International Journal of Analysis and Applications
ISSN 2291-8639
Volume 8, Number 2 (2015), 104-109
http://www.etamaths.com

SOME REMARKS CONCERNING THE JACOBI-DUNKL TRANSFORM IN THE SPACE $L^p(\mathbb{R}, A_{\alpha,\beta}(t)dt)$

R. DAHER, S. EL OUADIH* AND A. BELKHADIR

ABSTRACT. In this paper, using a generalized Jacobi-Dunkl translation operator, we obtain a generalization of Titchmarsh's theorem for the Dunkl transform for functions satisfying the (ϕ, p)-Lipschitz Jacobi-Dunkl condition in the space $L^p(\mathbb{R}, A_{\alpha,\beta}(t)dt), \alpha \geq \beta \geq \frac{-1}{2}, \alpha \neq \frac{-1}{2}$.

1. INTRODUCTION AND PRELIMINARIES

Titchmarsh's [8,Theorem 85] characterized the set of functions in $L^2(\mathbb{R})$ satisfying the Cauchy-Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transform, namely we have

Theorem 1.1. [8] Let $\alpha \in (0, 1)$ and assume that $f \in L^2(\mathbb{R})$. Then the following are equivalents
(a) $\quad \|f(t + h) - f(t)\| = 0(h^\alpha), \quad$ as $\quad h \to 0$
(b) $\quad \displaystyle\int_{|\lambda|\geq r} |\widehat{f}(\lambda)|^2 d\lambda = O(r^{-2\alpha}) \quad$ as $\quad r \to \infty,$
where $\widehat{f}$ stand for the Fourier transform of f.
In this paper, we prove a generalization of Theorem 1.1 for the Jacobi-Dunkl transform for functions satisfying the (ϕ, p)-Lipschitz Jacobi-Dunkl condition in the space $L^p(\mathbb{R}, A_{\alpha,\beta}(t)dt), 1 < p \leq 2$. For this purpose, we use the generalized Jacobi-Dunkl translation operator.
In this section, we recapitulate from [1,2,3,5,6] some results related to the harmonic analysis associated with Jacobi-Dunkl operator $\Lambda_{\alpha,\beta}$.
The Jacobi-Dunkl function with parameters $(\alpha, \beta), \alpha \geq \beta \geq \frac{-1}{2}, \alpha \neq \frac{-1}{2}$, defined by the formula:

$$\forall x \in \mathbb{R}, \psi_\lambda^{\alpha,\beta}(x) = \begin{cases} \varphi_\mu^{\alpha,\beta}(x) - \frac{i}{\lambda}\frac{d}{dx}\varphi_\mu^{\alpha,\beta}(x) & \text{if } \lambda \in \mathbb{C}\backslash\{0\} \\ 1 & \text{if } \lambda = 0 \end{cases}$$

with $\lambda^2 = \mu^2 + \rho^2$, $\rho = \alpha + \beta + 1$ and $\varphi_\mu^{\alpha,\beta}$ is the Jacobi function given by:

$$\varphi_\mu^{\alpha,\beta}(x) = F\left(\frac{\rho + i\mu}{2}, \frac{\rho - i\mu}{2}, \alpha + 1, -(\sinh(x))^2\right),$$

2010 *Mathematics Subject Classification.* 65R10.
Key words and phrases. Jacobi-Dunkl operator, Jacobi-Dunkl transform, generalized Jacobi-Dunkl translation.

F is the Gausse hypergeometric function (see [1,7]).
$\psi_\lambda^{\alpha,\beta}$ is the unique C^∞-solution on $\mathbb{R}$ of the differentiel-difference equation

$$\begin{cases} \Lambda_{\alpha,\beta}\mathcal{U} = i\lambda\mathcal{U} \quad , \lambda \in \mathbb{C} \\ \\ \mathcal{U}(0) = 1 \end{cases}$$

where $\Lambda_{\alpha,\beta}$ is the Jacobi-Dunkl operator given by:

$$\Lambda_{\alpha,\beta}\mathcal{U}(x) = \frac{d\mathcal{U}(x)}{dx} + [(2\alpha+1)\coth x + (2\beta+1)\tanh x] \times \frac{\mathcal{U}(x) - \mathcal{U}(-x)}{2}].$$

The operator $\Lambda_{\alpha,\beta}$ is a particular case of the operator D given by

$$D\mathcal{U}(x) = \frac{d\mathcal{U}(x)}{dx} + \frac{A'(x)}{A(x)} \times \left(\frac{\mathcal{U}(x) - \mathcal{U}(-x)}{2}\right),$$

where $A(x) = |x|^{2\alpha+1}B(x)$, and B a function of class C^∞ on $\mathbb{R}$, even and positive.
The operator $\Lambda_{\alpha,\beta}$ corresponds to the function

$$A(x) = A_{\alpha,\beta}(x) = 2^\rho(\sinh|x|)^{2\alpha+1}(\cosh|x|)^{2\beta+1}.$$

Using the relation

$$\frac{d}{dx}\varphi_\mu^{\alpha,\beta}(x) = -\frac{\mu^2+\rho^2}{4(\alpha+1)}\sinh(2x)\varphi_\mu^{\alpha+1,\beta+1}(x),$$

the function $\psi_\lambda^{\alpha,\beta}$ can be written in the form above (see [2])

$$\forall x \in \mathbb{R}, \psi_\lambda^{\alpha,\beta}(x) = \varphi_\mu^{\alpha,\beta}(x) + i\frac{\lambda}{4(\alpha+1)}\sinh(2x)\varphi_\mu^{\alpha+1,\beta+1}(x).$$

Denote $L^p_{\alpha,\beta}(\mathbb{R}) = L^p_{\alpha,\beta}(\mathbb{R}, A_{\alpha,\beta}(t)dt), 1 < p \leq 2$, the space of measurable functions f on $\mathbb{R}$ such that

$$\|f\|_{p,\alpha,\beta} = \left(\int_{\mathbb{R}} |f(t)|^p A_{\alpha,\beta}(t)dt\right)^{1/p} < +\infty.$$

Using the eigenfunctions $\psi_\lambda^{\alpha,\beta}$ of the operator $\Lambda_{\alpha,\beta}$ called the Jacobi-Dunkl kernels, we define the Jacobi-Dunkl transform by

$$\mathcal{F}_{\alpha,\beta}f(\lambda) = \int_{\mathbb{R}} f(t)\psi_\lambda^{\alpha,\beta}(t)A_{\alpha,\beta}(t)dt, \quad \lambda \in \mathbb{R},$$

and the inversion formula by

$$f(t) = \int_{\mathbb{R}} \mathcal{F}_{\alpha,\beta}f(\lambda)\psi_{-\lambda}^{\alpha,\beta}(t)d\sigma(\lambda),$$

where

$$d\sigma(\lambda) = \frac{|\lambda|}{8\pi\sqrt{\lambda^2-\rho^2}|C_{\alpha,\beta}(\sqrt{\lambda^2-\rho^2})|}\mathbb{I}_{\mathbb{R}\backslash]-\rho,\rho[}(\lambda)d\lambda.$$

Here,

$$C_{\alpha,\beta}(\mu) = \frac{2^{\rho-i\mu}\Gamma(\alpha+1)\Gamma(i\mu)}{\Gamma(\frac{1}{2}(\rho+i\mu))\Gamma(\frac{1}{2}(\alpha-\beta+1+i\mu))}, \quad \mu \in \mathbb{C}\backslash(i\mathbb{N})$$

and $\mathbb{I}_{\mathbb{R}\backslash]-\rho,\rho[}$ is the characteristic function of $\mathbb{R}\backslash]-\rho,\rho[$.
The Jacobi-Dunkl transform is a unitary isomorphism from $L^2_{\alpha,\beta}(\mathbb{R})$ onto $L^2(\mathbb{R}, d\sigma(\lambda))$, i.e.

$$\|f\|_{2,\alpha,\beta} = \|\mathcal{F}_{\alpha,\beta}(f)\|_{L^2(\mathbb{R},d\sigma(\lambda))}. \tag{1}$$

Plancherel's theorem (1) and the Marcinkiewics interpolation theorem (see [8]) we get for $f \in L^p_{\alpha,\beta}(\mathbb{R})$ with $1 < p \leq 2$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$,

$$\|\mathcal{F}_{\alpha,\beta}(f)\|_{L^q(\mathbb{R},d\sigma(\lambda))} \leq K\|f\|_{p,\alpha,\beta}, \tag{2}$$

where K is a positive constant (see [6]).
The operator of Jacobi-Dunkl translation is defined by

$$T_x f(y) = \int_{\mathbb{R}} f(z) d\nu^{\alpha,\beta}_{x,y}(z), \quad \forall x, y \in \mathbb{R}$$

where $\nu^{\alpha,\beta}_{x,y}(z), x, y \in \mathbb{R}$ are the signed measures given by

$$d\nu^{\alpha,\beta}_{x,y}(z) = \begin{cases} K_{\alpha,\beta}(x,y,z)A_{\alpha,\beta}(z)dz & \text{if } x, y \in \mathbb{R}^* \\ \\ \delta_x & \text{if } y = 0 \\ \delta_y & \text{if } x = 0 \end{cases}$$

Here, δ_x is the Dirac measure at x. And,

$$\begin{aligned} K_{\alpha,\beta}(x,y,z) &= M_{\alpha,\beta}(\sinh(|x|)\sinh(|y|)\sinh(|z|))^{-2\alpha}\mathbb{I}_{I_{x,y}} \times \int_0^\pi \rho_\theta(x,y,z) \\ &\times \ (g_\theta(x,y,z))_+^{\alpha-\beta-1}\sin^{2\beta}\theta d\theta \\ I_{x,y} &= [-|x|-|y|, -||x|-|y||] \cup [||x|-|y||, |x|+|y|] \\ \rho_\theta(x,y,z) &= 1 - \sigma^\theta_{x,y,z} + \sigma^\theta_{z,x,y} + \sigma^\theta_{z,y,x} \end{aligned}$$

$$\forall z \in \mathbb{R}, \theta \in [0,\pi], \sigma^\theta_{x,y,z} = \begin{cases} \frac{\cosh(x)+\cosh(y)-\cosh(z)\cos(\theta)}{\sinh(x)\sinh(y)} & \text{,if } xy \neq 0 \\ \\ 0 & \text{,if } xy = 0 \end{cases}$$

$$g_\theta(x,y,z) = 1 - \cosh^2(x) - \cosh^2(y) - \cosh^2(z) + 2\cosh(x)\cosh(y)\cosh(z)\cos\theta$$

$$t_+ = \begin{cases} t & \text{,if } t > 0 \\ \\ 0 & \text{,if } t \leq 0 \end{cases}$$

and,

$$M_{\alpha,\beta} = \begin{cases} \frac{2^{-2\rho}\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})} & \text{,if } \alpha > \beta \\ \\ 0 & \text{,if } \alpha = \beta \end{cases}$$

In [2], we have

$$\mathcal{F}_{\alpha,\beta}(T_h f)(\lambda) = \psi^{\alpha,\beta}_\lambda(h)\mathcal{F}_{\alpha,\beta}(f)(\lambda). \tag{3}$$

For $\alpha \geq \frac{-1}{2}$, we introduce the Bessel normalized function of the first kind defined by

$$j_\alpha(z) = \Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{z}{2})^{2n}}{n!\Gamma(n+\alpha+1)}, \quad z \in \mathbb{C}.$$

Moreover, we see that

$$\lim_{z\to 0}\frac{j_\alpha(z)-1}{z^2} \neq 0,$$

by consequence, there exists $C_1 > 0$ and $\eta > 0$ satisfying

$$|z| \leq \eta \Rightarrow |j_\alpha(z) - 1| \geq C_1|z|^2. \tag{4}$$

Lemma 1.1. Let $\alpha \geq \beta \geq \frac{-1}{2}, \alpha \neq \frac{-1}{2}$. Then for $|\nu| \leq \rho$, there exists a positive constant C_2 such that

$$|1-\varphi_{\mu+i\nu}^{\alpha,\beta}(t)| \geq C_2|1-j_\alpha(\mu t)|.$$

Proof. (See[4],Lemma 9).

2. MAIN RESULT

In this section we give the main result of this paper. We need first to define (ϕ, p)-Lipschitz Jacobi-Dunkl class.
Denote N_h by

$$N_h = T_h + T_{-h} - 2I$$

where I is the unit operator in the space $L^p_{\alpha,\beta}(\mathbb{R})$.

Definition 2.1. A function $f \in L^p_{\alpha,\beta}(\mathbb{R})$ is said to be in (ϕ, p)-Lipschitz Jacobi-Dunkl class, denoted by $Lip(\phi, p, \alpha, \beta)$, if

$$\|N_h f\|_{p,\alpha,\beta} = O(\phi(h)), \quad as \quad h \to 0,$$

where $\phi(t)$ is a continuous increasing function on $[0, \infty)$, $\phi(0) = 0$ and $\phi(ts) = \phi(t)\phi(s)$ for all $t, s \in [0, \infty)$.

Lemma 2.2. For $f \in L^p_{\alpha,\beta}(\mathbb{R})$, then

$$\left(\int_{\mathbb{R}} 2^q|\varphi_\mu^{\alpha,\beta}(h)-1|^q|\mathcal{F}_{\alpha,\beta}f(\lambda)|^q d\sigma(\lambda)\right)^{\frac{1}{q}} \leq K\|N_h f\|_{p,\alpha,\beta}$$

where $\frac{1}{p}+\frac{1}{q}=1$.
Proof. We us formula (3), we conclude that

$$\mathcal{F}_{\alpha,\beta}(N_h f)(\lambda) = (\psi_\lambda^{\alpha,\beta}(h) + \psi_\lambda^{\alpha,\beta}(-h) - 2)\mathcal{F}_{\alpha,\beta}(f)(\lambda),$$

Since

$$\psi_\lambda^{\alpha,\beta}(h) = \varphi_\mu^{\alpha,\beta}(h) + i\frac{\lambda}{4(\alpha+1)}\sinh(2h)\varphi_\mu^{\alpha+1,\beta+1}(h),$$

$$\psi_\lambda^{\alpha,\beta}(-h) = \varphi_\mu^{\alpha,\beta}(-h) - i\frac{\lambda}{4(\alpha+1)}\sinh(2h)\varphi_\mu^{\alpha+1,\beta+1}(-h),$$

and $\varphi_\mu^{\alpha,\beta}$ is even (see [2]), then

$$\mathcal{F}_{\alpha,\beta}(N_h f)(\lambda) = 2(\varphi_\mu^{\alpha,\beta}(h)-1)\mathcal{F}_{\alpha,\beta}(f)(\lambda).$$

By formula (2), we have the result.

Theorem 2.3. Let $f(x)$ belong to $Lip(\phi, p, \alpha, \beta)$. Then

$$\int_{|\lambda|\geq r} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) = O(\phi(r^{-q})), \quad as \quad r \to \infty,$$

where $\frac{1}{p}+\frac{1}{q}=1$.
Proof. Assume that $f \in Lip(\phi, p, \alpha, \beta)$, then we have

$$\|N_h f\|_{p,\alpha,\beta} = O(\phi(h)), \quad as \quad h \to 0.$$

From Lemma 2.2, we have

$$\int_{\mathbb{R}} |\varphi_\mu^{\alpha,\beta}(h)-1|^q|\mathcal{F}_{\alpha,\beta}f(\lambda)|^q d\sigma(\lambda) \le \frac{K^q}{2^q}\|N_h f\|_{p,\alpha,\beta}^q$$

By (4) and Lemma 1.1, we get

$$\int_{\frac{\eta}{2h}\le|\lambda|\le\frac{\eta}{h}} |1-\varphi_\mu^{\alpha,\beta}(h)|^q|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) \ge C_1^q C_2^q \int_{\frac{\eta}{2h}\le|\lambda|\le\frac{\eta}{h}} |\mu h|^{2q}|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda).$$

From $\frac{\eta}{2h} \le |\lambda| \le \frac{\eta}{h}$ we have

$$\begin{aligned}\left(\frac{\eta}{2h}\right)^2 &- \rho^2 \le \mu^2 \le \left(\frac{\eta}{h}\right)^2 - \rho^2 \\ &\Rightarrow \mu^2 h^2 \ge \frac{\eta^2}{4} - \rho^2 h^2.\end{aligned}$$

Take $h \le \frac{\eta}{3\rho}$, then we have $\mu^2 h^2 \ge C_3 = C_3(\eta)$.
So,

$$\int_{\frac{\eta}{2h}\le|\lambda|\le\frac{\eta}{h}} |1-\varphi_\mu^{\alpha,\beta}(h)|^q|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) \ge C_1^q C_2^q C_3^q \int_{\frac{\eta}{2h}\le|\lambda|\le\frac{\eta}{h}} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda).$$

There exists then a positives constants C and K_1 such that

$$\begin{aligned}\int_{\frac{\eta}{2h}\le|\lambda|\le\frac{\eta}{h}} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) &\le C\int_{\mathbb{R}} |1-\varphi_\mu^{\alpha,\beta}(h)|^q|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) \\ &\le K_1\phi^q(h) = K_1\phi(h^q).\end{aligned}$$

For all $0 < h < \frac{\eta}{3\rho}$. Then we have

$$\int_{r\le|\lambda|\le 2r} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) \le K_2\phi(r^{-q}), \quad r\to\infty,$$

where $K_2 = K_1\phi(\eta^q 2^{-q})$.
Furthermore, we obtain

$$\begin{aligned}&\int_{|\lambda|\ge r} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) \\ =& \left(\int_{r\le|\lambda|\le 2r} + \int_{2r\le|\lambda|\le 4r} + \int_{4r\le|\lambda|\le 8r} + \cdots\right)|\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) \\ \le& K_2\phi(r^{-q}) + K_2\phi((2r)^{-q}) + K_2\phi((4r)^{-q}) + \cdots \\ \le& K_2\phi(r^{-q}) + K_2\phi(2^{-q})\phi(r^{-q}) + K_2\phi((2^{-q})^2)\phi(r^{-q}) + \cdots \\ \le& K_2\phi(r^{-q})(1+\phi(2^{-q})+\phi((2^{-q})^2)+\cdots).\end{aligned}$$

We have $\phi(2^{-q}) < 1$, then

$$\int_{|\lambda|\ge r} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) \le K_3\phi(r^{-q}),$$

where $K_3 = K_2(1-\phi(2^{-q}))^{-1}$.
Finally, we get

$$\int_{|\lambda|\ge r} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) = O(\phi(r^{-q})), \quad as \quad r\to\infty.$$

Thus, the proof is finished.

References

[1] Ben Mohamed. H and Mejjaoli. H, Distributional Jacobi-Dunkl transform and applications, Afr.Diaspora J.Math 1(2004), 24-46.

[2] Ben Mohamed. H, The Jacobi-Dunkl transform on $\mathbb{R}$ and the convolution product on new space of distributions, Ramanujan J.21(2010), 145-175..

[3] Ben Salem. N and Ahmed Salem. A , Convolution structure associated with the Jacobi-Dunkl operator on $\mathbb{R}$, Ramanuy J.12(3) (2006), 359-378.

[4] Bray. W. O and Pinsky. M. A, Growth properties of Fourier transforms via module of continuity , Journal of Functional Analysis.255(288), 2256-2285.

[5] Chouchane. F, Mili. M and Trimche. K, Positivity of the intertwining operator and harmonic analysis associated with the Jacobi-Dunkl operator on $\mathbb{R}$, J.Anal. Appl.1(4)(2003), 387-412.

[6] Johansen. T. R, Remarks on the inverse cherednik-opdam transform on the real line, arxiv: 1502.01293v1 (2015).

[7] Koornwinder. T. H , A new proof of a Paley-Wiener type theorems for the Jacobi transform , Ark.Math.13(1975), 145-159.

[8] Titchmarsh. E. C, Introduction to the theory of Fourier integrals , Claredon, Oxford. (1948), Komkniga, Moscow, (2005).

Departement of Mathematics, Faculty of Sciences Aïn Chock, University Hassan II, Casablanca, Morocco

*Corresponding author