{-m} \int_{\mathbb{R}} f(x)\Lambda_{\alpha,n}^m \Psi_{-\lambda,\alpha,n}(x)|x|^{2\alpha+1}dx, \\
&= (-i\lambda)^{-m}(-1)^m \int_{\mathbb{R}} \Lambda_{\alpha,n}^m f(x)\Psi_{-\lambda,\alpha,n}(x)|x|^{2\alpha+1}dx,
\end{aligned}$$

hence, for a positive C,

$$\begin{aligned}
|\mathcal{F}_{\alpha,n} f(\lambda)| &= |(-i\lambda)^{-m}(-1)^m \int_{\mathbb{R}} \Lambda_{\alpha,n}^m f(x)\Psi_{-\lambda,\alpha,n}(x)|x|^{2\alpha+1}dx|, \\
&\leq |\lambda|^{-m} \int_{\mathbb{R}} |\Lambda_{\alpha,n}^m f(x)\Psi_{-\lambda,\alpha,n}(x)||x|^{2\alpha+1}dx|, \\
&\leq C|\lambda|^{-m} \int_{\mathbb{R}} R^m m^N (1+|x|)^{-N}|x|^{2\alpha+2n+1}dx|, \\
&= C(\frac{R}{|\lambda|})^m m^N \int_{\mathbb{R}} (1+|x|)^{-N}|x|^{2\alpha+2n+1}dx| \to 0 \ \text{ for } \ m \to \infty.
\end{aligned}$$

and thus $Supp\mathcal{F}_{\alpha,n} f \subset [-R, R]$.
Conversely, let $f \in \mathcal{C}_R^\infty(\mathbb{R})$. Fix $N \in \mathbb{N}_0$.

$$\mathcal{F}_{\alpha,n}^{-1} f(\lambda) := m_{\alpha+2n} \int_{\mathbb{R}} f(\lambda)\Psi_{\lambda,\alpha,n}(x)|\lambda|^{2\alpha+4n+1}d\lambda,$$

$$\begin{aligned}
x^N \Lambda_{\alpha,n}^m \mathcal{F}_{\alpha,n}^{-1} f(\lambda) &= m_{\alpha+2n} \int_{\mathbb{R}} f(\lambda) x^N \Lambda_{\alpha,n}^m \Psi_{\lambda,\alpha,n}(x)|\lambda|^{2\alpha+4n+1}d\lambda, \\
&= m_{\alpha+2n}(-i)^m \int_{\mathbb{R}} \lambda^m f(\lambda) x^N \ \frac{x^{2n}}{\lambda^{2n}} \Psi_{x,\alpha,n}(\lambda)|\lambda|^{2\alpha+4n+1}d\lambda, \\
&= (-i)^m m_{\alpha+2n} \int_{\mathbb{R}} \lambda^m f(\lambda) x^{N+2n} \Psi_{x,\alpha,n}(\lambda)|\lambda|^{2\alpha+2n+1}d\lambda, \\
&= (-i)^{m-N-2n} m_{\alpha+2n} \int_{\mathbb{R}} \lambda^m f(\lambda) \Lambda_{\alpha,n}^{N+2n} \Psi_{x,\alpha,n}(\lambda)|\lambda|^{2\alpha+2n+1}d\lambda, \\
&= (-i)^{m+N+2n} m_{\alpha+2n} \int_{\mathbb{R}} \Lambda_{\alpha,n}^{N+2n} (\lambda^m f(\lambda)) \Psi_{x,\alpha,n}(\lambda)|\lambda|^{2\alpha+2n+1}d\lambda.
\end{aligned}$$

a small calculation give

$$\Lambda_{\alpha,n}(\lambda^m f(\lambda) = m\lambda^{m-1}[f(\lambda) + \frac{1}{m}\lambda\frac{d}{d\lambda}f(\lambda) + \frac{1}{m}(\alpha + \frac{1}{2}(f(\lambda) - (-1)^m f(-\lambda)) - \frac{2n}{m}\frac{f(-\lambda)}{\lambda^m}]$$

Let $\widetilde{f}$ denote the function in square bracket. An induction argument with $f_1 = \widetilde{f}$ and $\widetilde{f}_{i+1} = \widetilde{f}_i$, show that we can write, for $m > N + 2n$

$$\Lambda_{\alpha,n}(\lambda^{N+2n}(\lambda^m f(\lambda)) = \lambda^{m-N-2n} m^{N+2n} \widetilde{f}_{N+2n}(\lambda),$$

where $\widetilde{f}_{N+2n} \in \mathcal{C}_R^\infty(\mathbb{R})$ with $supp\widetilde{f}_{N+2n} \subset suppf$, and

$$\| \widetilde{f}_{N+2n} \|_\infty \leq C \sum_{k=0}^{N+2n} \| \frac{d^k}{dx^k} f \|_\infty$$

where C is a positive constant only depending on f, α, n and N not on m. We get thus

$$|x^N \Lambda_{\alpha,n}^m \mathcal{F}_{\alpha,n}^{-1} f(x)| \leq C m_{\alpha+2n} R^{2\alpha+2n+1} m^N \sum_{k=0}^{N+2n} \| \frac{d^k}{dx^k} f \|_\infty$$

for all $x \in \mathbb{R}$, and $m > N + 2n$, and thus $\mathcal{F}_{\alpha,n}^{-1} f \in PW_R(\mathbb{R})$ ■

4. A characterization of the support of the generalized Dunkl transform on $\mathbb{R}$

Theorem 4.1. *Let $1 \leq p \leq \infty$. Let $f \in \mathcal{S}(\mathbb{R})$. Then*

$$\lim_{m\to\infty} \|\Lambda_{\alpha,n}^m f\|_{p,\alpha,n}^{\frac{1}{m}} = \sup\{|\lambda|, \lambda \in Supp\mathcal{F}_{\alpha,n}(f)\}.$$

Proof. Define $R_f = \sup\{|\lambda|, \lambda \in Supp\mathcal{F}_{\alpha,n}(f)\}$.
Assume that $\mathcal{F}_{\alpha,n}$ has a compact support. Then $f \in PW_R(\mathbb{R})$ by Theorem 3.1 and

$$\lim_{m\to\infty} \|\Lambda_{\alpha,n}^m f\|_{p,\alpha,n}^{\frac{1}{m}} \leq R_f \lim_{m\to\infty} m^{\frac{N}{m}} = R_f$$

for all $1 \leq p \leq \infty$, using (13) with $N \geq 2\alpha + 2m + 3$.
Now consider an arbitrary $f \in ..$, using (7)

$$\begin{aligned} \| \Lambda_{\alpha,n}^m f \|_{2,\alpha,n}^2 &= \int_{\mathbb{R}} |\Lambda_{\alpha,n}^m f(x)|^2 |x|^{2\alpha+1} dx, \\ &= \int_{\mathbb{R}} \Lambda_{\alpha,n}^m f(x) \overline{\Lambda_{\alpha,n}^m f(x)} |x|^{2\alpha+1} dx, \\ &= (-1)^m \int_{\mathbb{R}} \Lambda_{\alpha,n}^{2m} f(x) \overline{f(x)} |x|^{2\alpha+1} dx. \end{aligned}$$

Hölder's inequality with $\frac{1}{p} + \frac{1}{q} = 1$

$$\|\Lambda_{\alpha,n}^m f\|_{2,\alpha,n}^2 \leq \|\Lambda_{\alpha,n}^{2m} f\|_{p,\alpha,n} \|f\|_{q,\alpha,n}. \tag{14}$$

Similarly, we get

$$\|\Lambda_{\alpha,n}^{m+1} f\|_{2,\alpha,n}^2 \leq \|\Lambda_{\alpha,n}^{2m+1} f\|_{p,\alpha,n} \|\Lambda_{\alpha,n} f\|_{q,\alpha,n}.$$

Let $R < R_f$. From (11) and (12)

$$\begin{aligned} \| \Lambda_{\alpha,n}^m f \|_{2,\alpha,n}^2 &= \int_{\mathbb{R}} |\Lambda_{\alpha,n}^m f(x)|^2 |x|^{2\alpha+1} d\lambda, \\ &= m_{\alpha+2n} \int_{\mathbb{R}} |\mathcal{F}_{\alpha,n}(\Lambda_{\alpha,n}^m f(\lambda))|^2 |\lambda|^{2\alpha+4n+1} d\lambda, \\ &= m_{\alpha+2n} \int_{\mathbb{R}} |\lambda|^{2m} |\mathcal{F}_{\alpha,n} f(\lambda)|^2 |\lambda|^{2\alpha+4n+1} d\lambda, \\ &\geq m_{\alpha+2n} R^{2m} \int_{\mathbb{R}} |\mathcal{F}_{\alpha,n} f(\lambda)|^2 |\lambda|^{2\alpha+4n+1} d\lambda, \end{aligned}$$

where the last integral is positive.
Combining (14) with the above inequality yields

$$\liminf_{m\to\infty}\|\Lambda^{2m}_{\alpha,n}f\|^{\frac{1}{2m}}_{p,\alpha,n}\geq\liminf_{m\to\infty}\|\Lambda^{m}_{\alpha,n}f\|^{\frac{1}{m}}_{2,\alpha,n}\geq R$$

for any $1\leq p\leq\infty$, and similarly

$$\liminf_{m\to\infty}\|\Lambda^{2m+1}_{\alpha,n}f\|^{\frac{1}{2m+1}}_{p,\alpha,n}\geq R_f.$$

We thus conclude, for any $0<R<R_f$

$$R\leq\liminf_{m\to\infty}\|\Lambda^{m}_{\alpha,n}f\|^{\frac{1}{m}}_{p,\alpha,n}\leq\limsup_{m\to\infty}\|\Lambda^{m}_{\alpha,n}f\|^{\frac{1}{m}}_{p,\alpha,n}\leq R_f$$

this complect the proof of the theorem. ■

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