Int. J. Anal. Appl. (2026), 24:281

International Journal of Analysis and Applications

Inertial Bregman Subgradient Extragradient Methods for Solving Pseudomonotone Equilibrium and Fixed Point Problems in Reflexive Banach Spaces

Anchalee Kaewcharoen[1,2,*]

[1]*Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok, 65000, Thailand*

[2]*Center of Excellence in Mathematics, Faculty of Science, Naresuan University, Phitsanulok, 65000, Thailand*

[*]*Corresponding author:* anchaleeka@nu.ac.th

Abstract. This paper proposes an inertial Bregman subgradient extragradient algorithm for approximating a common solution of pseudomonotone equilibrium problems and fixed point problems associated with Bregman relatively nonexpansive mappings in reflexive Banach spaces. The proposed method incorporates an inertial extrapolation step into the Bregman subgradient extragradient framework and employs a self-adaptive stepsize rule. Under suitable assumptions on the bifunction and the underlying Banach space, strong convergence of the generated sequence to a common solution is established. Several consequences of the main theorem are also derived. Numerical experiments are presented to illustrate the convergence behavior and effectiveness of the proposed algorithm.

1. Introduction

Throughout the paper, unless otherwise stated, let E be a reflexive Banach space with E^* its dual, let C be a nonempty closed convex subset of E and denote by $\mathbb{R}$ the set of real numbers. Consider the following equilibrium problem: Find $x^* \in C$ such that

$$G(x^*, y) \geq 0, \quad \forall y \in C, \tag{1.1}$$

where $G : C \times C \to \mathbb{R}$ is a bifunction. We will write $EP(G)$ for the solution set of (1.1). Equilibrium problems play a fundamental role in nonlinear analysis and have become an active area of research because of their broad applicability in science, engineering, economics, finance, and game theory. The equilibrium framework offers a powerful and unified tool for modeling and analyzing a wide variety of mathematical problems. Indeed, many well-known models can be viewed as special

Received: Jun. 22, 2026.

2020 *Mathematics Subject Classification.* 47H05; 47J25; 65K15.

Key words and phrases. pseudomonotone equilibrium problems; Bregman relatively nonexpansive mappings; inertial subgradient extragradient methods; reflexive Banach spaces.

https://doi.org/10.28924/2291-8639-24-2026-281

ISSN: 2291-8639

instances of equilibrium problems, including mathematical programming problems, variational inclusion problems, variational inequality problems, complementarity problems, saddle-point problems, Nash equilibrium problems in noncooperative games, minimax inequality problems, minimization problems, and fixed point problems, see [4,10]. For example, if we set $G(x^*, y) = \langle Ax^*, y - x^*\rangle$, $\forall x^*, y \in C$, where $A : C \to E^*$ is a nonlinear mapping, then the equilibrium problem reduces to the variational inequality problem: Find $x^* \in C$ such that

$$\langle Ax^*, y - x^*\rangle \geq 0, \quad \forall y \in C,$$

which was introduced by Hartmann and Stampacchia [12]. The solution set of variational inequality problem is denoted by $VIP(A)$.

On the other hand, let $T : C \to C$ be an operator. A point $x \in C$ is called a fixed point of T if $Tx = x$, denote the set of fixed points of T by $F(T)$. Fixed point theory is one of the fundamental branches of nonlinear analysis and has been extensively studied over the past several decades. Owing to its theoretical significance and wide applicability, it has become an indispensable tool in many areas of mathematics and related disciplines. In particular, fixed point methods have been successfully applied to optimization, game theory, economics, variational inequalities, differential equations, and many other problems arising in science and engineering.

Tada and Takahashi [21] proposed the iterative method for finding a common solution of monotone equilibrium problem and the fixed point problem of nonexpansive mappings on Hilbert spaces in 2006 as:

$$\begin{cases} x_0 \in C, \\ z_n \in C \text{ such that } G(z_n, y) + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \ \forall y \in C, \\ w_n = \alpha_n x_n + (1 - \alpha_n)Tz_n, \\ C_n = \{v \in C : \|w_n - v\| \leq \|x_n - v\|\}, \\ Q_n = \{v \in C : \langle x_0 - x_n, v - x_n\rangle \leq 0\}, \\ x_{n+1} = \prod_{C_n \cap Q_n} x_0. \end{cases} \tag{1.2}$$

According to the above algorithm, at each step for determining the intermediate approximation z_n, we would like to solve a strongly monotone regularized equilibrium problem:

$$\text{Find } z_n \in C \text{ such that } G(z_n, y) + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \ \forall y \in C. \tag{1.3}$$

If the bifunction G is only pseudomonotone, then the subproblem (1.3) is not necessarily strongly monotone, even not pseudomonotone. In this case, algorithm (1.2) cannot be applied using the monotonicity of the subproblem. Ahn [2] proposed a hybrid extragradient method for finding the common solutions of the pseudomonotone equilibrium problem and the fixed point problem of nonexpansive mappings and studied the convergence analysis of the sequence $\{x_n\}$ generated by

$$\begin{cases} x_0 \in C, \\ y_n = argmin\{\lambda_n G(x_n, y) + \frac{1}{2}\|x_n - y\|^2 : y \in C\}, \\ t_n = argmin\{\lambda_n G(y_n, y) + \frac{1}{2}\|x_n - y\|^2 : y \in C\}, \\ z_n = \alpha_n x_n + (1-\alpha_n) T t_n, \\ C_n = \{v \in C : \|z_n - v\| \leq \|x_n - v\|\}, \\ Q_n = \{v \in C : \langle x_0 - x_n, v - x_n \rangle \leq 0\}, \\ x_{n+1} = \prod_{C_n \cap Q_n} x_0. \end{cases} \tag{1.4}$$

Since the pioneering work of Bregman [5] in 1967, Bregman distance functions have become an essential tool in modern optimization and nonlinear analysis. Their remarkable properties have led to the development of numerous iterative schemes and convergence techniques. In addition, Bregman distances have been extensively utilized in the study of optimization and feasibility problems, as well as in the approximation of solutions to equilibrium problems, fixed point problems, variational inequality problems, and other related mathematical models, etc.; for details, see [3,6,7].

In 2010, Reich et al. [18] introduced an iterative algorithm on Banach spaces involving maximal monotone operators. In the light of the Bregman projections, there were various iterative algorithms studied by researchers in this field, see [1,8].

In the setting of reflexive Banach spaces, Eskandani et al. [11] developed a Bregman hybrid extragradient algorithm for approximating a common solution of pseudomonotone equilibrium problems and common fixed point problems associated with a finite family of multivalued Bregman relatively nonexpansive mappings. To facilitate their convergence analysis, they introduced the notion of Bregman Lipschitz-type continuity for pseudomonotone bifunctions and employed a self-adaptive stepsize sequence satisfying:

$$\{\lambda_n\} \subset [a, b] \subset (0, p), \text{ where } p = \min\left\{\frac{1}{c_1}, \frac{1}{c_2}\right\}$$

where $c_1 = \max_{1 \leq i \leq N}\{c_{i,1}\}$, $c_2 = \max_{1 \leq i \leq N}\{c_{i,2}\}$, and $c_{i,1}$ and $c_{i,2}$ denote the Bregman Lipschitz-type constants of G_i for $i = 1, 2, ..., N$. Under these assumptions, the authors established the convergence of the sequence $\{x_n\}$ generated by

$$\begin{cases} x_0 \in C, \\ w_n^i = argmin\left\{\lambda_n G_i(x_n, y) + D_f(y, x_n), y \in C\right\}, i = 1, 2, ..., N, \\ z_n^i = argmin\left\{\lambda_n G_i(w_n^i, y) + D_f(y, x_n), y \in C\right\}, i = 1, 2, ..., N, \\ i_n \in Argmax\left\{D_f(z_n^i, x_n) : i = 1, 2, ..., N\right\}, \text{ set } \bar{z}_n := z_n^{i_n}, \\ y_n = \nabla f^*(\beta_{n,0} \nabla f(\bar{z}_n) + \sum_{r=1}^M \beta_{n,r} \nabla f(z_{n,r})), \ z_{n,r} \in T_r(\bar{z}_n), \\ x_{n+1} = proj_C^f(\nabla f^*(\alpha_n \nabla f(u_n) + (1-\alpha_n)\nabla f(y_n))). \end{cases} \tag{1.5}$$

Although several Bregman extragradient and inertial-type methods have been proposed for equilibrium and fixed point problems, strong convergence results for pseudomonotone equilibrium problems combined with Bregman relatively nonexpansive mappings in reflexive Banach spaces remain limited. Moreover, the incorporation of inertial extrapolation techniques into the Bregman subgradient extragradient framework has not been fully investigated in this setting.

Motivated by the above works and inspired by the recent studies of [13,22,23], we investigate the problem of approximating a common element of the solution set of a pseudomonotone equilibrium problem and the fixed point set of a Bregman relatively nonexpansive mapping. Specifically, we consider the problem of finding a point $x^* \in C$ satisfying

$$x^* \in EP(G) \cap F(T).$$

To this end, we develop a novel inertial Bregman subgradient extragradient algorithm in reflexive Banach spaces. We establish a strong convergence theorem for the proposed iterative scheme and derive several consequences of the main result. In addition, numerical examples are presented to demonstrate the efficiency and practical applicability of the proposed method.

2. Preliminaries

Assume $f : E \to (-\infty, +\infty]$ is a proper convex and lower semicontinuous mapping, the subdifferential of f at x is the convex set defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x\rangle \leq f(y),\ \forall y \in E\},$$

where the Fenchel conjugate of f is the function $f^* : E^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}.$$

And for any $x \in \mathrm{int}(\mathrm{dom} f)$, the interior of the domain of f and $y \in E$, the right-hand derivative of f at x in the direction y is

$$f^0(x, y) := \lim_{t\to 0^+} \frac{f(x + ty) - f(x)}{t}.$$

A mapping f is called Gâteaux differentiable at x if the above limit exists. So, $f^0(x, y)$ agrees with $\nabla f(x)$, the value of the gradient of f at x. It is called Fréchet differentiable at x, if the limit is attained uniformly in y, $\|y\| = 1$. It is called uniformly Fréchet differentiable on $C \subset E$, if the above limit is attained uniformly for $x \in C$ and $\|y\| = 1$.

The mapping f is called Legendre if the following statements hold [3]:

(i) $\mathrm{int}(\mathrm{dom} f) \neq \emptyset$, f is Gâteaux differentiable on $\mathrm{int}(\mathrm{dom} f)$, $\mathrm{dom}\nabla f = \mathrm{int}(\mathrm{dom} f)$;

(ii) $\mathrm{int}(\mathrm{dom} f^*) \neq \emptyset$, f^* is Gâteaux differentiable on $\mathrm{int}(\mathrm{dom} f^*)$, $\mathrm{dom}\nabla f^* = \mathrm{int}(\mathrm{dom} f^*)$.

We have the following [3]:

(i) f is Legendre if and only if f^* is the Legendre mapping;

(ii) $(\partial f)^{-1} = \partial f^*$;

(iii) $\nabla f = (\nabla f^*)^{-1}$, $\mathrm{ran}\nabla f = \mathrm{dom}\nabla f^* = \mathrm{int}(\mathrm{dom} f^*)$ and $\mathrm{ran}\nabla f^* = \mathrm{dom}\nabla f = \mathrm{int}(\mathrm{dom} f)$;

(iv) the mappings f and f^* are strictly convex on int(dom$f$) and int(domf^*).

Definition 2.1. *[5] Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable and convex function, D_f : $domf \times int(domf) \longrightarrow [0, +\infty)$ such that*

$$D_f(y, x) := f(y) - f(x) - \langle \nabla f(x), y - x\rangle, \tag{2.1}$$

is known as the Bregman distance with respect to f.

We have some important properties of D_f [5]: the three point property

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle \nabla f(z) - \nabla f(y), x - y\rangle. \tag{2.2}$$

Definition 2.2. *[5] Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable and convex function. The Bregman projection with respect to f of $x \in int(domf)$ onto a nonempty closed convex subset C of $int(domf)$, is the unique vector $proj^f_C(x) \in C$ satisfying*

$$D_f(proj^f_C(x), x) = \inf\{D_f(y, x) : y \in C\}. \tag{2.3}$$

Definition 2.3. *Let $T : C \to C$ be a mapping and $F(T) = \{x^* \in C : Tx^* = x^*\}$, the set of fixed points of T. Then*

(i) *a point $\bar{x} \in C$ is called an asymptotic fixed point if C contains a sequence $\{x_n\}$ with $x_n \rightharpoonup \bar{x}$ such that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. We denote by $\hat{F}(T)$ the set of all asymptotic fixed points of T;*

(ii) *T is called Bregman relatively nonexpansive if*

$$F(T) = \hat{F}(T) \neq \emptyset \ \text{ and } D_f(p, Tx) \leq D_f(p, x),\ \forall x \in C,\ p \in F(T);$$

(iii) *T is called Bregman firmly nonexpansive if*

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty\rangle \leq \langle \nabla f(x) - \nabla f(y), Tx - Ty\rangle,\ \forall x, y \in C.$$

The above inequality is equivalent to

$$\begin{aligned} &D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \\ &\qquad\qquad\qquad \leq D_f(Tx, y) + D_f(Ty, x),\ \forall x, y \in C. \end{aligned}$$

Definition 2.4. *[18] Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable and convex function. Then f is called:*

(i) *totally convex at $x \in int(domf)$ if its modulus of total convexity at x, i.e., the mapping v_f : $int(domf) \times [0, +\infty) \to [0, +\infty)$ such that*

$$v_f(x, t) := \inf\{D_f(y, x) : y \in domf, \|y - x\| = t\},$$

is positive for $t > 0$;

(ii) *totally convex if it is totally convex at each point of $int(domf)$;*

(iii) *totally convex on bounded sets if* $v_f : int(dom f) \times [0, +\infty) \to [0, +\infty)$ *such that*

$$v_f(B, t) = \inf\left\{v_f(x, t) : x \in B \cap dom f\right\},$$

is positive for any nonempty bounded subset B *of* E *and* $t > 0$.

Definition 2.5. *A mapping* $f : E \to (-\infty, +\infty]$ *is called:*

(i) *strongly coercive if* $\lim_{\|u\| \to +\infty} \frac{f(u)}{\|u\|} = +\infty$;

(ii) *sequentially consistent if for any* $\{x_n\}, \{y_n\} \subseteq E$ *where* $\{x_n\}$ *is bounded,*

$$\lim_{n \to \infty} D_f(y_n, x_n) = 0 \text{ implies } \lim_{n \to \infty} \|y_n - x_n\| = 0.$$

Lemma 2.1. *[15] Let* E *be a Banach space and* $f : E \to \mathbb{R}$ *be a Gâteaux differentiable and totally convex function which is uniformly convex on bounded subsets of* E*. Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in* E*. Then*

$$\lim_{n \to \infty} D_f(x_n, y_n) = 0 \text{ if and only if } \lim_{n \to \infty} \|x_n - y_n\| = 0. \tag{2.4}$$

Lemma 2.2. *[17] Let* $f : E \to (-\infty, +\infty]$ *be uniformly Fréchet differentiable and bounded on* $C \subseteq E$*, a bounded set. Then,* f *is uniformly continuous on* C *and* ∇f *is uniformly continuous on* C *from the strong topology of* E *to the strong topology of* E^*.

Lemma 2.3. *[18] Let* $f : E \to \mathbb{R}$ *be a totally convex function. If* $x_0 \in E$ *and the sequence* $\{D_f(x_0, x_n)\}$ *is bounded, then the sequence* $\{x_n\}$ *is bounded.*

Lemma 2.4. *[19] Let* $f : E \to (-\infty, +\infty]$ *be a Gâteaux differentiable and totally convex function on* $int(dom f)$*. Let* $x \in int(dom f)$ *and* $C \subseteq int(dom f)$*, a nonempty closed convex set. If* $v \in C$*, then the following statements are equivalent:*

(i) $v \in C$ *is the Bregman projection of* x *onto* C *with respect to* f*, i.e.,* $v = proj_C^f(x)$;

(ii) *the vector* v *is the unique solution of the variational inequality*

$$\langle \nabla f(x) - \nabla f(v), v - u \rangle \geq 0, \quad \forall u \in C;$$

(iii) *the vector* v *is the unique solution of the inequality*

$$D_f(u, v) + D_f(v, x) \leq D_f(u, x), \quad \forall u \in C.$$

Lemma 2.5. *[15] Let* E *be a reflexive Banach space, let* $f : E \to \mathbb{R}$ *be a strongly coercive function and let* $V_f : E \times E^* \to [0, +\infty)$ *be defined by*

$$V_f(x, x^*) = f(x) - \langle x, x^* \rangle + f^*(x^*), \quad \forall x \in E, x^* \in E^*.$$

Then the following assertions hold:

(1) $V_f(x, x^*) = D_f(x, \nabla f^*(x^*)), \quad \forall x \in E, x^* \in E^*$;

(2) $V_f(x, x^*) + \langle y^*, \nabla f^*(x^*) - x \rangle \leq V_f(x, x^* + y^*), \quad \forall x \in E, x^*, y^* \in E^*$.

Lemma 2.6. *[15] Let E be a Banach space, $t > 0$ be a constant and $f : E \to \mathbb{R}$ be a uniformly convex function on bounded subsets of E. Then*

$$f\left(\sum_{k=0}^{n} \alpha_k x_k\right) \le \sum_{k=0}^{n} \alpha_k f(x_k) - \alpha_i \alpha_j \rho_r\left(\|x_i - x_j\|\right),$$

for all $i, j \in \{0, 1, 2, ..., n\}$, $x_k \in rB$, $\alpha_k \in (0, 1)$ and $k = 0, 1, 2, ..., n$ with $\sum_{k=0}^{n} \alpha_k = 1$, where ρ_r is the gauge of uniform convexity of f.

Lemma 2.7. *[16] Let $f : E \to (-\infty, +\infty]$ be a proper lower semicontinuous and convex function, then $f^* : E^* \to (-\infty, +\infty]$ is proper weak* lower semicontinuous and convex. Hence, V_f is convex in the second variable. Thus, for all $z \in E$, we have*

$$D_f\left(z, \nabla f^*\left(\sum_{i=1}^{N} t_i \nabla f(x_i)\right)\right) \le \sum_{i=1}^{N} t_i D_f(z, x_i), \tag{2.5}$$

where $\{x_i\}_{i=1}^{N} \subset E$ and $\{t_i\}_{i=1}^{N} \subset (0, 1)$ with $\sum_{i=1}^{N} t_i = 1$.

Lemma 2.8. *[24] Let C be a nonempty convex subset of E and $f : C \to \mathbb{R}$ be a convex and subdifferentiable function on C. Subsequently, f attains its minimum at $x \in C$ if and only if $0 \in \partial f(x) + N_C(x)$, where $N_C(x)$ is the normal cone of C at x, that is*

$$N_C(x) := \{x^* \in E^* : \langle x - z, x^* \rangle \ge 0,\ \forall z \in C\}.$$

Lemma 2.9. *[11] Let $f : E \to \mathbb{R}$ be a convex, strongly coercive, lower semicontinuous, Gáteaux differentiable and cofinite function. Let $G : C \times C \to (-\infty, +\infty]$ be a function such that $G(x, \cdot)$ is proper convex and lower semicontinuous on C for every fixed $x \in C$. Then, for every $\lambda \in (0, \infty)$ and $x \in C$, there exists $z \in C$ such that*

$$z \in argmin\left\{\lambda G(x, y) + D_f(y, x) : y \in C\right\}.$$

Furthermore, if f is strictly convex, then this point is unique.

Let $CB(C)$ denote the family of nonempty closed bounded subsets of C.

Lemma 2.10. *[20] Let E be a reflexive Banach space, and let $f : E \to \mathbb{R}$ be uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let C be a nonempty closed and convex subset of $int(domf)$ and $T : C \to CB(C)$ be a Bregman relatively nonexpansive mapping. Then $F(T)$ is closed and convex.*

Lemma 2.11. *[18] Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable and totally convex function, x_0 be an element in E and C be a nonempty closed convex subset of E. Suppose that the sequence $\{x_n\}$ is bounded and the weak limits of any subsequence of a sequence $\{x_n\}$ belong to $C \subset E$. If $D_f(x_n, x_0) \le D_f(proj_C^f(x_0), x_0)$ for any $n \in \mathbb{N}$, then $\{x_n\}$ converges strongly to $proj_C^f(x_0)$.*

Lemma 2.12. *[9] Let $\{a_n\}$ and $\{b_n\}$ be sequences in $\mathbb{R}$ such that $a_n \le b_n$ for all $n \in \mathbb{N}$. Suppose that $\sigma, \delta \in (0, 1)$ and $0 < \mu < \delta$. Then, there exists a sequence $\{\alpha_n\}$ such that $\alpha_n a_n \le \mu b_n$ and $\sigma\mu < \alpha_n < \delta$.*

By the Lipschitz-type continuity of G on C and Lemma 2.12, from Corollary 2.6 in [9], we have the following Lemma.

Lemma 2.13. *Let G be Lipschitz-type continuous on C with two positive constants c_1 and c_2. Suppose that $\sigma \in (0,1), \delta < \min\left\{\frac{1}{c_1}, \frac{1}{c_2}\right\}$ and $0 < \mu < \delta$. Then, there exists a real number η such that*

$$\mu\left(D_f(x,y) + D_f(y,z)\right) \geq \eta\left(G(x,z) - G(x,y) - G(y,z)\right)$$

and $\sigma\mu < \eta < \delta$ where $x, y, z \in C$.

Assumption 2.1. *The bifunction $G : C \times C \to \mathbb{R}$ satisfies the following assumptions:*

(A1) *$G(x,x) = 0$ and G is pseudomonotone, i.e., $G(x,y) \geq 0$ implies $G(y,x) \leq 0$ for all $x, y \in C$;*

(A2) *G satisfies the Bregman Lipschitz-type condition, i.e., there exist two positive constants c_1, c_2 such that*

$$G(x,y) + G(y,z) \geq G(x,z) - c_1 D_f(y,x) - c_2 D_f(z,y), \ \forall x, y, z \in C; \tag{2.6}$$

(A3) *G is weakly sequentially continuous on $C \times C$ in the sense that if $\{x_n\}, \{y_n\} \subset C$ are two sequences converging weakly to x and y, respectively, then $G(x_n, y_n) \to G(x,y)$ as $n \to \infty$;*

(A4) *$G(x,\cdot)$ is convex lower semicontinuous and subdifferentiable on C for every fixed $x \in C$.*

3. Main Result

In this section, we propose a new inertial Bregman subgradient extragradient algorithm for approximating a common solution of the pseudomonotone equilibrium problem and the fixed point problem associated with a Bregman relatively nonexpansive mapping.

Motivated by the Bregman hybrid extragradient method of Eskandani et al. [11] and the inertial subgradient extragradient technique of Thong et al. [23], the proposed algorithm incorporates an inertial extrapolation step together with a self-adaptive stepsize rule. We establish the strong convergence of the generated sequence under suitable assumptions.

Let E be a reflexive Banach space and C be a nonempty closed and convex subset of E. Let $G : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4) and $T : C \to C$ be a Bregman relatively nonexpansive mapping. Let $f : E \to \mathbb{R}$ be uniformly Fréchet differentiable, strongly coercive, Legendre, uniformly convex and totally convex on bounded subsets of E, and bounded on bounded subsets of E. Assume further that f^* is uniformly convex on bounded subsets of E^*. Suppose that the solution set $\Omega := F(T) \cap EP(G) \neq \emptyset$.

Algorithm 3.1 (Inertial Bregman subgradient extragradient method)**.** *Pick $\lambda_1 > 0$, $\delta < \min\left\{\dfrac{1}{c_1}, \dfrac{1}{c_2}\right\}$, $0 < \mu < \delta$ and initial iterates $x_0, x_1 \in C = C_0 = Q_0$ and $x_0 = x_1$. Set $n = 1$, $\theta > 0$, $\varepsilon_n \in (0,1)$ and θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$ with $\bar{\theta}_n$ defined by*

$$\bar{\theta}_n = \begin{cases} \min\left\{\theta, \dfrac{\varepsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \theta, & \text{otherwise.} \end{cases}$$

Compute

$$\begin{cases} k_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = argmin\left\{\lambda_n G(k_n, z) + D_f(z, k_n) : z \in C\right\}, \\ z_n = argmin\left\{\lambda_n G(y_n, z) + D_f(z, k_n) : z \in C \cap D_n\right\}, \\ \qquad where\ D_n = \{v \in E : \langle \nabla f(k_n) - \lambda_n w_n - \nabla f(y_n), v - y_n\rangle \le 0\} \\ \qquad and\ w_n \in \partial_2 G(k_n, y_n), \\ t_n = \nabla f^* \left(\beta_n \nabla f(k_n) + (1-\beta_n)\nabla f(Tz_n)\right), \\ C_n = \left\{v \in C : D_f(v, t_n) \le D_f(v, k_n)\right\}, \\ Q_n = \{v \in C : \langle \nabla f(x_0) - \nabla f(x_n), v - x_n\rangle \le 0\}, \\ x_{n+1} = proj^f_{C_n \cap Q_n}(x_0), \end{cases} \tag{3.1}$$

and

$$\lambda_{n+1} = \begin{cases} \min\left\{\lambda_n, \dfrac{\mu\left(D_f(y_n, k_n) + D_f(z_n, y_n)\right)}{G(k_n, z_n) - G(k_n, y_n) - G(y_n, z_n)}\right\}, \\ \qquad\qquad if\ G(k_n, z_n) - G(k_n, y_n) - G(y_n, z_n) > 0 \\ \lambda_n, \quad otherwise, \end{cases} \tag{3.2}$$

for all $n \in \mathbb{N}$, *where* $\{\beta_n\} \subset (0,1)$ *with* $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$.

Lemma 3.1. *Let* $\{k_n\}, \{y_n\}$ *and* $\{z_n\}$ *be the sequences generated by Algorithm 3.1. Then*

(i) $C \subseteq D_n$, *for all* $n \in \mathbb{N}$,

(ii) *the following estimate is satisfied, for each* $x^* \in EP(G)$,

$$\begin{aligned} D_f(x^*, z_n) \le D_f(x^*, k_n) &- \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\mu\right) D_f(y_n, k_n) \\ &- \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\mu\right) D_f(z_n, y_n). \end{aligned} \tag{3.3}$$

Proof. (i) We first show the existence of w_n. Since

$$y_n = \arg\min\{\lambda_n G(k_n, y) + D_f(y, k_n) : y \in C\},$$

it follows from Lemma 2.8 that

$$0 \in \partial\left(\lambda_n G(k_n, y) + D_f(y, k_n)\right)(y_n) + N_C(y_n).$$

This implies that there exist $w_n \in \partial_2 G(k_n, y_n), \bar{w}_n \in N_C(y_n)$, such that

$$0 = \lambda_n w_n + \nabla f(y_n) - \nabla f(k_n) + \bar{w}_n.$$

Then by the definition of $N_C(\cdot)$, we get

$$\langle \nabla f(y_n) - \nabla f(k_n), y - y_n\rangle \ge \lambda_n \langle w_n, y_n - y\rangle, \quad \forall y \in C. \tag{3.4}$$

Hence

$$\langle \nabla f(k_n) - \nabla f(y_n), y - y_n \rangle - \langle \lambda_n w_n, y - y_n \rangle \leq 0, \quad \forall y \in C \tag{3.5}$$

and

$$\langle \nabla f(k_n) - \lambda_n w_n - \nabla f(y_n), y - y_n \rangle \leq 0, \quad \forall y \in C.$$

This shows that $C \subseteq D_n$, for all $n \in \mathbb{N}$.

(ii) Also, since

$$z_n = \arg\min\{\lambda_n G(y_n, y) + D_f(y, k_n) : y \in C \cap D_n\},$$

it follows from Lemma 2.8 that

$$0 \in \partial\Big(\lambda_n G(y_n, y) + D_f(y, k_n)\Big)(z_n) + N_{C\cap D_n}(z_n).$$

Then, there exists $s_n \in \partial_2 G(y_n, z_n)$ and $\bar{s}_n \in N_{C\cap D_n}(z_n)$ such that

$$0 = \lambda_n s_n + \nabla f(z_n) - \nabla f(k_n) + \bar{s}_n. \tag{3.6}$$

Then by the definition of $N_{C\cap D_n}(\cdot)$, we get

$$\lambda_n \langle s_n, y - z_n \rangle \geq \langle \nabla f(k_n) - \nabla f(z_n), y - z_n \rangle, \quad \forall y \in C \cap D_n. \tag{3.7}$$

Since $x^* \in EP(G) \subset C \subset D_n$, it follows from (3.7) that

$$\lambda_n \langle s_n, x^* - z_n \rangle \geq \langle \nabla f(k_n) - \nabla f(z_n), x^* - z_n \rangle. \tag{3.8}$$

Using the definition of $\partial_2 G(y_n, z_n)$, we get

$$G(y_n, x^*) - G(y_n, z_n) \geq \langle s_n, x^* - z_n \rangle. \tag{3.9}$$

This together with (3.8) yields

$$\lambda_n \left(G(y_n, x^*) - G(y_n, z_n) \right) \geq \langle \nabla f(k_n) - \nabla f(z_n), x^* - z_n \rangle. \tag{3.10}$$

Since $x^* \in EP(G)$, then $G(x^*, y_n) \geq 0$. Using the pseudomonotonicity of G, we get that $G(y_n, x^*) \leq 0$. Hence

$$\lambda_n G(y_n, z_n) \leq \langle \nabla f(z_n) - \nabla f(k_n), x^* - z_n \rangle. \tag{3.11}$$

From the fact that $w_n \in \partial_2 G(k_n, y_n)$ and the definition of subdifferential, we get

$$G(k_n, y) - G(k_n, y_n) \geq \langle w_n, y - y_n \rangle, \quad \forall y \in C. \tag{3.12}$$

Since $z_n \in D_n$, then

$$\langle \nabla f(k_n) - \lambda_n w_n - \nabla f(y_n), z_n - y_n \rangle \leq 0. \tag{3.13}$$

Hence

$$\langle \nabla f(k_n) - \nabla f(y_n), z_n - y_n \rangle \leq \lambda_n \langle w_n, z_n - y_n \rangle. \tag{3.14}$$

From (3.12) and (3.14), we obtain that

$$\lambda_n \left(G(k_n, z_n) - G(k_n, y_n) \right) \geq \langle \nabla f(k_n) - \nabla f(y_n), z_n - y_n \rangle. \tag{3.15}$$

Combining (3.11) and (3.15), we have

$$\begin{aligned}\langle \nabla f(k_n) - \nabla f(y_n), z_n - y_n\rangle + \langle \nabla f(k_n) - \nabla f(z_n), x^* - z_n\rangle \\ \le \lambda_n \left(G(k_n, z_n) - G(k_n, y_n) - G(y_n, z_n)\right).\end{aligned} \tag{3.16}$$

Using the three-point identity, we have

$$D_f(x^*, z_n) - D_f(x^*, k_n) = -D_f(z_n, k_n) + \langle \nabla f(k_n) - \nabla f(z_n), x^* - z_n\rangle.$$

Again, applying the three-point identity with $x = z_n,\ y = y_n,\ z = k_n$, we obtain

$$D_f(z_n, k_n) = D_f(y_n, k_n) + D_f(z_n, y_n) + \langle \nabla f(y_n) - \nabla f(k_n), z_n - y_n\rangle.$$

From the above two equalities and (3.16), we get that

$$\begin{aligned}D_f(x^*, z_n) \le D_f(x^*, k_n) - D_f(y_n, k_n) - D_f(z_n, y_n) \\ + \lambda_n \left(G(k_n, z_n) - G(k_n, y_n) - G(y_n, z_n)\right).\end{aligned} \tag{3.17}$$

Furthermore, by the definition of λ_{n+1}, we have

$$\begin{aligned}D_f(x^*, z_n) &\le D_f(x^*, k_n) - D_f(y_n, k_n) - D_f(z_n, y_n) \\ &\quad + \frac{\lambda_n}{\lambda_{n+1}} \lambda_{n+1} \left(G(k_n, z_n) - G(k_n, y_n) - G(y_n, z_n)\right) \\ &\le D_f(x^*, k_n) - D_f(y_n, k_n) - D_f(z_n, y_n) \\ &\quad + \frac{\lambda_n}{\lambda_{n+1}} \mu \left(D_f(y_n, k_n) + D_f(z_n, y_n)\right) \\ &= D_f(x^*, k_n) - \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\mu\right) D_f(y_n, k_n) \\ &\quad - \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\mu\right) D_f(z_n, y_n).\end{aligned} \tag{3.18}$$

□

Lemma 3.2. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1. Then $\Omega \subset C_n \cap Q_n,\ \forall n \ge 0$ and $\{x_n\}$ is well-defined.*

Proof. Let $p \in \Omega$, by Lemma 3.1, we obtain that

$$\begin{aligned}D_f(p, t_n) &= D_f(p, \nabla f^*(\beta_n \nabla f(k_n) + (1 - \beta_n) \nabla f(Tz_n))) \\ &\le \beta_n D_f(p, k_n) + (1 - \beta_n) D_f(p, Tz_n) \\ &\le \beta_n D_f(p, k_n) + (1 - \beta_n) D_f(p, z_n) \\ &\le D_f(p, k_n).\end{aligned} \tag{3.19}$$

Thus $p \in C_n$. Therefore $\Omega \subset C_n,\ \forall n \ge 0$. Further, by induction we show that $\Omega \subset C_n \cap Q_n,\ \forall n \ge 0$. As $Q_0 = C$, we have $\Omega \subset C_0 \cap Q_0$. Suppose that $\Omega \subset C_m \cap Q_m$, for some $m > 0$. Then $x_{m+1} \in C_m \cap Q_m$

and $x_{m+1} = proj^f_{C_m \cap Q_m}(x_0)$. From Lemma 2.4, we get

$$\langle \nabla f(x_0) - \nabla f(x_{m+1}), p - x_{m+1} \rangle \leq 0, \ \forall p \in \Omega, \tag{3.20}$$

which implies $p \in Q_{m+1}$. Hence, $\Omega \subset C_{m+1} \cap Q_{m+1}$ implies $\Omega \subset C_n \cap Q_n$, $\forall n \geq 0$. Moreover, C_n and Q_n are closed and convex. Indeed, Q_n is the intersection of C and a closed half-space. Also, C_n is closed and convex since

$$D_f(v, t_n) \leq D_f(v, k_n)$$

is equivalent to

$$f(v) - f(t_n) - \langle \nabla f(t_n), v - t_n \rangle \leq f(v) - f(k_n) - \langle \nabla f(k_n), v - k_n \rangle.$$

It follows that

$$\langle \nabla f(k_n) - \nabla f(t_n), v \rangle \leq f(t_n) - f(k_n) + \langle \nabla f(k_n), k_n \rangle - \langle \nabla f(t_n), t_n \rangle.$$

which defines a closed half-space. Therefore, $C_n \cap Q_n$ is a nonempty closed convex subset of C. Hence $x_{n+1} = proj^f_{C_n \cap Q_n}(x_0)$ is well-defined for all $n \geq 0$. This implies that $\{x_n\}$ is well-defined. □

Lemma 3.3. *The sequences $\{x_n\}, \{k_n\}, \{y_n\}$ and $\{z_n\}$ generated by Algorithm 3.1 are bounded.*

Proof. Let $p \in \Omega$. Since $x_n = proj^f_{C_{n-1} \cap Q_{n-1}}(x_0)$, by Lemma 2.4, we have

$$D_f(p, x_n) \leq D_f(p, x_0) - D_f(x_n, x_0) \leq D_f(p, x_0).$$

Hence $\{D_f(p, x_n)\}$ is bounded. By Lemma 2.3, $\{x_n\}$ is bounded. From the definition of k_n, we have

$$k_n = x_n + \theta_n(x_n - x_{n-1}).$$

Since $0 \leq \theta_n \leq \bar{\theta}_n \leq \theta$, it follows that $\{k_n\}$ is bounded. Consequently, since f and ∇f are bounded on bounded subsets, the sequence $\{D_f(p, k_n)\}$ is bounded. Next, by Lemma 3.1, for every $p \in \Omega$, we obtain

$$D_f(p, z_n) \leq D_f(p, k_n).$$

Thus $\{D_f(p, z_n)\}$ is bounded. By Lemma 2.3, $\{z_n\}$ is bounded. Moreover, Lemma 3.1 gives

$$\left(1 - \frac{\lambda_n}{\lambda_{n+1}}\mu\right) D_f(y_n, k_n) \leq D_f(p, k_n) - D_f(p, z_n) \leq D_f(p, k_n).$$

Since $\{\lambda_n\}$ is nonincreasing and bounded below by a positive number, there exists $\lambda > 0$ such that $\lambda_n \to \lambda$. Hence $\lambda_n / \lambda_{n+1} \to 1$. Since $0 < \mu < 1$, we have

$$1 - \frac{\lambda_n}{\lambda_{n+1}}\mu > 0,$$

for all sufficiently large n. This implies that the coefficient $1 - \frac{\lambda_n}{\lambda_{n+1}}\mu$ is eventually positive. Hence $\{D_f(y_n, k_n)\}$ is bounded. We now show that $\{y_n\}$ is bounded. Suppose, to the contrary, that $\{y_n\}$ is unbounded. Since $\{k_n\}$ is bounded and f is bounded on bounded subsets of E, the sequence $\{f(k_n)\}$ is bounded. Moreover, since f is uniformly Fréchet differentiable on bounded subsets of

E, Lemma 2.2 implies that ∇f is uniformly continuous on bounded subsets. Hence $\{\nabla f(k_n)\}$ is bounded. Then there exist constants $M, M_1 > 0$ such that

$$|f(k_n)| \le M, \quad \|\nabla f(k_n)\| \le M, \quad \|k_n\| \le M_1, \forall n \ge 1.$$

Thus, for all $n \ge 1$,

$$\begin{aligned} D_f(y_n, k_n) &= f(y_n) - f(k_n) - \langle \nabla f(k_n), y_n - k_n \rangle \\ &\ge f(y_n) - |f(k_n)| - \|\nabla f(k_n)\| \, \|y_n - k_n\| \\ &\ge f(y_n) - M - M(\|y_n\| + \|k_n\|) \\ &\ge f(y_n) - M - M\|y_n\| - MM_1. \end{aligned}$$

Since f is strongly coercive, the right-hand side tends to $+\infty$ whenever $\|y_n\| \to \infty$. Hence $D_f(y_n, k_n) \to +\infty$, which contradicts the boundedness of $\{D_f(y_n, k_n)\}$. Therefore $\{y_n\}$ is bounded. Consequently, $\{x_n\}, \{k_n\}, \{y_n\}$ and $\{z_n\}$ are bounded. □

Lemma 3.4. *Let $\{x_n\}, \{k_n\}, \{y_n\}, \{z_n\}$ and $\{t_n\}$ be sequences generated by Algorithm 3.1. Then, there hold the relations* $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, $\lim_{n\to\infty} \|x_n - k_n\| = 0$, $\lim_{n\to\infty} \|k_n - t_n\| = 0$, $\lim_{n\to\infty} \|k_n - y_n\| = 0$, $\lim_{n\to\infty} \|k_n - z_n\| = 0$ *and* $\lim_{n\to\infty} \|z_n - Tz_n\| = 0$.

Proof. Since $x_{n+1} \in C_n \cap Q_n$, we have $x_{n+1} \in Q_n$. Hence,

$$\langle \nabla f(x_0) - \nabla f(x_n), x_{n+1} - x_n \rangle \le 0.$$

By the three-point identity, we have

$$D_f(x_{n+1}, x_n) + D_f(x_n, x_0) - D_f(x_{n+1}, x_0) = \langle \nabla f(x_0) - \nabla f(x_n), x_{n+1} - x_n \rangle.$$

Therefore,

$$D_f(x_n, x_0) \le D_f(x_{n+1}, x_0) - D_f(x_{n+1}, x_n).$$

It follows that

$$D_f(x_n, x_0) \le D_f(x_{n+1}, x_0), \ \forall n \ge 0, \tag{3.21}$$

which implies $\{D_f(x_n, x_0)\}$ is nondecreasing. By the boundedness of $\{D_f(x_n, x_0)\}$, $\lim_{n\to\infty} D_f(x_n, x_0)$ exists and is finite. Further,

$$D_f(x_{n+1}, x_n) \le D_f(x_{n+1}, x_0) - D_f(x_n, x_0).$$

which yields $\lim_{n\to\infty} D_f(x_{n+1}, x_n) = 0$. Using Lemma 2.1, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.22}$$

From the definition of k_n,

$$\begin{aligned} \|k_n - x_n\| &= \|(x_n + \theta_n(x_n - x_{n-1})) - x_n\| \\ &= \|\theta_n(x_n - x_{n-1})\| \\ &\le \theta\|x_n - x_{n-1}\|, \end{aligned} \tag{3.23}$$

which implies by above that

$$\lim_{n\to\infty} \|k_n - x_n\| = 0. \tag{3.24}$$

Since $\|k_n - x_{n+1}\| \le \|k_n - x_n\| + \|x_n - x_{n+1}\|$, it follows from (3.22) and (3.24) that

$$\lim_{n\to\infty} \|k_n - x_{n+1}\| = 0. \tag{3.25}$$

Using Lemma 2.2 and the fact that f is uniformly Fréchet differentiable, we get

$$\lim_{n\to\infty} |f(k_n) - f(x_{n+1})| = 0 \tag{3.26}$$

and

$$\lim_{n\to\infty} \|\nabla f(k_n) - \nabla f(x_{n+1})\| = 0. \tag{3.27}$$

By the definition of $D_f(\cdot,\cdot)$, we get

$$D_f(x_{n+1}, k_n) = f(x_{n+1}) - f(k_n) - \langle \nabla f(k_n), x_{n+1} - k_n \rangle.$$

Since f is uniformly Fréchet differentiable, it is uniformly continuous on bounded subsets. Hence, by (3.25) and (3.26),

$$\lim_{n\to\infty} D_f(x_{n+1}, k_n) = 0. \tag{3.28}$$

As $x_{n+1} = proj^f_{C_n \cap Q_n}(x_0) \in C_n$, we have

$$D_f(x_{n+1}, t_n) \le D_f(x_{n+1}, k_n) \tag{3.29}$$

and hence by (3.28),

$$\lim_{n\to\infty} D_f(x_{n+1}, t_n) = 0. \tag{3.30}$$

By Lemma 2.1, we have $\lim_{n\to\infty} \|x_{n+1} - t_n\| = 0$. Using the fact above, we derive

$$\|k_n - t_n\| \le \|k_n - x_{n+1}\| + \|x_{n+1} - t_n\| \to 0, \text{ as } n \to \infty. \tag{3.31}$$

So, we have

$$\lim_{n\to\infty} |f(k_n) - f(t_n)| = 0 = \lim_{n\to\infty} \|\nabla f(k_n) - \nabla f(t_n)\|. \tag{3.32}$$

Moreover,

$$\begin{aligned} D_f(y, k_n) - D_f(y, t_n) &= f(y) - f(k_n) - \langle \nabla f(k_n), y - k_n \rangle \\ &\quad - (f(y) - f(t_n) - \langle \nabla f(t_n), y - t_n \rangle) \\ &= f(t_n) - f(k_n) - \langle \nabla f(k_n), y - k_n \rangle \\ &\quad + \langle \nabla f(t_n), y - t_n \rangle \\ &= f(t_n) - f(k_n) - \langle \nabla f(k_n), y - t_n \rangle \\ &\quad - \langle \nabla f(k_n), t_n - k_n \rangle + \langle \nabla f(t_n), y - t_n \rangle \\ &= f(t_n) - f(k_n) - \langle \nabla f(k_n) - \nabla f(t_n), y - t_n \rangle \\ &\quad - \langle \nabla f(k_n), t_n - k_n \rangle, \quad \forall y \in C. \end{aligned} \tag{3.33}$$

By the previous inequality, we get that

$$\lim_{n\to\infty}\Big(D_f(y,k_n)-D_f(y,t_n)\Big)=0,\quad \forall y\in C. \tag{3.34}$$

Let $p\in\Omega$. From the Algorithm 3.1 and Lemma 2.6 applied to f^*, we obtain that

$$\begin{aligned}
D_f(p,t_n) &= D_f\left(p,\nabla f^*\left(\beta_n\nabla f(k_n)+(1-\beta_n)\nabla f(Tz_n)\right)\right)\\
&= V_f\left(p,\beta_n\nabla f(k_n)+(1-\beta_n)\nabla f(Tz_n)\right)\\
&= f(p)-\langle p,\beta_n\nabla f(k_n)+(1-\beta_n)\nabla f(Tz_n)\rangle\\
&\qquad + f^*\left(\beta_n\nabla f(k_n)+(1-\beta_n)\nabla f(Tz_n)\right)\\
&\le f(p)-\beta_n\langle p,\nabla f(k_n)\rangle-(1-\beta_n)\langle p,\nabla f(Tz_n)\rangle\\
&\qquad +\beta_n f^*(\nabla f(k_n))+(1-\beta_n)f^*(\nabla f(Tz_n))\\
&\qquad\qquad -\beta_n(1-\beta_n)\rho_r^*\Big(\|\nabla f(k_n)-\nabla f(Tz_n)\|\Big)\\
&= \beta_n f(p)+(1-\beta_n)f(p)-\beta_n\langle p,\nabla f(k_n)\rangle-(1-\beta_n)\langle p,\nabla f(Tz_n)\rangle\\
&\qquad +\beta_n f^*(\nabla f(k_n))+(1-\beta_n)f^*(\nabla f(Tz_n))\\
&\qquad\qquad -\beta_n(1-\beta_n)\rho_r^*\Big(\|\nabla f(k_n)-\nabla f(Tz_n)\|\Big)\\
&= \beta_n V_f(p,\nabla f(k_n))+(1-\beta_n)V_f(p,\nabla f(Tz_n))\\
&\qquad -\beta_n(1-\beta_n)\rho_r^*\Big(\|\nabla f(k_n)-\nabla f(Tz_n)\|\Big)\\
&= \beta_n D_f(p,k_n)+(1-\beta_n)D_f(p,Tz_n)\\
&\qquad -\beta_n(1-\beta_n)\rho_r^*\Big(\|\nabla f(k_n)-\nabla f(Tz_n)\|\Big)\\
&\le D_f(p,k_n)-\beta_n(1-\beta_n)\rho_r^*\Big(\|\nabla f(k_n)-\nabla f(Tz_n)\|\Big).
\end{aligned} \tag{3.35}$$

This implies that

$$\beta_n(1-\beta_n)\rho_r^*\Big(\|\nabla f(k_n)-\nabla f(Tz_n)\|\Big)\le\ D_f(p,k_n)-D_f(p,t_n). \tag{3.36}$$

Since $\|k_n-t_n\|\to 0$ and the sequences involved are bounded, the uniform continuity of f and ∇f on bounded subsets implies $D_f(p,k_n)-D_f(p,t_n)\to 0$. Since

$$\liminf_{n\to\infty}\beta_n(1-\beta_n)>0,$$

it follows from (3.36) that

$$\rho_r^*\Big(\|\nabla f(k_n)-\nabla f(Tz_n)\|\Big)\to 0.$$

Since $\rho_r^*(s)>0$ for all $s>0$, we obtain

$$\|\nabla f(k_n)-\nabla f(Tz_n)\|\to 0,\ \text{as } n\to\infty. \tag{3.37}$$

Moreover, since $\{k_n\}$ and $\{Tz_n\}$ are bounded, we have

$$\begin{aligned}
D_f(Tz_n,k_n)+D_f(k_n,Tz_n) &= \langle\nabla f(Tz_n)-\nabla f(k_n),Tz_n-k_n\rangle\\
&\le \|\nabla f(Tz_n)-\nabla f(k_n)\|\|Tz_n-k_n\|\to 0.
\end{aligned}$$

Hence

$$D_f(Tz_n, k_n) \to 0.$$

By Lemma 2.1, we conclude that

$$\|Tz_n - k_n\| \to 0.$$

On the other hand, by Lemma 3.1, we have

$$\left(1 - \frac{\lambda_n}{\lambda_{n+1}}\mu\right) D_f(y_n, k_n) + \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\mu\right) D_f(z_n, y_n) \le D_f(p, k_n) - D_f(p, z_n). \tag{3.38}$$

Since the coefficient

$$1 - \frac{\lambda_n}{\lambda_{n+1}}\mu$$

is positive for all sufficiently large n, it follows from (3.38) that

$$0 \le D_f(p, k_n) - D_f(p, z_n)$$

for all sufficiently large n. Since $p \in F(T)$, we have

$$D_f(p, Tz_n) \le D_f(p, z_n).$$

Hence, for all sufficiently large n,

$$0 \le D_f(p, k_n) - D_f(p, z_n) \le D_f(p, k_n) - D_f(p, Tz_n).$$

Moreover, since $\|k_n - Tz_n\| \to 0$ and the sequences involved are bounded, the uniform continuity of f and ∇f on bounded subsets implies

$$D_f(p, k_n) - D_f(p, Tz_n) \to 0.$$

Therefore,

$$D_f(p, k_n) - D_f(p, z_n) \to 0.$$

Consequently,

$$D_f(y_n, k_n) \to 0 \quad \text{and} \quad D_f(z_n, y_n) \to 0.$$

Since f is totally convex on bounded subsets, Lemma 2.1 yields

$$\|y_n - k_n\| \to 0 \quad \text{and} \quad \|z_n - y_n\| \to 0. \tag{3.39}$$

Consequently,

$$\|z_n - k_n\| \le \|z_n - y_n\| + \|y_n - k_n\| \to 0. \tag{3.40}$$

Combining this with $\|k_n - Tz_n\| \to 0$, we obtain $\|z_n - Tz_n\| \le \|z_n - k_n\| + \|k_n - Tz_n\| \to 0$. Thus,

$$\lim_{n\to\infty} \|z_n - Tz_n\| = 0.$$

Finally, since $\|x_n - k_n\| \to 0$ and $\|z_n - k_n\| \to 0$, we also have $\|x_n - z_n\| \to 0$.
Therefore, all the asserted relations hold: $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, $\lim_{n\to\infty} \|x_n - k_n\| = 0$, $\lim_{n\to\infty} \|k_n - t_n\| = 0$, $\lim_{n\to\infty} \|k_n - y_n\| = 0$, and $\lim_{n\to\infty} \|k_n - z_n\| = 0$. Moreover, $\|z_n - Tz_n\| \to 0$. This completes the proof. □

Theorem 3.1. *Let $\{x_n\}, \{k_n\}, \{y_n\}$ and $\{z_n\}$ be sequences generated by Algorithm 3.1. Then there exists $x^* \in C$ such that $x^* \in \Omega := F(T) \cap EP(G)$ and $\{x_n\}$ converges strongly to $x^* = proj^f_\Omega(x_0)$.*

Proof. Let $\{x_{n_k}\}$ be an arbitrary weakly convergent subsequence of $\{x_n\}$, say $x_{n_k} \rightharpoonup \bar{x} \in C$. By Lemma 3.4, we have

$$\|x_n - k_n\| \to 0, \qquad \|k_n - z_n\| \to 0.$$

Hence

$$k_{n_k} \rightharpoonup \bar{x}, \qquad z_{n_k} \rightharpoonup \bar{x}.$$

Moreover, Lemma 3.4 gives

$$\|z_n - Tz_n\| \to 0.$$

Therefore,

$$\|z_{n_k} - Tz_{n_k}\| \to 0.$$

By the definition of an asymptotic fixed point, we obtain $\bar{x} \in \widehat{F}(T)$. Since T is Bregman relatively nonexpansive, $\widehat{F}(T) = F(T)$. Thus $\bar{x} \in F(T)$. Next, we show that $\bar{x} \in EP(G)$. Since

$$y_n = \arg\min\{\lambda_n G(k_n, y) + D_f(y, k_n) : y \in C\},$$

we have

$$\lambda_{n_k}\Big(G(k_{n_k}, y) - G(k_{n_k}, y_{n_k})\Big) \geq \langle \nabla f(y_{n_k}) - \nabla f(k_{n_k}), y_{n_k} - y\rangle$$

for all $y \in C$. By Lemma 3.4, we have

$$\|y_{n_k} - k_{n_k}\| \to 0.$$

Since $\{y_{n_k}\}$ and $\{k_{n_k}\}$ are bounded and f is uniformly Fréchet differentiable, it follows from Lemma 2.2 that ∇f is uniformly continuous on bounded subsets of E. Hence,

$$\|\nabla f(y_{n_k}) - \nabla f(k_{n_k})\| \to 0.$$

Moreover, $\{y_{n_k}\}$ is bounded and $y \in C$ is fixed, so $\{y_{n_k} - y\}$ is bounded. Therefore,

$$\Big|\Big\langle \nabla f(y_{n_k}) - \nabla f(k_{n_k}),\ y_{n_k} - y\Big\rangle\Big| \leq \|\nabla f(y_{n_k}) - \nabla f(k_{n_k})\|\,\|y_{n_k} - y\| \to 0.$$

Consequently,

$$\langle \nabla f(y_{n_k}) - \nabla f(k_{n_k}), y_{n_k} - y\rangle \to 0.$$

Using the weak sequential continuity of G, we obtain

$$G(k_{n_k}, y) \to G(\bar{x}, y) \quad \text{and} \quad G(k_{n_k}, y_{n_k}) \to G(\bar{x}, \bar{x}) = 0.$$

Passing to the limit in the above inequality yields

$$\lambda\Big(G(\bar{x}, y) - G(\bar{x}, \bar{x})\Big) \geq 0.$$

Since $G(\bar{x}, \bar{x}) = 0$, we obtain

$$G(\bar{x}, y) \geq 0, \qquad \forall y \in C.$$

Hence $\bar{x} \in EP(G)$. Therefore $\bar{x} \in F(T) \cap EP(G) = \Omega$. Since the weakly convergent subsequence $\{x_{n_k}\}$ was arbitrary, every weak cluster point of $\{x_n\}$ belongs to Ω. Since Ω is a nonempty closed convex subset of E, the Bregman projection $\text{proj}^f_\Omega(x_0)$ is well-defined. Let $x^* = proj^f_\Omega(x_0)$. Since $x_{n+1} = proj^f_{C_n\cap Q_n}(x_0)$ and $proj^f_\Omega(x_0) \in \Omega \subseteq C_n \cap Q_n$, by Lemma 2.4 we obtain that

$$D_f(x_{n+1}, x_0) \le D_f(proj^f_\Omega(x_0), x_0). \tag{3.41}$$

It follows from Lemma 2.11 that $\{x_n\}$ converges strongly to $x^* = \text{proj}^f_\Omega(x_0)$. This completes the proof. □

4. Consequences

Let E be a real Banach space and $1 < q \le 2 \le p$ with $\frac{1}{p} + \frac{1}{q} = 1$. For the p-uniformly convex space, the Bregman distance has the following relation [19]:

$$\tau\|x - y\|^p \le D_{\frac{1}{p}\|\cdot\|^p}(x, y) \le \langle J^p_E(x) - J^p_E(y), x - y\rangle, \tag{4.1}$$

where $\tau > 0$ is a fixed number and duality mapping $J^p_E : E \to 2^{E^*}$ is defined by

$$J^p_E(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^p, \|f\| = \|x\|^{p-1}\},$$

for all $x \in E$. We know that if E is a smooth strictly convex and reflexive Banach space, then J^p_E is a single-valued bijection and in this case, $J^p_E = (J^q_{E^*})^{-1}$, where $J^q_{E^*}$ is the duality mapping of E^*.

For $p = 2$, the duality mapping J^p_E, is called the normalized duality and is denoted by J. The function $\phi : E^2 \to \mathbb{R}$ is defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2, \tag{4.2}$$

for all $x, y \in E$. The generalized projection Π_C from E onto C is defined by

$$\Pi_C(x) = argmin_{y\in C}\phi(y, x),\ \forall x \in E,$$

where C is a nonempty closed convex subset of E.

We consider the case $p = 2$ and the Lyapunov function (4.2). Furthermore, if A_i is L_i-Lipschitz continuous, then G_i is Bregman-Lipschitz-type continuous with $c_{i,1} = c_{i,2} = \frac{L}{2\tau}$. Let E be a uniformly smooth and uniformly convex Banach space and $f(\cdot) = \frac{1}{2}\|\cdot\|^2$. Therefore, $\nabla f = J, D_{\frac{1}{2}\|\cdot\|^2}(x, y) = \frac{1}{2}\phi(x, y)$ and $proj^{\frac{1}{2}\|\cdot\|^2}_C = \Pi_C$. In particular if E is a Hilbert space, then $\nabla f = I, D_{\frac{1}{2}\|\cdot\|^2}(x, y) = \frac{1}{2}\|x - y\|^2$ and $proj^{\frac{1}{2}\|\cdot\|^2}_C = P_C$, where P_C is the metric projection.

As a direct consequence of Theorem 3.1, we obtain the following result for uniformly smooth and 2-uniformly convex Banach spaces. For the following corollary, let C be a nonempty closed convex subset of a uniformly smooth and 2-uniformly convex Banach space E. Let

$$f(x) = \frac{1}{2}\|x\|^2,\ x \in E.$$

Then $\nabla f = J, D_f(x, y) = \frac{1}{2}\phi(x, y)$, and $proj^f_C = \Pi_C$. Moreover, when

$$f(x) = \frac{1}{2}\|x\|^2,$$

the notion of a Bregman relatively nonexpansive mapping coincides with that of a relatively nonexpansive mapping.

Corollary 4.1. *Let $G : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 2.1. Let T be a relatively nonexpansive mapping on C. Let $\Omega = F(T) \cap EP(G) \neq \emptyset$. Assume that $\lambda_1 > 0$, $\delta < \min\{1/c_1, 1/c_2\}$, and $0 < \mu < \delta$. Suppose that $\{x_n\}, \{k_n\}, \{y_n\}$ and $\{z_n\}$ are generated by*

$$\begin{cases} x_0, x_1 \in C = C_0 = Q_0, x_0 = x_1, \\ k_n = x_n + \theta_n(x_n - x_{n-1}) \\ y_n = argmin\left\{\lambda_n G(k_n, z) + \frac{1}{2}\phi(z, k_n) : z \in C\right\} \\ z_n = argmin\left\{\lambda_n G(y_n, z) + \frac{1}{2}\phi(z, k_n) : z \in C \cap D_n\right\} \\ \qquad where\ D_n = \{v \in E : \langle J(k_n) - \lambda_n w_n - J(y_n), v - y_n\rangle \leq 0\} \\ \qquad and\ w_n \in \partial_2 G(k_n, y_n) \\ t_n = J^{-1}\left(\beta_n J(k_n) + (1-\beta_n)J(Tz_n)\right) \\ C_n = \left\{v \in C : \phi(v, t_n) \leq \phi(v, k_n)\right\} \\ Q_n = \{v \in C : \langle J(x_0) - J(x_n), v - x_n\rangle \leq 0\} \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \end{cases} \tag{4.3}$$

and

$$\lambda_{n+1} = \begin{cases} \min\left\{\lambda_n, \dfrac{\mu\left(\phi(y_n, k_n) + \phi(z_n, y_n)\right)}{2\left(G(k_n, z_n) - G(k_n, y_n) - G(y_n, z_n)\right)}\right\}, \\ \qquad\qquad if\ G(k_n, z_n) - G(k_n, y_n) - G(y_n, z_n) > 0 \\ \lambda_n, \quad otherwise \end{cases} \tag{4.4}$$

where $\{\theta_n\}$ satisfies the condition in Algorithm 3.1 and $\{\beta_n\} \subset (0,1)$ such that $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. Then, $\{x_n\}$ converges strongly to $\Pi_\Omega(x_0)$.

Next, we discuss an application of Theorem 3.1 to a variational inequality problem for a pseudomonotone mapping.

Let C be a nonempty closed convex subset of a reflexive Banach space E, E^* be the dual of E and $A : C \to E^*$ be a nonlinear mapping. The mapping A is said to be pseudomonotone on C if

$$\langle Ax, y - x\rangle \geq 0 \Longrightarrow \langle Ay, x - y\rangle \leq 0, \ \forall x, y \in C.$$

We state the variational inequality problem as follows:

$$\text{Find } \ x^* \in C \ \text{ such that } \ \langle Ax^*, x - x^*\rangle \geq 0, \ \forall x \in C. \tag{4.5}$$

If we take the bifunction $G(x, y) = \langle Ax, y - x\rangle$ for all $x, y \in C$, then the equilibrium problem reduces to the above variational inequality problem. If E is a real Hilbert space, it is well-known that x^* is a solution of (4.5) if and only if x^* solves the fixed point equation

$$x^* = P_C(x^* - \lambda Ax^*), \ \lambda > 0.$$

It was shown in [11] that if A is a pseudomonotone and L-Lipschitz continuous mapping from C to E^*, then $G(x,y) = \langle Ax, y-x\rangle$ is pseudomonotone and Bregman Lipschitz continuous. The following lemma follows from Lemma 4.1 in [11].

Lemma 4.1. *Let C be a nonempty closed convex subset of a reflexive Banach space E, $A : C \to E^*$ be a mapping, and $f : E \to \mathbb{R}$ be a Legendre function. Then*

$$proj_C^f(\nabla f^*(\nabla f(x) - \lambda A(y))) = argmin_{w\in C}\left\{\lambda\langle Ay, w-y\rangle + D_f(w,x)\right\}, \tag{4.6}$$

for all $x \in E, y \in C$ and $\lambda \in (0,+\infty)$.

Corollary 4.2. *Let A be a weakly sequentially continuous, pseudomonotone and L-Lipschitz continuous mapping from C to E^*. Let T be a Bregman relatively nonexpansive mapping on C. Let $\Omega = F(T) \cap VIP(A,C) \neq \emptyset$. Assume that $\lambda_1 > 0$, $\delta < \min\{1/c_1, 1/c_2\}$, and $0 < \mu < \delta$. Suppose that $\{x_n\}, \{k_n\}, \{y_n\}$ and $\{z_n\}$ are generated by*

$$\begin{cases} x_0, x_1 \in C = C_0 = Q_0, x_0 = x_1, \\ k_n = x_n + \theta_n(x_n - x_{n-1}) \\ y_n = proj_C^f\left(\nabla f^*\left(\nabla f(k_n) - \lambda_n A(k_n)\right)\right) \\ z_n = proj_{C\cap D_n}^f\left(\nabla f^*\left(\nabla f(k_n) - \lambda_n A(y_n)\right)\right) \\ \qquad \text{where } D_n = \{v \in E : \langle \nabla f(k_n) - \lambda_n A(k_n) - \nabla f(y_n), v - y_n\rangle \leq 0\} \\ t_n = \nabla f^*\left(\beta_n \nabla f(k_n) + (1-\beta_n)\nabla f(Tz_n)\right) \\ C_n = \left\{v \in C : D_f(v, t_n) \leq D_f(v, k_n)\right\} \\ Q_n = \{v \in C : \langle \nabla f(x_0) - \nabla f(x_n), v - x_n\rangle \leq 0\} \\ x_{n+1} = proj_{C_n\cap Q_n}^f x_0, \end{cases} \tag{4.7}$$

and

$$\lambda_{n+1} = \begin{cases} \min\left\{\lambda_n, \dfrac{\mu\left(D_f(y_n,k_n) + D_f(z_n,y_n)\right)}{\langle A(k_n) - A(y_n), z_n - y_n\rangle}\right\}, \\ \qquad\qquad \text{if } \langle A(k_n) - A(y_n), z_n - y_n\rangle > 0 \\ \lambda_n, \quad \text{otherwise} \end{cases} \tag{4.8}$$

where $\{\theta_n\}$ satisfies the condition in Algorithm 3.1 and $\{\beta_n\} \subset (0,1)$ such that $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. Then, $\{x_n\}$ converges strongly to $proj_\Omega^f(x_0)$.

5. Numerical Example

To illustrate the convergence behavior of the proposed algorithm, we present numerical examples in Hilbert spaces.

We use $\|x_{n+1} - x_n\| < 10^{-4}$ as the stopping criterion, with a maximum number of iterations equal to 25000.

Example 5.1. *In this example, we take $E = \mathbb{R}^{10}$ and*

$$f(x) = \frac{1}{2}\|x\|^2.$$

Then $\nabla f = I$,

$$D_f(x,y) = \frac{1}{2}\|x-y\|^2,$$

and the Bregman projection reduces to the metric projection. We suppose that the set C has the form

$$C = \left\{x \in \mathbb{R}^{10} : -2 \le x_i \le 5,\ i = 1,2,...,10\right\}.$$

For each numerical experiment, the matrices $P, Q \in \mathbb{R}^{10\times 10}$ are generated randomly such that

$$Q = A^T A, \qquad P = Q + B^T B,$$

where $A, B \in \mathbb{R}^{10\times 10}$ are random matrices. Hence, P and Q are positive semidefinite and

$$P - Q \ge 0.$$

We consider the bifunction G defined by

$$G(x,y) = \langle Px + Qy + q, y - x\rangle,\ \forall x, y \in C.$$

The positive semidefiniteness of $P-Q$ implies that G is monotone and hence, pseudomonotone. Indeed, $G(x,y) + G(y,x) = -\langle (P-Q)(x-y), x-y\rangle \le 0$. It is known (see Muu and Oettli [14]) that the bifunction

$$G(x,y) = \langle Px + Qy + q, y - x\rangle$$

satisfies the Lipschitz-type condition with

$$c_1 = c_2 = \frac{\|P-Q\|}{2}.$$

The mapping $T : \mathbb{R}^{10} \to \mathbb{R}^{10}$ is defined as the projection P_C and the following initial points were considered:

$$\begin{aligned} x^{(1)} &= (0.8, -0.4, 0.8, -0.4, \ldots, 0.8, -0.4) \\ x^{(2)} &= (1, -0.5, 1, -0.5, \ldots, 1, -0.5) \\ x^{(3)} &= (1.5, -1, 0, \ldots, 0). \end{aligned}$$

For Algorithm 3.1, we choose

$$\lambda_1 = 0.36, \qquad \delta = 0.9 \min\left\{\frac{1}{c_1}, \frac{1}{c_2}\right\}, \qquad \mu = 0.3\,\delta,$$

which guarantees that

$$0 < \mu < \delta < \min\left\{\frac{1}{c_1}, \frac{1}{c_2}\right\}.$$

Moreover, we set

$$\theta = 5, \qquad \varepsilon_n = \frac{5}{(n+1)^4}, \qquad \beta_n = 0.5,$$

and select θ_n according to the rule in Algorithm 3.1.

The performance of Algorithm 3.1 for different initial points is reported in Table 1. For each initial point, we record the number of iterations, the CPU time, and the final error corresponding to the stopping criterion $\|x_{n+1} - x_n\| < 10^{-4}$.

Table 1. Performance of Algorithm 3.1 for different initial points in Example 5.1

Initial point	*Iterations*	*CPU time (s)*	*Final error*
$x^{(1)}$	*12551*	*4.4290*	9.9008×10^{-5}
$x^{(2)}$	*21830*	*7.8503*	9.9937×10^{-5}
$x^{(3)}$	*15537*	*5.9327*	9.9198×10^{-5}

As shown in Table 1, Algorithm 3.1 converges for all tested initial points. Although the number of iterations varies depending on the starting point, the obtained final errors remain of the same order of magnitude and satisfy the prescribed stopping criterion.

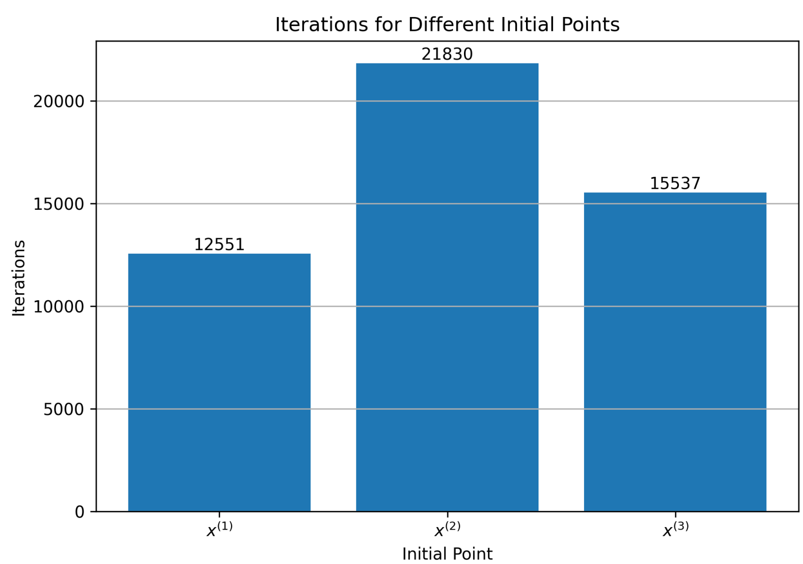

Figure 1. Number of iterations required by Algorithm 3.1 for different initial points in Example 5.1.

Figure 1 illustrates the influence of the initial point on the convergence behavior of Algorithm 3.1. Although the number of iterations varies for different starting points, the algorithm converges successfully in all tested cases. The obtained final errors remain below the prescribed tolerance, demonstrating the stability of the proposed method.

Example 5.2. *Let* $E = \ell^2(\mathbb{R})$ *be the Hilbert space whose elements are all 2-summable sequences* $\{x_i\}_{i=1}^{\infty}$ *of scalars in* $\mathbb{R}$*, that is*

$$\ell^2(\mathbb{R}) := \left\{ x = (x_1, x_2, \cdots, x_i, \cdots),\ x_i \in \mathbb{R} \text{ and } \sum_{i=1}^{\infty} |x_i|^2 < \infty \right\}$$

with inner product $\langle\cdot,\cdot\rangle : \ell^2 \times \ell^2 \to \mathbb{R}$ *and* $\|\cdot\| : \ell^2 \to \mathbb{R}$ *defined by* $\langle x, y\rangle := \sum_{i=1}^{\infty} x_i y_i$ *and* $\|x\| = \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{\frac{1}{2}}$*, where* $x = \{x_i\}_{i=1}^{\infty}, y = \{y_i\}_{i=1}^{\infty}$*. Let* $C = \{x \in E : \|x\| \le 1\}$*. Define the bifunction* $G : C \times C \to \mathbb{R}$ *by*

$$G(x, y) = (3 - \|x\|)\langle x, y - x\rangle, \ \ \forall x, y \in C.$$

It is easy to show that G is a pseudomonotone bifunction that is not monotone and G satisfies conditions (A1) - (A4) with Bregman Lipschitz-type constants $c_1 = c_2 = \frac{5}{2}$*. We define the mapping* $T : \ell^2 \to \ell^2$ *by* $T(x) = P_C(x)$*, where:*

$$P_C(x) = \begin{cases} \frac{x}{\|x\|}, & \text{if } \|x\| > 1, \\ x, & \text{otherwise.} \end{cases}$$

Then, T is a Bregman relatively nonexpansive mapping. For Algorithm 3.1, we choose

$$\lambda_1 = 0.01, \qquad c_1 = c_2 = \frac{5}{2},$$

and set

$$\delta = 0.9 \min\left\{\frac{1}{c_1}, \frac{1}{c_2}\right\}, \qquad \mu = 0.3\delta.$$

Moreover, we take

$$\theta = 5, \qquad \varepsilon_n = \frac{5}{(n+1)^4}, \qquad \beta_n = 0.5,$$

and select θ_n *according to the rule in Algorithm 3.1. For the numerical implementation, we approximate the infinite-dimensional space* $\ell^2(\mathbb{R})$ *by its finite-dimensional subspace* $\mathbb{R}^{100}$*. The following initial points were considered:*

$$\begin{aligned} x^{(1)} &= \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\right), \\ x^{(2)} &= \left(\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \dots\right), \\ x^{(3)} &= 0.8\left(1, \frac{1}{2}, \frac{1}{3}, \dots\right) / \left\|\left(1, \frac{1}{2}, \frac{1}{3}, \dots\right)\right\|, \\ x^{(4)} &= (0.8, 0, 0, \dots), \\ x^{(5)} &= (0.6, 0.4, 0, \dots). \end{aligned}$$

All initial points belong to the feasible set

$$C = \{x \in \ell^2(\mathbb{R}) : \|x\| \le 1\}.$$

Table 2. Performance of Algorithm 3.1 for different initial points in Example 5.2.

Initial point	*Iterations*	*CPU time (s)*	*Final error*
$x^{(1)}$	*540*	*0.2128*	9.9578×10^{-5}
$x^{(2)}$	*540*	*0.1875*	9.9578×10^{-5}
$x^{(3)}$	*7971*	*2.7866*	9.9254×10^{-5}
$x^{(4)}$	*597*	*0.1983*	9.9699×10^{-5}
$x^{(5)}$	*5919*	*2.0334*	5.9851×10^{-5}

The numerical results demonstrate that Algorithm 3.1 converges for all tested initial points. Although the number of iterations varies significantly depending on the choice of the initial point, the obtained final errors remain of the same order of magnitude. This indicates the robustness of the proposed algorithm with respect to different starting points.

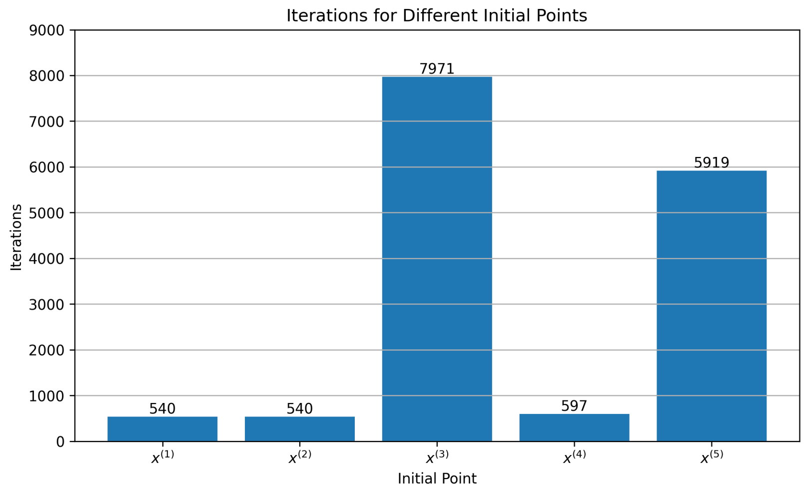

FIGURE 2. Number of iterations required by Algorithm 3.1 for different initial points in Example 5.2.

Figure 2 illustrates the influence of the initial point on the convergence behavior of Algorithm 3.1. Although the number of iterations varies considerably for different starting points, the algorithm converges successfully in all tested cases. The obtained final errors remain below the prescribed tolerance, indicating the robustness of the proposed method.

Acknowledgement: The author would like to express her sincere gratitude to the Faculty of Science and the Center of Excellence in Mathematics, Naresuan University, for their support.

Conflicts of Interest: The author declares that there are no conflicts of interest regarding the publication of this paper.

REFERENCES

[1] R.P. Agarwal, J.-W. Chen, Y.J. Cho, Strong Convergence Theorems for Equilibrium Problems and Weak Bregman Relatively Nonexpansive Mappings in Banach Spaces, J. Inequal. Appl. 2013 (2013), 119. https://doi.org/10.1186/1029-242X-2013-119.

[2] P.N. Anh, A Hybrid Extragradient Method for Pseudomonotone Equilibrium Problems and Fixed Point Problems, Bull. Malays. Math. Sci. Soc. 36 (2013), 107–116. https://math.usm.my/bulletin/pdf/v36n1/v36n1p10.pdf.

[3] H.H. Bauschke, J.M. Borwein, P.L. Combettes, Essential Smoothness, Essential Strict Convexity, and Legendre Functions in Banach Spaces, Commun. Contemp. Math. 3 (2001), 615–647. https://doi.org/10.1142/S0219199701000524.

[4] E. Blum, W. Oettli, From Optimization and Variational Inequalities to Equilibrium Problems, Math. Stud. 63 (1994), 123–145. https://zbmath.org/1090020.

[5] L.M. Bregman, The Relaxation Method of Finding the Common Point of Convex Sets and Its Application to the Solution of Problems in Convex Programming, USSR Comput. Math. Math. Phys. 7 (1967), 200–217. https://doi.org/10.1016/0041-5553(67)90040-7.

[6] D. Butnariu, A.N. Iusem, Totally Convex Functions for Fixed Points Computation and Infinite Dimensional Optimization, Springer Netherlands, 2000. https://doi.org/10.1007/978-94-011-4066-9.

[7] D. Butnariu, E. Resmerita, Bregman Distances, Totally Convex Functions, and a Method for Solving Operator Equations in Banach Spaces, Abstr. Appl. Anal. 2006 (2006), 084919. https://doi.org/10.1155/AAA/2006/84919.

[8] J. Chen, Z. Wan, L. Yuan, Y. Zheng, Approximation of Fixed Points of Weak Bregman Relatively Nonexpansive Mappings in Banach Spaces, Int. J. Math. Math. Sci. 2011 (2011), 420192. https://doi.org/10.1155/2011/420192.

[9] V. Dadashi, O.S. Iyiola, Y. Shehu, The Subgradient Extragradient Method for Pseudomonotone Equilibrium Problems, Optimization 69 (2020), 901–923. https://doi.org/10.1080/02331934.2019.1625899.

[10] P. Daniele, F. Giannessi, A. Maugeri, Equilibrium Problems and Variational Models, Springer, 2003. https://doi.org/10.1007/978-1-4613-0239-1.

[11] G.Z. Eskandani, M. Raeisi, T.M. Rassias, A Hybrid Extragradient Method for Solving Pseudomonotone Equilibrium Problems Using Bregman Distance, J. Fixed Point Theory Appl. 20 (2018), 132. https://doi.org/10.1007/s11784-018-0611-9.

[12] P. Hartman, G. Stampacchia, On Some Non-linear Elliptic Differential-functional Equations, Acta Math. 115 (1966), 271–310. https://doi.org/10.1007/BF02392210.

[13] L.O. Jolaoso, The Subgradient Extragradient Method for Solving Pseudomonotone Equilibrium and Fixed Point Problems in Banach Spaces, Optimization 71 (2022), 4051–4081. https://doi.org/10.1080/02331934.2021.1935935.

[14] L.D. Muu, W. Oettli, Convergence of an Adaptive Penalty Scheme for Finding Constrained Equilibria, Nonlinear Anal. Theory Methods Appl. 18 (1992), 1159–1166. https://doi.org/10.1016/0362-546X(92)90159-C.

[15] E. Naraghirad, J.C. Yao, Bregman Weak Relatively Nonexpansive Mappings in Banach Spaces, Fixed Point Theory Appl. 2013 (2013), 141. https://doi.org/10.1186/1687-1812-2013-141.

[16] R.R. Phelps, Convex Functions, Monotone Operators and Differentiability, Springer Berlin Heidelberg, 1993. https://doi.org/10.1007/978-3-540-46077-0.

[17] S. Reich, S. Sabach, A Strong Convergence Theorem for a Proximal-Type Algorithm in Reflexive Banach Spaces, J. Nonlinear Convex Anal. 10 (2009), 471–485. https://zbmath.org/5653524.

[18] S. Reich, S. Sabach, Two Strong Convergence Theorems for a Proximal Method in Reflexive Banach Spaces, Numer. Funct. Anal. Optim. 31 (2010), 22–44. https://doi.org/10.1080/01630560903499852.

[19] F. Schöpfer, T. Schuster, A.K. Louis, An Iterative Regularization Method for the Solution of the Split Feasibility Problem in Banach Spaces, Inverse Probl. 24 (2008), 055008. https://doi.org/10.1088/0266-5611/24/5/055008.

[20] N. Shahzad, H. Zegeye, Convergence Theorem for Common Fixed Points of a Finite Family of Multi-Valued Bregman Relatively Nonexpansive Mappings, Fixed Point Theory Appl. 2014 (2014), 152. https://doi.org/10.1186/1687-1812-2014-152.

[21] A. Tada, W. Takahashi, Strong Convergence Theorem for an Equilibrium Problem and a Nonexpansive Mapping, in: W. Takahashi, T. Tanaka (Eds.), Nonlinear Analysis and Convex Analysis, Yokohama Publishers, 2007, 609–617. https://zbmath.org/1122.47055.

[22] A. Taiwo, L.O. Jolaoso, O.T. Mewomo, Parallel Hybrid Algorithm for Solving Pseudomonotone Equilibrium and Split Common Fixed Point Problems, Bull. Malays. Math. Sci. Soc. 43 (2020), 1893–1918. https://doi.org/10.1007/s40840-019-00781-1.

[23] D.V. Thong, P. Cholamjiak, M.T. Rassias, Y.J. Cho, Strong Convergence of Inertial Subgradient Extragradient Algorithm for Solving Pseudomonotone Equilibrium Problems, Optim. Lett. 16 (2022), 545–573. https://doi.org/10.1007/s11590-021-01734-z.

[24] J. van Tiel, Convex Analysis: An Introductory Text, Wiley, 1984. ISBN: 978-0-471-90263-8.