

Qualitative Analysis of Positive Solutions for Coupled Atangana-Baleanu-Caputo Fractional Systems with Nonlocal Integral Conditions

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Abstract. This article investigates a coupled system of nonlinear FDEs involving the ABC fractional derivative with IBCs. By employing the upper and lower solution approach in conjunction with Schauder's FPT, we establish existence of positive solutions without imposing any monotonicity restrictions on the nonlinear terms. The uniqueness of solution is demonstrated via the Banach contraction principle under suitable Lipschitz conditions. Furthermore, we analyze the stability properties of the obtained solutions using the concepts of Ulam-Hyers (UH), Generalized UH (GUH) and UH-Rassias (UHR) stability. Two numerical examples are provided to illustrate theoretical results. The proposed framework is particularly useful for handling non-monotone nonlinearities in coupled ABC fractional systems with nonlocal boundary conditions.

1. INTRODUCTION

FDEs have been extensively studied in recent times mainly due to the ability to model systems whose states depend on history which is known as memory and hereditary effects in scientific literature. As such properties are inherent in nature and man-made systems, FDEs find applications in various fields including but not limited to physics and engineering, and other fields such as biology, chemistry and quantitative finance. For further details on theory of FDEs and their applications, see the pioneering works of Hilfer, Kilbas, Miller, Podlubny, and Samko [1–5]. As the nonlocal nature of FDEs allows us to describe many physical processes such as anomalous

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diffusion and viscoelasticity better, as well as many other areas such as signal processing and control theory [4,6,7].

Even though many types of fractional derivative definitions have been introduced, Caputo and Riemann-Liouville derivatives are still prevailing due to their mathematical robustness and wide range of applications in different fields of science. However, both types of definition include singular kernels, which can be a restriction in some practical cases when a nonsingular one is needed [8]. Therefore, in order to solve this problem, Atangana and Baleanu (AB) proposed a new kind of fractional derivative with the help of Mittag-Leffler kernel, which is called AB derivative [9]. Within the scope of the Caputo type (AB derivative) the kernel is nonsingular and has such features as a constant derivative equal to zero and fractional integral that returns the initial function with respect to its initial condition [9,10]

There have been many studies on the uniqueness and existence of positive solutions of fractional boundary value problems (BVPs), especially integral boundary value problems, using classical derivatives Nan and Wang [14] studied the following nonlinear FDE with Riemann-Liouville derivative:

$$\begin{cases} \mathfrak{D}^{\aleph} \omega_1(\wp) = \mathfrak{f}(\wp, \omega_1(\wp)), & \wp \in [0, 1], \\ \omega_1(0) = 0, \end{cases}$$

where $0 < \aleph < 1$, $\mathfrak{D}^{\aleph}$ denotes the Riemann-Liouville derivative, & $\mathfrak{f} : [0, 1] \times \mathcal{R}^+ \rightarrow \mathcal{R}^+$ is continuous. Using the upper-lower solution technique and Schauder's FPT, they showed that positive solutions exist without imposing any monotonicity condition on the nonlinearity.

Subsequently, Wang et al. [15] extended the analysis to fractional differential equations (FDEs) with integral boundary conditions (IBCs):

$$\begin{cases} \mathfrak{D}^{\aleph} \omega_1(\wp) = \mathfrak{f}(\wp, \omega_1(\wp)), & \wp \in [0, \mathfrak{T}], \\ \omega_1(0) = \chi \int_0^T \omega_1(\zeta) \mathfrak{d}\zeta + \mathfrak{c}, \end{cases}$$

where $0 < \aleph < 1$, $\chi \geq 0$, and $\mathfrak{f} : [0, \mathfrak{T}] \times \mathcal{R} \rightarrow \mathcal{R}$ is continuous. The proof used the upper and lower solutions technique along with monotone iterative technique. Uniqueness and existence of solutions to the same problem was proved by Nanware and Dhaigude [16] without assuming any local Hölder continuity condition.

In the past few years, coupled systems of FDEs have gained a lot of attention owing to their importance in describing real-world situations involving interacting subsystems having memory property [17–23]. Coupled systems can occur in many practical situations such as chemical reactions, biological systems, epidemiological models, and coupled oscillators [24–28]. In analysis of coupled fractional order systems, more mathematical complications arise than in case of scalar fractional order systems, mainly in proving uniqueness, existence and stability of solutions.

Motivated by these developments and the growing interest in ABC fractional derivatives, we investigate following coupled system of nonlinear FDEs involving the ABC derivative

$$\begin{cases} {}^{\text{ABC}}\mathfrak{D}_{0+}^{\aleph} \omega_1(\wp) = \mathfrak{f}(\wp, \omega_1(\wp), \omega_2(\wp)), & \wp \in [0, 1], \\ {}^{\text{ABC}}\mathfrak{D}_{0+}^{\aleph} \omega_2(\wp) = \mathfrak{g}(\wp, \omega_1(\wp), \omega_2(\wp)), & \wp \in [0, 1], \end{cases} \quad (1.1)$$

with integral boundary conditions:

$$\omega_1(0) = \chi_1 \int_0^1 \omega_1(\zeta) \mathfrak{d}\zeta + \mathfrak{c}_1, \quad \omega_2(0) = \chi_2 \int_0^1 \omega_2(\zeta) \mathfrak{d}\zeta + \mathfrak{c}_2, \quad (1.2)$$

where $0 < \aleph, \aleph < 1$, $\chi_1, \chi_2 \geq 0$, $\mathfrak{c}_1, \mathfrak{c}_2 \in \mathcal{R}^+$, and the nonlinearities $\mathfrak{f}, \mathfrak{g} : [0, 1] \times \mathcal{R}^+ \times \mathcal{R}^+ \rightarrow \mathcal{R}^+$ are continuous functions.

The novelty of our study in the integration of different features.

- The application of the ABC nonsingular kernel in the setting of a coupled system provides an edge over the classical singular kernel operator due to some of its properties.
- The use of the integral type boundary conditions of the type $\omega_1(0) = \chi_1 \int_0^1 \omega_1(\zeta) \mathfrak{d}\zeta + \mathfrak{c}_1$ and $\omega_2(0) = \chi_2 \int_0^1 \omega_2(\zeta) \mathfrak{d}\zeta + \mathfrak{c}_2$, which include nonlocal conditions as particular cases;
- The development of lower and upper control functions without imposing the condition of monotonicity on the nonlinear terms $\mathfrak{f}$ and $\mathfrak{g}$, which generalizes the method developed in [14] for the coupled ABC case;
- Stability analysis through the use of the UH, GUH and UHR theory.

Unlike earlier research concerning ABC equations, which generally demand the use of monotonicity or certain growth conditions [29–35], our technique is more generalized. In addition, whereas recent literature [36–42] has mainly concentrated on the use of FPT approaches under Lipschitz or compactness assumptions for individual equations, we have developed lower and upper solutions without relying on monotonicity in systems.

The outline of the article is given below. In Section 2, some definitions and lemmas of ABC FC are stated. Section 3 presents the main results: integral representation, compactness of the operator, existence via upper/lower solutions, and uniqueness via contraction. We also establish UH, GUH and UHR stability of the obtained solution in Section 4. Section 5 provides two illustrative examples that confirm the theoretical findings. Finally, Section 6 draws conclusions and outlines possible future extensions.

2. PRELIMINARIES

This section introduces basic concepts, definitions, & preliminary results concerning fractional calculus (FC) and FPTs, which serve as the foundation for the developments presented in this paper.

Definition 2.1. [2, 4] For $\mathfrak{g} \in \mathcal{C}[0, 1]$, the Riemann-Liouville integral of order $\aleph > 0$ is expressed by

$$I^{\aleph} \mathfrak{g}(\wp) = \frac{1}{\Gamma(\aleph)} \int_0^{\wp} (\wp - \zeta)^{\aleph-1} \mathfrak{g}(\zeta) \mathfrak{d}\zeta, \quad \wp > 0,$$

where Γ denotes the Gamma function.

Definition 2.2. [2, 4] Let $g : [0, 1] \rightarrow \mathcal{R}$ be a given function and assume that $n - 1 < \aleph < n$. Then, the Riemann-Liouville derivative of order $\aleph$ is defined by

$$\mathfrak{D}^{\aleph} g(\wp) = \frac{\mathfrak{d}^n}{\mathfrak{d}\wp^n} I^{n-\aleph} g(\wp), \quad \wp > 0,$$

where $n = [\aleph] + 1$ and $[\aleph]$ denotes the integer part of $\aleph$.

Definition 2.3. [9] Let $0 < \aleph < 1$. The ABC fractional derivative of order $\aleph$ of a function $g \in \mathcal{C}^1[0, 1]$ is defined by

$${}^{\text{ABC}}\mathfrak{D}^{\aleph} g(\wp) = \frac{B(\aleph)}{1-\aleph} \int_0^{\wp} E_{\aleph} \left(-\frac{\aleph}{1-\aleph} (\wp - \zeta)^{\aleph} \right) g'(\zeta) \mathfrak{d}\zeta, \quad \wp > 0,$$

where $B(\aleph) > 0$ is a normalization function satisfying $B(0) = B(1) = 1$, and

$$E_{\aleph}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\aleph k + 1)}$$

is the Mittag-Leffler function.

The corresponding ABC fractional integral is

$${}^{\text{AB}}I^{\aleph} g(\wp) = \frac{1-\aleph}{B(\aleph)} g(\wp) + \frac{\aleph}{B(\aleph)\Gamma(\aleph)} \int_0^{\wp} (\wp - \zeta)^{\aleph-1} g(\zeta) \mathfrak{d}\zeta, \quad \wp > 0.$$

Lemma 2.1. [9] Let $g \in \mathcal{C}^1[0, 1]$ and $0 < \aleph < 1$. Then

$${}^{\text{AB}}I^{\aleph} \left({}^{\text{ABC}}\mathfrak{D}^{\aleph} g(\wp) \right) = g(\wp) - g(0),$$

and

$${}^{\text{ABC}}\mathfrak{D}^{\aleph} \left({}^{\text{AB}}I^{\aleph} g(\wp) \right) = g(\wp).$$

Theorem 2.1. [Banach Contraction Principle] [43] Let $(\mathfrak{X}, \mathfrak{d})$ be a complete metric space and suppose that $\mathfrak{P} : \mathfrak{X} \rightarrow \mathfrak{X}$ is a contraction. Then, by Banach contraction principle, $\exists$ a unique FP x in $\mathfrak{X}$ for which

$$\mathfrak{P}(x) = x.$$

Theorem 2.2 (Schauder FPT). [44] Let $\mathfrak{X}$ be a closed, bounded and convex subset of Banach space. If $\mathfrak{P} : \mathfrak{N} \rightarrow \mathfrak{N}$ is a continuous map s.t. the set $\{\mathfrak{P}x : x \in \mathfrak{N}\}$ is relatively compact in $\mathfrak{X}$, then $\mathfrak{P}$ has at least one FP.

Theorem 2.3. (Arzelà–Ascoli Theorem [45]). A subset of $\mathcal{C}[0, 1]$ is relatively compact if and only if it is equicontinuous and uniformly bounded.

Definition 2.4. A pair of functions $(\omega_1, \omega_2) \in \mathcal{C}[0, 1] \cap \mathcal{C}^1[0, 1] \times \mathcal{C}[0, 1] \cap \mathcal{C}^1[0, 1]$ is called a solution of the coupled system (1.1)-(1.2) if it satisfies both ABC differential equations and the integral boundary conditions.

Definition 2.5. A solution (ω_1, ω_2) of the coupled system (1.1)-(1.2) is called a positive solution if $\omega_1(\wp) \geq 0$ $\omega_2(\wp) \geq 0$ for all $\wp \in [0, 1]$.

Definition 2.6. [14] Consider real numbers a_1, a_2, b_1, b_2 satisfying $a_1 < b_1$ and $a_2 < b_2$. For any $(\omega_1, \omega_2) \in [a_1, b_1] \times [a_2, b_2]$, the upper-control functions are defined as

$$\mathcal{H}_f(\varphi, \omega_1, \omega_2) = \sup_{a_1 \leq \xi \leq \omega_1, a_2 \leq \eta \leq \omega_2} f(\varphi, \xi, \eta),$$

$$\mathcal{H}_g(\varphi, \omega_1, \omega_2) = \sup_{a_1 \leq \xi \leq \omega_1, a_2 \leq \eta \leq \omega_2} g(\varphi, \xi, \eta),$$

and the lower-control functions as

$$h_f(\varphi, \omega_1, \omega_2) = \inf_{\omega_1 \leq \xi \leq b_1, \omega_2 \leq \eta \leq b_2} f(\varphi, \xi, \eta),$$

$$h_g(\varphi, \omega_1, \omega_2) = \inf_{\omega_1 \leq \xi \leq b_1, \omega_2 \leq \eta \leq b_2} g(\varphi, \xi, \eta).$$

Obviously, $\mathcal{H}_f(\varphi, \omega_1, \omega_2)$, $\mathcal{H}_g(\varphi, \omega_1, \omega_2)$, $h_f(\varphi, \omega_1, \omega_2)$, and $h_g(\varphi, \omega_1, \omega_2)$ are monotonically non-decreasing in ω_1 and ω_2 , and

$$h_f(\varphi, \omega_1, \omega_2) \leq f(\varphi, \omega_1, \omega_2) \leq \mathcal{H}_f(\varphi, \omega_1, \omega_2),$$

$$h_g(\varphi, \omega_1, \omega_2) \leq g(\varphi, \omega_1, \omega_2) \leq \mathcal{H}_g(\varphi, \omega_1, \omega_2).$$

Definition 2.7. Let $(\widetilde{\omega}_1, \widetilde{\omega}_2), (\underline{\omega}_1, \underline{\omega}_2) \in \mathcal{K}$ with

$$\underline{\omega}_1(\varphi) \leq \widetilde{\omega}_1(\varphi), \quad \underline{\omega}_2(\varphi) \leq \widetilde{\omega}_2(\varphi), \quad \varphi \in [0, 1],$$

satisfy

$${}^{ABC}\mathfrak{D}^{\aleph} \widetilde{\omega}_1(\varphi) \geq \mathcal{H}_f(\varphi, \widetilde{\omega}_1(\varphi), \widetilde{\omega}_2(\varphi)), \quad \varphi \in [0, 1],$$

$${}^{ABC}\mathfrak{D}^{\aleph} \widetilde{\omega}_2(\varphi) \geq \mathcal{H}_g(\varphi, \widetilde{\omega}_1(\varphi), \widetilde{\omega}_2(\varphi)), \quad \varphi \in [0, 1],$$

$$\widetilde{\omega}_1(0) \geq \chi_1 \int_0^1 \widetilde{\omega}_1(\zeta) d\zeta + c_1, \quad \widetilde{\omega}_2(0) \geq \chi_2 \int_0^1 \widetilde{\omega}_2(\zeta) d\zeta + c_2,$$

or equivalently,

$$\begin{aligned} \widetilde{\omega}_1(\varphi) &\geq \frac{1-\aleph}{B(\aleph)} \mathcal{H}_f(\varphi, \widetilde{\omega}_1(\varphi), \widetilde{\omega}_2(\varphi)) + \frac{\aleph}{B(\aleph)\Gamma(\aleph)} \int_0^\varphi (\varphi - \zeta)^{\aleph-1} \mathcal{H}_f(\zeta, \widetilde{\omega}_1(\zeta), \widetilde{\omega}_2(\zeta)) d\zeta \\ &\quad + \chi_1 \int_0^1 \widetilde{\omega}_1(\zeta) d\zeta + c_1, \end{aligned}$$

$$\begin{aligned} \widetilde{\omega}_2(\varphi) &\geq \frac{1-\aleph}{B(\aleph)} \mathcal{H}_g(\varphi, \widetilde{\omega}_1(\varphi), \widetilde{\omega}_2(\varphi)) + \frac{\aleph}{B(\aleph)\Gamma(\aleph)} \int_0^\varphi (\varphi - \zeta)^{\aleph-1} \mathcal{H}_g(\zeta, \widetilde{\omega}_1(\zeta), \widetilde{\omega}_2(\zeta)) d\zeta \\ &\quad + \chi_2 \int_0^1 \widetilde{\omega}_2(\zeta) d\zeta + c_2, \end{aligned}$$

and

$${}^{ABC}\mathfrak{D}^{\aleph} \underline{\omega}_1(\varphi) \leq h_f(\varphi, \underline{\omega}_1(\varphi), \underline{\omega}_2(\varphi)), \quad \varphi \in [0, 1],$$

$${}^{ABC}\mathfrak{D}^{\aleph} \underline{\omega}_2(\varphi) \leq h_g(\varphi, \underline{\omega}_1(\varphi), \underline{\omega}_2(\varphi)), \quad \varphi \in [0, 1],$$

$$\underline{\omega}_1(0) \leq \chi_1 \int_0^1 \underline{\omega}_1(\zeta) d\zeta + c_1, \quad \underline{\omega}_2(0) \leq \chi_2 \int_0^1 \underline{\omega}_2(\zeta) d\zeta + c_2,$$

or equivalently,

$$\begin{aligned}\underline{\omega}_1(\wp) &\leq \frac{1-\aleph}{\mathbf{B}(\aleph)} \mathfrak{h}_{\mathfrak{f}}(\wp, \underline{\omega}_1(\wp), \underline{\omega}_2(\wp)) + \frac{\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph)} \int_0^{\wp} (\wp - \zeta)^{\aleph-1} \mathfrak{h}_{\mathfrak{f}}(\zeta, \underline{\omega}_1(\zeta), \underline{\omega}_2(\zeta)) \mathfrak{d}\zeta \\ &\quad + \chi_1 \int_0^1 \underline{\omega}_1(\zeta) \mathfrak{d}\zeta + \mathfrak{c}_1, \\ \underline{\omega}_2(\wp) &\leq \frac{1-\aleph}{\mathbf{B}(\aleph)} \mathfrak{h}_{\mathfrak{g}}(\wp, \underline{\omega}_1(\wp), \underline{\omega}_2(\wp)) + \frac{\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph)} \int_0^{\wp} (\wp - \zeta)^{\aleph-1} \mathfrak{h}_{\mathfrak{g}}(\zeta, \underline{\omega}_1(\zeta), \underline{\omega}_2(\zeta)) \mathfrak{d}\zeta \\ &\quad + \chi_2 \int_0^1 \underline{\omega}_2(\zeta) \mathfrak{d}\zeta + \mathfrak{c}_2.\end{aligned}$$

Then the pair $(\widetilde{\omega}_1, \widetilde{\omega}_2)$ is called an upper solution and the pair $(\underline{\omega}_1, \underline{\omega}_2)$ is called a lower solution of the coupled system (1.1)-(1.2).

3. MAIN RESULTS

This section is devoted to analysis of existence and uniqueness for the coupled system (1.1)-(1.2). To facilitate the proofs of the main results, we begin with the following lemma.

Lemma 3.1. Assume that $\mathfrak{f}(\wp, \omega_1(\wp), \omega_2(\wp)), \mathfrak{g}(\wp, \omega_1(\wp), \omega_2(\wp)) \in \mathfrak{C}([0, 1], \mathcal{R}^+)$. Then $(\omega_1, \omega_2) \in \mathfrak{C}[0, 1] \cap \mathfrak{C}^1[0, 1] \times \mathfrak{C}[0, 1] \cap \mathfrak{C}^1[0, 1]$ is a solution of the coupled BVP (1.1)-(1.2) if and only if (ω_1, ω_2) is a solution of the integral equations

$$\begin{aligned}\omega_1(\wp) &= \frac{1-\aleph}{\mathbf{B}(\aleph)} \mathfrak{f}(\wp, \omega_1(\wp), \omega_2(\wp)) + \frac{\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph)} \int_0^{\wp} (\wp - \zeta)^{\aleph-1} \mathfrak{f}(\zeta, \omega_1(\zeta), \omega_2(\zeta)) \mathfrak{d}\zeta \\ &\quad + \chi_1 \int_0^1 \omega_1(\zeta) \mathfrak{d}\zeta + \mathfrak{c}_1,\end{aligned}\tag{3.1}$$

$$\begin{aligned}\omega_2(\wp) &= \frac{1-\aleph}{\mathbf{B}(\aleph)} \mathfrak{g}(\wp, \omega_1(\wp), \omega_2(\wp)) + \frac{\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph)} \int_0^{\wp} (\wp - \zeta)^{\aleph-1} \mathfrak{g}(\zeta, \omega_1(\zeta), \omega_2(\zeta)) \mathfrak{d}\zeta \\ &\quad + \chi_2 \int_0^1 \omega_2(\zeta) \mathfrak{d}\zeta + \mathfrak{c}_2.\end{aligned}\tag{3.2}$$

Proof. Suppose (ω_1, ω_2) is a solution of coupled system (1.1)-(1.2). Applying the integral operator ${}^{\text{AB}}\mathcal{I}^{\aleph}$ to both sides of the eqn in (1.1) yields

$${}^{\text{AB}}\mathcal{I}^{\aleph}({}^{\text{ABC}}\mathfrak{D}^{\aleph}\omega_1(\wp)) = {}^{\text{AB}}\mathcal{I}^{\aleph}\mathfrak{f}(\wp, \omega_1(\wp), \omega_2(\wp)).$$

By Lemma 2.1 and the integral boundary condition (1.2), we obtain

$$\omega_1(\wp) - \omega_1(0) = \frac{1-\aleph}{\mathbf{B}(\aleph)} \mathfrak{f}(\wp, \omega_1(\wp), \omega_2(\wp)) + \frac{\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph)} \int_0^{\wp} (\wp - \zeta)^{\aleph-1} \mathfrak{f}(\zeta, \omega_1(\zeta), \omega_2(\zeta)) \mathfrak{d}\zeta.$$

Substituting $\omega_1(0) = \chi_1 \int_0^1 \omega_1(\zeta) \mathfrak{d}\zeta + \mathfrak{c}_1$ yields (3.1). Similarly, applying ${}^{\text{AB}}\mathcal{I}^{\aleph}$ to the second equation in (1.1) and using the boundary condition for ω_2 yields (3.2).

Conversely, suppose (ω_1, ω_2) satisfies the integral equations (3.1)-(3.2). Using Definition 2.3 together with Lemma 2.1, we obtain for $\wp \in [0, 1]$:

$$\begin{aligned} & {}^{\text{ABC}}\mathfrak{D}^{\aleph}\omega_1(\wp) \\ &= {}^{\text{ABC}}\mathfrak{D}^{\aleph}\left(\frac{1-\aleph}{\mathbf{B}(\aleph)}\mathfrak{f}(\wp, \omega_1(\wp), \omega_2(\wp)) + \frac{\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph)} \int_0^{\wp} (\wp - \zeta)^{\aleph-1} \mathfrak{f}(\zeta, \omega_1(\zeta), \omega_2(\zeta)) \mathfrak{d}\zeta \right. \\ &\quad \left. + \chi_1 \int_0^1 \omega_1(\zeta) \mathfrak{d}\zeta + \mathbf{c}_1\right) \\ &= {}^{\text{ABC}}\mathfrak{D}^{\aleph}\left(\frac{\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph)} \int_0^{\wp} (\wp - \zeta)^{\aleph-1} \mathfrak{f}(\zeta, \omega_1(\zeta), \omega_2(\zeta)) \mathfrak{d}\zeta\right) + 0 \\ &= \mathfrak{f}(\wp, \omega_1(\wp), \omega_2(\wp)). \end{aligned}$$

Similarly,

$${}^{\text{ABC}}\mathfrak{D}^{\aleph}\omega_2(\wp) = \mathfrak{g}(\wp, \omega_1(\wp), \omega_2(\wp)).$$

Moreover, the integral boundary conditions $\omega_1(0) = \chi_1 \int_0^1 \omega_1(\zeta) \mathfrak{d}\zeta + \mathbf{c}_1$ and $\omega_2(0) = \chi_2 \int_0^1 \omega_2(\zeta) \mathfrak{d}\zeta + \mathbf{c}_2$ hold. $\square$

Now, define the operator $\mathfrak{P} : \mathcal{K} \rightarrow \mathcal{K}$ by

$$\mathfrak{P}(\omega_1, \omega_2)(\wp) = (\mathfrak{P}_1(\omega_1, \omega_2)(\wp), \mathfrak{P}_2(\omega_1, \omega_2)(\wp)),$$

where

$$\begin{aligned} \mathfrak{P}_1(\omega_1, \omega_2)(\wp) &= \frac{1-\aleph}{\mathbf{B}(\aleph)}\mathfrak{f}(\wp, \omega_1(\wp), \omega_2(\wp)) + \frac{\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph)} \int_0^{\wp} (\wp - \zeta)^{\aleph-1} \mathfrak{f}(\zeta, \omega_1(\zeta), \omega_2(\zeta)) \mathfrak{d}\zeta \\ &\quad + \chi_1 \int_0^1 \omega_1(\zeta) \mathfrak{d}\zeta + \mathbf{c}_1, \end{aligned} \quad (3.3)$$

$$\begin{aligned} \mathfrak{P}_2(\omega_1, \omega_2)(\wp) &= \frac{1-\aleph}{\mathbf{B}(\aleph)}\mathfrak{g}(\wp, \omega_1(\wp), \omega_2(\wp)) + \frac{\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph)} \int_0^{\wp} (\wp - \zeta)^{\aleph-1} \mathfrak{g}(\zeta, \omega_1(\zeta), \omega_2(\zeta)) \mathfrak{d}\zeta \\ &\quad + \chi_2 \int_0^1 \omega_2(\zeta) \mathfrak{d}\zeta + \mathbf{c}_2. \end{aligned} \quad (3.4)$$

We first establish that the operator $\mathfrak{P} : \mathcal{K} \rightarrow \mathcal{K}$ is compact.

Lemma 3.2. Suppose that $\mathfrak{f}, \mathfrak{g} \in \mathfrak{C}([0, 1] \times \mathcal{R}^+ \times \mathcal{R}^+, \mathcal{R}^+)$. Under these assumptions, the operator $\mathfrak{P} : \mathcal{K} \rightarrow \mathcal{K}$ is compact.

Proof. The continuity of the operator $\mathfrak{P} : \mathcal{K} \rightarrow \mathcal{K}$ follows directly from the nonnegativity and continuity assumptions imposed on $\mathfrak{f}$ and $\mathfrak{g}$.

Consider a bounded set $\mathbf{S} \subset \mathcal{K}$. Then $\exists$ a constant $\mathcal{M} > 0$ s.t. $\|(\omega_1, \omega_2)\| \leq \mathcal{M} \forall (\omega_1, \omega_2) \in \mathbf{S}$. In addition, suppose $\exists$ a constant $\mathbf{L}_{\mathfrak{f}} > 0$ s.t.

$$\mathbf{L}_{\mathfrak{f}} = \max_{(\wp, \omega_1, \omega_2) \in [0, 1] \times [0, \mathcal{M}] \times [0, \mathcal{M}]} \mathfrak{f}(\wp, \omega_1(\wp), \omega_2(\wp)) + 1,$$

and similarly,

$$L_g = \max_{(\wp, \omega_1, \omega_2) \in [0,1] \times [0, \mathcal{M}] \times [0, \mathcal{M}]} g(\wp, \omega_1(\wp), \omega_2(\wp)) + 1.$$

Using (3.3), it follows that for every $(\omega_1, \omega_2) \in \mathbf{S}$ and each $\wp \in [0, 1]$, we obtain

$$\begin{aligned} |\mathfrak{P}_1(\omega_1, \omega_2)(\wp)| &\leq \frac{1-\aleph}{B(\aleph)} |\mathfrak{f}(\wp, \omega_1(\wp), \omega_2(\wp))| + \frac{\aleph}{B(\aleph)\Gamma(\aleph)} \int_0^\wp (\wp - \zeta)^{\aleph-1} |\mathfrak{f}(\zeta, \omega_1(\zeta), \omega_2(\zeta))| d\zeta \\ &\quad + \chi_1 \int_0^1 |\omega_1(\zeta)| d\zeta + c_1 \\ &\leq \frac{1-\aleph}{B(\aleph)} L_f + \frac{\aleph L_f}{B(\aleph)\Gamma(\aleph)} \cdot \frac{\wp^\aleph}{\aleph} + \chi_1 \mathcal{M} + c_1 \\ &\leq \frac{1-\aleph}{B(\aleph)} L_f + \frac{L_f}{B(\aleph)\Gamma(\aleph+1)} + \chi_1 \mathcal{M} + c_1. \end{aligned}$$

Thus,

$$\|\mathfrak{P}_1(\omega_1, \omega_2)\| \leq \frac{1-\aleph}{B(\aleph)} L_f + \frac{L_f}{B(\aleph)\Gamma(\aleph+1)} + \chi_1 \mathcal{M} + c_1.$$

Similarly,

$$\|\mathfrak{P}_2(\omega_1, \omega_2)\| \leq \frac{1-\aleph}{B(\aleph)} L_g + \frac{L_g}{B(\aleph)\Gamma(\aleph+1)} + \chi_2 \mathcal{M} + c_2.$$

Consequently, $\mathfrak{P}(\mathbf{S})$ forms a uniformly bounded family of functions.

We now proceed to establish the equicontinuity of the operator $\mathfrak{P}$. For any $(\omega_1, \omega_2) \in \mathbf{S}$, let $\wp_1, \wp_2 \in [0, 1]$ with $\wp_1 < \wp_2$. Then

$$\begin{aligned} &|\mathfrak{P}_1(\omega_1, \omega_2)(\wp_1) - \mathfrak{P}_1(\omega_1, \omega_2)(\wp_2)| \\ &= \left| \frac{1-\aleph}{B(\aleph)} (\mathfrak{f}(\wp_1, \omega_1(\wp_1), \omega_2(\wp_1)) - \mathfrak{f}(\wp_2, \omega_1(\wp_2), \omega_2(\wp_2))) \right. \\ &\quad \left. + \frac{\aleph}{B(\aleph)\Gamma(\aleph)} \left(\int_0^{\wp_1} (\wp_1 - s)^{\aleph-1} \mathfrak{f}(\zeta, \omega_1(\zeta), \omega_2(\zeta)) d\zeta - \int_0^{\wp_2} (\wp_2 - s)^{\aleph-1} \mathfrak{f}(\zeta, \omega_1(\zeta), \omega_2(\zeta)) d\zeta \right) \right| \\ &\leq \frac{1-\aleph}{B(\aleph)} |\mathfrak{f}(\wp_1, \omega_1(\wp_1), \omega_2(\wp_1)) - \mathfrak{f}(\wp_2, \omega_1(\wp_2), \omega_2(\wp_2))| \\ &\quad + \frac{\aleph}{B(\aleph)\Gamma(\aleph)} \left| \int_0^{\wp_1} [(\wp_1 - s)^{\aleph-1} - (\wp_2 - s)^{\aleph-1}] \mathfrak{f}(\zeta, \omega_1(\zeta), \omega_2(\zeta)) d\zeta \right. \\ &\quad \left. - \int_{\wp_1}^{\wp_2} (\wp_2 - s)^{\aleph-1} \mathfrak{f}(\zeta, \omega_1(\zeta), \omega_2(\zeta)) d\zeta \right| \\ &\leq \frac{1-\aleph}{B(\aleph)} |\mathfrak{f}(\wp_1, \omega_1(\wp_1), \omega_2(\wp_1)) - \mathfrak{f}(\wp_2, \omega_1(\wp_2), \omega_2(\wp_2))| \\ &\quad + \frac{\aleph L_f}{B(\aleph)\Gamma(\aleph)} \int_0^{\wp_1} [(\wp_1 - s)^{\aleph-1} - (\wp_2 - s)^{\aleph-1}] d\zeta + \frac{\aleph L_f}{B(\aleph)\Gamma(\aleph)} \int_{\wp_1}^{\wp_2} (\wp_2 - s)^{\aleph-1} d\zeta \\ &\leq \frac{1-\aleph}{B(\aleph)} |\mathfrak{f}(\wp_1, \omega_1(\wp_1), \omega_2(\wp_1)) - \mathfrak{f}(\wp_2, \omega_1(\wp_2), \omega_2(\wp_2))| \end{aligned}$$

$$\begin{aligned}
& + \frac{L_f}{B(N)\Gamma(N+1)} (\wp_1^N + (\wp_2 - \wp_1)^N - \wp_2^N + (\wp_2 - \wp_1)^N) \\
& \leq \frac{1-N}{B(N)} |\mathfrak{f}(\wp_1, \omega_1(\wp_1), \omega_2(\wp_1)) - \mathfrak{f}(\wp_2, \omega_1(\wp_2), \omega_2(\wp_2))| + \frac{2L_f}{B(N)\Gamma(N+1)} (\wp_2 - \wp_1)^N.
\end{aligned} \quad (3.5)$$

Since $\mathfrak{f}$ is uniformly continuous on compact set $[0, 1] \times [0, \mathcal{M}] \times [0, \mathcal{M}]$, the first term tends to zero as $\wp_1 \rightarrow \wp_2$. Also,

$$\frac{2L_f}{B(N)\Gamma(N+1)} (\wp_2 - \wp_1)^N \rightarrow 0 \text{ as } \wp_1 \rightarrow \wp_2.$$

Hence, the RHS of inequality tends to zero independently of $(\omega_1, \omega_2) \in \mathbf{S}$. Therefore, $\mathfrak{P}_1(\mathbf{S})$ is equicontinuous. Similarly, $\mathfrak{P}_2(\mathbf{S})$ is equicontinuous. Thus, the operator $\mathfrak{P}$ is compact, by the Arzelà-Ascoli theorem 2.3. $\square$

The following result establishes existence of a solution to the coupled system (1.1)-(1.2) by applying Schauder's FPT.

Theorem 3.1. Assume that the functions $\mathfrak{f}, \mathfrak{g} : [0, 1] \times \mathcal{R}^+ \times \mathcal{R}^+ \rightarrow \mathcal{R}^+$ are continuous and that $(\widetilde{\omega}_1, \widetilde{\omega}_2)$ and (ω_1, ω_2) are upper and lower solutions of the coupled system (1.1)-(1.2), respectively. Then the coupled system (1.1)-(1.2) admits at least one positive solution (ω_1, ω_2) . Furthermore,

$$\underline{\omega}_1(\wp) \leq \omega_1(\wp) \leq \widetilde{\omega}_1(\wp), \quad \underline{\omega}_2(\wp) \leq \omega_2(\wp) \leq \widetilde{\omega}_2(\wp), \quad \wp \in [0, 1].$$

Proof. Define the set

$$\Theta = \{(\omega_1, \omega_2) \in \mathcal{K} : \underline{\omega}_1(\wp) \leq \omega_1(\wp) \leq \widetilde{\omega}_1(\wp), \underline{\omega}_2(\wp) \leq \omega_2(\wp) \leq \widetilde{\omega}_2(\wp), \wp \in [0, 1]\}.$$

Equipped with the norm $\|(\omega_1, \omega_2)\| = \max_{\wp \in [0, 1]} |\omega_1(\wp)| + \max_{\wp \in [0, 1]} |\omega_2(\wp)| \leq b_1 + b_2$, the set Θ forms a convex, closed and bounded subset of the Banach space $\mathfrak{X} = \mathcal{C}[0, 1] \times \mathcal{C}[0, 1]$. Furthermore, Lemma 3.2 ensures that the operator $\mathfrak{P} : \mathcal{K} \rightarrow \mathcal{K}$ is compact. Consequently, in order to invoke Schauder's FPT, it suffices to verify that $\mathfrak{P}$ maps Θ into itself, that is, $\mathfrak{P}(\Theta) \subseteq \Theta$.

Let $(\omega_1, \omega_2) \in \Theta$ be arbitrary. Then

$$\underline{\omega}_1(\wp) \leq \omega_1(\wp) \leq \widetilde{\omega}_1(\wp), \quad \underline{\omega}_2(\wp) \leq \omega_2(\wp) \leq \widetilde{\omega}_2(\wp).$$

Utilizing Definitions 2.6 and 2.7, it follows that

$$\begin{aligned}
\mathfrak{P}_1(\omega_1, \omega_2)(\wp) &= \frac{1-N}{B(N)} \mathfrak{f}(\wp, \omega_1(\wp), \omega_2(\wp)) + \frac{N}{B(N)\Gamma(N)} \int_0^\wp (\wp - \zeta)^{N-1} \mathfrak{f}(\zeta, \omega_1(\zeta), \omega_2(\zeta)) \mathfrak{d}\zeta \\
&\quad + \chi_1 \int_0^1 \omega_1(\zeta) \mathfrak{d}\zeta + \mathfrak{c}_1 \\
&\leq \frac{1-N}{B(N)} \mathcal{H}_f(\wp, \omega_1(\wp), \omega_2(\wp)) + \frac{N}{B(N)\Gamma(N)} \int_0^\wp (\wp - \zeta)^{N-1} \mathcal{H}_f(\zeta, \omega_1(\zeta), \omega_2(\zeta)) \mathfrak{d}\zeta \\
&\quad + \chi_1 \int_0^1 \omega_1(\zeta) \mathfrak{d}\zeta + \mathfrak{c}_1 \\
&\leq \frac{1-N}{B(N)} \mathcal{H}_f(\wp, \widetilde{\omega}_1(\wp), \widetilde{\omega}_2(\wp)) + \frac{N}{B(N)\Gamma(N)} \int_0^\wp (\wp - \zeta)^{N-1} \mathcal{H}_f(\zeta, \widetilde{\omega}_1(\zeta), \widetilde{\omega}_2(\zeta)) \mathfrak{d}\zeta
\end{aligned}$$

$$\begin{aligned}
& + \chi_1 \int_0^1 \widetilde{\omega}_1(\zeta) \, \mathrm{d}\zeta + \mathbf{c}_1 \\
& \leq \widetilde{\omega}_1(\wp),
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{P}_1(\omega_1, \omega_2)(\wp) & \geq \frac{1-\aleph}{\mathbf{B}(\aleph)} \mathfrak{h}_{\mathfrak{f}}(\wp, \omega_1(\wp), \omega_2(\wp)) + \frac{\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph)} \int_0^\wp (\wp - \zeta)^{\aleph-1} \mathfrak{h}_{\mathfrak{f}}(\zeta, \omega_1(\zeta), \omega_2(\zeta)) \, \mathrm{d}\zeta \\
& \quad + \chi_1 \int_0^1 \omega_1(\zeta) \, \mathrm{d}\zeta + \mathbf{c}_1 \\
& \geq \frac{1-\aleph}{\mathbf{B}(\aleph)} \mathfrak{h}_{\mathfrak{f}}(\wp, \underline{\omega}_1(\wp), \underline{\omega}_2(\wp)) + \frac{\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph)} \int_0^\wp (\wp - \zeta)^{\aleph-1} \mathfrak{h}_{\mathfrak{f}}(\zeta, \underline{\omega}_1(\zeta), \underline{\omega}_2(\zeta)) \, \mathrm{d}\zeta \\
& \quad + \chi_1 \int_0^1 \underline{\omega}_1(\zeta) \, \mathrm{d}\zeta + \mathbf{c}_1 \\
& \geq \underline{\omega}_1(\wp).
\end{aligned}$$

Similarly,

$$\underline{\omega}_2(\wp) \leq \mathfrak{P}_2(\omega_1, \omega_2)(\wp) \leq \widetilde{\omega}_2(\wp).$$

Consequently,

$$\underline{\omega}_1(\wp) \leq \mathfrak{P}_1(\omega_1, \omega_2)(\wp) \leq \widetilde{\omega}_1(\wp), \quad \underline{\omega}_2(\wp) \leq \mathfrak{P}_2(\omega_1, \omega_2)(\wp) \leq \widetilde{\omega}_2(\wp).$$

Hence, $\mathfrak{P}(\omega_1, \omega_2)(\wp) \in \Theta$ for every $\wp \in [0, 1]$, which shows that $\mathfrak{P}$ maps Θ into itself. Therefore, $\mathfrak{P} : \Theta \rightarrow \Theta$.

By Theorem 2.2, the operator $\mathfrak{P}$ possesses at least one FP $(\omega_1, \omega_2) \in \Theta$ for $\wp \in [0, 1]$. Consequently, the coupled BVP (1.1)-(1.2) admits at least one positive solution $(\omega_1, \omega_2) \in \mathfrak{C}[0, 1] \times \mathfrak{C}[0, 1]$. Moreover,

$$\underline{\omega}_1(\wp) \leq \omega_1(\wp) \leq \widetilde{\omega}_1(\wp), \quad \underline{\omega}_2(\wp) \leq \omega_2(\wp) \leq \widetilde{\omega}_2(\wp), \quad \wp \in [0, 1].$$

□

□

The following theorem is derived using Banach's Contraction Principle.

Theorem 3.2. Assume that $\mathfrak{f}, \mathfrak{g} : [0, 1] \times \mathcal{R}^+ \times \mathcal{R}^+ \rightarrow \mathcal{R}^+$ are continuous and there exist constants $\ell_1, \ell_2 > 0$ s.t.

$$|\mathfrak{f}(\wp, \omega_{11}, \omega_{21}) - \mathfrak{f}(\wp, \omega_{12}, \omega_{22})| \leq \ell_1(|\omega_{11} - \omega_{12}| + |\omega_{21} - \omega_{22}|),$$

$$|\mathfrak{g}(\wp, \omega_{11}, \omega_{21}) - \mathfrak{g}(\wp, \omega_{12}, \omega_{22})| \leq \ell_2(|\omega_{11} - \omega_{12}| + |\omega_{21} - \omega_{22}|),$$

for all $\wp \in [0, 1]$, $\omega_{11}, \omega_{12}, \omega_{21}, \omega_{22} \in \mathcal{R}^+$.

If

$$\left(\frac{\ell_1(1-\aleph)}{\mathbf{B}(\aleph)} + \frac{\ell_1\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph+1)} + \chi_1 \right) + \left(\frac{\ell_2(1-\aleph)}{\mathbf{B}(\aleph)} + \frac{\ell_2\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph+1)} + \chi_2 \right) < 1,$$

then the coupled system (1.1)-(1.2) has a unique positive solution $(\omega_1, \omega_2) \in \mathfrak{C}[0, 1] \times \mathfrak{C}[0, 1]$.

Proof. As established in Theorem 3.1, the coupled system (1.1)-(1.2) admits at least one positive solution. We now focus on the uniqueness of the solution. For this purpose, it is sufficient to show that the operator $\mathfrak{P}$ associated with (3.3)-(3.4) satisfies the contraction condition on $\mathfrak{X} = \mathfrak{C}[0, 1] \times \mathfrak{C}[0, 1]$.

Accordingly, for arbitrary functions $(\omega_{11}, \omega_{21}), (\omega_{12}, \omega_{22}) \in \mathfrak{X}$ and any $\wp \in [0, 1]$, equations (3.3)-(3.4) yield

$$\begin{aligned} & |\mathfrak{P}_1(\omega_{11}, \omega_{21})(\wp) - \mathfrak{P}_1(\omega_{12}, \omega_{22})(\wp)| \\ & \leq \frac{1 - \aleph}{\mathbf{B}(\aleph)} |\mathfrak{f}(\wp, \omega_{11}(\wp), \omega_{21}(\wp)) - \mathfrak{f}(\wp, \omega_{12}(\wp), \omega_{22}(\wp))| \\ & \quad + \frac{\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph)} \int_0^\wp (\wp - \zeta)^{\aleph-1} |\mathfrak{f}(\zeta, \omega_{11}(\zeta), \omega_{21}(\zeta)) - \mathfrak{f}(\zeta, \omega_{12}(\zeta), \omega_{22}(\zeta))| \mathfrak{d}\zeta \\ & \quad + \chi_1 \int_0^1 |\omega_{11}(\zeta) - \omega_{12}(\zeta)| \mathfrak{d}\zeta \\ & \leq \frac{\ell_1(1 - \aleph)}{\mathbf{B}(\aleph)} \|(\omega_{11} - \omega_{12}, \omega_{21} - \omega_{22})\| + \frac{\ell_1 \aleph}{\mathbf{B}(\aleph)\Gamma(\aleph)} \|(\omega_{11} - \omega_{12}, \omega_{21} - \omega_{22})\| \\ & \quad \times \int_0^\wp (\wp - \zeta)^{\aleph-1} \mathfrak{d}\zeta + \chi_1 \|\omega_{11} - \omega_{12}\| \\ & \leq \left(\frac{\ell_1(1 - \aleph)}{\mathbf{B}(\aleph)} + \frac{\ell_1}{\mathbf{B}(\aleph)\Gamma(\aleph + 1)} + \chi_1 \right) \|(\omega_{11} - \omega_{12}, \omega_{21} - \omega_{22})\|. \end{aligned}$$

Using $\frac{1}{\Gamma(\aleph)} = \frac{\aleph}{\Gamma(\aleph+1)}$, we obtain

$$\|\mathfrak{P}_1(\omega_{11}, \omega_{21}) - \mathfrak{P}_1(\omega_{12}, \omega_{22})\| \leq \left(\frac{\ell_1(1 - \aleph)}{\mathbf{B}(\aleph)} + \frac{\ell_1 \aleph}{\mathbf{B}(\aleph)\Gamma(\aleph + 1)} + \chi_1 \right) \|(\omega_{11} - \omega_{12}, \omega_{21} - \omega_{22})\|.$$

Similarly,

$$\|\mathfrak{P}_2(\omega_{11}, \omega_{21}) - \mathfrak{P}_2(\omega_{12}, \omega_{22})\| \leq \left(\frac{\ell_2(1 - \aleph)}{\mathbf{B}(\aleph)} + \frac{\ell_2 \aleph}{\mathbf{B}(\aleph)\Gamma(\aleph + 1)} + \chi_2 \right) \|(\omega_{11} - \omega_{12}, \omega_{21} - \omega_{22})\|.$$

Therefore,

$$\begin{aligned} & \|\mathfrak{P}(\omega_{11}, \omega_{21}) - \mathfrak{P}(\omega_{12}, \omega_{22})\| = \|\mathfrak{P}_1(\omega_{11}, \omega_{21}) - \mathfrak{P}_1(\omega_{12}, \omega_{22})\| + \|\mathfrak{P}_2(\omega_{11}, \omega_{21}) - \mathfrak{P}_2(\omega_{12}, \omega_{22})\|. \\ & \leq \left[\left(\frac{\ell_1(1 - \aleph)}{\mathbf{B}(\aleph)} + \frac{\ell_1 \aleph}{\mathbf{B}(\aleph)\Gamma(\aleph + 1)} + \chi_1 \right) + \left(\frac{\ell_2(1 - \aleph)}{\mathbf{B}(\aleph)} + \frac{\ell_2 \aleph}{\mathbf{B}(\aleph)\Gamma(\aleph + 1)} + \chi_2 \right) \right] \|(\omega_{11} - \omega_{12}, \omega_{21} - \omega_{22})\|. \end{aligned}$$

Since

$$\left(\frac{\ell_1(1 - \aleph)}{\mathbf{B}(\aleph)} + \frac{\ell_1 \aleph}{\mathbf{B}(\aleph)\Gamma(\aleph + 1)} + \chi_1 \right) + \left(\frac{\ell_2(1 - \aleph)}{\mathbf{B}(\aleph)} + \frac{\ell_2 \aleph}{\mathbf{B}(\aleph)\Gamma(\aleph + 1)} + \chi_2 \right) < 1,$$

it follows from Theorem 2.1 that $\mathfrak{P}$ possesses a unique FP. Consequently, the coupled system (1.1)-(1.2) admits a unique positive solution on the interval $[0, 1]$. $\square$

4. STABILITY ANALYSIS

This section investigates the UH stability, GUH stability and UHR stability of the coupled system (1.1)-(1.2). These concepts describe the robustness of the solution under small perturbations and demonstrate that an approximate solution remains close to the exact solution.

Definition 4.1 (UH Stability). *The coupled BVP (1.1)-(1.2) is said to be UH stable if $\exists$ a constant $C_f > 0$ s.t. for every pair of functions $(\omega_1, \omega_2) \in C[0, 1] \times C[0, 1]$ satisfying*

$$|{}^{ABC}\mathfrak{D}^{\mathfrak{N}}\omega_1(\wp) - \mathfrak{f}(\wp, \omega_1(\wp), \omega_2(\wp))| \leq \epsilon_1,$$

$$|{}^{ABC}\mathfrak{D}^{\mathfrak{X}}\omega_2(\wp) - \mathfrak{g}(\wp, \omega_1(\wp), \omega_2(\wp))| \leq \epsilon_2,$$

for all $\wp \in [0, 1]$, for some $\epsilon_1, \epsilon_2 > 0$, $\exists$ an exact solution (ω_1^*, ω_2^*) of (1.1)-(1.2) satisfying

$$|\omega_1(\wp) - \omega_1^*(\wp)| \leq C_{f,1}\epsilon_1, \quad |\omega_2(\wp) - \omega_2^*(\wp)| \leq C_{g,2}\epsilon_2.$$

Definition 4.2 (GUH Stability). *The coupled BVP (1.1)-(1.2) is said to be GUH stable if $\exists$ functions $\Phi_f, \Phi_g : \mathcal{R}^+ \rightarrow \mathcal{R}^+$ with $\Phi_f(0) = \Phi_g(0) = 0$ s.t. for every pair of functions $(\omega_1, \omega_2) \in \mathcal{C}[0, 1] \times \mathcal{C}[0, 1]$ satisfying*

$$|{}^{ABC}\mathfrak{D}^{\mathfrak{N}}\omega_1(\wp) - \mathfrak{f}(\wp, \omega_1(\wp), \omega_2(\wp))| \leq \Phi_f(\epsilon_1),$$

$$|{}^{ABC}\mathfrak{D}^{\mathfrak{X}}\omega_2(\wp) - \mathfrak{g}(\wp, \omega_1(\wp), \omega_2(\wp))| \leq \Phi_g(\epsilon_2),$$

for all $\wp \in [0, 1]$ and some $\epsilon_1, \epsilon_2 > 0$, there exists an exact solution (ω_1^*, ω_2^*) of (1.1)-(1.2) satisfying

$$|\omega_1(\wp) - \omega_1^*(\wp)| \leq C_{f,1}\Phi_f(\epsilon_1), \quad |\omega_2(\wp) - \omega_2^*(\wp)| \leq C_{g,2}\Phi_g(\epsilon_2).$$

Definition 4.3 (UHR Stability). *Let $\phi, \psi : [0, 1] \rightarrow \mathcal{R}^+$ be continuous functions. The coupled system (1.1)-(1.2) is said to be UHR stable with respect to (ϕ, ψ) if there exist constants $C_{f,\phi}, C_{g,\psi} > 0$ s.t. every pair of functions $(\omega_1, \omega_2) \in C[0, 1] \times C[0, 1]$ satisfying*

$$|{}^{ABC}\mathfrak{D}^{\mathfrak{N}}\omega_1(\wp) - \mathfrak{f}(\wp, \omega_1(\wp), \omega_2(\wp))| \leq \epsilon_1\phi(\wp),$$

$$|{}^{ABC}\mathfrak{D}^{\mathfrak{X}}\omega_2(\wp) - \mathfrak{g}(\wp, \omega_1(\wp), \omega_2(\wp))| \leq \epsilon_2\psi(\wp),$$

for all $\wp \in [0, 1]$, fulfills

$$|\omega_1(\wp) - \omega_1^*(\wp)| \leq C_{f,\phi}\epsilon_1\phi(\wp), \quad |\omega_2(\wp) - \omega_2^*(\wp)| \leq C_{g,\psi}\epsilon_2\psi(\wp),$$

where (ω_1^*, ω_2^*) is the exact solution of the coupled system (1.1)-(1.2).

Lemma 4.1. *Assume that $(\omega_1, \omega_2) \in C[0, 1] \times C[0, 1]$ satisfies*

$$|{}^{ABC}\mathfrak{D}^{\mathfrak{N}}\omega_1(\wp) - \mathfrak{f}(\wp, \omega_1(\wp), \omega_2(\wp))| \leq \epsilon_1,$$

$$|{}^{ABC}\mathfrak{D}^{\mathfrak{X}}\omega_2(\wp) - \mathfrak{g}(\wp, \omega_1(\wp), \omega_2(\wp))| \leq \epsilon_2,$$

for all $\wp \in [0, 1]$. Then there exist functions $\eta_f(\wp)$ and $\eta_g(\wp)$ s.t.

$$|\eta_f(\wp)| \leq \epsilon_1, \quad |\eta_g(\wp)| \leq \epsilon_2,$$

and

$${}^{ABC}\mathfrak{D}^{\aleph}\omega_1(\wp) = \mathfrak{f}(\wp, \omega_1(\wp), \omega_2(\wp)) + \eta_{\mathfrak{f}}(\wp),$$

$${}^{ABC}\mathfrak{D}^{\aleph}\omega_2(\wp) = \mathfrak{g}(\wp, \omega_1(\wp), \omega_2(\wp)) + \eta_{\mathfrak{g}}(\wp).$$

Applying the ABC integral operator to both sides yields

$$\begin{aligned}\omega_1(\wp) &= \frac{1-\aleph}{\mathbf{B}(\aleph)}[\mathfrak{f}(\wp, \omega_1(\wp), \omega_2(\wp)) + \eta_{\mathfrak{f}}(\wp)] + \frac{\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph)} \int_0^{\wp} (\wp - \zeta)^{\aleph-1} [\mathfrak{f}(\zeta, \omega_1(\zeta), \omega_2(\zeta)) + \eta_{\mathfrak{f}}(\zeta)] \mathfrak{d}\zeta \\ &\quad + \chi_1 \int_0^1 \omega_1(\zeta) \mathfrak{d}\zeta + \mathfrak{c}_1,\end{aligned}$$

$$\begin{aligned}\omega_2(\wp) &= \frac{1-\aleph}{\mathbf{B}(\aleph)}[\mathfrak{g}(\wp, \omega_1(\wp), \omega_2(\wp)) + \eta_{\mathfrak{g}}(\wp)] + \frac{\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph)} \int_0^{\wp} (\wp - \zeta)^{\aleph-1} [\mathfrak{g}(\zeta, \omega_1(\zeta), \omega_2(\zeta)) + \eta_{\mathfrak{g}}(\zeta)] \mathfrak{d}\zeta \\ &\quad + \chi_2 \int_0^1 \omega_2(\zeta) \mathfrak{d}\zeta + \mathfrak{c}_2.\end{aligned}$$

Consequently,

$$|\omega_1(\wp) - \mathfrak{P}_1(\omega_1, \omega_2)(\wp)| \leq \epsilon_1 \left(\frac{1-\aleph}{\mathbf{B}(\aleph)} + \frac{\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph+1)} \right),$$

$$|\omega_2(\wp) - \mathfrak{P}_2(\omega_1, \omega_2)(\wp)| \leq \epsilon_2 \left(\frac{1-\aleph}{\mathbf{B}(\aleph)} + \frac{\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph+1)} \right).$$

Theorem 4.1 (UH Stability). *Suppose that*

$$|\mathfrak{f}(\wp, \omega_{11}, \omega_{21}) - \mathfrak{f}(\wp, \omega_{12}, \omega_{22})| \leq \ell_1(|\omega_{11} - \omega_{12}| + |\omega_{21} - \omega_{22}|),$$

$$|\mathfrak{g}(\wp, \omega_{11}, \omega_{21}) - \mathfrak{g}(\wp, \omega_{12}, \omega_{22})| \leq \ell_2(|\omega_{11} - \omega_{12}| + |\omega_{21} - \omega_{22}|),$$

for all $\wp \in [0, 1]$ and $\omega_{11}, \omega_{12}, \omega_{21}, \omega_{22} \in \mathcal{R}^+$, where $\ell_1, \ell_2 > 0$. Furthermore, assume that

$$\mathfrak{q} = \left(\frac{\ell_1(1-\aleph)}{\mathbf{B}(\aleph)} + \frac{\ell_1\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph+1)} + \chi_1 \right) + \left(\frac{\ell_2(1-\aleph)}{\mathbf{B}(\aleph)} + \frac{\ell_2\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph+1)} + \chi_2 \right) < 1.$$

Then the coupled system (1.1)-(1.2) is UH stable.

Proof. Let (ω_1^*, ω_2^*) be the unique solution of the coupled system (1.1)-(1.2) and let (ω_1, ω_2) be an approximate solution satisfying

$$|{}^{ABC}\mathfrak{D}^{\aleph}\omega_1(\wp) - \mathfrak{f}(\wp, \omega_1(\wp), \omega_2(\wp))| \leq \epsilon_1,$$

$$|{}^{ABC}\mathfrak{D}^{\aleph}\omega_2(\wp) - \mathfrak{g}(\wp, \omega_1(\wp), \omega_2(\wp))| \leq \epsilon_2.$$

Using Lemma 4.1 and the operator $\mathfrak{P}$, we obtain

$$|\omega_1(\wp) - \omega_1^*(\wp)| \leq |\omega_1(\wp) - \mathfrak{P}_1(\omega_1, \omega_2)(\wp)| + |\mathfrak{P}_1(\omega_1, \omega_2)(\wp) - \mathfrak{P}_1(\omega_1^*, \omega_2^*)(\wp)|,$$

$$|\omega_2(\wp) - \omega_2^*(\wp)| \leq |\omega_2(\wp) - \mathfrak{P}_2(\omega_1, \omega_2)(\wp)| + |\mathfrak{P}_2(\omega_1, \omega_2)(\wp) - \mathfrak{P}_2(\omega_1^*, \omega_2^*)(\wp)|.$$

From Lemma 4.1,

$$|\omega_1(\wp) - \mathfrak{P}_1(\omega_1, \omega_2)(\wp)| \leq \epsilon_1 \left(\frac{1-\aleph}{\mathbf{B}(\aleph)} + \frac{\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph+1)} \right),$$

$$|\omega_2(\wp) - \mathfrak{P}_2(\omega_1, \omega_2)(\wp)| \leq \epsilon_2 \left(\frac{1 - \mathfrak{R}}{\mathbf{B}(\mathfrak{R})} + \frac{\mathfrak{R}}{\mathbf{B}(\mathfrak{R})\Gamma(\mathfrak{R} + 1)} \right).$$

Since $\mathfrak{P}$ is a contraction,

$$|\mathfrak{P}_1(\omega_1, \omega_2)(\wp) - \mathfrak{P}_1(\omega_1^*, \omega_2^*)(\wp)| \leq \left(\frac{\ell_1(1 - \mathfrak{N})}{\mathbf{B}(\mathfrak{N})} + \frac{\ell_1 \mathfrak{N}}{\mathbf{B}(\mathfrak{N})\Gamma(\mathfrak{N} + 1)} + \chi_1 \right) \|(\omega_1 - \omega_1^*, \omega_2 - \omega_2^*)\|,$$

$$|\mathfrak{P}_2(\omega_1, \omega_2)(\wp) - \mathfrak{P}_2(\omega_1^*, \omega_2^*)(\wp)| \leq \left(\frac{\ell_2(1 - \mathfrak{R})}{\mathbf{B}(\mathfrak{R})} + \frac{\ell_2 \mathfrak{R}}{\mathbf{B}(\mathfrak{R})\Gamma(\mathfrak{R} + 1)} + \chi_2 \right) \|(\omega_1 - \omega_1^*, \omega_2 - \omega_2^*)\|.$$

Therefore,

$$\begin{aligned} \|\omega_1 - \omega_1^*\| &\leq \epsilon_1 \left(\frac{1 - \mathfrak{N}}{\mathbf{B}(\mathfrak{N})} + \frac{\mathfrak{N}}{\mathbf{B}(\mathfrak{N})\Gamma(\mathfrak{N} + 1)} \right) + \left(\frac{\ell_1(1 - \mathfrak{N})}{\mathbf{B}(\mathfrak{N})} + \frac{\ell_1 \mathfrak{N}}{\mathbf{B}(\mathfrak{N})\Gamma(\mathfrak{N} + 1)} + \chi_1 \right) \\ &\quad \times \|(\omega_1 - \omega_1^*, \omega_2 - \omega_2^*)\|, \end{aligned}$$

$$\begin{aligned} \|\omega_2 - \omega_2^*\| &\leq \epsilon_2 \left(\frac{1 - \mathfrak{R}}{\mathbf{B}(\mathfrak{R})} + \frac{\mathfrak{R}}{\mathbf{B}(\mathfrak{R})\Gamma(\mathfrak{R} + 1)} \right) + \left(\frac{\ell_2(1 - \mathfrak{R})}{\mathbf{B}(\mathfrak{R})} + \frac{\ell_2 \mathfrak{R}}{\mathbf{B}(\mathfrak{R})\Gamma(\mathfrak{R} + 1)} + \chi_2 \right) \\ &\quad \times \|(\omega_1 - \omega_1^*, \omega_2 - \omega_2^*)\|. \end{aligned}$$

Adding these inequalities and using $q < 1$, we get

$$(1 - q) \|(\omega_1 - \omega_1^*, \omega_2 - \omega_2^*)\| \leq \epsilon_1 \left(\frac{1 - \mathfrak{N}}{\mathbf{B}(\mathfrak{N})} + \frac{\mathfrak{N}}{\mathbf{B}(\mathfrak{N})\Gamma(\mathfrak{N} + 1)} \right) + \epsilon_2 \left(\frac{1 - \mathfrak{R}}{\mathbf{B}(\mathfrak{R})} + \frac{\mathfrak{R}}{\mathbf{B}(\mathfrak{R})\Gamma(\mathfrak{R} + 1)} \right).$$

Thus,

$$\|(\omega_1 - \omega_1^*, \omega_2 - \omega_2^*)\| \leq \mathbf{C}_{\mathfrak{f}} \max\{\epsilon_1, \epsilon_2\},$$

where

$$\mathbf{C}_{\mathfrak{f}} = \frac{\frac{1 - \mathfrak{N}}{\mathbf{B}(\mathfrak{N})} + \frac{\mathfrak{N}}{\mathbf{B}(\mathfrak{N})\Gamma(\mathfrak{N} + 1)} + \frac{1 - \mathfrak{R}}{\mathbf{B}(\mathfrak{R})} + \frac{\mathfrak{R}}{\mathbf{B}(\mathfrak{R})\Gamma(\mathfrak{R} + 1)}}{1 - q}.$$

Therefore, the coupled system (1.1)-(1.2) is UH stable. $\square$

Lemma 4.2. Assume that $(\omega_1, \omega_2) \in \mathfrak{C}[0, 1] \times \mathfrak{C}[0, 1]$ satisfies

$$|\text{ABC} \mathfrak{D}^{\mathfrak{N}} \omega_1(\wp) - \mathfrak{f}(\wp, \omega_1(\wp), \omega_2(\wp))| \leq \Phi_{\mathfrak{f}}(\epsilon_1),$$

$$|\text{ABC} \mathfrak{D}^{\mathfrak{R}} \omega_2(\wp) - \mathfrak{g}(\wp, \omega_1(\wp), \omega_2(\wp))| \leq \Phi_{\mathfrak{g}}(\epsilon_2),$$

for all $\wp \in [0, 1]$. Then there exist functions $\eta_{\mathfrak{f}}(\wp)$ and $\eta_{\mathfrak{g}}(\wp)$ such that

$$|\eta_{\mathfrak{f}}(\wp)| \leq \Phi_{\mathfrak{f}}(\epsilon_1), \quad |\eta_{\mathfrak{g}}(\wp)| \leq \Phi_{\mathfrak{g}}(\epsilon_2),$$

and

$$\text{ABC} \mathfrak{D}^{\mathfrak{N}} \omega_1(\wp) = \mathfrak{f}(\wp, \omega_1(\wp), \omega_2(\wp)) + \eta_{\mathfrak{f}}(\wp),$$

$$\text{ABC} \mathfrak{D}^{\mathfrak{R}} \omega_2(\wp) = \mathfrak{g}(\wp, \omega_1(\wp), \omega_2(\wp)) + \eta_{\mathfrak{g}}(\wp).$$

Applying the ABC integral operator to both sides yields

$$\omega_1(\wp) = \frac{1 - \mathfrak{N}}{\mathbf{B}(\mathfrak{N})} [\mathfrak{f}(\wp, \omega_1(\wp), \omega_2(\wp)) + \eta_{\mathfrak{f}}(\wp)]$$

$$\begin{aligned}
& + \frac{\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph)} \int_0^\wp (\wp - \zeta)^{\aleph-1} [\mathfrak{f}(\zeta, \omega_1(\zeta), \omega_2(\zeta)) + \eta_{\mathfrak{f}}(\zeta)] \mathfrak{d}\zeta \\
& + \chi_1 \int_0^1 \omega_1(\zeta) \mathfrak{d}\zeta + \mathfrak{c}_1, \\
\omega_2(\wp) & = \frac{1 - \aleph}{\mathbf{B}(\aleph)} [\mathfrak{g}(\wp, \omega_1(\wp), \omega_2(\wp)) + \eta_{\mathfrak{g}}(\wp)] \\
& + \frac{\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph)} \int_0^\wp (\wp - \zeta)^{\aleph-1} [\mathfrak{g}(\zeta, \omega_1(\zeta), \omega_2(\zeta)) + \eta_{\mathfrak{g}}(\zeta)] \mathfrak{d}\zeta \\
& + \chi_2 \int_0^1 \omega_2(\zeta) \mathfrak{d}\zeta + \mathfrak{c}_2.
\end{aligned}$$

Consequently,

$$\begin{aligned}
|\omega_1(\wp) - \mathfrak{P}_1(\omega_1, \omega_2)(\wp)| & \leq \Phi_{\mathfrak{f}}(\epsilon_1) \left(\frac{1 - \aleph}{\mathbf{B}(\aleph)} + \frac{\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph + 1)} \right), \\
|\omega_2(\wp) - \mathfrak{P}_2(\omega_1, \omega_2)(\wp)| & \leq \Phi_{\mathfrak{g}}(\epsilon_2) \left(\frac{1 - \aleph}{\mathbf{B}(\aleph)} + \frac{\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph + 1)} \right).
\end{aligned}$$

Theorem 4.2 (GUH Stability). *Suppose that*

$$|\mathfrak{f}(\wp, \omega_{11}, \omega_{21}) - \mathfrak{f}(\wp, \omega_{12}, \omega_{22})| \leq \ell_1 (|\omega_{11} - \omega_{12}| + |\omega_{21} - \omega_{22}|),$$

$$|\mathfrak{g}(\wp, \omega_{11}, \omega_{21}) - \mathfrak{g}(\wp, \omega_{12}, \omega_{22})| \leq \ell_2 (|\omega_{11} - \omega_{12}| + |\omega_{21} - \omega_{22}|),$$

for all $\wp \in [0, 1]$ and $\omega_{11}, \omega_{12}, \omega_{21}, \omega_{22} \in \mathcal{R}^+$, where $\ell_1, \ell_2 > 0$. Furthermore, assume that

$$q = \left(\frac{\ell_1(1 - \aleph)}{\mathbf{B}(\aleph)} + \frac{\ell_1 \aleph}{\mathbf{B}(\aleph)\Gamma(\aleph + 1)} + \chi_1 \right) + \left(\frac{\ell_2(1 - \aleph)}{\mathbf{B}(\aleph)} + \frac{\ell_2 \aleph}{\mathbf{B}(\aleph)\Gamma(\aleph + 1)} + \chi_2 \right) < 1.$$

Then the coupled system (1.1)-(1.2) is generalized UH stable.

Proof. Let (ω_1^*, ω_2^*) be the unique solution of the coupled system (1.1)-(1.2) and let (ω_1, ω_2) be an approximate solution satisfying

$$\left| {}^{\text{ABC}}\mathfrak{D}^{\aleph} \omega_1(\wp) - \mathfrak{f}(\wp, \omega_1(\wp), \omega_2(\wp)) \right| \leq \Phi_{\mathfrak{f}}(\epsilon_1),$$

$$\left| {}^{\text{ABC}}\mathfrak{D}^{\aleph} \omega_2(\wp) - \mathfrak{g}(\wp, \omega_1(\wp), \omega_2(\wp)) \right| \leq \Phi_{\mathfrak{g}}(\epsilon_2).$$

Using Lemma 4.2 and the operator $\mathfrak{P}$, we obtain

$$|\omega_1(\wp) - \omega_1^*(\wp)| \leq |\omega_1(\wp) - \mathfrak{P}_1(\omega_1, \omega_2)(\wp)| + |\mathfrak{P}_1(\omega_1, \omega_2)(\wp) - \mathfrak{P}_1(\omega_1^*, \omega_2^*)(\wp)|,$$

$$|\omega_2(\wp) - \omega_2^*(\wp)| \leq |\omega_2(\wp) - \mathfrak{P}_2(\omega_1, \omega_2)(\wp)| + |\mathfrak{P}_2(\omega_1, \omega_2)(\wp) - \mathfrak{P}_2(\omega_1^*, \omega_2^*)(\wp)|.$$

From Lemma 4.2,

$$|\omega_1(\wp) - \mathfrak{P}_1(\omega_1, \omega_2)(\wp)| \leq \Phi_{\mathfrak{f}}(\epsilon_1) \left(\frac{1 - \aleph}{\mathbf{B}(\aleph)} + \frac{\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph + 1)} \right),$$

$$|\omega_2(\wp) - \mathfrak{P}_2(\omega_1, \omega_2)(\wp)| \leq \Phi_{\mathfrak{g}}(\epsilon_2) \left(\frac{1 - \aleph}{\mathbf{B}(\aleph)} + \frac{\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph + 1)} \right).$$

Since $\mathfrak{P}$ is a contraction,

$$|\mathfrak{P}_1(\omega_1, \omega_2)(\wp) - \mathfrak{P}_1(\omega_1^*, \omega_2^*)(\wp)| \leq \left(\frac{\ell_1(1-\aleph)}{\mathbf{B}(\aleph)} + \frac{\ell_1\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph+1)} + \chi_1 \right) \|(\omega_1 - \omega_1^*, \omega_2 - \omega_2^*)\|,$$

$$|\mathfrak{P}_2(\omega_1, \omega_2)(\wp) - \mathfrak{P}_2(\omega_1^*, \omega_2^*)(\wp)| \leq \left(\frac{\ell_2(1-\aleph)}{\mathbf{B}(\aleph)} + \frac{\ell_2\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph+1)} + \chi_2 \right) \|(\omega_1 - \omega_1^*, \omega_2 - \omega_2^*)\|.$$

Therefore,

$$\begin{aligned} \|\omega_1 - \omega_1^*\| &\leq \Phi_{\mathfrak{f}}(\epsilon_1) \left(\frac{1-\aleph}{\mathbf{B}(\aleph)} + \frac{\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph+1)} \right) + \left(\frac{\ell_1(1-\aleph)}{\mathbf{B}(\aleph)} + \frac{\ell_1\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph+1)} + \chi_1 \right) \\ &\quad \times \|(\omega_1 - \omega_1^*, \omega_2 - \omega_2^*)\|, \end{aligned}$$

$$\begin{aligned} \|\omega_2 - \omega_2^*\| &\leq \Phi_{\mathfrak{g}}(\epsilon_2) \left(\frac{1-\aleph}{\mathbf{B}(\aleph)} + \frac{\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph+1)} \right) + \left(\frac{\ell_2(1-\aleph)}{\mathbf{B}(\aleph)} + \frac{\ell_2\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph+1)} + \chi_2 \right) \\ &\quad \times \|(\omega_1 - \omega_1^*, \omega_2 - \omega_2^*)\|. \end{aligned}$$

Adding these inequalities and using $q < 1$, we get

$$(1-q)\|(\omega_1 - \omega_1^*, \omega_2 - \omega_2^*)\| \leq \Phi_{\mathfrak{f}}(\epsilon_1) \left(\frac{1-\aleph}{\mathbf{B}(\aleph)} + \frac{\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph+1)} \right) + \Phi_{\mathfrak{g}}(\epsilon_2) \left(\frac{1-\aleph}{\mathbf{B}(\aleph)} + \frac{\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph+1)} \right).$$

Thus,

$$\|(\omega_1 - \omega_1^*, \omega_2 - \omega_2^*)\| \leq \mathbf{C}_{\mathfrak{f}} \max\{\Phi_{\mathfrak{f}}(\epsilon_1), \Phi_{\mathfrak{g}}(\epsilon_2)\},$$

where

$$\mathbf{C}_{\mathfrak{f}} = \frac{\frac{1-\aleph}{\mathbf{B}(\aleph)} + \frac{\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph+1)} + \frac{1-\aleph}{\mathbf{B}(\aleph)} + \frac{\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph+1)}}{1-q}.$$

Therefore, the coupled system (1.1)-(1.2) is generalized UH stable. $\square$

Theorem 4.3 (UHR Stability). Assume that all conditions of Theorem 4.1 hold. Let $\phi, \psi : [0, 1] \rightarrow \mathcal{R}^+$ be continuous and suppose there exist constants $\zeta_{\phi}, \zeta_{\psi} > 0$ s.t.

$$\begin{aligned} \frac{\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph)} \int_0^{\wp} (\wp - \zeta)^{\aleph-1} \phi(\zeta) \, \mathfrak{d}\zeta &\leq \zeta_{\phi} \phi(\wp), \\ \frac{\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph)} \int_0^{\wp} (\wp - \zeta)^{\aleph-1} \psi(\zeta) \, \mathfrak{d}\zeta &\leq \zeta_{\psi} \psi(\wp), \end{aligned}$$

for all $\wp \in [0, 1]$. Then the coupled system (1.1)-(1.2) is UHR stable.

Proof. Assume

$$\begin{aligned} \left| {}^{\text{ABC}}\mathfrak{D}^{\aleph} \omega_1(\wp) - \mathfrak{f}(\wp, \omega_1(\wp), \omega_2(\wp)) \right| &\leq \epsilon_1 \phi(\wp), \\ \left| {}^{\text{ABC}}\mathfrak{D}^{\aleph} \omega_2(\wp) - \mathfrak{g}(\wp, \omega_1(\wp), \omega_2(\wp)) \right| &\leq \epsilon_2 \psi(\wp). \end{aligned}$$

Proceeding as in Theorem 4.1, we obtain

$$\begin{aligned} |\omega_1(\wp) - \mathfrak{P}_1(\omega_1, \omega_2)(\wp)| &\leq \epsilon_1 \left(\frac{1-\aleph}{\mathbf{B}(\aleph)} + \zeta_{\phi} \right) \phi(\wp), \\ |\omega_2(\wp) - \mathfrak{P}_2(\omega_1, \omega_2)(\wp)| &\leq \epsilon_2 \left(\frac{1-\aleph}{\mathbf{B}(\aleph)} + \zeta_{\psi} \right) \psi(\wp). \end{aligned}$$

Using the contraction property,

$$\begin{aligned}\|\omega_1 - \omega_1^*\| &\leq \epsilon_1 \left(\frac{1-\aleph}{\mathbf{B}(\aleph)} + \zeta_\phi \right) \phi(\wp) + \left(\frac{\ell_1(1-\aleph)}{\mathbf{B}(\aleph)} + \frac{\ell_1\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph+1)} + \chi_1 \right) \|(\omega_1 - \omega_1^*, \omega_2 - \omega_2^*)\|, \\ \|\omega_2 - \omega_2^*\| &\leq \epsilon_2 \left(\frac{1-\aleph}{\mathbf{B}(\aleph)} + \zeta_\psi \right) \psi(\wp) + \left(\frac{\ell_2(1-\aleph)}{\mathbf{B}(\aleph)} + \frac{\ell_2\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph+1)} + \chi_2 \right) \|(\omega_1 - \omega_1^*, \omega_2 - \omega_2^*)\|.\end{aligned}$$

Hence,

$$\|(\omega_1 - \omega_1^*, \omega_2 - \omega_2^*)\| \leq \mathbf{C}_{\mathfrak{f},\phi} \epsilon_1 \phi(\wp) + \mathbf{C}_{\mathfrak{g},\psi} \epsilon_2 \psi(\wp),$$

where

$$\mathbf{C}_{\mathfrak{f},\phi} = \frac{\frac{1-\aleph}{\mathbf{B}(\aleph)} + \zeta_\phi}{1-q}, \quad \mathbf{C}_{\mathfrak{g},\psi} = \frac{\frac{1-\aleph}{\mathbf{B}(\aleph)} + \zeta_\psi}{1-q}.$$

Therefore, the coupled system (1.1)-(1.2) is UHR stable. $\square$

5. NUMERICAL EXAMPLES

To validate theoretical findings, two examples are provided in this section.

Example 5.1. Consider the following coupled system of FDEs with IBCs involving the ABC derivative:

$$\begin{cases} {}^{\text{ABC}}\mathfrak{D}_{0+}^{1/2} \omega_1(\wp) = 1 + \frac{\omega_1(\wp)}{1+\sin(\omega_1(\wp))} + \frac{\omega_2(\wp)}{2+\cos(\omega_2(\wp))}, & \wp \in [0, 1], \\ {}^{\text{ABC}}\mathfrak{D}_{0+}^{1/2} \omega_2(\wp) = 2 + \frac{\omega_2(\wp)}{1+\sin(\omega_2(\wp))} + \frac{\omega_1(\wp)}{3+\cos(\omega_1(\wp))}, & \wp \in [0, 1], \end{cases} \quad (5.1)$$

subject to the integral boundary conditions:

$$\omega_1(0) = \frac{2}{3} \int_0^1 \omega_1(\zeta) \mathfrak{d}\zeta + \frac{1}{2}, \quad \omega_2(0) = \frac{1}{2} \int_0^1 \omega_2(\zeta) \mathfrak{d}\zeta + \frac{1}{3}. \quad (5.2)$$

Here $\aleph = \aleph = \frac{1}{2}$, $\chi_1 = \frac{2}{3}$, $\chi_2 = \frac{1}{2}$, $\mathbf{c}_1 = \frac{1}{2}$, $\mathbf{c}_2 = \frac{1}{3}$, and $\mathbf{B}(\aleph) = \mathbf{B}(\aleph) = 1$.

For $0 < \omega_{11}, \omega_{12}, \omega_{21}, \omega_{22} < +\infty$ and $\wp \in [0, 1]$, we have

$$\begin{aligned}|\mathfrak{f}(\wp, \omega_{11}, \omega_{21}) - \mathfrak{f}(\wp, \omega_{12}, \omega_{22})| &\leq \frac{1}{4} |\omega_{11} - \omega_{12}| + \frac{1}{2} |\omega_{21} - \omega_{22}| \leq \frac{3}{4} (|\omega_{11} - \omega_{12}| + |\omega_{21} - \omega_{22}|), \\ |\mathfrak{g}(\wp, \omega_{11}, \omega_{21}) - \mathfrak{g}(\wp, \omega_{12}, \omega_{22})| &\leq \frac{1}{4} |\omega_{21} - \omega_{22}| + \frac{1}{3} |\omega_{11} - \omega_{12}| \leq \frac{7}{12} (|\omega_{11} - \omega_{12}| + |\omega_{21} - \omega_{22}|).\end{aligned}$$

Thus, $\ell_1 = \frac{3}{4}$ and $\ell_2 = \frac{7}{12}$.

Now,

$$\begin{aligned}\frac{\ell_1(1-\aleph)}{\mathbf{B}(\aleph)} + \frac{\ell_1\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph+1)} + \chi_1 &= \frac{3}{4} \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1/2}{\Gamma(3/2)} + \frac{2}{3} \approx 0.375 + 0.2116 + 0.6667 = 1.2533, \\ \frac{\ell_2(1-\aleph)}{\mathbf{B}(\aleph)} + \frac{\ell_2\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph+1)} + \chi_2 &= \frac{7}{12} \cdot \frac{1}{2} + \frac{7}{12} \cdot \frac{1/2}{\Gamma(3/2)} + \frac{1}{2} \approx 0.2917 + 0.1646 + 0.5 = 0.9563.\end{aligned}$$

Therefore,

$$\left(\frac{\ell_1(1-\aleph)}{\mathbf{B}(\aleph)} + \frac{\ell_1\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph+1)} + \chi_1 \right) + \left(\frac{\ell_2(1-\aleph)}{\mathbf{B}(\aleph)} + \frac{\ell_2\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph+1)} + \chi_2 \right) \approx 2.2096 \neq 1.$$

Since the contraction condition is not satisfied, we cannot apply Theorem 3.2 to guarantee uniqueness. However, we can still establish existence using Theorem 3.1 if we can construct appropriate upper and lower solutions.

Let us verify that $\widetilde{\omega}_1(\wp) = \widetilde{\omega}_2(\wp) = \mathcal{M}$ and $\underline{\omega}_1(\wp) = \underline{\omega}_2(\wp) = 0$ are upper and lower solutions, respectively, for sufficiently large $\mathcal{M} > 0$.

For the upper solutions, we need:

$$\begin{aligned}\mathcal{M} &\geq \frac{1-\aleph}{\mathbf{B}(\aleph)} \left(1 + \frac{\mathcal{M}}{1} + \frac{\mathcal{M}}{2}\right) + \frac{\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph)} \int_0^\wp (\wp - \zeta)^{\aleph-1} \left(1 + \frac{\mathcal{M}}{1} + \frac{\mathcal{M}}{2}\right) \mathfrak{d}\zeta + \chi_1 \mathcal{M} + \mathfrak{c}_1, \\ \mathcal{M} &\geq \frac{1-\aleph}{\mathbf{B}(\aleph)} \left(2 + \frac{\mathcal{M}}{1} + \frac{\mathcal{M}}{3}\right) + \frac{\aleph}{\mathbf{B}(\aleph)\Gamma(\aleph)} \int_0^\wp (\wp - \zeta)^{\aleph-1} \left(2 + \frac{\mathcal{M}}{1} + \frac{\mathcal{M}}{3}\right) \mathfrak{d}\zeta + \chi_2 \mathcal{M} + \mathfrak{c}_2.\end{aligned}$$

For sufficiently large $\mathcal{M}$, these inequalities hold, so $\widetilde{\omega}_1(\wp) = \widetilde{\omega}_2(\wp) = \mathcal{M}$ are upper solutions. Also, $\underline{\omega}_1(\wp) = \underline{\omega}_2(\wp) = 0$ are lower solutions since $\mathfrak{f}, \mathfrak{g} \geq 0$.

By Theorem 3.1, the coupled system (5.1)-(5.2) has at least one positive solution.

Example 5.2. Consider the following coupled system:

$$\begin{cases} {}^{\text{ABC}}\mathfrak{D}_{0+}^{1/3} \omega_1(\wp) = \frac{1}{2} \sin t + \frac{\omega_1(\wp)}{4(1+\omega_1(\wp))}, & \wp \in [1/2, 1], \\ {}^{\text{ABC}}\mathfrak{D}_{0+}^{1/4} \omega_2(\wp) = \frac{1}{3} \cos t + \frac{\omega_2(\wp)}{5(1+\omega_2(\wp))}, & \wp \in [1/2, 1], \end{cases} \quad (5.3)$$

with integral boundary conditions:

$$\omega_1(0) = \frac{1}{4} \int_{1/2}^1 \omega_1(\zeta) \mathfrak{d}\zeta, \quad \omega_2(0) = \frac{1}{5} \int_{1/2}^1 \omega_2(\zeta) \mathfrak{d}\zeta. \quad (5.4)$$

Here $\chi_1 = \frac{1}{4}$, $\chi_2 = \frac{1}{5}$, $\mathfrak{c}_1 = \mathfrak{c}_2 = 0$, $\aleph = \frac{1}{3}$, $\aleph = \frac{1}{4}$, and $\mathbf{B}(\aleph) = \mathbf{B}(\aleph) = 1$.

For $0 < \omega_{11}, \omega_{12}, \omega_{21}, \omega_{22} < +\infty$ and $\wp \in [1/2, 1]$, we have

$$\begin{aligned}|\mathfrak{f}(\wp, \omega_{11}, \omega_{21}) - \mathfrak{f}(\wp, \omega_{12}, \omega_{22})| &\leq \frac{1}{4} |\omega_{11} - \omega_{12}| \leq \frac{1}{4} (|\omega_{11} - \omega_{12}| + |\omega_{21} - \omega_{22}|), \\ |\mathfrak{g}(\wp, \omega_{11}, \omega_{21}) - \mathfrak{g}(\wp, \omega_{12}, \omega_{22})| &\leq \frac{1}{5} |\omega_{21} - \omega_{22}| \leq \frac{1}{5} (|\omega_{11} - \omega_{12}| + |\omega_{21} - \omega_{22}|).\end{aligned}$$

Thus, $\ell_1 = \frac{1}{4}$ and $\ell_2 = \frac{1}{5}$.

Now,

$$\begin{aligned}\frac{\ell_1(1-\aleph)}{\mathbf{B}(\aleph)} + \frac{\ell_1 \aleph}{\mathbf{B}(\aleph)\Gamma(\aleph+1)} + \chi_1 &= \frac{1}{4} \cdot \frac{2}{3} + \frac{1}{4} \cdot \frac{1/3}{\Gamma(4/3)} + \frac{1}{4} \approx 0.1667 + 0.0723 + 0.25 = 0.4890, \\ \frac{\ell_2(1-\aleph)}{\mathbf{B}(\aleph)} + \frac{\ell_2 \aleph}{\mathbf{B}(\aleph)\Gamma(\aleph+1)} + \chi_2 &= \frac{1}{5} \cdot \frac{3}{4} + \frac{1}{5} \cdot \frac{1/4}{\Gamma(5/4)} + \frac{1}{5} \approx 0.15 + 0.0442 + 0.2 = 0.3942.\end{aligned}$$

Therefore,

$$\left(\frac{\ell_1(1-\aleph)}{\mathbf{B}(\aleph)} + \frac{\ell_1 \aleph}{\mathbf{B}(\aleph)\Gamma(\aleph+1)} + \chi_1 \right) + \left(\frac{\ell_2(1-\aleph)}{\mathbf{B}(\aleph)} + \frac{\ell_2 \aleph}{\mathbf{B}(\aleph)\Gamma(\aleph+1)} + \chi_2 \right) \approx 0.8832 < 1.$$

By Theorem 3.2, the coupled system (5.3)-(5.4) has a unique positive solution $(\omega_1, \omega_2) \in \mathbf{C}[1/2, 1] \times \mathbf{C}[1/2, 1]$.

Using the Laplace transform of the ABC fractional derivative, we can obtain an explicit solution. The Mittag-Leffler function

$$\mathbf{E}_{\aleph, \aleph}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\aleph k + \aleph)}, \quad z \in \mathbb{C},$$

appears in the solution representation. The unique solution satisfies

$$\underline{\omega}_1(\varphi) \leq \omega_1(\varphi) \leq \widetilde{\omega}_1(\varphi), \quad \underline{\omega}_2(\varphi) \leq \omega_2(\varphi) \leq \widetilde{\omega}_2(\varphi),$$

where appropriate lower and upper solutions can be constructed explicitly.

5.1. Graphical Representation of the Results. The three-dimensional graphical simulations provide visual support for the analytical results established in this work. Figures 1 and 2 illustrate the behavior of the positive solution components $(\omega_1(\varphi))$ and $(\omega_2(\varphi))$, respectively. The obtained surfaces remain positive throughout the domain, confirming the positivity results derived theoretically.

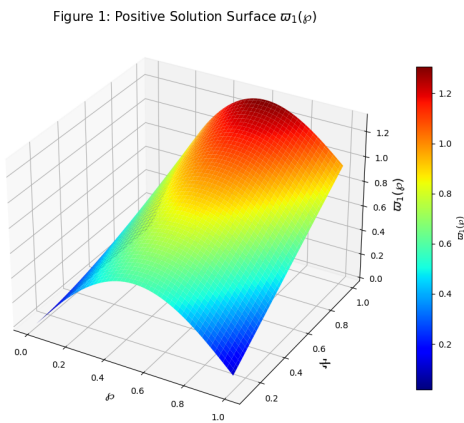


FIGURE 1. Three-dimensional profile of the positive solution component $\omega_1(\varphi)$.

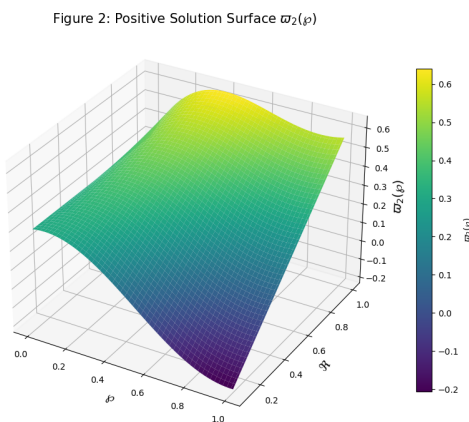


FIGURE 2. Three-dimensional profile of the positive solution component $\omega_2(\varphi)$.

Figure 3: Coupled Solution Interaction Surface

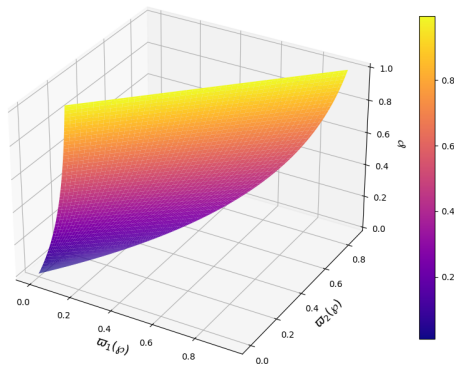
FIGURE 3. Interaction surface between the coupled positive solutions $\varphi_1(\rho)$ and $\varphi_2(\rho)$.

Figure 4: Ulam-Hyers Stability Surface

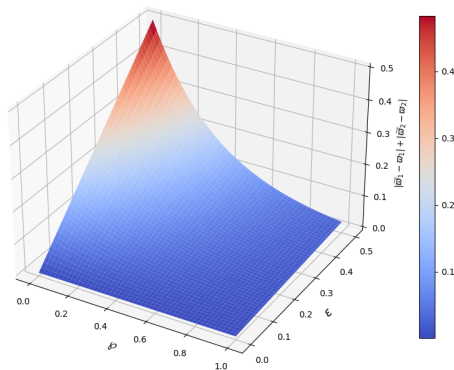


FIGURE 4. UH stability surface of the coupled ABC fractional system.

Figure 5: Stability and Uniqueness Region

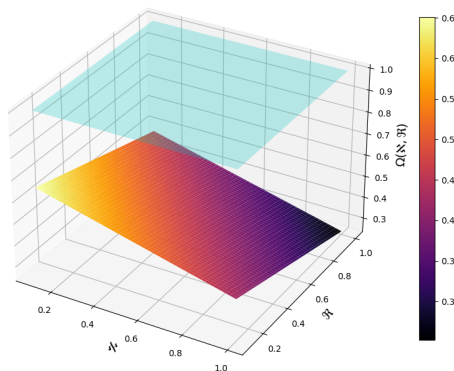
FIGURE 5. Contraction Region with $\Omega < 1$ for uniqueness of positive solution.

Figure 3 shows the interrelationship between the two components of the coupled solution and highlights the dependence of one on the other within the nonlinear ABC fractional system. Figure 4 presents the behavior of UH stability, revealing that minor disturbances will lead to bounded differences in the trajectories of the solution. This observation agrees with the UH, GUH and UHR stability results proved in the paper.

Figure 5 presents the contraction region associated with the Banach FP condition. The graphical profile confirms that the contraction quantity satisfies ($\Omega < 1$), with the numerical value approximately equal to (0.8832). Consequently, the coupled ABC fractional system admits a unique positive solution, in agreement with the theoretical findings.

6. CONCLUSIONS

This paper studied coupled system of nonlinear ABC FDEs with IBCs. By constructing lower and upper control functions without requiring monotonicity, we proved existence via Schauder's FPT and uniqueness via Banach's contraction principle. We further showed that the unique positive solution is UH, GUH and UHR stable, meaning small perturbations cause only small changes in the solution. Two examples confirmed our results. Generally, this paper presents a reliable and stable method for solving coupled ABC FDEs with nonlocal boundary conditions. The future scope of work includes generalization of the present results for higher-order systems, systems with delay, and application to particular physical/biological models.

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