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Hardy-Rogers Type Interpolative Contractions in Fuzzy Strong b-Metric Spaces with Application to a Duffing Oscillator

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Abstract. We study fixed point theory in fuzzy strong b-metric spaces for a class of interpolative contractions driven by a pair of auxiliary functions (Υ, F). After recalling the necessary background, we introduce Banach, Kannan, Chatterjea, Reich-Rus-Ciric, and Hardy-Rogers type fuzzy strong b-interpolative contractions and establish existence and uniqueness of fixed points for the corresponding (Υ, F) versions. A recurring theme is that these contractions are strictly weaker than their classical analogues: for each type we exhibit a mapping that satisfies the interpolative condition while violating the classical one, and we identify explicitly the range of the contraction parameter over which the classical inequality fails. As an application, we recast a modified Duffing oscillator as a nonlinear Volterra integral equation of the second kind and use the Hardy-Rogers (Υ, F) theorem to prove that it admits a unique continuous solution. We also derive a geometric a posteriori error bound for the associated Picard iteration, and numerical experiments together with phase-space diagrams illustrate both the convergence and the underlying nonlinear dynamics.

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1. Introduction

Fixed point theory is one of the central tools of nonlinear analysis: a point ς with $T\varsigma = \varsigma$ for a self-map T of a metric space $(\mathbb{C}, d)$ encodes the solution of an equation, and the Banach contraction principle turns the search for such a point into a convergent iteration. Much of the subsequent development has proceeded along two complementary lines: relaxing the contraction inequality, and enlarging the ambient space in which it is posed. Both directions are motivated by concrete problems in which the available information is imprecise or only partially known, and for which a rigid classical metric is too restrictive. Fuzzy metric spaces address exactly this situation by measuring nearness through a degree of membership rather than a single real number, and they carry over a large part of the metric and topological machinery to a setting better suited to uncertainty.

The foundations were laid over several decades. Schweizer and Sklar [1] introduced statistical metric spaces and the notion of a continuous t-norm; Zadeh [2] introduced fuzzy sets; and Kramosil and Michálek [3] combined these ideas to define fuzzy metric spaces. George and Veeramani [4] later modified this definition to obtain a Hausdorff topology, and Grabiec [5] proved the first fixed point theorems in the Kramosil–Michálek framework. Gregori and Sapena [6] further advanced fixed point theory in the fuzzy setting by establishing fixed point theorems for complete fuzzy metric spaces. Their results addressed fuzzy metric spaces in the sense of George and Veeramani as well as Kramosil–Michálek fuzzy metric spaces endowed with the completeness notion introduced by Grabiec [5].

A second strand of the theory concerns weaker contraction conditions. Karapınar [7–9] introduced *interpolative* contractions, which admit discontinuous mappings by combining several distances through a weighted geometric mean. Auxiliary (control) functions provide a further layer of flexibility: de la Sen, Fulga, Karapınar and Shahzad [10] introduced Proinov-type auxiliary functions in fuzzy metric spaces, and Zhou et al. [11] refined this construction and established the corresponding fixed point theorems and corollaries. Iterative mappings in fuzzy metric settings were studied by Ishtiaq et al. [12] and by Javed et al. [13], both of whom used fixed point arguments to prove existence and uniqueness for associated models; related applications to differential and integral models appear in [14–19].

The metric side of the space has been generalized in parallel. Replacing the triangle inequality by a relaxed version gives b-metric spaces, in which Debnath and de la Sen [20] proved fixed point theorems for interpolative Ćirić–Reich–Rus contractions. Strengthening one term of that inequality yields *strong* b-metric spaces, studied by Doan [21]. In the fuzzy category, Hussain, Salimi and Parvaneh [22] obtained fixed point results for fuzzy b-metric spaces, while Oner [23,24] introduced fuzzy strong b-metric spaces (FSBMS) and established their basic topological properties. Kanwal, Kattan, Perveen, Islam and Shagari [25] subsequently proved fixed point theorems in FSBMS for interpolative Banach- and Reich–Rus–Ćirić-type contractions with fixed exponents.

Building on [11, 12, 23–25], our aim is to develop fixed point theory in FSBMS for interpolative contractions governed by a pair of auxiliary functions (Υ, F), thereby replacing fixed numerical exponents by functional control. The remainder of the paper is organized as follows. Section 2 collects the definitions and known results that we use. Section 3 contains our main contribution: we define Banach, Kannan, Chatterjea, Reich–Rus–Ćirić and Hardy–Rogers type fuzzy strong b-interpolative contractions, first with fixed exponents and then in their (Υ, F) form, prove existence and uniqueness of fixed points, and give examples that separate the interpolative conditions from the classical ones. Section 4 applies the Hardy–Rogers (Υ, F) theorem to a modified Duffing oscillator written as a Volterra integral equation, together with a numerical illustration. Section 5 compares our results with the literature, and Section 6 concludes.

2. Preliminaries

Throughout, $\mathbb{C}$ denotes a nonempty set and $\mathfrak{R}$ a fuzzy (strong b-)metric on it; the self-mapping under study is written Π. We collect here the definitions and known results needed later, following mainly Oner [23, 24], Kanwal et al. [25] and Zhou et al. [11]. We begin with the metric-type notions.

Definition 2.1 ([24]). *Let $\mathbb{C}$ be a nonempty set and ≥ 1. A map $\mathfrak{R} : \mathbb{C} \times \mathbb{C} \to \mathbb{R}^+$ is a b-*metric *on $\mathbb{C}$, and $(\mathbb{C}, \mathfrak{R},)$ a b-*metric space*, if for all $\varrho, \varsigma, \mathsf{y} \in \mathbb{C}$:*

(i) $\mathfrak{R}(\varrho, \varsigma) = 0$ *if and only if* $\varrho = \varsigma$;
(ii) $\mathfrak{R}(\varrho, \varsigma) = \mathfrak{R}(\varsigma, \varrho)$;
(iii) $\mathfrak{R}(\varrho, \varsigma) \leq \left[\mathfrak{R}(\varrho, \mathsf{y}) + \mathfrak{R}(\mathsf{y}, \varsigma)\right]$.

Definition 2.2 ([24]). *Let $\mathbb{C}$ be a nonempty set and ≥ 1. A map $\mathfrak{R} : \mathbb{C} \times \mathbb{C} \to \mathbb{R}^+$ is a* strong b-metric *(SBM) on $\mathbb{C}$, and $(\mathbb{C}, \mathfrak{R},)$ a* strong b-metric space *(SBMS), if for all $\varrho, \varsigma, \mathsf{y} \in \mathbb{C}$:*

(i) $\mathfrak{R}(\varrho, \varsigma) = 0$ *if and only if* $\varrho = \varsigma$;
(ii) $\mathfrak{R}(\varrho, \varsigma) = \mathfrak{R}(\varsigma, \varrho)$;
(iii) $\mathfrak{R}(\varrho, \varsigma) \leq \mathfrak{R}(\varrho, \mathsf{y}) + \ \mathfrak{R}(\mathsf{y}, \varsigma)$.

Remark 2.1. *In a strong b-metric space $(\mathbb{C}, \mathfrak{R},)$ the metric $\mathfrak{R}$ is continuous, and every open ball is an open set.*

Definition 2.3 ([25]). *A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a* continuous t-norm *(ctn) if it is associative, commutative, continuous, has 1 as identity and is monotone; that is, for all $\varrho_1, \varrho_2, \varrho_3, \varrho_4 \in [0,1]$:*

(1) $\varrho_1 * \varrho_2 = \varrho_2 * \varrho_1$;
(2) $\varrho_1 * (\varrho_2 * \varrho_3) = (\varrho_1 * \varrho_2) * \varrho_3$;
(3) $\varrho_1 * 1 = \varrho_1$;
(4) $\varrho_1 * \varrho_2 \leq \varrho_3 * \varrho_4$ *whenever* $\varrho_1 \leq \varrho_3$ *and* $\varrho_2 \leq \varrho_4$.

Example 2.1. *The three basic continuous t-norms are the minimum, product and Łukasiewicz t-norms:*

$$\varrho * \varsigma = \min(\varrho, \varsigma), \qquad \varrho * \varsigma = \varrho\varsigma, \qquad \varrho * \varsigma = \max(\varrho + \varsigma - 1,\ 0).$$

Definition 2.4 ([11]). *A* fuzzy metric space *(FMS) is a triple* $(\mathbb{C}, \mathfrak{R}, *)$ *where* $\mathbb{C}$ *is a nonempty set,* $*$ *is a ctn, and* $\mathfrak{R}$ *is a fuzzy set on* $\mathbb{C} \times \mathbb{C} \times (0, \infty)$ *such that for all* $\varrho, \varsigma, \mathsf{y} \in \mathbb{C}$ *and* $\upsilon, \varpi > 0$*:*

(i) $\mathfrak{R}(\varrho, \varsigma, \upsilon) > 0$;
(ii) $\mathfrak{R}(\varrho, \varsigma, \upsilon) = 1$ *if and only if* $\varrho = \varsigma$;
(iii) $\mathfrak{R}(\varrho, \varsigma, \upsilon) = \mathfrak{R}(\varsigma, \varrho, \upsilon)$;
(iv) $\mathfrak{R}(\varrho, \mathsf{y}, \max\{\upsilon, \varpi\}) \geq \mathfrak{R}(\varrho, \varsigma, \upsilon) * \mathfrak{R}(\varsigma, \mathsf{y}, \varpi)$;
(v) $\mathfrak{R}(\varrho, \varsigma, \cdot) : (0, \infty) \to [0, 1]$ *is left continuous.*

Example 2.2. *Let* $\mathbb{C} = \mathbb{R}^+$ *and* $\mathfrak{R}(\varrho, \varsigma, \upsilon) = \dfrac{\upsilon}{\upsilon + |\varrho - \varsigma|}$ *with the product ctn. Then* $(\mathbb{C}, \mathfrak{R}, *)$ *is an FMS.*

Definition 2.5 ([25]). *Let* $\mathbb{C} \neq \emptyset$*, let* ≥ 1*, and let* $*$ *be a ctn. A fuzzy set* $\mathfrak{R} : \mathbb{C} \times \mathbb{C} \times (0, \infty) \to (0, 1]$ *is a* fuzzy strong b-metric *(FSBM) on* $\mathbb{C}$*, and* $(\mathbb{C}, \mathfrak{R}, *,)$ *a* fuzzy strong b-metric space *(FSBMS), if for all* $\varrho, \varsigma, \mathsf{y} \in \mathbb{C}$ *and* $\upsilon, \varpi > 0$*:*

(i) $\mathfrak{R}(\varrho, \varsigma, \upsilon) > 0$;
(ii) $\mathfrak{R}(\varrho, \varsigma, \upsilon) = 1$ *if and only if* $\varrho = \varsigma$;
(iii) $\mathfrak{R}(\varrho, \varsigma, \upsilon) = \mathfrak{R}(\varsigma, \varrho, \upsilon)$;
(iv) $\mathfrak{R}(\varrho, \mathsf{y}, \upsilon + \varpi) \geq \mathfrak{R}(\varrho, \varsigma, \upsilon) * \mathfrak{R}(\varsigma, \mathsf{y}, \varpi)$;
(v) $\mathfrak{R}(\varrho, \varsigma, \cdot) : (0, \infty) \to [0, 1]$ *is left continuous.*

Example 2.3. *Let* $\mathbb{C} = \mathbb{R}$ *and*

$$\mathfrak{R}(\varrho, \varsigma, \upsilon) = \frac{\upsilon}{\upsilon + \mathfrak{R}(\varrho, \varsigma)}$$

for $\upsilon > 0$ *and* $\varrho, \varsigma \in \mathbb{C}$ *with product ctn. Then,* $(\mathbb{C}, \mathfrak{R}, *,)$ *is a FSBMS. Hence, (i)–(iii) and (v) are obvious; now we show (iv)*

$$\begin{aligned}
\mathfrak{R}(\varrho, \varsigma, \upsilon) . \mathfrak{R}(\varsigma, \mathsf{y}, \varpi) &= \frac{\upsilon}{\upsilon + \mathfrak{R}(\varrho, \varsigma)} . \frac{\varpi}{\varpi + \mathfrak{R}(\varsigma, \mathsf{y})} \\
&= \frac{1}{1 + \frac{\mathfrak{R}(\varrho, \varsigma)}{\upsilon}} . \frac{1}{1 + \frac{h\mathfrak{R}(\varsigma, \mathsf{y})}{\varpi}} \\
&\leq \frac{1}{1 + \frac{\mathfrak{R}(\varrho, \varsigma)}{\upsilon + \varpi}} . \frac{1}{1 + \frac{\mathfrak{R}(\varsigma, \mathsf{y})}{\upsilon + \varpi}} \\
&\leq \frac{1}{1 + \frac{\mathfrak{R}(\varrho, \varsigma) + \mathfrak{R}(\varsigma, \mathsf{y})}{\upsilon + \varpi}} \\
&\leq \frac{1}{1 + \mathfrak{R}(\varrho, \mathsf{y})} \\
&= \frac{\upsilon + \varpi}{\upsilon + \varpi + \mathfrak{R}(\varrho, \mathsf{y})} \\
&= \mathfrak{R}(\varrho, \mathsf{y}, \upsilon + \varpi).
\end{aligned}$$

Hence, all the conditions are satisfied with ≥ 1*. Thus,* $\mathfrak{R}(\varrho, \varsigma, \upsilon)$ *is an FSBMS.*

Example 2.4. *Let* $\mathbb{C} = \mathbb{R}$ *and*

$$\mathfrak{R}(\varrho, \varsigma, \upsilon) = e^{-\frac{\mathfrak{R}(\varrho, \varsigma)}{\upsilon}}$$

for $\upsilon > 0$ and $\varrho, \varsigma \in \mathbb{C}$, with $\ \geq 1$ and product ctn. Then, $(\mathbb{C}, \mathfrak{R}, *,)$ is a FSBMS. Hence, (i)–(iii) and (v) are obvious, and we show (iv)

$$\begin{aligned}
\mathfrak{R}(\varrho,\varsigma,\upsilon).\mathfrak{R}(\varsigma,\mathsf{y},\varpi) &= e^{-\frac{\mathfrak{R}(\varrho,\varsigma)}{\upsilon}}.e^{-\frac{\mathfrak{R}(\varsigma,\mathsf{y})}{\varpi}} \\
&= e^{-\frac{\mathfrak{R}(\varrho,\varsigma)}{\upsilon}}.e^{-\frac{\mathfrak{R}(\varsigma,\mathsf{y})}{\varpi}} \\
&\leq e^{-\frac{\mathfrak{R}(\varrho,\varsigma)}{\upsilon+\varpi}}.e^{-\frac{\mathfrak{R}(\varsigma,\mathsf{y})}{\upsilon+\varpi}} \\
&\leq e^{-\left(\frac{\mathfrak{R}(\varrho,\varsigma)}{\upsilon+\varpi}+\frac{\mathfrak{R}(\varsigma,\mathsf{y})}{\upsilon+\varpi}\right)} \\
&\leq e^{-\frac{\mathfrak{R}(\varrho,\varsigma)+\mathfrak{R}(\varsigma,\mathsf{y})}{\upsilon+\varpi}} \\
&= e^{-\frac{\mathfrak{R}(\varrho,\mathsf{y})}{\upsilon+\varpi}} \\
&= \mathfrak{R}(\varrho,\mathsf{y},\upsilon+\varpi).
\end{aligned}$$

Hence, all the conditions are satisfied. Thus, $\mathfrak{R}(\varrho,\varsigma,\upsilon)$ is an FSBMS.

Remark 2.2. *Every FMS is an FSBMS, but the converse fails.*

Indeed, let $\mathbb{C} = \{0,1,2\}$ and let d be the symmetric map with

$$d(0,0) = d(1,1) = d(2,2) = 0, \qquad d(0,1) = d(0,2) = 1, \qquad d(1,2) = 5,$$

and define the exponential fuzzy set $\mathfrak{R}(\varrho,\varsigma,\upsilon) = e^{-d(\varrho,\varsigma)/\upsilon}$ with the product ctn. Then $(\mathbb{C}, \mathfrak{R}, *,)$ is an FSBMS for a suitable $\ \geq 1$, but it is *not* an FMS. To see the latter, take $\varrho = 1$, $\varsigma = 0$, $\mathsf{y} = 2$ and $\upsilon = \varpi = 1$; the non-Archimedean triangle inequality 2.4(iv) would require

$$\mathfrak{R}\Big(1,2,\max\{1,1\}\Big) \ \geq \ \mathfrak{R}(1,0,1) * \mathfrak{R}(0,2,1), \qquad \text{i.e.} \qquad e^{-5} \ \geq \ e^{-2},$$

which is false. Hence $(\mathbb{C}, \mathfrak{R}, *)$ is not an FMS.

Definition 2.6 ([25])**.** *Let $(\mathbb{C}, \mathfrak{R}, *,)$ be an FSBMS and $\{\varrho_n\}$ a sequence in $\mathbb{C}$.*

(1) $\{\varrho_n\}$ converges *to $\varrho \in \mathbb{C}$ if $\lim_{n\to\infty} \mathfrak{R}(\varrho_n, \varrho, \upsilon) = 1$ for every $\upsilon > 0$.*
(2) $\{\varrho_n\}$ *is* Cauchy *if, for every $\varepsilon \in (0,1)$ and every $\upsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $\mathfrak{R}(\varrho_n, \varrho_m, \upsilon) > 1 - \varepsilon$ for all $n, m \geq n_0$.*
(3) $(\mathbb{C}, \mathfrak{R}, *,)$ *is* complete *if every Cauchy sequence converges.*

Definition 2.7 ([12])**.** *A self-mapping $\Pi : \mathbb{C} \to \mathbb{C}$ on an FSBMS is said to be asymptotically regular at a point in $\mathbb{C}$, if*

$$\lim_{n\to\infty} \mathfrak{R}\left(\Pi^n \varrho, \Pi^{n+1}\varrho, \upsilon\right) = 1.$$

Definition 2.8. *[25] A mapping $\Pi : \mathbb{C} \to \mathbb{C}$ satisfying*

$$\mathfrak{R}(\Pi\varrho, \Pi\varsigma, k\upsilon) \geq \mathfrak{R}(\varrho,\varsigma,\upsilon) \ \ \forall\, \varrho,\varsigma \in \mathbb{C},$$

is called a fuzzy contraction, with $k \in [0,1)$.

Definition 2.9. *[25] A mapping $\Pi : \mathbb{C} \to \mathbb{C}$ satisfying the following inequality*

$$\mathfrak{R}(\Pi\varrho, \Pi\varsigma, k\upsilon) \geq \mathfrak{R}(\varrho,\varsigma,\upsilon) \ \ \forall \varrho,\varsigma \in \mathbb{C}, \tag{2.1}$$

is called Banach contraction, where $(\mathbb{C}, \Re, *,)$ *is an FSBMS and* k *is a number in* $[0,1)$.

Theorem 2.1. *[25] Assume that* $(\mathbb{C}, \Re, *,)$ *is a complete FSBMS and* $*$ *is a ctn,* $\Re(\varrho, \varsigma, v)$ *be strictly increasing in variable* v, *and*

$$\lim_{n\to\infty} \Re\left(\Pi^n \varrho, \Pi\varsigma, v\right) = 1 \text{ for all } \varrho, \varsigma \in \mathbb{C}.$$

Suppose $\Pi : \mathbb{C} \to \mathbb{C}$ *is a mapping satisfying 2.1 for all* $\varrho, \varsigma \in \mathbb{C}$ *where* $k \in (0,1)$. *Thus* Π *has a unique fixed point in* $\mathbb{C}$.

Definition 2.10. *[25] A mapping* $\Pi : \mathbb{C} \to \mathbb{C}$ *satisfying the following inequality,*

$$\Re\left(\Pi\varrho, \Pi\varsigma, kv\right) \geq \Re\left(\varrho, \Pi\varrho, v\right) * \Re\left(\varsigma, \Pi\varsigma, v\right) \ \forall \varrho, \varsigma \in \mathbb{C}, \tag{2.2}$$

is called a Kannan type contraction, where $(\mathbb{C}, \Re, *,)$ *is an FSBMS and* $k \in (0,1)$.

Theorem 2.2. *[25] Assume* $(\mathbb{C}, \Re, *,)$ *is a complete FSBMS and* $*$ *is a ctn,* $\Re(\varrho, \varsigma, v)$ *is strictly increasing in variable,* v, *and*

$$\lim_{n\to\infty} \Re\left(\Pi^n \varrho, \Pi\varsigma, v\right) = 1 \text{ for all } \varrho, \varsigma \in \mathbb{C}.$$

Suppose $\Pi : \mathbb{C} \to \mathbb{C}$ *is a mapping satisfying 2.2 for all* $\varrho, \varsigma \in \mathbb{C}$ *where* $k \in (0,1)$. *Thus* Π *has a unique fixed point in* $\mathbb{C}$.

Definition 2.11. *[25] A mapping* $\Pi : \mathbb{C} \to \mathbb{C}$ *satisfying the following inequality,*

$$\Re\left(\Pi\varrho, \Pi\varsigma, kv\right) \geq \Re\left(\varrho, \Pi\varsigma, v\right) * \Re\left(\varsigma, \Pi\varrho, v\right) \ \forall \varrho, \varsigma \in \mathbb{C}, \tag{2.3}$$

is called a Chatterjea type contraction, where $(\mathbb{C}, \Re, *,)$ *is an FSBMS and* k *is a number in* $(0,1)$.

Theorem 2.3. *[25] Assume that* $(\mathbb{C}, \Re, *,)$ *is a complete FSBMS and* $*$ *is a ctn,* $\Re(\varrho, \varsigma, v)$ *be strictly increasing in variable,* v, *and*

$$\lim_{n\to\infty} \Re\left(\Pi^n \varrho, \Pi\varsigma, v\right) = 1 \text{ for all } \varrho, \varsigma \in \mathbb{C}.$$

Suppose $\Pi : \mathbb{C} \to \mathbb{C}$ *is a mapping satisfying (2.3) for all* $\varrho, \varsigma \in \mathbb{C}$ *where* $k \in (0,1)$. *Thus* Π *has a unique fixed point (in short, UFP) in* $\mathbb{C}$.

Proposition 2.1. *[24] Let* $(\mathbb{C}, \Re, *,)$ *be a FSBMS and* $\{\varrho_n\} \subset \mathbb{C}$ *be a sequence. If* $\{\varrho_n\}$ *is convergent, then the limit point* $\{\varrho_n\}$ *is unique.*

Proposition 2.2. *[24] Let* $(\mathbb{C}, \Re, *,)$ *be a FSBMS and* $\{\varrho_n\} \subset \mathbb{C}$ *be a sequence. If* $\{\varrho_n\}$ *is convergent, then* $\{\varrho_n\}$ *is Cauchy.*

Definition 2.12. *[11] The* $\mathcal{L}$ *family of pairs* (Υ, F) *of functions* $\Upsilon, F : (0,1] \to \mathbb{R}$ *satisfied the following assertions:*

(p1) Υ is nondecreasing (in short, ND);
(p2) $F(\varrho) > \Upsilon(\varrho)$ for any $\varrho \in (0,1)$;

(p3) $\lim_{\varrho\to L^-}\inf F(g) > \lim_{\varrho\to L^-}\Upsilon(g)$ for any $L\in(0,1)$;
(p4) if $\varrho\in(0,1)$ is such that $F(\varrho)\geq\Upsilon(1)$, then $p=1$.
Examples:
(1) $\Upsilon(\varrho)=\varrho$ and $F(\varrho)=\sqrt{\varrho}$ for all $\varrho\in(0,1)$.
(2) $\Upsilon(\varrho)=\frac{1}{\ln\varrho}$ and $F(\varrho)=\frac{1}{\ln\varrho^2}$ for all $\varrho\in(0,1)$.
(3) $\Upsilon(g)=\frac{1}{2^{\ln 2\varrho}}$ and $F(g)=\frac{1}{2^{\ln\varrho}}$ for all $\varrho\in(0,1)$.

3. Main Results

We now develop our fixed point theory for interpolative contractions in fuzzy strong b-metric spaces. The section is organized in two stages. First we introduce the fuzzy strong b-interpolative contractions (FSBIC) of Banach, Kannan, Chatterjea, Reich–Rus–Ćirić and Hardy–Rogers type, and for each we give a worked example together with a remark showing that the interpolative condition is strictly weaker than the corresponding classical fuzzy strong b-contraction. In the second stage we attach a pair of auxiliary functions (Υ, F) to each of these conditions, prove the associated existence and uniqueness theorems, and record the corollaries obtained by specializing the auxiliary pair. Together these results recover the theorems of [25] as special cases while covering a strictly larger class of mappings.

Definition 3.1. *Let $(\mathbb{C},\mathfrak{R},*,)$ be an FSBMS. A mapping $\Pi:\mathbb{C}\to\mathbb{C}$ is called a Banach-type FSBIC if*

$$\mathfrak{R}(\Pi\varrho,\Pi\varsigma,kv)\geq(\mathfrak{R}(\varrho,\varsigma,v))^{\nu},\tag{3.1}$$

for all $\varrho,\varsigma\in\mathbb{C}\setminus\mathrm{Fix}(\Pi)$, where $\nu\in(0,1]$ and $k\in(0,1)$.

Example 3.1. *Let $\mathbb{C}=[0,2]$ and*

$$\mathfrak{R}(\varrho,\varsigma,v)=e^{-\frac{|\varrho-\varsigma|}{v}}\ \textit{where } v>0.$$

is an FSBMS with ≥ 1 and the product ctn. Let $\Pi:\mathbb{C}\to\mathbb{C}$ be defined by $\Pi(\varrho)=\frac{\varrho+2}{3}$. For all $\varrho,\varsigma\in\mathbb{C}$,

$$\left|\frac{\varrho+2}{3}-\frac{\varsigma+2}{3}\right|=\frac{|\varrho-\varsigma|}{3},$$

so that

$$\mathfrak{R}(\Pi\varrho,\Pi\varsigma,kv)=e^{-\frac{|\varrho-\varsigma|}{3kv}},\qquad \textit{while}\qquad \Big(\mathfrak{R}(\varrho,\varsigma,v)\Big)^{\nu}=\left(e^{-\frac{|\varrho-\varsigma|}{v}}\right)^{\nu}=e^{-\frac{\nu|\varrho-\varsigma|}{v}}.$$

Since $e^{-a}\geq e^{-b}$ is equivalent to $a\leq b$, the interpolative inequality $\mathfrak{R}(\Pi\varrho,\Pi\varsigma,kv)\geq(\mathfrak{R}(\varrho,\varsigma,v))^{\nu}$ holds precisely when $\frac{1}{3kv}\leq\frac{\nu}{v}$, that is, when $k\nu\geq\frac{1}{3}$. Hence, for any $k,\nu\in(0,1)$ with $k\nu\geq\frac{1}{3}$ (for instance $k,\nu\in(\frac{1}{\sqrt{3}},1)$), the mapping Π is a Banach-type fuzzy strong b-interpolative contraction on $(\mathbb{C},\mathfrak{R},*,)$, with unique fixed point $\varrho=1$.

Remark 3.1. *But on the other hand, for the mapping, $\Pi(\varrho)=\frac{\varrho+2}{3}$. choose any $k\in(0,1)$, we have*

$$\mathfrak{R}(\Pi\varrho,\Pi\varsigma,kv)=e^{-\frac{|\varrho-\varsigma|}{3kv}}.$$

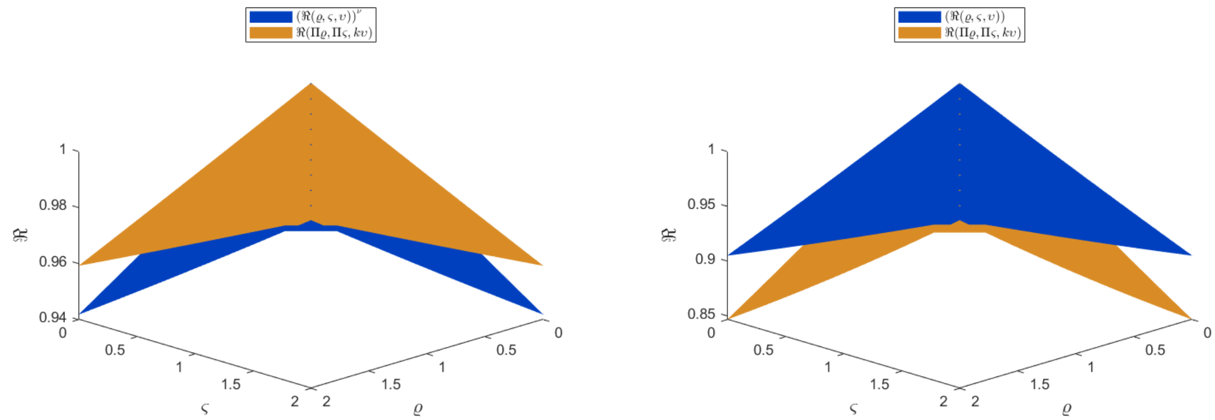

FIGURE 1. Comparison between Banach-type fuzzy strong b-interpolative contraction and the failure of Banach-type fuzzy strong b-contraction. The left figure confirms the interpolative contraction condition, while the right figure shows that the Banach-type fuzzy strong b-contraction condition fails for $k < \frac{1}{3} \in [0,1)$.

On the other hand,

$$\Re\left(\varrho,\varsigma,\upsilon)\right) = e^{-\frac{|\varrho-\varsigma|}{\upsilon}} = e^{-\frac{|\varrho-\varsigma|}{\upsilon}}$$

Since $\frac{1}{3k\upsilon} \leq \frac{1}{\upsilon}$, *it follows that* $k \geq \frac{1}{3}$. *Therefore, for every* $k < \frac{1}{3}$, *the inequality*

$$\Re(\Pi\varrho,\Pi\varsigma,k\upsilon) \geq \Re(\varrho,\varsigma,\upsilon)$$

does not hold.

Hence, Π *is not a Banach-type fuzzy strong b-contraction for* $k < \frac{1}{3}$ *on* $(\mathbb{C},\Re,*,)$.

Definition 3.2. *Let* $(\mathbb{C},\Re,*,)$ *be an FSBMS. A mapping* $\Pi : \mathbb{C} \to \mathbb{C}$ *is called Kannan-type FSBIC, if*

$$\Re\left(\Pi\varrho,\Pi\varsigma,k\upsilon\right) \geq \left(\Re\left(\varrho,\Pi\varrho,\upsilon\right)\right)^{\nu}\left(\Re\left(\varsigma,\Pi\varsigma,\upsilon\right)\right)^{1-\nu}, \tag{3.2}$$

for all $\varrho,\varsigma \in \mathbb{C} \setminus \mathrm{Fix}(\Pi)$, *where* $\nu \in (0,1)$ *and* $k \in (0,1)$.

Example 3.2. *Let* $\mathbb{C} = [0,2]$ *and define a mapping*

$$\Re : \mathbb{C} \times \mathbb{C} \times (0,\infty) \to (0,1]$$

by

$$\Re(\varrho,\varsigma,\upsilon) = \frac{\upsilon}{\upsilon + |\varrho-\varsigma|^2}, \qquad \upsilon > 0.$$

Then $(\mathbb{C},\Re,*,)$ *is a CFSBMS with* $= 2$, *where* $*$ *is a product ctn.*

Now define the self-mapping $\Pi : \mathbb{C} \to \mathbb{C}$ *by*

$$\Pi(\varrho) = \frac{\varrho}{4}, \qquad \forall\, \varrho \in \mathbb{C}.$$

For arbitrary $\varrho,\varsigma \in \mathbb{C}$, *we have*

$$|\Pi\varrho - \Pi\varsigma| = \left|\frac{\varrho}{4} - \frac{\varsigma}{4}\right| = \frac{1}{4}|\varrho-\varsigma|.$$

Hence,

$$\mathfrak{R}(\Pi\varrho, \Pi\varsigma, kv) = \frac{kv}{kv + \frac{1}{16}|\varrho - \varsigma|^2}.$$

Moreover,

$$|\varrho - \Pi\varrho| = \left|\varrho - \frac{\varrho}{4}\right| = \frac{3}{4}|\varrho|,$$

and similarly,

$$|\varsigma - \Pi\varsigma| = \frac{3}{4}|\varsigma|.$$

Therefore,

$$\mathfrak{R}(\varrho, \Pi\varrho, v) = \frac{v}{v + \frac{9}{16}|\varrho|^2},$$

and

$$\mathfrak{R}(\varsigma, \Pi\varsigma, v) = \frac{v}{v + \frac{9}{16}|\varsigma|^2}.$$

Let $k = 0.6$ *and* $\nu \in (0, 1)$. *Then*

$$\mathfrak{R}(\Pi\varrho, \Pi\varsigma, kv) = \frac{kv}{kv + \frac{1}{16}|\varrho - \varsigma|^2}.$$

Since

$$|\varrho - \varsigma|^2 \leq 2\left(|\varrho|^2 + |\varsigma|^2\right),$$

it follows that

$$\frac{1}{16}|\varrho - \varsigma|^2 \leq \frac{1}{8}\left(|\varrho|^2 + |\varsigma|^2\right).$$

Consequently,

$$\mathfrak{R}(\Pi\varrho, \Pi\varsigma, kv) \geq [\mathfrak{R}(\varrho, \Pi\varrho, v)]^{\nu}\,[\mathfrak{R}(\varsigma, \Pi\varsigma, v)]^{1-\nu}.$$

Thus, Π *satisfies the interpolative Kannan-type FSBIC. Therefore,* Π *has a unique fixed point in* $\mathbb{C}$. *Hence,* 0 *is the unique fixed point of* Π.

Remark 3.2. *We now show that the mapping* Π *does not satisfy the classical Kannan-type fuzzy contraction condition.*

Take $\varrho = 1$ *and* $\varsigma = 0$. *Then*

$$\Pi\varrho = \frac{1}{4}, \qquad \Pi\varsigma = 0.$$

Hence,

$$|\Pi\varrho - \Pi\varsigma| = \left|\frac{1}{4} - 0\right| = \frac{1}{4},$$

which yields

$$\mathfrak{R}(\Pi\varrho, \Pi\varsigma, kv) = \frac{kv}{kv + \frac{1}{16}}.$$

Further,

$$|\varrho - \Pi\varrho| = \left|1 - \frac{1}{4}\right| = \frac{3}{4},$$

and

$$|\varsigma - \Pi\varsigma| = |0 - 0| = 0.$$

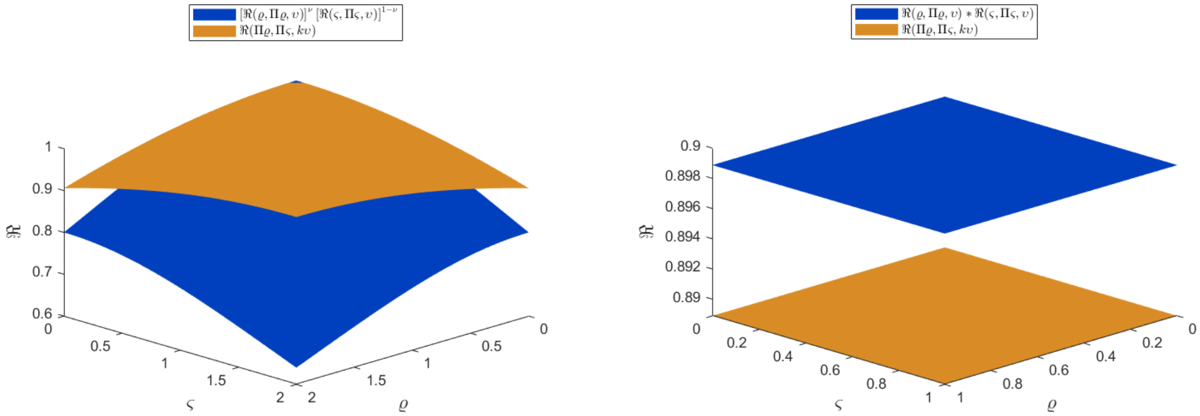

FIGURE 2. Comparison between the Kannan-type FSBIC and the classical Kannan-type FSBC for the mapping $\Pi(\varrho) = \frac{\varrho}{4}$. The left figure illustrates that the interpolative Kannan-type contraction condition is satisfied on $\mathbb{C} = [0, 2]$ for $k = 0.6$ and $\nu \in (0, 1)$. The right figure demonstrates the failure of the classical Kannan-type fuzzy contraction condition for $k = 0.1 \in [0, \frac{1}{3})$.

Therefore,

$$\mathfrak{R}(\varrho, \Pi\varrho, \upsilon) = \frac{\upsilon}{\upsilon + \frac{9}{16}},$$

while

$$\mathfrak{R}(\varsigma, \Pi\varsigma, \upsilon) = 1.$$

Hence,

$$\mathfrak{R}(\varrho, \Pi\varrho, \upsilon) * \mathfrak{R}(\varsigma, \Pi\varsigma, \upsilon) = \frac{\upsilon}{\upsilon + \frac{9}{16}}.$$

Choosing $k = 0.1$, we obtain

$$\mathfrak{R}(\Pi\varrho, \Pi\varsigma, k\upsilon) = \frac{0.1\upsilon}{0.1\upsilon + \frac{1}{16}}.$$

For sufficiently small $\upsilon > 0$, one has

$$\frac{0.1\upsilon}{0.1\upsilon + \frac{1}{16}} < \frac{\upsilon}{\upsilon + \frac{9}{16}}.$$

Thus,

$$\mathfrak{R}(\Pi\varrho, \Pi\varsigma, k\upsilon) < \mathfrak{R}(\varrho, \Pi\varrho, \upsilon) * \mathfrak{R}(\varsigma, \Pi\varsigma, \upsilon).$$

Therefore, Π fails to satisfy the classical Kannan-type fuzzy contraction condition for $k \in [0, \frac{1}{3})$, although it satisfies the Kannan-type FSBIC.

Definition 3.3. *Let $(\mathbb{C}, \mathfrak{R}, *,)$ be an FSBMS. A mapping $\Pi : \mathbb{C} \to \mathbb{C}$ is called Reich-Rus-Ciric-type FSBIC, if*

$$\mathfrak{R}\left(\Pi\varrho, \Pi\varsigma, k\upsilon\right) \geq \left(\mathfrak{R}\left(\varrho, \varsigma, \upsilon\right)\right)^{\nu} * \left(\mathfrak{R}\left(\varrho, \Pi\varrho, \upsilon\right)\right)^{\eta} * \left(\mathfrak{R}\left(\varsigma, \Pi\varsigma, \upsilon\right)\right)^{1-\nu-\eta} \tag{3.3}$$

for all $\varrho, \varsigma \in \mathbb{C} \setminus \mathrm{Fix}(\Pi)$, where $\nu, \eta, \mathrm{u} \in (0, 1)$ and $k \in (0, 1)$.

Example 3.3. *Let $\mathbb{C} = [0, 4]$ and define a mapping*

$$\Re : \mathbb{C} \times \mathbb{C} \times (0, \infty) \to (0, 1]$$

by

$$\Re(\varrho, \varsigma, \upsilon) = e^{-\frac{|\varrho-\varsigma|^2}{\upsilon}}, \qquad \upsilon > 0.$$

Then $(\mathbb{C}, \Re, *,)$ forms a CFSBMS with $= 2$, where $*$ denotes the usual product ctn.

Now define the self-mapping $\Pi : \mathbb{C} \to \mathbb{C}$ by

$$\Pi(\varrho) = \frac{\varrho + 4}{5}, \qquad \forall\, \varrho \in \mathbb{C}.$$

For arbitrary $\varrho, \varsigma \in \mathbb{C}$, we have

$$|\Pi\varrho - \Pi\varsigma| = \left|\frac{\varrho + 4}{5} - \frac{\varsigma + 4}{5}\right| = \frac{1}{5}|\varrho - \varsigma|.$$

Hence,

$$\Re(\Pi\varrho, \Pi\varsigma, k\upsilon) = e^{-\frac{|\varrho-\varsigma|^2}{25k\upsilon}}.$$

Moreover,

$$|\varrho - \Pi\varrho| = \left|\varrho - \frac{\varrho + 4}{5}\right| = \frac{4}{5}|\varrho - 1|,$$

and similarly,

$$|\varsigma - \Pi\varsigma| = \frac{4}{5}|\varsigma - 1|.$$

Therefore,

$$\Re(\varrho, \Pi\varrho, \upsilon) = e^{-\frac{16|\varrho-1|^2}{25\upsilon}},$$

and

$$\Re(\varsigma, \Pi\varsigma, \upsilon) = e^{-\frac{16|\varsigma-1|^2}{25\upsilon}}.$$

For $\nu, \eta \in (0, 1)$ with $\nu + \eta < 1$, the right-hand side of the interpolative condition becomes

$$\begin{aligned}\Big(\Re(\varrho, \varsigma, \upsilon)\Big)^{\nu} \cdot \Big(\Re(\varrho, \Pi\varrho, \upsilon)\Big)^{\eta} \cdot \Big(\Re(\varsigma, \Pi\varsigma, \upsilon)\Big)^{1-\nu-\eta} &= \Big(e^{-\frac{|\varrho-\varsigma|^2}{\upsilon}}\Big)^{\nu}\Big(e^{-\frac{16|\varrho-1|^2}{25\upsilon}}\Big)^{\eta}\Big(e^{-\frac{16|\varsigma-1|^2}{25\upsilon}}\Big)^{1-\nu-\eta} \\ &= \exp\left[-\left(\frac{\nu|\varrho-\varsigma|^2}{\upsilon} + \frac{16\eta|\varrho-1|^2}{25\upsilon} + \frac{16(1-\nu-\eta)|\varsigma-1|^2}{25\upsilon}\right)\right].\end{aligned}$$

Using the exponential form of the metric, the Reich–Rus–Ćirić type FSBIC inequality $\Re(\Pi\varrho, \Pi\varsigma, k\upsilon) \geq$ (RHS) is equivalent to

$$\frac{1}{25k\upsilon}|\varrho - \varsigma|^2 \leq \frac{\nu|\varrho-\varsigma|^2}{\upsilon} + \frac{16\eta|\varrho-1|^2}{25\upsilon} + \frac{16(1-\nu-\eta)|\varsigma-1|^2}{25\upsilon}.$$

Dividing by $|\varrho - \varsigma|^2$ ($\varrho \neq \varsigma$), we obtain

$$\frac{1}{25k\upsilon} \leq \frac{\nu}{\upsilon} + \frac{16\eta}{25\upsilon}\,\frac{|\varrho-1|^2}{|\varrho-\varsigma|^2} + \frac{16(1-\nu-\eta)}{25\upsilon}\,\frac{|\varsigma-1|^2}{|\varrho-\varsigma|^2}.$$

Since all terms on the right-hand side are nonnegative, the inequality holds whenever $\dfrac{1}{25k} \leq \nu$*, that is,*

$$k \geq \frac{1}{25\nu}.$$

Therefore Π *satisfies the Reich–Rus–Ćirić type FSBIC condition whenever* $k \in \left[\frac{1}{25\nu}, 1\right)$ *with* $\nu, \eta \in (0,1)$ *and* $\nu + \eta < 1$*. Since* $\Pi(1) = 1$*, the point* 1 *is the unique fixed point of* Π*.*

Remark 3.3. *On the other hand, if we consider,* $\varrho = 0$*,* $\varsigma = 2$*. Then*

$$\Pi\varrho - \Pi\varsigma| = |\frac{4}{5} - \frac{6}{5}| = \frac{2}{5}.$$

Therefore,

$$\mathfrak{R}(\Pi\varrho, \Pi\varsigma, kv) = e^{-\frac{4}{25kv}}.$$

$$\mathfrak{R}(0, 2, v) = e^{-\frac{4}{v}},\ \mathfrak{R}(\varrho, \Pi\varrho, v) = e^{-\frac{16}{25v}},\ \mathfrak{R}(\varsigma, \Pi\varsigma, v) = e^{-\frac{16}{25v}}.$$

Hence,

$$\mathfrak{R}\left(\varrho, \varsigma, v\right) * \left(\mathfrak{R}\left(\varrho, \Pi\varrho, v\right)\right) * \left(\mathfrak{R}\left(\varsigma, \Pi\varsigma, v\right)\right) = e^{-\frac{4}{v}}.e^{-\frac{16}{25v}}.e^{-\frac{16}{25v}} = e^{-\frac{132}{25v}}.$$

Thus,

$$\mathfrak{R}(\Pi\varrho, \Pi\varsigma, kv) \geq \mathfrak{R}\left(\varrho, \varsigma, v\right) * \left(\mathfrak{R}\left(\varrho, \Pi\varrho, v\right)\right) * \left(\mathfrak{R}\left(\varsigma, \Pi\varsigma, v\right)\right).$$

Substituting the above quantities yields

$$e^{-\frac{4}{25kv}} \geq e^{-\frac{132}{25v}}.$$

So, we have $\frac{4}{25k} \leq \frac{132}{25}$ *this, implies that* $k \geq \frac{1}{33}$*. Hence, the mapping* Π *does not satisfy the classical Reich-Rus-Ciric-type FSBC for* $k < \frac{1}{33}$*.*

Definition 3.4. *Let* $(\mathbb{C}, \mathfrak{R}, *,)$ *be an FSBMS. A mapping* $\Pi : \mathbb{C} \to \mathbb{C}$ *is called Chatterjea-type FSBIC, if*

$$\mathfrak{R}\left(\Pi\varrho, \Pi\varsigma, kv\right) \geq \sqrt{\left(\mathfrak{R}\left(\varrho, \Pi\varsigma, v\right)\right) * \left(\mathfrak{R}\left(\varsigma, \Pi\varrho, v\right)\right)} \tag{3.4}$$

for all $\varrho, \varsigma \in \mathbb{C} \setminus \mathrm{Fix}(\Pi)$ *, where* $k \in (0,1)$*.*

Example 3.4. *Let* $\mathbb{C} = [0,4]$ *and define a mapping* $\mathfrak{R} : \mathbb{C} \times \mathbb{C} \times (0,\infty) \to (0,1]$ *by*

$$\mathfrak{R}(\varrho, \varsigma, v) = \frac{v}{v + |\varrho - \varsigma|^2}, \qquad v > 0.$$

Then $(\mathbb{C}, \mathfrak{R}, *,)$ *forms a CFSBMS with* $\ = 2$*, where* $*$ *denotes the usual product ctn.*

Now define the self-mapping $\Pi : \mathbb{C} \to \mathbb{C}$ *by*

$$\Pi(\varrho) = \frac{3\varrho + 2}{7}, \qquad \forall\, \varrho \in \mathbb{C}.$$

Therefore, the mapping Π *satisfies the Chatterjea-type FSBIC condition. Thus,*

$$\Pi(\varrho) = \varrho$$

$$\frac{3\varrho + 2}{7} = \varrho$$

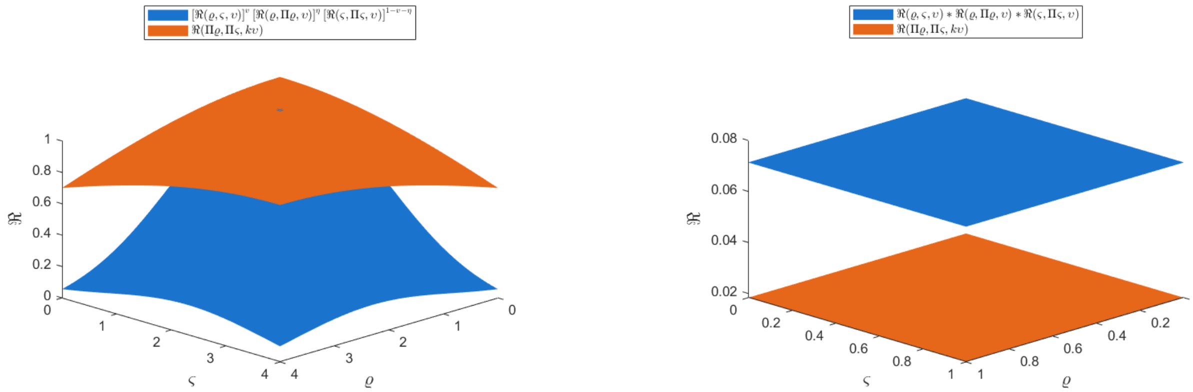

FIGURE 3. Comparison between the Reich–Rus–Ćirić type FSBIC condition and the classical Reich-Rus-Ćirić type FSBC for the mapping $\Pi(\varrho) = \dfrac{\varrho+4}{5}$. The left figure illustrates that the Reich-Rus-Ćirić type FSBIC condition is satisfied on $\mathbb{C} = [0,4]$ for admissible parameters $k \in \left[\dfrac{1}{25\upsilon}, 1\right)$, $\upsilon, \eta \in (0,1)$, and $\upsilon + \eta < 1$. The right figure demonstrates the failure of the classical Reich-Rus-Ćirić type FSBC condition for $k < \dfrac{1}{33}$.

$$
\begin{aligned}
4\varrho &= 2 \\
\varrho &= \frac{1}{2}.
\end{aligned}
$$

Hence, $\frac{1}{2}$ is the unique fixed point of Π.

Remark 3.4. *On the other hand, for $\varrho = 0$, $\varsigma = 4$. Then, $\Pi(0) = \frac{2}{7}$ and $\Pi(4) = 2$. Therefore,*

$$\mathfrak{R}(\Pi\varrho, \Pi\varsigma, k\upsilon) = \frac{k\upsilon}{k\upsilon + |\varrho-\varsigma|^2} = \frac{k\upsilon}{k\upsilon + |\frac{2}{7}-2|^2} = \frac{k\upsilon}{k\upsilon + \frac{144}{49}}.$$

On the other hand,

$$\mathfrak{R}\left(\varrho, \Pi\varsigma, \upsilon\right) * \mathfrak{R}\left(\varsigma, \Pi\varrho, \upsilon\right) = \frac{\upsilon}{\upsilon + |0-2|^2} \cdot \frac{\upsilon}{\upsilon + |4-\frac{2}{7}|^2} = \frac{\upsilon}{\upsilon+4} \cdot \frac{\upsilon}{\upsilon + \frac{676}{49}}$$

Hence,

$$\mathfrak{R}(\Pi\varrho, \Pi\varsigma, k\upsilon) \geq \mathfrak{R}\left(\varrho, \Pi\varsigma, \upsilon\right) * \mathfrak{R}\left(\varsigma, \Pi\varrho, \upsilon\right)$$

The mapping Π fails to satisfy the classical Chatterjea-type FSBC condition for $k = 0.1 \in [0, \frac{1}{2})$, although it satisfies the Chatterjea-type FSBIC condition.

Definition 3.5. *Let $(\mathbb{C}, \mathfrak{R}, *,)$ be an FSBMS. A mapping $\Pi : \mathbb{C} \to \mathbb{C}$ is called a Hardy-Roger's-type FSBIC, if*

$$\mathfrak{R}(\Pi\varrho, \Pi\varsigma, k\upsilon) \;\geq\; (\mathfrak{R}(\varrho,\varsigma,\upsilon))^{\nu} * (\mathfrak{R}(\varrho,\Pi\varrho,\upsilon))^{\eta} * (\mathfrak{R}(\varsigma,\Pi\varsigma,\upsilon))^{\mathsf{u}} * (\mathfrak{R}(\varrho,\Pi\varsigma,\upsilon))^{\mathsf{l}} * (\mathfrak{R}(\varsigma,\Pi\varrho,\upsilon))^{1-\nu-\eta-\mathsf{u}-\mathsf{l}} \quad (3.5)$$

for all $\varrho, \varsigma \in \mathbb{C} \setminus \mathrm{Fix}(\Pi)$, where $\nu, \eta, \mathsf{u}, \mathsf{l} \in (0,1)$ with $\nu + \eta + \mathsf{u} + \mathsf{l} < 1$ and $k \in (0,1)$.

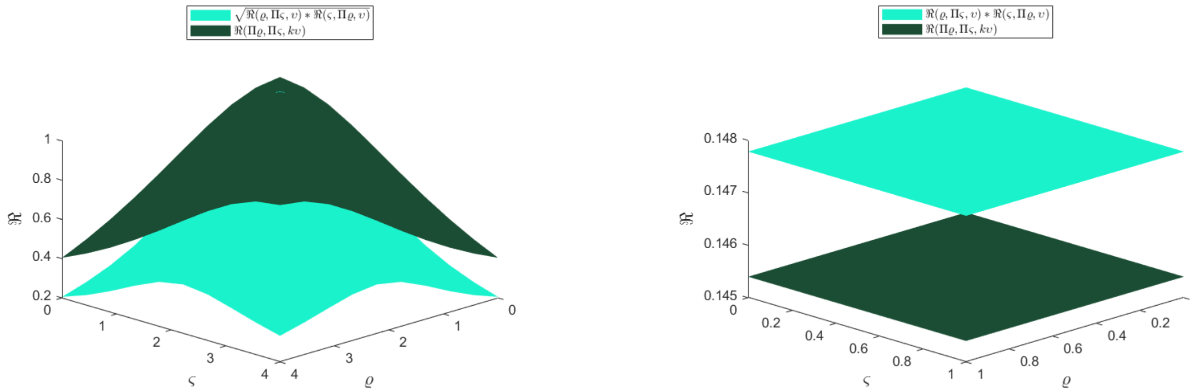

FIGURE 4. Comparison between the Chatterjea-type FSBIC condition and the classical Chatterjea-type FSBC for the mapping $\Pi(\varrho) = \dfrac{3\varrho+2}{7}$. The left figure illustrates that the Chatterjea-type FSBIC condition is satisfied on $\mathbb{C} = [0,4]$ for admissible parameters $k \in (0.4, 1)$. The right figure demonstrates the failure of the classical Chatterjea-type FSBC condition for $k < \dfrac{1}{5}$.

Example 3.5. *Let* $\mathbb{C} = [0,1]$ *and define*

$$\Re(\varrho,\varsigma,\upsilon) = \frac{\upsilon}{\upsilon + |\varrho-\varsigma|^2}, \qquad \upsilon > 0.$$

Then $(\mathbb{C},\Re,*,)$ *is a complete fuzzy strong b-metric space with* $=2$*, where* $*$ *is the product t-norm. Define* $\Pi : \mathbb{C} \to \mathbb{C}$ *by*

$$\Pi(\varrho) = \begin{cases} \dfrac{\varrho^2}{4}, & \varrho \in [0,1), \\ \dfrac{1}{2}, & \varrho = 1. \end{cases}$$

Moreover, for every $\varrho,\varsigma \in [0,1]$*,*

$$|\Pi(\varrho) - \Pi(\varsigma)| = \frac{1}{4}|\varrho^2 - \varsigma^2| = \frac{1}{4}|\varrho-\varsigma||\varrho+\varsigma| \le \frac{1}{2}|\varrho-\varsigma|,$$

since $\varrho + \varsigma \le 2$*. Consequently,*

$$|\Pi(\varrho) - \Pi(\varsigma)|^2 \le \frac{1}{4}|\varrho-\varsigma|^2.$$

Therefore,

$$\Re(\Pi\varrho,\Pi\varsigma,k\upsilon) = \frac{k\upsilon}{k\upsilon + |\Pi(\varrho)-\Pi(\varsigma)|^2} \ge \frac{k\upsilon}{k\upsilon + \frac{1}{4}|\varrho-\varsigma|^2}.$$

Hence, for suitable parameters $\nu,\eta,\mathsf{u},\mathsf{l} \in (0,1)$ *satisfying* $\nu+\eta+\mathsf{u}+\mathsf{l} < 1$*, and for* $k \in \left(\frac{1}{4},1\right)$*, the mapping* Π *satisfies the Hardy-Rogers type FSBIC (3.5). Thus* Π *is a Hardy-Rogers type FSBIC and possesses the fixed point* 0*.*

Remark 3.5. *Consider* $\varrho = 0$ *and* $\varsigma = 1$*. Then* $\Pi(0) = 0, \quad \Pi(1) = \frac{1}{2}$*. Hence*

$$|\Pi(0) - \Pi(1)| = \frac{1}{2}, \quad |0 - \Pi(0)| = 0,$$

$$|1-\Pi(1)|=\frac{1}{2}, \qquad |1-\Pi(0)|=1,$$

and

$$|0-\Pi(1)|=\frac{1}{2}.$$

Therefore,

$$\mathfrak{R}(\Pi\varrho,\Pi\varsigma,kv)=\frac{kv}{kv+\frac{1}{4}},$$
$$\mathfrak{R}(\varrho,\varsigma,v)=\frac{v}{v+1},$$
$$\mathfrak{R}(\varrho,\Pi\varrho,v)=1,$$
$$\mathfrak{R}(\varsigma,\Pi\varsigma,v)=\frac{v}{v+\frac{1}{4}},$$
$$\mathfrak{R}(\varrho,\Pi\varsigma,v)=\frac{v}{v+\frac{1}{4}},$$

and

$$\mathfrak{R}(\varsigma,\Pi\varrho,v)=\frac{v}{v+1}.$$

Consequently,

$$\mathfrak{R}(\varrho,\varsigma,v)*\mathfrak{R}(\varrho,\Pi\varrho,v)*\mathfrak{R}(\varsigma,\Pi\varsigma,v)*\mathfrak{R}(\varrho,\Pi\varsigma,v)*\mathfrak{R}(\varsigma,\Pi\varrho,v)=\left(\frac{v}{v+1}\right)^2\left(\frac{v}{v+\frac{1}{4}}\right)^2.$$

For sufficiently small values of k,

$$\frac{kv}{kv+\frac{1}{4}}<\left(\frac{v}{v+1}\right)^2\left(\frac{v}{v+\frac{1}{4}}\right)^2.$$

Hence the classical Hardy–Rogers fuzzy strong b-contraction condition fails.

Therefore, Π is an example of a Hardy–Rogers type FSBIC which does not satisfy the corresponding classical Hardy–Rogers type FSBC.

3.1. **Hardy–Rogers type (Υ,F) fuzzy strong b-interpolative contraction.** In this subsection, we establish the existence and uniqueness of fixed points for Hardy–Rogers type fuzzy strong b-interpolative contractions involving a pair of auxiliary functions (Υ,F). The obtained result generalizes several existing contraction principles in the setting of fuzzy strong b-metric spaces. As consequences of our main theorem, we derive Reich–Rus–Ćirić, Kannan, Chatterjea and Banach type results.

Definition 3.6. *Let $(\mathbb{C},\mathfrak{R},*,)$ be an FSBMS and let $\Upsilon,F:(0,1]\to\mathbb{R}$ be two functions such that $F(t)>\Upsilon(t)$ for all $t\in(0,1)$. A mapping $\Pi:\mathbb{C}\to\mathbb{C}$ is said to be a Hardy-Rogers type (Υ,F)-FSBIC if there exist $\nu,\eta,\mathsf{u},\mathsf{l}\in(0,1)$ with $\nu+\eta+\mathsf{u}+\mathsf{l}<1$ such that*

$$\begin{aligned}\Upsilon\left(\mathfrak{R}\left(\Pi\varrho,\Pi\varsigma,v\right)\right)\geq F\Big(&\left(\mathfrak{R}(\varrho,\varsigma,v)\right)^{\nu}*\left(\mathfrak{R}(\varrho,\Pi\varrho,v)\right)^{\eta}*\left(\mathfrak{R}(\varsigma,\Pi\varsigma,v)\right)^{\mathsf{u}}\\&*\left(\mathfrak{R}(\varrho,\Pi\varsigma,v)\right)^{\mathsf{l}}*\left(\mathfrak{R}(\varsigma,\Pi\varrho,v)\right)^{1-\nu-\eta-\mathsf{u}-\mathsf{l}}\Big)\end{aligned} \tag{3.6}$$

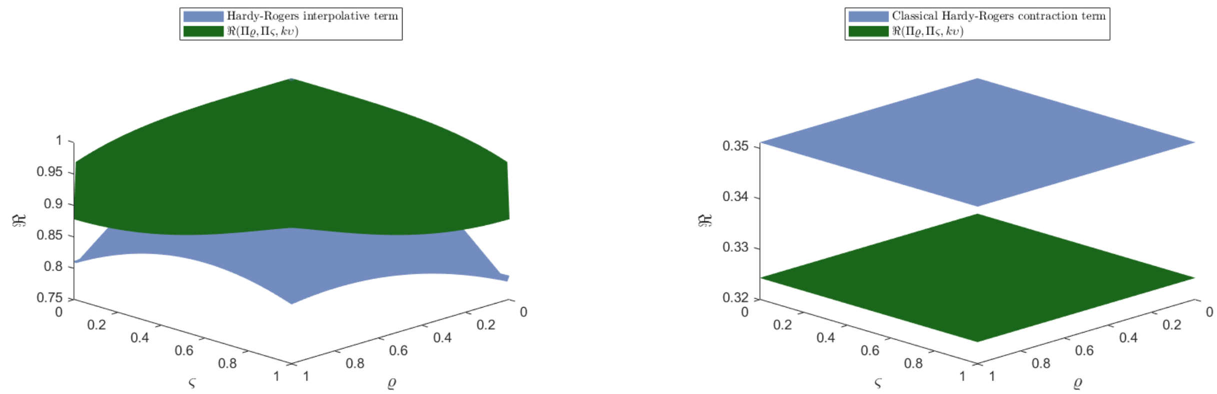

FIGURE 5. Comparison between the Hardy–Rogers type FSBIC condition and the classical Hardy–Rogers type FSBC for the mapping $\Pi(\varrho) = \begin{cases} \frac{\varrho^2}{4}, & \varrho \in [0,1), \\ \frac{1}{2}, & \varrho = 1. \end{cases}$ The left figure illustrates that the Hardy-Rogers type FSBIC condition is satisfied on $\mathbb{C} = [0,1]$ for admissible parameters $k \in \left(\frac{1}{4}, 1\right)$, with $\nu, \eta, \mathsf{u}, \mathsf{l} \in (0,1)$ and $\nu + \eta + \mathsf{u} + \mathsf{l} < 1$. The right figure demonstrates the failure of the classical Hardy-Rogers type FSBC condition for $k < 0.01$.

for all $\varrho, \varsigma \in \mathbb{C} \setminus \mathrm{Fix}(\Pi), \quad v > 0.$

Example 3.6. *Let* $\mathbb{C} = \{g, h, i, j\}$ *and define* $d : \mathbb{C} \times \mathbb{C} \to [0, \infty)$ *by*

d	g	h	i	j
g	0	1	10	100
h	1	0	9	99
i	10	9	0	90
j	100	99	90	0

Then d *is a metric on* $\mathbb{C}$. *Define*

$$\mathfrak{R}(\varrho, \varsigma, v) = e^{-\frac{d(\varrho,\varsigma)^2}{v}}, \qquad v > 0.$$

Then $(\mathbb{C}, \mathfrak{R}, *,)$ *is a complete FSBMS with product t-norm and* ≥ 1. *Define* $\Pi : \mathbb{C} \to \mathbb{C}$ *by*

$$\Pi(g) = g, \qquad \Pi(h) = g, \qquad \Pi(i) = h, \qquad \Pi(j) = i.$$

Define $\Upsilon, F : (0, 1] \to \mathbb{R}$ *by*

$$\Upsilon(t) = t, \quad F(t) = \sqrt{t}, \quad 0 < t \leq 1.$$

Then $F(t) > \Upsilon(t)$ *for all* $t \in (0,1)$. *Choose* $\nu = 0.05, \quad \eta = 0.05, \quad \mathsf{u} = 0.05, \quad \mathsf{l} = 0.40.$ *Then* $\nu + \eta + \mathsf{u} + \mathsf{l} = 0.55 < 1$. *For* $\varrho, \varsigma \in \mathbb{C} \setminus \mathrm{Fix}(\Pi)$ *therefore,*

$$\begin{aligned} &\Upsilon\left(\mathfrak{R}(\Pi\varrho, \Pi\varsigma, v)\right) \\ &\geq F\Big((\mathfrak{R}(\varrho, \varsigma, v))^{\nu} * (\mathfrak{R}(\varrho, \Pi\varrho, v))^{\eta} * (\mathfrak{R}(\varsigma, \Pi\varsigma, v))^{\mathsf{u}} * (\mathfrak{R}(\varrho, \Pi\varsigma, v))^{\mathsf{l}} * (\mathfrak{R}(\varsigma, \Pi\varrho, v))^{1-\nu-\eta-\mathsf{u}-\mathsf{l}} \Big). \end{aligned}$$

Thus, Π is a Hardy-Rogers type (Υ, F)-FSBIC. Moreover, $\Pi(g) = g$, so g is a fixed point of Π.

Remark 3.6. *The above mapping Π does not satisfy the corresponding classical Hardy–Rogers type FSBIC condition for $k = 0.1$. Consider $\varrho = i$, $\quad \varsigma = h$. Then $\Pi i = h$, $\quad \Pi h = g$ and $\nu = 0.05$, $\quad \eta = 0.05$, $\quad \mathsf{u} = 0.05$, $\quad \mathsf{l} = 0.40$, Thus*

$$\Re(\Pi i, \Pi h, kv) \geq (\Re(i,h,v))^{\nu} * (\Re(i,\Pi i,v))^{\eta} * (\Re(h,\Pi h,v))^{\mathsf{u}} * (\Re(i,\Pi h,v))^{\mathsf{l}} * (\Re(h,\Pi i,v))^{1-\nu-\eta-\mathsf{u}-\mathsf{l}}$$

$$e^{-\frac{1}{kv}} = e^{-\frac{81\nu+81\eta+\mathsf{u}+100\mathsf{l}}{v}}.$$

Hence,

$$e^{-\frac{1}{kv}} \geq e^{-\frac{48.15}{v}},$$

Therefore, the classical Hardy-Rogers type FSBIC condition fails for $0 < k < \frac{1}{48.15}$. Thus, Π satisfies the Hardy-Rogers type (Υ, F)-FSBIC, but it does not satisfy the corresponding classical Hardy-Rogers type FSBIC for sufficiently small k.

Definition 3.7. *A mapping $\Pi : \mathbb{C} \to \mathbb{C}$ defined on FSBMS $(\mathbb{C}, \Re, *,)$ is said to be Banach type (Υ, F)-fuzzy strong b-interpolative contraction (in short, FSBIC), if there exists $\nu \in (0,1]$ such that the following inequality holds:*

$$\Upsilon(\Re(\Pi\varrho, \Pi\varsigma, v)) \geq F(\Re(\varrho, \varsigma, v))^{v}, \tag{3.7}$$

for all $(\varrho, \varsigma) \in \mathbb{C}$, $\Re(\Pi\varrho, \Pi\varsigma, v) > 0$, and where $F(\varrho) > \Upsilon(\varrho)$.

Example 3.7. *This example illustrates a mapping that is a Banach-type (Υ, F)-fuzzy interpolative contraction but not a standard interpolative Banach contraction.*

Let $\mathbb{C} = \mathbb{R}$ and

$$\Re(\varrho, \varsigma, v) = \frac{v}{v + |\varrho - \varsigma|}.$$

Then, $(\mathbb{C}, \Re, *,)$ is an FSBMS with product ctn and ≥ 1. Define $\Pi : \mathbb{C} \to \mathbb{C}$ by $\Pi(\varrho) = \varrho^2$. Also, define $\Upsilon, F : (0,1] \to \mathbb{R}$ by

$$\Upsilon(i) = \begin{cases} \frac{1}{2^{\ln(2i)}} & \text{if } 0 < i < 1, \\ 1 & \text{if } i = 1, \end{cases} \quad \text{and} \quad F(i) = \begin{cases} \frac{1}{2^{\ln(i)}} & \text{if } 0 < i < 1, \\ 2 & \text{if } i = 1. \end{cases}$$

Thus, Π is a Banach-type (Υ, F)-FSBIC. All other definition requirements are also met. Also, observe that Π is not a Banach-type FSBIC for $k = 0.5$, since,

$$\Re(\Pi(2), \Pi(3), kv) = 0.0909 \leq 0.7071 = (\Re(2,3,v))^{v}.$$

This contradicts the contraction condition.

Therefore, the Banach contraction is fulfilled.

For self-mapping Π, and two functions $(\Upsilon, F) : (0,1] \to \mathbb{R}$, we state the following conditions:

(i) for each $\varrho_0 \in \mathbb{C}$, there is $\varrho_1 = \Pi(\varrho_0)$,

(i) Υ is non-decreasing and for every $1 > \varrho \geq t > 0$, one has $F(\varrho) > \Upsilon(\varsigma)$.

(ii) $\lim_{s \to \Pi^-} \inf F(s) > \lim_{s \to \Pi^-} \sup(\Upsilon(s))$.

(iii) if $t \in (0,1]$ such that $\Upsilon(t) \geq F(1)$.

(iv) $\sup_{\varrho > \varepsilon} \Upsilon(\varrho) > -\infty$.

(v) $\lim_{\varrho \to \mathrm{l}} \inf F(\varrho) > \Upsilon(\mathrm{l})\ \forall \mathrm{l} \in (0,1)$.

(vi) if $\{\Upsilon(y_n)\}$ and $\{F(y_n)\}$ are converging to same limit and $\{\Upsilon(y_n)\}$ is strictly increasing, then $\lim_{n\to\infty} y_n = 1$.

(vii) $\Upsilon(r^{v} l^{\eta}) \geq \Upsilon(\varrho)$ and $F(\varrho) > \Upsilon(\varrho)\ \forall \varrho \in (0,1)$.

Lemma 3.1. *Assume* $(\mathbb{C}, \mathfrak{R}, *,)$ *is an FSBMS and* $\{\varrho_n\} \subset \mathbb{C}$. *Let*$\varrho_n$ *be a sequence that verifies* $\lim_{n\to\infty} \mathfrak{R}(\varrho_n, \varrho_{n+1}, v) = 1$. *If* $\{\varrho_n\}$ *is not Cauchy, then there are* $\left\{\varrho_{n_k}\right\}, \left\{\varrho_{m_k}\right\}$, *and* $\varepsilon \geq 0$ *such that*

$$\lim_{k\to\infty} \mathfrak{R}\left(\varrho_{n_K+1}, \varrho_{m_k+1}, v\right) = (1-\varepsilon). \tag{3.8}$$

$$\lim_{k\to\infty} \mathfrak{R}(\varrho_{n_K}, \varrho_{m_K}, v) = \lim_{k\to\infty} \mathfrak{R}(\varrho_{n_K+1}, \varrho_{m_K}, v) = \lim_{k\to\infty} \mathfrak{R}\left(\varrho_{n_K}, \varrho_{m_k+1}, v\right) = 1-\varepsilon. \tag{3.9}$$

Proof. Assume, $(\mathbb{C}, \mathfrak{R}, *,)$ is an FSBMS and $\{\varrho_n\}$ is not Cauchy and $\lim_{n\to\infty} \mathfrak{R}(\varrho_n, \varrho_{n+1}, v) = 1$. Thus, for any $\varepsilon > 0$. There is a natural integer k_0 such that for the least $m \geq n$,

$$\mathfrak{R}(\varrho_{n+1}, \varrho_m, v) \geq 1-\varepsilon \text{ and } \mathfrak{R}(\varrho_{n+1}, \varrho_m, v) < 1-\varepsilon\ \forall n, m \geq k_0.$$

We establish two subsequences of $\{\varrho_n\}$ of $\left\{\varrho_{n_k}\right\}$ and $\left\{\varrho_{m_k}\right\}$ to verify the following inequalities.

$$\mathfrak{R}(\varrho_{n_K+1}, \varrho_{m_K}, v) > 1-\varepsilon \text{ and } \mathfrak{R}\left(\varrho_{n_K+1}, \varrho_{m_k+1}, v\right) < 1-\varepsilon\ \forall n_k, m_k > k_0.$$

By axiom (iv) of the FSBMS, we have the following information:

$$\begin{aligned} 1-\varepsilon &> \mathfrak{R}\left(\varrho_{n_K+1}, \varrho_{m_k+1}, v\right) \\ &\geq \mathfrak{R}\left(\varrho_{n_K+1}, \varrho_{m_K}, \frac{v}{2}\right) * \mathfrak{R}\left(\varrho_{m_K}, \varrho_{m_k+1}, \frac{v}{2}\right) \\ &\geq (1-\varepsilon) * \mathfrak{R}\left(\varrho_{m_K}, \varrho_{m_k+1}, \frac{v}{2}\right). \end{aligned}$$

This implies that,

$$\lim_{k\to\infty} \mathfrak{R}\left(\varrho_{n_K+1}, \varrho_{m_k+1}, \frac{v}{2}\right) = (1-\varepsilon).$$

Again, by axiom (iv) of the FSBMS, we have the following information:

$$\frac{\mathfrak{R}\left(\varrho_{n_K+1}, \varrho_{m_k+1}, \frac{v}{2}\right)}{\mathfrak{R}\left(\varrho_{m_K}, \varrho_{m_k+1}, \frac{v}{2}\right)} \geq \mathfrak{R}\left(\varrho_{n_k+1}, \varrho_{m_K}, v\right) \geq 1+ \in.$$

This leads to

$$\lim_{k\to\infty} \mathfrak{R}(\varrho_{n_K+1}, \varrho_{m_K}, v) = 1-\varepsilon.$$

Since,

$$\mathfrak{R}(\varrho_{n_K+1}, \varrho_{m_K}, v) \geq \mathfrak{R}\left(\varrho_{n_k+1}, \varrho_{n_k}, \frac{v}{2}\right) * \mathfrak{R}\left(\varrho_{n_k}, \varrho_{m_K}, \frac{v}{2}\right).$$

The inequality is as follows:

$$\begin{aligned}\frac{\Re\left(\varrho_{n_K+1},\varrho_{m_K},v\right)}{\Re\left(\varrho_{n_k+1},\varrho_{n_k},\frac{v}{2}\right)} &\geq \Re\left(\varrho_{n_k},\varrho_{m_K},\frac{v}{2}\right)\\ &\geq \Re\left(\varrho_{m_K},\varrho_{n_k+1},\frac{v}{2}\right)*\Re\left(\varrho_{n_K+1},\varrho_{n_k},\frac{v}{2}\right).\end{aligned}$$

This suggests that

$$\lim_{k\to\infty}\Re\left(\varrho_{n_K+1},\varrho_{m_k+1},v\right)=1-\varepsilon.$$

Since,

$$1-\varepsilon>\Re\left(\varrho_{n_K+1},\varrho_{m_k+1},v\right)\geq\Re\left(\varrho_{n_k+1},\varrho_{n_k},\frac{v}{2}\right)*\Re\left(\varrho_{n_k},\varrho_{m_k+1},\frac{v}{2h}\right)$$

$$\begin{aligned}\frac{1-\varepsilon}{\Re\left(\varrho_{n_K+1},\varrho_{n_k},\frac{v}{2}\right)} &\geq \Re\left(\varrho_{n_K},\varrho_{m_k+1},\frac{v}{2}\right)\\ &\geq \Re\left(\varrho_{n_K},\varrho_{n_k+1},\frac{v}{2}\right)*\Re\left(\varrho_{n_K+1},\varrho_{m_k+1},\frac{v}{2}\right).\end{aligned}$$

This suggests that

$$\Re\left(\varrho_{n_K},\varrho_{m_K},v\right)=(1-\varepsilon).$$

This completes the proof. □

The next two theorems deal with Banach type (Υ, F)-fuzzy interpolative contraction.

Theorem 3.1. *Suppose $(\mathbb{C},\Re,*,)$ is a complete FSBMS, $*$ is a ctn, and $\Re(\varrho,\varsigma,v)$ is strictly increasing in v and $\lim_{v\to\infty}\Re(\varrho,\varsigma,v)=1$, for all $\varrho,\varsigma\in\mathbb{C}$. Let Π be a self-mapping satisfying condition (3.7) and assumption (i)-(iv). Then Π admits a fixed point and the iterative sequence converges to a point in $\mathbb{C}$.*

Proof. Start by selecting an initial guess $\varrho_0\in\mathbb{C}$, and define an sequence $\{\varrho_n\}$ recursively by

$$\varrho_n=\Pi\left(\varrho_{n-1}\right)=\Pi^n\left(\varrho_0\right)\ \text{ for each } n\in\mathbb{N}.$$

Observe that, if for some n, we have , $\varrho_n=\Pi\left(\varrho_n\right)$ then ϱ_n is a fixed point of Π for all $n\geq 0$. We assume $\varrho_n\neq\varrho_{n+1}$ for all $n\in\mathbb{N}\cup\{0\}$. Next, define $g_n=\Re\left(\varrho_n,\varrho_{n+1},v\right)$ for all $n\geq 0$. Applying the first part of assumption (ii) together with equation (3.7), we obtain

$$\Upsilon\left(\mathsf{g}_n\right)\geq\Upsilon\left(\Re\left(\Pi\varrho_{n-1},\Pi\varrho_n,v\right)\right)\geq F\left(\left(\Re\left(\varrho_{n-1},\varrho_n,v\right)\right)^{v}\right).$$

By assumption (i), we obtain

$$\Upsilon\left(\mathsf{g}_n\right)\geq F\left(\left(\mathsf{g}_{n-1}\right)^{v}\right)>\Upsilon\left(\left(\mathsf{g}_{n-1}\right)^{v}\right)\tag{3.10}$$

Because Υ is non-decreasing $\mathsf{g}_n>\mathsf{g}_{n-1}$ for each $n\geq 1$, implying that $\mathsf{G}<1$, resulting in $\lim_{n\to\infty}\mathsf{g}_n=\mathsf{G}+$. If $\mathsf{G}<1$, we can extract the following information from (3.10):

$$\Upsilon\left(\mathsf{G}+\right)=\lim_{n\to\infty}\Upsilon\left(\mathsf{g}_n\right)\geq\lim_{n\to\infty}\inf F\left(\left(\mathsf{g}_{n-1}\right)^{v}\right)\geq\lim_{\varrho\to\Pi+}\inf F\left(\varrho\right).$$

This contradicts assumption (iii), hence,$\mathsf{G}=1$.

The sequence $\{\varrho_n\}$ is Cauchy: Suppose that $\{\varrho_n\}$ is not a Cauchy sequence. Then, according to lemma (3.1), there exist two subsequences $\left\{\varrho_{n_k}\right\}$, $\left\{\varrho_{m_k}\right\}$ of $\{\varrho_n\}$ with $\varepsilon > 0$ that satisfy (3.8) and (3.9). Using (3.8), we may conclude that

$$\Re\left(\varrho_{n_K+1}, \varrho_{m_k+1}, \upsilon\right) > (1-\varepsilon)$$

for all $n \geq 0$, we have

$$\Upsilon\left(\Re\left(\varrho_{n_K+1}, \varrho_{m_k+1}, \upsilon\right)\right) \geq \Upsilon\left(\Re\left(\Pi\varrho_{n_K}, \Pi\varrho_{m_K}, \upsilon\right)\right) \geq F\left(\left(\Re\left(\varrho_{n_K}, \varrho_{m_K}, \upsilon\right)\right)^{\upsilon}\right).$$

If $\varrho_k = \Re\left(\varrho_{n_K+1}, \varrho_{m_k+1}, \upsilon\right)$, $\varsigma_k = \Re\left(\varrho_{n_k}, \varrho_{m_K}, \upsilon\right)$, we have

$$\Upsilon(\varrho_k) \geq F\left((\varsigma_k)^{\upsilon}\right) \text{ for all } k \geq 1. \tag{3.11}$$

By (3.8), we have $\lim_{k\to\infty} \varrho_k = (1-\varepsilon)$ and (3.11) implies

$$\lim_{\varrho\to(1-\varepsilon)} \sup \Upsilon(\varrho) \geq \lim_{k\to\infty} \sup \Upsilon(\varrho_k) \geq \lim_{k\to\infty} \inf F\left((\varsigma_k)^{\upsilon}\right) \geq \lim_{\varrho\to 0} \inf F(\varrho). \tag{3.12}$$

The information in (3.12) contradicts assumption (ii), indicating that the sequence $\{\varrho_n\}$ is Cauchy in the complete FSBMS. As $n \to \infty$, $\varrho_n \to \varrho$ in $(\mathbb{C}, \Re, *,)$. Since $(\mathbb{C}, \Re, *,)$ is an AR. We state that $\Re(\varrho, \Pi\varrho, \upsilon) = 1$. If $\Re(\varrho_{n+1}, \Pi\varrho, \upsilon) > 1$, then (3.7).

$$\begin{aligned} \Upsilon\left(\Re\left(\varrho_{n+1}, \Pi\varrho, \upsilon\right)\right) &\geq \Upsilon\left(\Re\left(\Pi\varrho_n, \Pi\varrho, \upsilon\right)\right) \geq F\left(\left(\Re\left(\varrho_n, \varrho, \upsilon\right)\right)^{\upsilon}\right) \\ &> \Upsilon\left(\left(\Re\left(\varrho_n, \varrho, \upsilon\right)\right)^{\upsilon}\right) \end{aligned}$$

for every $n \in \mathbb{N}$ and $\upsilon > 0$ and thus for any integer, $\upsilon > 0$, by using (4), we get

$$\begin{aligned} \Re\left(\varrho_n, \Pi\varrho, \upsilon\right) &\geq \Re\left(\varrho, \varrho_{n+1}, \frac{\upsilon}{2}\right) * \Re\left(\varrho_{n+1}, \Pi\varrho, \frac{\upsilon}{2}\right) \geq \Re\left(\varrho, \varrho_{n+1}, \frac{\upsilon}{2}\right) * \Re\left(\varrho_{n+1}, \varrho_n, \frac{\upsilon}{4}\right) * \Re\left(\varrho_n, \Pi\varrho, \frac{\upsilon}{4^2}\right) \\ \Re\left(\varrho_n, \Pi\varrho, \upsilon\right) &\geq \Re\left(\varrho_n, \varrho_{n+1}, \frac{\upsilon}{2}\right) * \Re\left(\varrho_n, \varrho, \frac{\upsilon}{2k}\right). \end{aligned}$$

By the first part of (ii), we get

$$\Re\left(\varrho_{n+1}, \Pi\varrho, \upsilon\right) \geq \left(\Re\left(\varrho_n, \varrho, \frac{\upsilon}{2k}\right)\right)^{\upsilon},$$

applying the limit $n \to \infty$, we get $\Re(\varrho, \Pi\varrho, \upsilon) \geq 1$. This means that $\Re(\varrho, \Pi\varrho, \upsilon) = 1$. Hence, $\varrho = \Pi\varrho$.

Uniqueness: Assume, ϱ, ϱ^* are two distinct fixed points of the mapping Π.

$$\Re\left(\varrho, \varrho^*, \upsilon\right) = \Re\left(\Pi\varrho, \Pi\varrho^*, k\upsilon\right) \geq \Re\left(\varrho, \varrho^*, \upsilon\right)^{\upsilon},$$

This contradicts the fact that $\Re(\varrho, \varrho^*, \upsilon)$ is strictly increasing υ. Therefore, $\varrho = \varrho^*$. □

Theorem 3.2. *Suppose $(\mathbb{C}, \Re, *,)$ is a complete FSBMS, $*$ is a ctn, and $\Re(\varrho, l, \upsilon)$ is strictly increasing in υ and $\lim_{\upsilon\to\infty} \Re(\varrho, l, \upsilon) = 1$, for all $\varrho, l \in \mathbb{C}$. Let Π be a self-mapping satisfying condition (3.7) and assumptions (i), (iii), and (iv)-(viii). Then Π admits a fixed point and the iterative sequence converges to a point in $\mathbb{C}$.*

Proof. The initial steps of the proof of Theorem 3.2 are the same as those of Theorem 3.1. By gathering the first part of assumption (i) with condition (3.7), we get

$$\begin{aligned} \Upsilon\left(\Re\left(\varrho_n, \varrho_{n+1}, v\right)\right) &\geq \Upsilon\left(\Re\left(\Pi\varrho_{n-1}, \Pi\varrho_n, v\right)\right) \geq F\left(\left(\Re\left(\varrho_{n-1}, \varrho_n, v\right)\right)^v\right) \\ &> \Upsilon\left(\Re\left(\varrho_{n-1}, \varrho_n, v\right)\right)^v \\ &\geq \Upsilon\left(\Re\left(\varrho_{n-1}, \varrho_n, v\right)\right) \end{aligned} \tag{3.13}$$

The inequality shows that (3.13) shows that $\{\Upsilon\left(\Re\left(\varrho_{n-1}, \varrho_n, v\right)\right)\}$ is strictly increasing. If it is not bounded above, given (v), we get $\sup_{\Re(\varrho_{n-1},\varrho_n,v)>\varepsilon} \Upsilon\left(\Re\left(\varrho_{n-1}, \varrho_n, v\right)\right) > -\infty$. This implies that

$$\lim_{\Re(\varrho_{n-1},\varrho_n,v)\to\varepsilon+} \sup \Upsilon\left(\Re\left(\varrho_{n-1}, \varrho_n, v\right)\right) > -\infty$$

Thus, $\lim_{n\to\infty} \Re\left(\varrho_{n-1}, \varrho_n, v\right) = 1$, we get

$$\lim_{\Re(\varrho_{n-1},\varrho_n,v)\to\varepsilon+} \sup \Upsilon\left(\Re\left(\varrho_{n-1}, \varrho_n, v\right)\right) = -\infty$$

(This is a contradiction (v).) If $\{\Upsilon\left(\Re\left(\varrho_{n-1}, \varrho_n, v\right)\right)\}$ is bounded above, then $\{F\left(\Re\left(\varrho_{n-1}, \varrho_n, v\right)\right)\}$ also converges to the same limit (3.13). Thus, by (iii), we get $\lim_{n\to\infty} \Re\left(\varrho_{n-1}, \varrho_n, v\right) = 1$. Hence, Π is asymptotically regular (AR).

Assuming $\{\varrho_n\}$ is a Cauchy sequence, lemma 3.1 implies the existence of $\left\{\varrho_{n_k}\right\}$. Given $\left\{\varrho_{m_k}\right\}$ and $\varepsilon > 0$, we may conclude that $\Re\left(\varrho_{n_k+1}, \varrho_{m_k+1}, v\right) > (1-\varepsilon)$. Using $\mathsf{g} = \varrho_{n_k}$ and $\mathsf{e} = \varrho_{m_k}$ in (3.7), we can write for every $k \geq 1$

$$\Upsilon\left(\Re\left(\varrho_{n_k+1}, \varrho_{m_k+1}, v\right)\right) \geq \Upsilon\left(\Re\left(\Pi\varrho_{n_k}, \Pi\varrho_{m_k}, v\right)\right) \geq F\left(\left(\Re\left(\varrho_{n_k}, \varrho_{m_k}, v\right)\right)^v\right)$$

If $\varrho_k = \Re\left(\varrho_{n_K+1}, \varrho_{m_k+1}, v\right)$, $\varsigma_k = \Re\left(\varrho_{n_K}, \varrho_{m_K}, v\right)$, we have

$$\Upsilon\left(\varrho_k\right) \geq F\left(\left(\varsigma_k\right)^v\right) \text{ for all } k \geq 1. \tag{3.14}$$

By (3.8), we get $\lim_{k\to\infty} \varrho_k = (1-\varepsilon)$ and (3.14) implies

$$\lim_{\varrho\to(1-\varepsilon)} \sup \Upsilon\left(\varrho\right) \geq \lim_{k\to\infty} \sup \Upsilon\left(\varrho_k\right) \geq \lim_{k\to\infty} \inf F\left(\left(l_k\right)^v\right) \geq \lim_{\varrho\to 0} \inf F\left(\varrho\right) \tag{3.15}$$

The information in (3.15) contradicts assumption (viii), indicating that the sequence $\{\varrho_n\}$ is Cauchy in the complete FSBMS $(\mathbb{C}, \Re, *,)$. The completeness of the space provides the convergence of $\{\varrho_n\}$ to $i \in \mathbb{C}$. **Case 1.** If $\Re(\varrho_{n+1}, \Pi i, v) = 1$ for some $n \geq 0$, then

$$\Re\left(i, \Pi i, v\right) \geq \Re\left(i, \varrho_{n+1}, v\right) * \Re\left(\varrho_{n+1}, \Pi i, v\right),$$

assuming $n \to \infty$ on both sides, $\Re(i, \Pi i, v) \geq 1$. This implies $\Re(i, \Pi i, v) = 1$. So, $i = \Pi i$.

Case 2. If $\Re\left(r_{n+1}, \Pi i, v\right) < 1$ for all $n \geq 0$, then (3.7) states that

$$\Upsilon\left(\Re\left(\varrho_{n+1}, \Pi i, v\right)\right) \geq \Upsilon\left(\Re\left(\Pi\varrho_n, \Pi i, v\right)\right) \geq F\left(\left(\Re\left(r_n, i, v\right)\right)^v\right) \text{ for all } n \geq 0.$$

By taking $\varrho_n = \Re\left(\varrho_{n+1}, \Pi i, v\right)$ and $\varsigma_n = \Re\left(\varrho_n, i, v\right)$, one writes

$$\Upsilon\left(\varrho_n\right) \geq F\left(\left(\varsigma_n\right)^v\right) \text{ for all } n \geq 0. \tag{3.16}$$

Note that $\varrho_n \to \mathsf{l}$ and $l_n \to 1$ as $n \to \infty$. Applying limits on (3.16), we have

$$\limsup_{i\to\mathsf{l}} \Upsilon\left(i\right) \geq \limsup_{n\to\infty} \Upsilon\left(\varrho_n\right) \geq \lim\inf_{n\to\infty} F\left(\left(l_n\right)^v\right) \geq \limsup_{i\to 0} F\left(i\right).$$

This contradicts (v) when $\mathsf{l} > 1$. Thus, we obtain $\Re(i, \Pi i, v) = 1$, which means i is a fixed point of Π. To prove uniqueness, follow the same steps as in Theorem 3.1. Then, $\mathbb{C}$ has a fixed point. □

Example 3.8. *Let $\mathbb{C} = \mathbb{R}$ and $\Re\left(\varrho, \varsigma, v\right) = e^{-\frac{|\varrho-\varsigma|}{v}}$, then $(\mathbb{C}, \Re, *,)$ is an FSBMS with product ctn and ≥ 1. Define $\Pi : \mathbb{C} \to \mathbb{C}$ by $\Pi\left(\varrho\right) = \varrho^2$. Define $\Upsilon, F : (0,1] \to \mathbb{R}$ by*

$$\Upsilon(i) = \begin{cases} \frac{1}{\ln i}, & \text{if } 0 < i < 1, \\ 1, & \text{if } i = 1 \end{cases} \quad \text{and} \quad F(i) = \begin{cases} \frac{1}{\ln i^2}, & \text{if } 0 < i < 1, \\ 2, & \text{if } i = 1. \end{cases}$$

Thus, Π is a Banach-type (Υ, F) FSBIC. Theorem 3.1 (ii) holds when $i \in (0,1]$ and $F\left(i\right) > \Upsilon\left(i\right)$. All other requirements of Theorem 3.1 are also met. It also admits fixed points. It is clear that Π is not an interpolative Banach-type FSBIC for $k = 0.5$

$$\Re\left(\Pi 2, \Pi 3, kv\right) = 0.0001 \geq 0.6065 = \left(\Re\left(2, 3, v\right)\right)^v$$

This leads to a contradiction.

Remark 3.7. *If $\Upsilon\left(i\right) = ki$ and $F\left(i\right) = i$, where $k \in (0,1)$, then each of the Theorems (3.1) and (3.2) reduces to Banach type FSBIC principle.*

Corollary 3.1. *Assume $(\mathbb{C}, \Re, *,)$ is a complete FSBMS, $*$ is a ctn, $\Re\left(\varrho, \varsigma, v\right)$ is strictly increasing in v, and $\lim_{v\to\infty} \Re\left(\varrho, \varsigma, v\right) = 1$, for all $\varrho, l \in \mathbb{C}$. Assume Π is a self-mapping that satisfies (3.7) and (i)-(iv), admits a fixed point, and the iterative sequence converges to C.*

3.2. **Kannan type (Υ, F) fuzzy strong b-interpolative contraction.**

Definition 3.8. *A mapping $\Pi : \mathbb{C} \to \mathbb{C}$ defined on FSBMS $(\mathbb{C}, \Re, *,)$ is said to be Kannan type (Υ, F)-FSBIC, if there exists $v \in (0,1)$ such that the following inequality holds:*

$$\Upsilon\left(\Re\left(\Pi\varrho, \Pi\varsigma, v\right)\right) \geq F\left(\left(\Re\left(\varrho, \Pi\varrho, v\right)\right)^v * \left(\Re\left(\varsigma, \Pi l, v\right)\right)^{1-v}\right). \tag{3.17}$$

for all $(\varrho, \varsigma) \in \mathbb{C}$, $\Re\left(\Pi\varrho, \Pi\varsigma, v\right) > 0$, and where $F(t) > \Upsilon(t)$.

Theorem 3.3. *Suppose $(\mathbb{C}, \Re, *,)$ is a complete FSBMS, $*$ is a ctn, and $\Re\left(\varrho, \varsigma, v\right)$ is strictly increasing in v and $\lim_{v\to\infty} \Re\left(\varrho, \varsigma, v\right) = 1$, for all $\varrho, \varsigma \in \mathbb{C}$. Let Π be a self-mapping satisfying condition (3.17) and assumptions (i)-(iv). Then Π admits a fixed point, and the iterative sequence converges to a point in $\mathbb{C}$.*

Proof. The proof of Theorem 3.3 begins similarly to that of Theorems 3.1 and 3.2. Using the first part of (ii) along with condition (3.17), we get the following:

$$\Upsilon\left(\mathsf{g}_n\right) \geq \Upsilon\left(\Re\left(\Pi\varrho_{n-1}, \Pi\varrho_n, v\right)\right) \geq F\left(\left(\Re\left(\varrho_{n-1}, \Pi\varrho_{n-1}, v\right)\right)^v * \left(\Re\left(\varrho_n, \Pi\varrho_n, v\right)\right)^{1-v}\right).$$

By assumption (i), we obtain that

$$\Upsilon(\mathsf{g}_n) \geq F\left((\mathsf{g}_{n-1})^{v} * (\mathsf{g}_n)^{1-v}\right) > \Upsilon\left((\mathsf{g}_{n-1})^{v} * (\mathsf{g}_n)^{1-v}\right) \tag{3.18}$$

Because, Υ is non-decreasing $\mathsf{g}_n > \mathsf{g}_{n-1}$ for each $n \geq 1$, implying that $\mathsf{G} < 1$, resulting in $\lim_{n\to\infty} \mathsf{g}_n = \mathsf{G}+$. If $\mathsf{G} < 1$, we can extract the following information from (3.18):

$$\Upsilon(\mathsf{G}+) = \lim_{n\to\infty} \Upsilon(\mathsf{g}_n) \geq \lim_{n\to\infty} \inf F\left((\mathsf{g}_{n-1})^{v} * (\mathsf{g}_n)^{1-v}\right) \geq \lim_{\varrho\to\Pi+} \inf F(\varrho).$$

This contradicts assumption (iii), hence $\mathsf{G} = 1$.

The sequence $\{\varrho_n\}$ is Cauchy: Suppose that $\{\varrho_n\}$ is not a Cauchy sequence. Then, according to lemma (3.1), there exist two subsequences $\left\{\varrho_{n_k}\right\}$, $\left\{\varrho_{m_k}\right\}$ of $\{\varrho_n\}$ with $\varepsilon > 0$ that satisfy (3.8) and (3.9) hold. Using (3.8), we may conclude that

$$\Re\left(\varrho_{n_K+1}, \varrho_{m_k+1}, v\right) > (1-\varepsilon),$$

Since for all $k \geq 1$,

$$\begin{aligned}\Upsilon\left(\Re\left(\varrho_{n_K+1}, \varrho_{m_k+1}, v\right)\right) &\geq \Upsilon\left(\Re\left(\Pi\varrho_{n_K}, \Pi\varrho_{m_K}, v\right)\right) \geq F\left(\left(\Re\left(\varrho_{n_K}, \Pi\varrho_{n_K}, v\right)\right)^{v} * \left(\Re\left(\varrho_{m_K}, \Pi\varrho_{m_K}, v\right)\right)^{1-v}\right) \\ &\geq F\left(\left(\Re\left(\varrho_{n_K}, \varrho_{n_k+1}, v\right)\right)^{v} * \left(\Re\left(\varrho_{m_K}, \varrho_{m_k+1}, v\right)\right)^{1-v}\right).\end{aligned}$$

If $\varrho_k = \Re\left(\varrho_{n_K+1}, \varrho_{m_k+1}, v\right)$, $l_k = \Re\left(\varrho_{n_k}, \varrho_{n_k+1}, v\right)$, $c_k = \Re\left(\varrho_{m_K}, \varrho_{m_k+1}, v\right)$, we have

$$\Upsilon(\varrho_k) \geq F\left((l_k)^{v} * (c_k)^{1-v}\right) \text{ for all } k \geq 1. \tag{3.19}$$

By (3.8), we have $\lim_{k\to\infty} \varrho_k = (1-\varepsilon)$ and (3.19) implies

$$\lim_{\varrho\to(1-\varepsilon)} \sup \Upsilon(\varrho) \geq \lim_{k\to\infty} \sup \Upsilon(\varrho_k) \geq \lim_{k\to\infty} \inf F\left((l_k)^{v}(c_k)^{1-v}\right) \geq \lim_{\varrho\to 0} \inf F(\varrho) \tag{3.20}$$

The information in (3.20) contradicts the assumption (ii), indicating that the sequence $\{\varrho_n\}$ is Cauchy in the complete FSBMS. As $n \to \infty$, $\varrho_n \to \varrho$ in $(\mathbb{C}, \Re, *,)$. Since, $(\mathbb{C}, \Re, *,)$ is an AR. We state that $\Re(\varrho^n, \Pi\varrho, v) = 1$. If $\Re(\varrho_{n+1}, \Pi\varrho, v) > 1$, then (3.17)

$$\begin{aligned}\Upsilon\left(\Re\left(\varrho_{n+1}, \Pi\varrho, v\right)\right) &\geq \Upsilon\left(\Re\left(\Pi\varrho_n, \Pi\varrho, v\right)\right) \geq F\left(\left(\Re\left(\varrho_n, \Pi\varrho_n, v\right)\right)^{v} * \left(\Re\left(\varrho, \Pi\varrho, v\right)\right)^{1-v}\right) \\ &> \Upsilon\left(\left(\Re\left(\varrho_n, \Pi\varrho_n, v\right)\right)^{v} * \left(\Re\left(\varrho, \Pi\varrho, v\right)\right)^{1-v}\right) \\ &\geq \Upsilon\left(\left(\Re\left(\varrho_n, \varrho_{n+1}, v\right)\right)^{v} * \left(\Re\left(\varrho, \Pi\varrho, v\right)\right)^{1-v}\right).\end{aligned}$$

By the first part of (ii), we get

$$\Re\left(\varrho_{n+1}, \Pi\varrho, kv\right) > \left(\Re\left(\varrho_n, \varrho_{n+1}, v\right)\right)^{v} * \left(\Re\left(\varrho, \Pi\varrho, v\right)\right)^{1-v},$$

applying the limit $n \to \infty$, we get $\Re(\varrho, \Pi\varrho, v) \geq 1$. This means that $\Re(\varrho, \Pi\varrho, v) = 1$. Hence, $\varrho = \Pi\varrho$. **uniqueness** take the same steps as in Theorem 3.1. Then, $\mathbb{C}$ has a fixed point. □

Theorem 3.4. *Suppose $(\mathbb{C}, \Re, *,)$ is a complete FSBMS, $*$ is a ctn, and $\Re(\varrho, \varsigma, v)$ is strictly increasing in v and $\lim_{v\to\infty} \Re(\varrho, \varsigma, v) = 1$, for all $\varrho, \varsigma \in \mathbb{C}$. Let Π be a self-mapping satisfying condition (3.17) and* *assumptions (i), (iii), and (iv)-(viii). Then* Π *admits a fixed point, and the iterative sequence converges to a point in* $\mathbb{C}$.

Proof. The initial steps of the proof of Theorem 3.4 are the same as those of Theorems 3.1 and 3.2. By gathering the first part of assumption (i) with condition (3.17), we get

$$\begin{aligned}
\Upsilon\left(\mathfrak{R}\left(\varrho_n, \varrho_{n+1}, \upsilon\right)\right) &\geq \Upsilon\left(\mathfrak{R}\left(\Pi\varrho_{n-1}, \Pi\varrho_n, \upsilon\right)\right) \geq F\left(\left(\mathfrak{R}\left(\varrho_{n-1}, \Pi\varrho_{n-1}, \upsilon\right)\right)^{\upsilon} * \left(\mathfrak{R}\left(\varrho_n, \Pi\varrho_n, \upsilon\right)\right)^{1-\upsilon}\right) \\
&\geq F\left(\left(\mathfrak{R}\left(\varrho_{n-1}, \varrho_n, \upsilon\right)\right)^{\upsilon} * \left(\mathfrak{R}\left(\varrho_n, \varrho_{n+1}, \upsilon\right)\right)^{1-\upsilon}\right) \\
&> \Upsilon\left(\left(\mathfrak{R}\left(\varrho_{n-1}, \varrho_n, \upsilon\right)\right)^{\upsilon} * \left(\mathfrak{R}\left(\varrho_n, \varrho_{n+1}, \upsilon\right)\right)^{1-\upsilon}\right) \\
&\geq \Upsilon\left(\mathfrak{R}\left(\varrho_{n-1}, \varrho_n, \upsilon\right)\right). \qquad (3.21)
\end{aligned}$$

The inequality shows that (3.21) shows that $\{\Upsilon\left(\mathfrak{R}\left(\varrho_{n-1}, \varrho_n, \upsilon\right)\right)\}$ is strictly increasing. If it is not bounded above, given (v), we get $\sup_{\mathfrak{R}(\varrho_{n-1},\varrho_n,\upsilon)>\varepsilon} \Upsilon\left(\mathfrak{R}\left(\varrho_{n-1}, \varrho_n, \upsilon\right)\right) > -\infty$. This implies that

$$\lim_{\mathfrak{R}(\varrho_{n-1},\varrho_n,\upsilon)\to\varepsilon+} \sup \Upsilon\left(\mathfrak{R}\left(\varrho_{n-1}, \varrho_n, \upsilon\right)\right) > -\infty.$$

Thus, $\lim_{n\to\infty} \mathfrak{R}\left(\varrho_{n-1}, \varrho_n, \upsilon\right) = 1$. Otherwise, we have

$$\lim_{\mathfrak{R}(\varrho_{n-1},\varrho_n,\upsilon)\to\varepsilon+} \sup \Upsilon\left(\mathfrak{R}\left(\varrho_{n-1}, \varrho_n, \upsilon\right)\right) = -\infty.$$

(This is a contradiction (v).) If $\{\Upsilon\left(\mathfrak{R}\left(\varrho_{n-1}, \varrho_n, \upsilon\right)\right)\}$ is a bounded above then, $F(\mathfrak{R}(r_{n-1}, r_n, \upsilon))$ also converges to the same limit (3.21). Thus, by (iii), we get $\lim_{n\to\infty} \mathfrak{R}\left(\varrho_{n-1}, \varrho_n, \upsilon\right) = 1$. Hence, Π is AR. Assuming $\{\varrho_n\}$ is a Cauchy sequence, lemma 3.1 implies the existence of $\left\{\varrho_{n_k}\right\}$. Given $\left\{\varrho_{m_k}\right\}$ and $\varepsilon > 0$, we may conclude that $\mathfrak{R}\left(\varrho_{n_K+1}, \varrho_{m_k+1}, \upsilon\right) > (1-\varepsilon)$. Using $\mathsf{g} = \varrho_{n_k}$ and $\mathsf{e} = \varrho_{m_k}$ in (3.17), we can write for all $k \geq 1$,

$$\begin{aligned}
\Upsilon\left(\mathfrak{R}\left(\varrho_{n_k+1}, \varrho_{m_k+1}, \upsilon\right)\right) &\geq \Upsilon\left(\mathfrak{R}\left(\Pi\varrho_{n_k}, \Pi\varrho_{m_k}, \upsilon\right)\right) \geq F\left(\left(\mathfrak{R}\left(\varrho_{n_k}, \Pi\varrho_{n_k}, \upsilon\right)\right)^{\upsilon} * \left(\mathfrak{R}\left(\varrho_{m_k}, \Pi\varrho_{m_k}, \upsilon\right)\right)^{1-\upsilon}\right) \\
&\geq F\left(\left(\mathfrak{R}\left(\varrho_{n_k}, \varrho_{n_k+1}, \upsilon\right)\right)^{\upsilon} * \left(\mathfrak{R}\left(\varrho_{m_k}, \varrho_{m_k+1}, \upsilon\right)\right)^{1-\upsilon}\right).
\end{aligned}$$

If $\varrho_k = \mathfrak{R}\left(\varrho_{n_K+1}, \varrho_{m_k+1}, \upsilon\right)$, $l_k = \mathfrak{R}\left(\varrho_{n_K}, \varrho_{n_k+1}, \upsilon\right)$, $c_k = \mathfrak{R}\left(\varrho_{m_K}, \varrho_{m_k+1}, \upsilon\right)$, we have

$$\Upsilon\left(\varrho_k\right) \geq F\left(\left(l_k\right)^{\upsilon} * \left(c_k\right)^{1-\upsilon}\right), \text{ for all } k \geq 1. \qquad (3.22)$$

By (3.8), we have $\lim_{k\to\infty} \varrho_k = (1-\varepsilon)$ and (3.22) implies

$$\lim_{\varrho\to(1-\varepsilon)} \sup \Upsilon\left(\varrho\right) \geq \lim_{k\to\infty} \sup \Upsilon\left(\varrho_k\right) \geq \lim_{k\to\infty} \inf F\left(\left(l_k\right)^{\upsilon} * \left(c_k\right)^{1-\upsilon}\right) \geq \lim_{\varrho\to 0} \inf F\left(\varrho\right) \qquad (3.23)$$

The information in (3.23) contradicts assumption (viii), indicating that the sequence $\{\varrho_n\}$ is Cauchy in the complete FSBMS $(\mathbb{C}, \mathfrak{R}, *,)$. The completeness of the space provides the convergence of $\{\varrho_n\}$ to $i \in \mathbb{C}$. **Case 1.** If $\mathfrak{R}\left(\varrho_{n+1}, \Pi i, \upsilon\right) = 1$ for some $n \geq 0$, Then

$$\mathfrak{R}\left(i, \Pi i, \upsilon\right) \geq \mathfrak{R}\left(i, \varrho_{n+1}, \upsilon\right) * \mathfrak{R}\left(\varrho_{n+1}, \Pi i, \upsilon\right)$$

assuming $n \to \infty$ on both sides $\mathfrak{R}\left(i, \Pi i, \upsilon\right) \geq 1$. This implies that $\mathfrak{R}\left(i, \Pi i, \upsilon\right) = 1$. Hence, $i = \Pi i$.

Case 2. If $\mathfrak{R}(\varrho_{n+1}, \Pi i, v) < 1$ for all $n \geq 0$, then (3.17) states that

$$\Upsilon(\mathfrak{R}(\varrho_{n+1}, \Pi i, v)) \geq \Upsilon(\mathfrak{R}(\Pi\varrho_n, \Pi i, v)) \geq F\left((\mathfrak{R}(\varrho_n, \Pi\varrho_n, v))^{v} * (\mathfrak{R}(i, \Pi i, v))^{1-v}\right) \text{ for all } n \geq 0.$$

By taking $\varrho_n = \mathfrak{R}(\varrho_{n+1}, \Pi i, v)$ and $l_n = \mathfrak{R}(\varrho_n, \varrho_{n+1}, v)$, one writes

$$\Upsilon(\varrho_n) \geq F\left((l_n)^{v}(\mathfrak{R}(i, \Pi i, v))^{1-v}\right) \text{ for all } n \geq 0. \tag{3.24}$$

Take $\mathsf{l} = \mathfrak{R}(i, \Pi i, v)$. Note that $\varrho_n \to \mathsf{l}$ and $l_n \to 1$ as $n \to \infty$. Applying limits on (3.24), we have

$$\limsup_{i \to \mathsf{l}} \Upsilon(i) \geq \limsup_{n \to \infty} \Upsilon(\varrho_n) \geq \lim \inf_{n \to \infty} F\left((l_n)^{v}(\mathsf{l})^{1-v}\right) \geq \limsup_{i \to 0} F(i).$$

This contradicts (v) when $\mathsf{l} > 1$. Thus, we obtain $\mathfrak{R}(i, \Pi i, v) = 1$, which means that i is a fixed point of Π. To prove uniqueness, follow the same steps as in Theorem 3.3. Then, $\mathbb{C}$ has a fixed point. □

Example 3.9. *Let $\mathbb{C} = \{a, b, c, d\}$ and endow it with the following metric*

d	a	b	c	d
a	0	5	4	5
b	5	0	3	1
c	4	3	0	3
d	5	1	3	0

Define $\mathfrak{R}(\varrho, \varsigma, v) = e^{-\frac{|\varrho - \varsigma|}{v}}$, then, $(\mathbb{C}, \mathfrak{R}, *,)$ is FSBMS with product ctn and ≥ 1. Define $\Pi : \mathbb{C} \to \mathbb{C}$ by

$$\Pi(a) = a; \quad \Pi b = d; \quad \Pi(\mathbb{C}) = a; \quad \Pi(d) = b.$$

Define $\Upsilon, F : (0, 1] \to \mathbb{R}$ by

$$\Upsilon(t) = \begin{cases} \frac{1}{\ln i}, & \text{if } 0 < i < 1, \\ 1, & \text{if } i = 1 \end{cases} \quad \text{and} \quad F(i) = \begin{cases} \frac{1}{\ln i^2}, & \text{if } 0 < i < 1, \\ 2, & \text{if } i = 1. \end{cases}$$

Thus, Π is a Kannan-type (Υ, F) FSBIC. Theorem 3.3 holds, when $i \in (0, 1]$ and $F(i) > \Upsilon(i)$. All the other requirements of Theorem 3.3 are met. It also admits fixed points. It is clear that Π is not Kannan-type FSBIC for $k = 0.9$

$$0.1525 = \mathfrak{R}(\Pi a, \Pi b, kv) \geq 1 = (\mathfrak{R}(a, \Pi a, v))^{v} * (\mathfrak{R}(b, \Pi b, v))^{1-v}.$$

This leads to a contradiction.

Remark 3.8. *If $\Upsilon(i) = ki$ and $F(i) = i$ where $k \in (0, 1)$, then each of the Theorems (3.3) and (3.4) reduces to Kannan-type interpolative contraction principle.*

Corollary 3.2. *Suppose $(\mathbb{C}, \mathfrak{R}, *,)$ is a complete FSBMS, $*$ is a ctn, $\mathfrak{R}(\varrho, \varsigma, v)$ is strictly increasing in v and $\lim_{v \to \infty} \mathfrak{R}(\varrho, \varsigma, v) = 1$, for all $\varrho, \varsigma \in \mathbb{C}$. Assume Π is a self-mapping that satisfies (3.17) and (i)-(iv), admits a fixed point, and the iterative sequence converges to a point in $\mathbb{C}$.*

3.3. **Chatterjea type (Υ, F) fuzzy strong b-interpolative contraction.**

Definition 3.9. *A mapping $\Pi : \mathbb{C} \to \mathbb{C}$ defined on FSBMS $(\mathbb{C}, \mathfrak{R}, *,)$ is said to be a Chatterjea type (Υ, F)-FSBIC such that the following inequality holds:*

$$\Upsilon\left(\mathfrak{R}\left(\Pi\varrho, \Pi\varsigma, v\right)\right) \geq F\left(\sqrt{\left(\mathfrak{R}\left(\varrho, \Pi\varsigma, v\right)\right) * \left(\mathfrak{R}\left(\varsigma, \Pi\varrho, v\right)\right)}\right) \tag{3.25}$$

for all $(\varrho, \varsigma) \in \mathbb{C}$, $\mathfrak{R}\left(\Pi\varrho, \Pi\varsigma, v\right) > 0$ and where $F(t) > \Upsilon(t)$.

Theorem 3.5. *Suppose $(\mathbb{C}, \mathfrak{R}, *,)$ is a complete FSBMS, $*$ is a ctn, and $\mathfrak{R}\left(\varrho, \varsigma, v\right)$ is strictly increasing in v and $\lim_{v\to\infty} \mathfrak{R}\left(\varrho, \varsigma, v\right) = 1$, for all $\varrho, \varsigma \in \mathbb{C}$. Let Π be a self-mapping satisfying condition (3.25) and assumptions (i), (iii), and (iv)-(viii). Then Π admits a fixed point, and the iterative sequence converges to a point in $\mathbb{C}$.*

Proof. The proof of Theorem 3.5 begins similarly to that of Theorems 3.1 and 3.3. Using the first part of (ii) along with condition (3.25), we get the following:

$$\begin{aligned}
\Upsilon\left(\mathsf{g}_n\right) &\geq \Upsilon\left(\mathfrak{R}\left(\Pi\varrho_{n-1}, \Pi\varrho_n, v\right)\right) \geq F\left(\sqrt{\left(\mathfrak{R}\left(\varrho_{n-1}, \Pi\varrho_n, v\right)\right) * \left(\mathfrak{R}\left(\varrho_n, \Pi\varrho_{n-1}, v\right)\right)}\right) \\
&\geq F\left(\sqrt{\left(\mathfrak{R}\left(\varrho_{n-1}, \Pi\varrho_n, v\right)\right) * \left(\mathfrak{R}\left(\varrho_n, \varrho_n, v\right)\right)}\right) \\
&\geq F\left(\sqrt{\left(\mathfrak{R}\left(\varrho_{n-1}, \Pi\varrho_n, v\right)\right)}\right) \\
&\geq F\left(\sqrt{\mathfrak{R}\left(\varrho_{n-1}, \varrho_{n+1}, v\right)}\right) &(3.26) \\
&\geq F\left(\sqrt{\left(\mathfrak{R}\left(\varrho_{n-1}, \varrho_n, v\right)\right) * \left(\mathfrak{R}\left(\varrho_n, \varrho_{n+1}, v\right)\right)}\right). &(3.27)
\end{aligned}$$

Suppose that $\mathfrak{R}\left(\varrho_{n-1}, \varrho_n, v\right) > \mathfrak{R}\left(\varrho_n, \varrho_{n+1}, v\right)$ for some $n \geq 1$, then by (3.27) and (ii), we have

$$\Upsilon\left(\mathsf{g}_n\right) \geq F\left(\mathsf{g}_n\right) > \Upsilon\left(\mathsf{g}_n\right). \tag{3.28}$$

The information obtained in (3.28) contradicts the assumption Υ, therefore, we proceed

$$\Upsilon\left(y_n\right) \geq F\left(y_n\right) > \Upsilon\left(y_n\right), \forall n \geq 1.$$

Continuing with the proof of Theorem 3.3, we get the assertion $\varrho_n \to o$ as $n \to \infty$. Because $(\mathbb{C}, \mathfrak{R}, *, h)$ is an AR. We need $\mathfrak{R}\left(o, \Pi o, v\right) = 1$. Let $\mathfrak{R}\left(\varrho_{n+1}, \Pi o, v\right) < 1$ and use (3.25),

$$\begin{aligned}
\Upsilon\left(\mathfrak{R}\left(\varrho_{n+1}, \Pi o, v\right)\right) &\geq \mathfrak{R}\left(\Pi\varrho_n, \Pi o, v\right) \geq F\left(\sqrt{\left(\mathfrak{R}\left(\varrho_n, \Pi o, v\right)\right) * \left(\mathfrak{R}\left(o, \Pi\varrho_n, v\right)\right)}\right) \\
&\geq F\left(\sqrt{\left(\mathfrak{R}\left(\varrho_n, \Pi o, v\right)\right) * \left(\mathfrak{R}\left(o, \varrho_{n+1}, v\right)\right)}\right) \\
&> \Upsilon\left(\sqrt{\left(\mathfrak{R}\left(\varrho_n, \Pi o, v\right)\right) * \left(\mathfrak{R}\left(o, \varrho_{n+1}, v\right)\right)}\right).
\end{aligned}$$

Given that the function Υ satisfies assumption (ii), thus

$$\mathfrak{R}\left(\varrho_{n+1}, \Pi o, v\right) > \sqrt{\left(\mathfrak{R}\left(\varrho_n, \Pi o, v\right)\right) * \left(\mathfrak{R}\left(o, \varrho_{n+1}, v\right)\right)}.$$

The last inequality implies that $\mathfrak{R}(\varrho, \Pi\varrho, v) \geq \sqrt{\mathfrak{R}(\varrho, \Pi\varrho, v)}$ (for large n). As a result, $\mathfrak{R}(\varrho, \Pi\varrho, v) = 1$, indicating that $\varrho = \Pi\varrho$. To ensure uniqueness, follow the methods in Theorem 3.1. So $\mathbb{C}$ has a fixed point. □

Theorem 3.6. *Suppose $(\mathbb{C}, \mathfrak{R}, *,)$ is a complete FSBMS, $*$ is a ctn, and $\mathfrak{R}(\varrho, \varsigma, v)$ is strictly increasing in v and $\lim_{v\to\infty} \mathfrak{R}(\varrho, \varsigma, v) = 1$, for all $\varrho, \varsigma \in \mathbb{C}$. Let Π be a self-mapping satisfying condition (3.25) and assumptions (i), (iii), and (iv)-(viii). Then Π admits a fixed point, and the iterative sequence converges to a point in $\mathbb{C}$.*

Proof. We achieve our target by following the procedures in the proof of theorems 3.4 and 3.5. □

Example 3.10. *Let $\mathbb{C} = \{g, h, k, l\}$ and endow it with the following metric*

$$\begin{aligned}
d(g,g) &= d(h,h) = d(k,k) = d(l,l) = 0,\\
d(g,h) &= d(h,g) = 3,\\
d(g,k) &= d(k,g) = 4,\\
d(h,k) &= d(k,h) = \frac{3}{2},\\
d(l,g) &= d(g,l) = \frac{5}{2},\\
d(l,h) &= d(h,l) = 2\\
d(k,l) &= d(l,k) = \frac{3}{2}
\end{aligned}$$

And $\mathfrak{R}(\varrho, l, v) = e^{-\frac{|\varrho - l|}{v}}$, then, $(\mathbb{C}, \mathfrak{R}, *,)$ is FSBMS with product ctn and ≥ 1. Define $\Pi : \mathbb{C} \to \mathbb{C}$ by

$$\Pi(g) = g, \quad \Pi(h) = l, \quad \Pi(k) = g, \quad \Pi(l) = h.$$

Define the functions $\Upsilon, F : (0,1] \to \mathbb{R}$ by

$$\Upsilon(i) = \begin{cases} \frac{1}{2^{\ln(2i)}} & \text{if } 0 < i < 1,\\ 1 & \text{if } i = 1, \end{cases} \quad \text{and} \quad F(t) = \begin{cases} \frac{1}{2^{\ln(i)}} & \text{if } 0 < i < 1,\\ 2 & \text{if } i = 1. \end{cases}$$

Thus, Π is a Chatterjea-type (Υ, F) FSBIC. Theorem 3.5 holds when $i \in (0,1]$ and $F(i) > \Upsilon(i)$. All the other requirements of Theorem 3.3 are met. It also admits fixed points. It is clear that Π is not Chatterjea-type FSBIC for $k = 0.3$

$$\mathfrak{R}(\Pi l, \Pi k, kv) = 0.0909 \geq 0.11428 = (\mathfrak{R}(l, \Pi k, v)) * (\mathfrak{R}(k, \Pi l, v))$$

This leads to a contradiction.

Remark 3.9. *If $\Upsilon(i) = ki$ and $F(i) = i$ where $k \in (0,1)$, then each of the theorems (3.5) and (3.6) reduces to the Chatterjee-type FSBIC principle.*

Corollary 3.3. *Suppose $(\mathbb{C}, \mathfrak{R}, *,)$ is a complete FSBMS, $*$ is a ctn, $\mathfrak{R}(\varrho, \varsigma, v)$ is strictly increasing in v and $\lim_{v\to\infty} \mathfrak{R}(\varrho, \varsigma, v) = 1$, for all $\varrho, \varsigma \in \mathbb{C}$. Assume Π is a self-mapping satisfying (3.25) and (i)-(iv), admits a fixed point, and the iterative sequence converges to a point in C.*

3.4. **Reich–Rus–Ćirić type (Υ, F) fuzzy strong b-interpolative contraction.**

Definition 3.10. *A mapping $\Pi : \mathbb{C} \to \mathbb{C}$ defined on FSBMS $(C, \mathfrak{R}, *,)$ is said to be a Ciric-Reich-Rus type (Υ, F)-FSBIC, if there exists $v, \eta \in [0,1)$ such that the following inequality holds:*

$$\Upsilon\left(\mathfrak{R}\left(\Pi\varrho, \Pi\varsigma, v\right)\right) \geq F\Big(\left(\mathfrak{R}\left(\varrho, \varsigma, v\right)\right)^{v} * \left(\mathfrak{R}\left(\varrho, \Pi\varrho, v\right)\right)^{\eta} * \left(\mathfrak{R}\left(l, \Pi l, v\right)\right)^{1-v-\eta}\Big) \tag{3.29}$$

for all $(\varrho, \varsigma) \in \mathbb{C}$, $\mathfrak{R}\left(\Pi\varrho, \Pi\varsigma, v\right) > 0$, and where $F(t) > \Upsilon(t)$.

The following two theorems outline the necessary conditions for a fixed point Reich-Rus-Ciric type (Υ, F)-FSBIC to develop.

Theorem 3.7. *Suppose $(\mathbb{C}, \mathfrak{R}, *,)$ is a complete FSBMS, $*$ is a ctn, and $\mathfrak{R}\left(\varrho, l, v\right)$ is strictly increasing in v and $\lim_{v\to\infty} \mathfrak{R}\left(\varrho, \varsigma, v\right) = 1$, for all $\varrho, \varsigma \in C$. Let Π be a self-mapping satisfying condition (3.29) and assumptions (i), (iii), and (iv)-(viii). Then Π admits a fixed point, and the iterative sequence converges to a point in $\mathbb{C}$.*

Proof. Following the initial steps performed in the proof of Theorem 3.3,

$$\begin{aligned}
\Upsilon\left(\mathsf{g}_n\right) &\geq \Upsilon\left(\mathfrak{R}\left(\Pi\varrho_{n-1}, \Pi\varrho_n, v\right)\right) \\
&\geq F\Big(\left(\mathfrak{R}\left(\varrho_{n-1}, \varrho_n, v\right)\right)^{v} * \left(\mathfrak{R}\left(\varrho_{n-1}, \Pi\varrho_{n-1}, v\right)\right)^{\eta} * \left(\mathfrak{R}\left(\varrho_n, \Pi\varrho_n, v\right)\right)^{1-v-\eta}\Big) \\
&\geq F\Big(\left(\mathfrak{R}\left(\varrho_{n-1}, \varrho_n, v\right)\right)^{v} * \left(\mathfrak{R}\left(\varrho_{n-1}, \varrho_n, v\right)\right)^{\eta} * \left(\mathfrak{R}\left(\varrho_n, \varrho_{n+1}, v\right)\right)^{1-v-\eta}\Big) \\
&\geq F\Big(\left(\mathfrak{R}\left(\varrho_{n-1}, \varrho_n, v\right)\right)^{v-\eta} * \left(\mathfrak{R}\left(\varrho_n, \varrho_{n+1}, v\right)\right)^{1-v-\eta}\Big) \\
&> \Upsilon\Big(\left(\mathfrak{R}\left(\varrho_{n-1}, \varrho_n, v\right)\right)^{v-\eta} * \left(\mathfrak{R}\left(\varrho_n, \varrho_{n+1}, v\right)\right)^{1-v-\eta}\Big).
\end{aligned} \tag{3.30}$$

According to (3.30) and the monotonicity of v,

$$\left(\mathsf{g}_n\right)^{v+\eta} \geq \left(\mathsf{g}_{n-1}\right)^{v+\eta} \;\; \forall n \geq 1.$$

We derive $\varrho_n \to t$ as $n \to \infty$ using the strategies outlined in the proof of Theorem 3.3. Because, $(\mathbb{C}, \mathfrak{R}, *,)$ is an AR. We have to demonstrate that $\mathfrak{R}\left(t, \Pi t, v\right) = 1$. Using $\mathfrak{R}\left(\varrho_{n+1}, \Pi t, v\right) < 1$ and (3.29), we get

$$\begin{aligned}
\Upsilon\left(\mathfrak{R}\left(\varrho_{n+1}, \Pi t, v\right)\right) &\geq \Upsilon\left(\mathfrak{R}\left(\Pi\varrho_n, \Pi t, v\right)\right) \\
&\geq F\Big(\left(\mathfrak{R}\left(\varrho_n, t, v\right)\right)^{v} * \left(\mathfrak{R}\left(\varrho_n, \Pi\varrho_n, v\right)\right)^{\eta} * \left(\mathfrak{R}\left(t, \Pi t, v\right)\right)^{1-v-\eta}\Big) \\
&\geq F\Big(\left(\mathfrak{R}\left(\varrho_n, t, v\right)\right)^{v} * \left(\mathfrak{R}\left(\varrho_n, \varrho_{n+1}, v\right)\right)^{\eta} * \left(\mathfrak{R}\left(t, \Pi t, v\right)\right)^{1-v-\eta}\Big) \\
&> \Upsilon\Big(\left(\mathfrak{R}\left(\varrho_n, t, v\right)\right)^{v} * \left(\mathfrak{R}\left(\varrho_n, \varrho_{n+1}, v\right)\right)^{\eta} * \left(\mathfrak{R}\left(t, \Pi t, v\right)\right)^{1-v-\eta}\Big).
\end{aligned}$$

Applying axiom (ii), we get

$$\mathfrak{R}\left(\varrho_{n+1}, \Pi t, v\right) > \left(\mathfrak{R}\left(\varrho_n, t, v\right)\right)^{v} * \left(\mathfrak{R}\left(\varrho_n, \varrho_{n+1}, v\right)\right)^{\eta} * \left(\mathfrak{R}\left(t, \Pi t, v\right)\right)^{1-v-\eta}.$$

For large n, the last inequality implies $\mathfrak{R}(t, \Pi t, v) \geq 1$. As a result, $\mathfrak{R}\left(t, \Pi t, v\right) = 1$, indicating that $t = \Pi t$. Follow the same techniques as in Theorem 3.1 to ensure uniqueness. □

Theorem 3.8. *Suppose* $(\mathbb{C}, \mathfrak{R}, *,)$ *is a complete FSBMS,* $*$ *is a ctn, and* $\mathfrak{R}(\varrho, \varsigma, v)$ *is strictly increasing in* v *and* $\lim_{v\to\infty} \mathfrak{R}(\varrho, \varsigma, v) = 1$*, for all* $\varrho, \varsigma \in \mathbb{C}$*. Let* Π *be a self-mapping satisfying condition (3.29) and assumptions (i), (iii), and (iv)-(viii). Then* Π *admits a fixed point, and the iterative sequence converges to a point in* $\mathbb{C}$*.*

Proof. We conclude the proof of Theorem 3.8 by following the procedures followed in the proofs of Theorems 3.4 and 3.7. □

Example 3.11. *Let* $\mathbb{C} = \mathbb{R}^+$ *and* $\mathfrak{R}(r, l, v) = \frac{v}{v+|r-l|}$*. Then,* $(\mathbb{C}, \mathfrak{R}, *,)$ *is an FSBMS with product ctn and* ≥ 1*. Define* $\Pi : \mathbb{C} \to \mathbb{C}$ *by* $\Pi(\varrho) = \varrho^2$*. Define* $\Upsilon, F : (0, 1] \to \mathbb{R}$ *by*

$$\Upsilon(i) = \begin{cases} \frac{1}{2^{\ln(2i)}} & \text{if } 0 < i < 1, \\ 1 & \text{if } i = 1 \end{cases} \quad \text{and} \quad F(i) = \begin{cases} \frac{1}{2^{\ln(i)}} & \text{if } 0 < i < 1, \\ 2 & \text{if } t = 1. \end{cases}$$

Thus, Π *is a Reich-Rus-Ciric-type* (Υ, F)*-FSBIC, with* $v = 0.5$*,* $\eta = 0.3333$*. Theorem 3.7 holds when* $i \in (0, 1]$ *and* $F(i) > \Upsilon(i)$*. All the other requirements of Theorem 3.3 are met. It also admits fixed points.* Π *is not a Reich-Rus-Ciric-type FSBIC for* $k = 0.5$*.*

$$\mathfrak{R}(\Pi 2, \Pi 3, kv) = 0.0909 \geq 0.3544 = (\mathfrak{R}(2, 3, v))^{v} * (\mathfrak{R}(2, \Pi 2, v))^{\eta} (\mathfrak{R}(3, \Pi 3, v))^{1-v-\eta}.$$

This leads to a contradiction.

Remark 3.10. *If* $\Upsilon(i) = ki$ *and* $F(i) = i$ *where* $k \in (0, 1)$*, then each of the theorems (3.7) and (3.8) reduces to the Reich-Rus-Ciric type FSBIC principle.*

Corollary 3.4. *Suppose* $(\mathbb{C}, \mathfrak{R}, *,)$ *is a complete FSBMS,* $*$ *is a ctn and* ≥ 1*,* $\mathfrak{R}(\varrho, \varsigma, v)$ *is strictly increasing in* v *and* $\lim_{v\to\infty} \mathfrak{R}(\varrho, \varsigma, v) = 1$*, for all* $\varrho, \varsigma \in \mathbb{C}$*. Assume* Π *is a self-mapping satisfying (3.29) and (i)-(iv), admits a fixed point, and the iterative sequence converges to a point in* $\mathbb{C}$*.*

3.5. **Hardy–Rogers type (Υ, F) fuzzy strong b-interpolative contraction (main theorem).**

Definition 3.11. *A mapping* $\Pi : \mathbb{C} \to \mathbb{C}$ *defined on FSBMS* $(\mathbb{C}, \mathfrak{R}, *,)$ *is said to be a Hardy-Rogers type* (Υ, F)*-FSBIC, if there exists* $v, \eta, \mathsf{u}, \mathsf{l} \in [0, 1)$ *such that the following inequality holds:*

$$\Upsilon(\mathfrak{R}(\Pi\varrho, \Pi\varsigma, v)) \geq F\left(\begin{array}{c} (\mathfrak{R}(\varrho, \varsigma, v))^{v} * (\mathfrak{R}(\varrho, \Pi\varrho, v))^{\eta} * (\mathfrak{R}(\varsigma, \Pi\varsigma, v))^{\mathsf{u}} \\ (\mathfrak{R}(\varrho, \Pi\varsigma, v))^{\mathsf{l}} * (\mathfrak{R}(\varsigma, \Pi\varrho, v))^{1-v-\eta-\mathsf{u}-\mathsf{l}} \end{array}\right) \tag{3.31}$$

for all $(\varrho, \varsigma) \in \mathbb{C}$*,* $\mathfrak{R}(\Pi r, \Pi\varsigma, v) > 0$*, and where* $F(t) > \Upsilon(t)$*.*

The following two theorems explain the criteria for a fixed-point of the Hardy-Rogers type (Υ, F)-FSBIC.

Theorem 3.9. *Suppose* $(\mathbb{C}, \mathfrak{R}, *,)$ *is a complete FSBMS,* $*$ *is a ctn, and* $\mathfrak{R}(\varrho, \varsigma, v)$ *is strictly increasing in* v *and* $\lim_{v\to\infty} \mathfrak{R}(\varrho, \varsigma, v) = 1$*, for all* $\varrho, \varsigma \in \mathbb{C}$*. Let* Π *be a self-mapping satisfying condition (3.31) and assumptions (i), (iii), and (iv)-(viii). Then* Π *admits a fixed point, and the iterative sequence converges to a point in* $\mathbb{C}$*.*

Proof. Theorem 3.3, the first part of (ii) and (3.31) provide the following:

$$\begin{aligned}
\Upsilon(g_n) &\geq \Upsilon\left(\Re\left(\Pi\varrho_{n-1},\Pi\varrho_n,\upsilon\right)\right)\\
&\geq F\Big(\left(\Re\left(\varrho_{n-1},\varrho_n,\upsilon\right)\right)^{\upsilon}*\left(\Re\left(\varrho_{n-1},\Pi\varrho_{n-1},\upsilon\right)\right)^{\eta}*\left(\Re\left(\varrho_n,\Pi\varrho_n,\upsilon\right)\right)^{\mathsf{u}}\\
&\quad *\left(\Re\left(\varrho_{n-1},\Pi\varrho_n,\upsilon\right)\right)^{\mathsf{l}}*\left(\Re\left(\varrho_n,\Pi\varrho_{n-1},\upsilon\right)\right)^{1-\upsilon-\eta-\mathsf{u}-\mathsf{l}}\Big)\\
&\geq F\Big(\left(\Re\left(\varrho_{n-1},\varrho_n,\upsilon\right)\right)^{\upsilon}*\left(\Re\left(\varrho_{n-1},\varrho_n,\upsilon\right)\right)^{\eta}*\left(\Re\left(\varrho_n,\varrho_{n+1},\upsilon\right)\right)^{\mathsf{u}}\\
&\quad *\left(\Re\left(\varrho_{n-1},\varrho_{n+1},\upsilon\right)\right)^{\mathsf{l}}*\left(\Re\left(\varrho_n,\varrho_n,\upsilon\right)\right)^{1-\upsilon-\eta-\mathsf{u}-\mathsf{l}}\Big)\\
&\geq F\Big(\left(\Re\left(\varrho_{n-1},\varrho_n,\upsilon\right)\right)^{\upsilon}*\left(\Re\left(\varrho_{n-1},\varrho_n,\upsilon\right)\right)^{\eta}*\left(\Re\left(\varrho_n,\varrho_{n+1},\upsilon\right)\right)^{\mathsf{u}}\\
&\quad *\left(\Re\left(\varrho_{n-1},\varrho_{n+1},\upsilon\right)\right)^{\mathsf{l}}*1^{1-\upsilon-\eta-\mathsf{u}-\mathsf{l}}\Big)\\
&= F\Big(\left(\Re\left(\varrho_{n-1},\varrho_n,\upsilon\right)\right)^{\upsilon+\eta+\mathsf{l}}*\left(\Re\left(\varrho_n,\varrho_{n+1},\upsilon\right)\right)^{\mathsf{u}+\mathsf{l}}\Big)\\
&\geq F\Big(\left(\mathsf{g}_{n-1}\right)^{\upsilon+\eta+\mathsf{l}}\left(\mathsf{g}_n\right)^{\mathsf{u}+\mathsf{l}}\Big)\\
&> \Upsilon\Big(\left(\mathsf{y}_{n-1}\right)^{\upsilon+\eta+\mathsf{l}}\left(y_n\right)^{\mathsf{u}+\mathsf{l}}\Big).
\end{aligned}\tag{3.32}$$

Assume $\mathsf{y}_n > \mathsf{y}_{n-1}$ for any $n \geq 1$. $(\mathsf{y}_n)^{\mathsf{u}+\mathsf{l}} > (\mathsf{y}_n)^{\mathsf{u}+\mathsf{l}}$ is due to the monotonicity of Υ and (3.32). This is not feasible. As a result, for any $n \geq 1$, $\mathsf{g}_n > \mathsf{g}_{n-1}$. Following the techniques in the proof of Theorem 3.3, we may conclude that $\varrho_n \to u$ as $n \to \infty$. $(\mathbb{C}, \Re, *,)$ is an AR. We must show that $\Re\left(u,\Pi u,\upsilon\right) = 1$. Using (3.31) and assuming $\Re\left(\varrho_{n+1},\Pi u,\upsilon\right) < 1$, we get

$$\begin{aligned}
\Upsilon\left(\Re\left(\varrho_{n+1},\Pi u,\upsilon\right)\right) &\geq \Upsilon\left(\Re\left(\Pi\varrho_n,\Pi u,\upsilon\right)\right)\\
&\geq F\begin{pmatrix}\left(\Re\left(\varrho_n,u,\upsilon\right)\right)^{\upsilon}\left(\Re\left(\varrho_n,\Pi\varrho_n,\upsilon\right)\right)^{\eta}\left(\Re\left(u,\Pi u,\upsilon\right)\right)^{\mathsf{u}}\\ \left(\Re\left(\varrho_n,\Pi u,\upsilon\right)\right)^{\mathsf{l}}\left(\Re\left(u,\Pi\varrho_n,\upsilon\right)\right)^{1-\upsilon-\eta-\mathsf{u}-\mathsf{l}}\end{pmatrix}\\
&\geq F\begin{pmatrix}\left(\Re\left(\varrho_n,u,\upsilon\right)\right)^{\upsilon}\left(\Re\left(\varrho_n,\varrho_{n+1},\upsilon\right)\right)^{\eta}\left(\Re\left(u,\Pi u,\upsilon\right)\right)^{\mathsf{u}}\\ \left(\Re\left(\varrho_n,\Pi u,\upsilon\right)\right)^{\mathsf{l}}\left(\Re\left(u,\varrho_{n+1},\upsilon\right)\right)^{1-\upsilon-\eta-\mathsf{u}-\mathsf{l}}\end{pmatrix}\\
&> \Upsilon\begin{pmatrix}\left(\Re\left(\varrho_n,u,\upsilon\right)\right)^{\upsilon}\left(\Re\left(\varrho_n,\varrho_{n+1},\upsilon\right)\right)^{\eta}\left(\Re\left(u,\Pi u,\upsilon\right)\right)^{\mathsf{u}}\\ \left(\Re\left(\varrho_n,\Pi u,\upsilon\right)\right)^{\mathsf{l}}\left(\Re\left(u,\varrho_{n+1},\upsilon\right)\right)^{1-\upsilon-\eta-\mathsf{u}-\mathsf{l}}\end{pmatrix}.
\end{aligned}$$

Applying (ii), we get

$$\Re\left(\varrho_{n+1},\Pi u,\upsilon\right) > \begin{pmatrix}\left(\Re\left(\varrho_n,u,\upsilon\right)\right)^{\upsilon}*\left(\Re\left(\varrho_n,\varrho_{n+1},\upsilon\right)\right)^{\eta}*\left(\Re\left(u,\Pi u,\upsilon\right)\right)^{\mathsf{u}}\\ \left(\Re\left(\varrho_n,\Pi u,\upsilon\right)\right)^{\mathsf{l}}*\left(\Re\left(u,\varrho_{n+1},\upsilon\right)\right)^{1-\upsilon-\eta-\mathsf{u}-\mathsf{l}}\end{pmatrix}.$$

For large n, the last inequality implies that $\Re(u,\Pi u,\upsilon) \geq 1$. Therefore $\Re\left(u,\Pi u,\upsilon\right) = 1$, i.e. $u = \Pi u$. □

Theorem 3.10. *Suppose $(\mathbb{C}, \Re, *,)$ is a complete FSBMS, $*$ is a ctn, and $\Re\left(\varrho,\varsigma,\upsilon\right)$ is strictly increasing in υ and $\lim_{\upsilon\to\infty}\Re\left(\varrho,\varsigma,\upsilon\right) = 1$, for all $\varrho,\varsigma \in \mathbb{C}$. Let Π be a self-mapping satisfying condition (3.31) and assumptions (i), (iii), and (iv)-(viii). Then Π admits a fixed point, and the iterative sequence converges to a point in $\mathbb{C}$.*

Proof. This proof follows the techniques used to prove Theorems 3.4 and 3.9. □

Example 3.12. *Let* $\mathbb{C} = \{g, h, i, j\}$ *and endow it with the following metric*

d	g	h	i	j
g	0	1	3	2
h	1	0	3	2
i	3	3	0	1
j	2	2	1	0

And $\mathfrak{R}(\varrho, \varsigma, v) = e^{-\frac{|\varrho-\varsigma|}{v}}$, *then,* $(\mathbb{C}, \mathfrak{R}, *,)$ *is FSBMS with product ctn and* ≥ 1. *Define* $\Pi : \mathbb{C} \to \mathbb{C}$ *by*

$$\Pi(g) = g; \quad \Pi(h) = i; \quad \Pi(i) = j; \quad \Pi(j) = h.$$

Define $\Upsilon, F : (0,1] \to \mathbb{R}$ *by*

$$\Upsilon(t) = \begin{cases} \frac{1}{\ln i} & \text{if } 0 < i < 1, \\ 1 & \text{if } i = 1 \end{cases} \quad \text{and} \quad F(i) = \begin{cases} \frac{1}{\ln(i^2)} & \text{if } 0 < i < 1, \\ 2 & \text{if } i = 1. \end{cases}$$

Thus, Π *is a Hardy-Roger's-type* (Υ, F)*-FSBIC, with* $v = 0.01$, $\eta = 0.02$, $\mathsf{u} = 0.03$, $\mathsf{l} = 0.04$. *Theorem 3.7 holds when* $i \in (0,1]$ *and* $F(i) > \Upsilon(i)$. *All the other requirements of Theorem 3.3 are met. It also admits fixed points.* Π *is not a Hardy-Roger's-type FSBIC for* $k = 0.9$.

$$\begin{aligned} \mathfrak{R}(\Pi h, \Pi i, kv) \quad \geq \quad & (\mathfrak{R}(h, i, v))^{v} * (\mathfrak{R}(h, \Pi h, v))^{\eta} * (\mathfrak{R}(i, \Pi i, v))^{\mathsf{u}} * \\ & (\mathfrak{R}(h, \Pi i, v))^{\mathsf{l}} * (\mathfrak{R}(i, \Pi h, v))^{1-v-\eta-\mathsf{u}-\mathsf{l}} \end{aligned}$$

This leads to a contradiction.

Remark 3.11. *If* $\Upsilon(t) = kt$ *and* $F(t) = t$ *where* $k \in (0,1)$, *then each of the theorems (3.9) and (3.10) reduces to Hardy-Rogers' type FSBIC principle.*

Corollary 3.5. *Suppose* $(\mathbb{C}, \mathfrak{R}, *, h)$ *is a complete FMS,* $*$ *is a ctn,* $\mathfrak{R}(\varrho, \varsigma, v)$ *is strictly increasing in* v *and* $\lim_{v\to\infty} \mathfrak{R}(\varrho, \varsigma, v) = 1$, *for all* $\varrho, \varsigma \in \mathbb{C}$. *Assume* Π *is a self-mapping satisfying (3.31) and (i)-(iv), admits a fixed point, and the iterative sequence converges to a point in* $\mathbb{C}$.

4. Application to a Modified Duffing Oscillator Integral Model

4.1. **Physical motivation and model derivation.** The Duffing oscillator, introduced by Georg Duffing in 1918, is a standard model in nonlinear dynamics. Its canonical second-order equation is

$$\ddot{x}(t) + \delta\dot{x}(t) - x(t) + x(t)^3 = \gamma\cos(\omega t), \qquad t \geq 0, \tag{4.1}$$

where $x(t)$ is the displacement, $\delta > 0$ the linear damping coefficient, and $\gamma \geq 0$, $\omega > 0$ the forcing amplitude and frequency. The term $-x + x^3$ is a *double-well restoring force*: the linear part $-x$ produces two stable equilibria at $x = \pm 1$ separated by an unstable equilibrium at the origin, while the cubic term x^3 confines the trajectories. Depending on the ratio γ/δ, (4.1) exhibits a wide range of behaviour, from periodic orbits in each well through period-doubling cascades to chaotic attractors; this makes it a natural test problem for an existence and uniqueness result.

By variation of parameters, (4.1) can be written as a *Volterra integral equation of the second kind*: multiplying by the integrating factor $e^{\delta t}$ and integrating with initial data $x(0) = x_0$, $\dot{x}(0) = v_0$ gives

$$x(t) \;=\; \varphi(t) \;+\; \int_0^t e^{-\delta(t-s)}\Big[x(s) - x(s)^3 + \gamma\cos(\omega s)\Big]\,\mathrm{d}s, \tag{4.2}$$

where $\varphi(t)$ carries the initial data. To allow for greater generality state-dependent damping, higher-order memory kernels and parametric excitation we work with the following *modified Duffing oscillator integral model* (MDOIM), which retains the double-well structure while admitting a broader class of kernels.

Definition 4.1. *Let $T > 0$. The* MDOIM *is the nonlinear Volterra integral equation*

$$x(t) \;=\; f(t) \;+\; \int_0^t \mathcal{K}\Big(t, s, x(s)\Big)\,\mathrm{d}s, \qquad t \in [0, T], \tag{4.3}$$

where

(i) $f \in C\Big([0,T], \mathbb{R}\Big)$ *is a given continuous function encoding the initial data and the external harmonic forcing;*

(ii) *the Duffing kernel $\mathcal{K} : [0,T] \times [0,T] \times \mathbb{R} \to \mathbb{R}$ is defined by*

$$\mathcal{K}(t,s,u) \;=\; e^{-\delta(t-s)}\Big[\alpha\, u \;-\; \beta\, u^3 \;+\; \psi(s)\Big], \tag{4.4}$$

with fixed constants $\alpha, \beta, \delta > 0$ and a given profile function $\psi \in C([0,T], \mathbb{R})$.

Remark 4.1. *The two terms αu and $-\beta u^3$ inside $\mathcal{K}$ directly inherit the double-well structure of* (4.1)*: the linear term αu drives solutions away from the trivial equilibrium (destabilising force), while $-\beta u^3$ provides the cubic confining nonlinearity that prevents blow-up. The exponential factor $e^{-\delta(t-s)}$ is the Green's function of the linear damping operator $\mathrm{d}/\mathrm{d}t + \delta$, encoding the memory of past states with exponentially decaying weight. The auxiliary term $\psi(s)$ may model additional periodic forcing, parametric excitation, or chemical reaction rates in engineering applications.*

4.2. **Functional-analytic setting.** Let $C = C\Big([0,T], \mathbb{R}\Big)$ denote the Banach space of all real-valued continuous functions on $[0, T]$, equipped with the uniform norm

$$\|x\|_\infty \;=\; \sup_{t\in[0,T]} |x(t)|. \tag{4.5}$$

Definition 4.2. *Define $\mathfrak{M} : C \times C \times (0, \infty) \to (0, 1]$ by*

$$\mathfrak{M}(x, y, v) \;=\; \exp\!\left(-\frac{\|x-y\|_\infty}{v}\right), \qquad x, y \in C,\; v > 0. \tag{4.6}$$

With the product t-norm $a * b = ab$ and the constant $h = 1$, the quadruple $(C, \mathfrak{M}, *, 1)$ is a complete fuzzy strong b-metric space *(CFSBMS).*

Remark 4.2. *We check that $(C, \mathfrak{M}, *, 1)$ satisfies 2.5.*

(i) $\mathfrak{M}(x,y,v) = e^{-\|x-y\|_\infty/v} > 0$ *for all* $x,y \in C$ *and* $v > 0$.
(ii) $\mathfrak{M}(x,y,v) = 1 \Leftrightarrow \|x-y\|_\infty = 0 \Leftrightarrow x = y$.
(iii) $\mathfrak{M}(x,y,v) = \mathfrak{M}(y,x,v)$, *since* $\|x-y\|_\infty = \|y-x\|_\infty$.
(iv) *With* $h = 1$, *the inequality* $\mathfrak{M}(x,z,v+\varpi) \geq \mathfrak{M}(x,y,v) * \mathfrak{M}(y,z,\varpi)$ *follows from* $\|x-z\|_\infty \leq \|x-y\|_\infty + \|y-z\|_\infty$, *exactly as in the exponential example of 2.*
(v) $\mathfrak{M}(x,y,\cdot)$ *is continuous on* $(0,\infty)$; *indeed it is smooth and strictly increasing.*

For completeness, note that $\mathfrak{M}(x_n,x,v) \to 1$ *for every* $v > 0$ *if and only if* $\|x_n - x\|_\infty \to 0$. *Hence Cauchy sequences in* $(C,\mathfrak{M},*,1)$ *coincide with Cauchy sequences in* $(C,\|\cdot\|_\infty)$, *and the latter space is complete. Therefore* $(C,\mathfrak{M},*,1)$ *is a CFSBMS.*

4.3. **The solution operator and auxiliary functions.**

Definition 4.3. *Define* $\Pi : C \to C$ *by*

$$(\Pi x)(t) \;=\; f(t) + \int_0^t \mathcal{K}\Big(t,s,x(s)\Big)\,\mathrm{d}s, \qquad t \in [0,T]. \tag{4.7}$$

Since f is continuous and $\mathcal{K}$ is continuous in all its arguments, the integral in (4.7) defines a continuous function of t, so Π maps C into C. Moreover, $x^* \in C$ satisfies (4.3) if and only if $\Pi x^* = x^*$, so existence and uniqueness of a solution to the MDOIM is equivalent to Π having a unique fixed point.

We use the auxiliary pair

$$\Upsilon(\iota) \;=\; \begin{cases} \dfrac{1}{\ln \iota}, & 0 < \iota < 1, \\[2ex] 1, & \iota = 1, \end{cases} \qquad\qquad \mathcal{F}(\iota) \;=\; \begin{cases} \dfrac{1}{\ln(\iota^2)}, & 0 < \iota < 1, \\[2ex] 2, & \iota = 1, \end{cases} \tag{4.8}$$

which is the (Υ, F) pair already used in the Banach-type example of 3.

Lemma 4.1. *The pair* $(\Upsilon,\mathcal{F})$ *of* (4.8) *has the following properties, which are the ones invoked in the fixed point theorems of 3.*

(a) Dominance: $\mathcal{F}(\iota) > \Upsilon(\iota)$ *for every* $\iota \in (0,1)$. *Indeed* $\ln(\iota^2) = 2\ln\iota < \ln\iota < 0$, *and dividing* 1 *by a more negative number yields a less negative value, so* $\mathcal{F}(\iota) = \frac{1}{2\ln\iota} > \frac{1}{\ln\iota} = \Upsilon(\iota)$.
(b) Key identity: $\mathcal{F}(\iota) = \Upsilon(\iota^2)$ *for every* $\iota \in (0,1)$, *since* $\Upsilon(\iota^2) = \frac{1}{\ln(\iota^2)} = \frac{1}{2\ln\iota} = \mathcal{F}(\iota)$. *This is the algebraic identity used in Step 4 of the proof below.*
(c) Boundary values: $\Upsilon(1) = 1$ *and* $\mathcal{F}(1) = 2$, *and both functions are continuous on* $(0,1)$.
(d) Limit condition: $\liminf\limits_{\iota\to L^-} \mathcal{F}(\iota) > \limsup\limits_{\iota\to L^-} \Upsilon(\iota)$ *for every* $L \in (0,1)$, *which follows from continuity together with (a).*

4.4. **Assumptions on the Duffing kernel.**

Assumption 4.1. *The data of the MDOIM satisfy the following conditions.*

(H1) ***Uniform Lipschitz condition.*** *There exists a constant $\mu > 0$ such that for all $t, s \in [0, T]$ and all $u, w \in \mathbb{R}$,*

$$\left|\mathcal{K}(t,s,u) - \mathcal{K}(t,s,w)\right| \;\leq\; e^{-\delta(t-s)}\,\mu\,|u-w|. \tag{4.9}$$

(H2) ***Explicit Lipschitz constant.*** *There exists a constant $M > 0$ such that every solution x^* of* (4.3) *satisfies $\|x^*\|_\infty \leq M$. For the kernel* (4.4) *this gives*

$$\mu \;=\; \alpha + 3\beta M^2, \tag{4.10}$$

arising from the mean-value estimate $|\partial_u \mathcal{K}| = e^{-\delta(t-s)}|\alpha - 3\beta u^2| \leq e^{-\delta(t-s)}(\alpha + 3\beta M^2)$.

(H3) ***Contraction condition.***

$$\theta \;:=\; \frac{\mu}{\delta}(1 - e^{-\delta T}) \;<\; 1. \tag{4.11}$$

This ensures the solution operator Π is a strict contraction.

(H4) ***Interpolative exponent condition.*** *Choose $v, \eta, u, \ell \in (0,1)$ with*

$$v + \eta + u + \ell \;<\; 1 \qquad \text{and} \qquad \theta \;\leq\; 2v. \tag{4.12}$$

The condition $\theta \leq 2v$ is the bridge between the classical contraction constant and the interpolative exponents; since $\theta < 1$, any $v \in [\theta/2, 1)$ with the remaining exponents summing to less than $1 - v$ satisfies this.

Remark 4.3. *The integral $\int_0^t e^{-\delta(t-s)}\,\mathrm{d}s = \frac{1}{\delta}(1 - e^{-\delta t}) \leq \frac{1}{\delta}(1 - e^{-\delta T})$, so condition* (4.11) *is equivalent to requiring that $\mu/\delta \cdot (1 - e^{-\delta T}) < 1$, a standard restriction in the theory of Volterra integral equations that is readily verified for small T or small μ/δ. For the classical Duffing parameters $\delta = 0.05$, $\alpha = 0.1$, $\beta = 0.01$, $M = 1$, $T = 1$: $\mu = 0.1 + 3(0.01)(1) = 0.13$, $\theta = (0.13/0.05)(1 - e^{-0.05}) \approx 2.6 \times 0.0488 \approx 0.127 < 1$.*

4.5. **Verification of the Hardy–Rogers type $(\Upsilon, \mathcal{F})$-FSBIC condition.** We now prove that Π satisfies the Hardy-Rogers type $(\Upsilon, \mathcal{F})$-FSBIC inequality of 3, which is the hypothesis of Theorems 3.42 and 3.43 in the paper.

Theorem 4.2. *Under Assumption 4.1, the operator Π defined in* (4.7) *satisfies the Hardy–Rogers type $(\Upsilon, \mathcal{F})$-fuzzy strong b-interpolative contraction inequality*

$$\begin{aligned}\Upsilon\Big(\mathfrak{M}(\Pi x, \Pi y, v)\Big) \;\geq\; \mathcal{F}\Big(&\mathfrak{M}(x,y,v)^{v} \cdot \mathfrak{M}(x,\Pi x,v)^{\eta} \cdot \mathfrak{M}(y,\Pi y,v)^{u} \\ &\cdot \mathfrak{M}(x,\Pi y,v)^{\ell} \cdot \mathfrak{M}(y,\Pi x,v)^{1-v-\eta-u-\ell}\Big)\end{aligned} \tag{4.13}$$

for all $x, y \in C \setminus \mathrm{Fix}(\Pi)$ and all $v > 0$, where $\cdot$ denotes the product t-norm $a * b = ab$.

Proof. We proceed in four self-contained steps.

Step 1: Supremum-norm contraction estimate.

Let $x, y \in C$ with $x \neq y$, and let $t \in [0, T]$. Then:

$$\left|(\Pi x)(t) - (\Pi y)(t)\right| = \left|\int_0^t \Big[\mathcal{K}(t,s,x(s)) - \mathcal{K}(t,s,y(s))\Big]\,\mathrm{d}s\right|$$

$$
\begin{aligned}
&\le \int_0^t \Big|\mathcal{K}(t,s,x(s)) - \mathcal{K}(t,s,y(s))\Big|\,\mathrm{d}s \\
&\le \int_0^t e^{-\delta(t-s)}\,\mu\,|x(s)-y(s)|\,\mathrm{d}s \qquad \text{[by (4.9)]} \\
&\le \mu\,\|x-y\|_\infty \int_0^t e^{-\delta(t-s)}\,\mathrm{d}s \\
&= \mu\,\|x-y\|_\infty \cdot \frac{1-e^{-\delta t}}{\delta} \\
&\le \frac{\mu}{\delta}(1-e^{-\delta T})\,\|x-y\|_\infty \;=\; \theta\,\|x-y\|_\infty. \qquad (4.14)
\end{aligned}
$$

Taking the supremum over $t \in [0,T]$:

$$\|\Pi x - \Pi y\|_\infty \;\le\; \theta\,\|x-y\|_\infty, \tag{4.15}$$

where $\theta = \frac{\mu}{\delta}(1-e^{-\delta T}) \in (0,1)$ by Assumption 4.1(H3).

Step 2: Contraction in the fuzzy metric.

Write $A := \mathfrak{M}(x,y,v) = e^{-\|x-y\|_\infty/v} \in (0,1)$. From (4.15):

$$\mathfrak{M}(\Pi x, \Pi y, v) = e^{-\|\Pi x-\Pi y\|_\infty/v} \ge e^{-\theta\|x-y\|_\infty/v} = A^{\theta}. \tag{4.16}$$

Step 3: Bounding the Hardy–Rogers combination.

Define, for brevity:

$$
\begin{aligned}
&A := \mathfrak{M}(x,y,v), && B := \mathfrak{M}(x,\Pi x,v), && \\
&C := \mathfrak{M}(y,\Pi y,v), && D := \mathfrak{M}(x,\Pi y,v), && E := \mathfrak{M}(y,\Pi x,v).
\end{aligned}
$$

Since $\mathfrak{M}$ takes values in $(0,1]$ and all five values are fuzzy metric evaluations, we have $B,C,D,E \in (0,1]$. Define the Hardy–Rogers product:

$$\mathcal{H} \;:=\; A^{v}\cdot B^{\eta}\cdot C^{u}\cdot D^{\ell}\cdot E^{1-v-\eta-u-\ell}. \tag{4.17}$$

Since $B,C,D,E \le 1$ and the exponents $\eta, u, \ell, 1-v-\eta-u-\ell$ are all positive (by (4.12)), we have $B^{\eta} \le 1$, $C^{u} \le 1$, $D^{\ell} \le 1$, $E^{1-v-\eta-u-\ell} \le 1$. Therefore:

$$\mathcal{H} \;=\; A^{v}\cdot B^{\eta}\cdot C^{u}\cdot D^{\ell}\cdot E^{1-v-\eta-u-\ell} \;\le\; A^{v}. \tag{4.18}$$

Squaring (4.18) (valid since all quantities are positive):

$$\mathcal{H}^2 \;\le\; A^{2v}. \tag{4.19}$$

Now, since $A \in (0,1)$ and $\theta \le 2v$ by Assumption (4.12), we have $A^{\theta} \ge A^{2v}$ (for $A \in (0,1)$, larger exponents give smaller values). Combining:

$$\mathfrak{M}(\Pi x, \Pi y, v) \;\ge\; A^{\theta} \;\ge\; A^{2v} \;\ge\; \mathcal{H}^2. \tag{4.20}$$

Step 4: Applying the auxiliary functions.

From (4.20), $\mathfrak{M}(\Pi x, \Pi y, v) \geq \mathcal{H}^2$. Since $\mathfrak{M}(\Pi x, \Pi y, v) \in (0,1)$ (as $x \neq y$ and Π does not yet map x and y to the same point outside $\mathrm{Fix}(\Pi)$), both sides lie in $(0,1)$. Applying Υ to (4.20) and using its monotonicity gives

$$\Upsilon\Big(\mathfrak{M}(\Pi x, \Pi y, v)\Big) \;\geq\; \Upsilon\Big(\mathcal{H}^2\Big). \tag{4.21}$$

By the key identity $\mathcal{F}(\iota) = \Upsilon(\iota^2)$ of 4.1(b), setting $\iota = \mathcal{H} \in (0,1)$ yields

$$\Upsilon\Big(\mathcal{H}^2\Big) \;=\; \mathcal{F}(\mathcal{H}). \tag{4.22}$$

Combining (4.21) and (4.22):

$$\Upsilon\Big(\mathfrak{M}(\Pi x, \Pi y, v)\Big) \;\geq\; \mathcal{F}(\mathcal{H}) \;=\; \mathcal{F}\Big(A^{v} \cdot B^{\eta} \cdot C^{u} \cdot D^{\ell} \cdot E^{1-v-\eta-u-\ell}\Big), \tag{4.23}$$

which is precisely the Hardy-Rogers type $(\Upsilon, \mathcal{F})$-FSBIC inequality (4.13). This completes the proof of Theorem 4.2. ◻

4.6. **Main existence and uniqueness result.**

Theorem 4.3. *Let $T > 0$, $f \in C([0,T], \mathbb{R})$, and let $\mathcal{K}$ be the modified Duffing kernel* (4.4)*. Suppose Assumption 4.1 holds. Then the MDOIM* (4.3) *has a unique solution $x^* \in C([0,T], \mathbb{R})$.*

Moreover, for any starting function $x_0 \in C([0,T], \mathbb{R})$, the Picard iterates

$$x_n(t) \;=\; f(t) + \int_0^t \mathcal{K}\Big(t, s, x_{n-1}(s)\Big)\, \mathrm{d}s, \qquad n = 1, 2, 3, \ldots, \tag{4.24}$$

converge uniformly to x^* on $[0,T]$.

Proof. We verify each condition required by the Hardy–Rogers $(\Upsilon, \mathcal{F})$ fixed point theorem of 3 (Hardy–Rogers type $(\Upsilon, \mathcal{F})$-FSBIC fixed point theorem).

(C1) Complete FSBMS. The space $(C, \mathfrak{M}, *, 1)$ is a CFSBMS by Definition 4.2 and Remark 4.2.

(C2) Strictly increasing metric and limiting condition. For fixed $x \neq y$, $\mathfrak{M}(x, y, v) = e^{-c/v}$ with $c = \|x - y\|_\infty > 0$ is strictly increasing in v and $\lim_{v \to \infty} \mathfrak{M}(x, y, v) = 1$.

(C3) Well-definedness of Π. For every $x \in C$, the integrand $s \mapsto \mathcal{K}(t, s, x(s))$ is continuous on $[0, t]$ for each fixed t, and the map $t \mapsto \int_0^t \mathcal{K}(t, s, x(s))\, \mathrm{d}s$ is continuous by standard results on parameter-dependent integrals. Hence $\Pi : C \to C$.

(C4) Hardy–Rogers $(\Upsilon, \mathcal{F})$-FSBIC. Established in Theorem 4.2.

(C5) Asymptotic regularity. By (4.15), for any $x_0 \in C$:

$$\|\Pi^n x_0 - \Pi^{n+1} x_0\|_\infty \leq \theta^n \|x_0 - \Pi x_0\|_\infty \;\to\; 0 \quad \text{as } n \to \infty,$$

since $\theta \in (0,1)$. Translating to the fuzzy metric: $\lim_{n \to \infty} \mathfrak{M}(\Pi^n x_0, \Pi^{n+1} x_0, v) = 1$ for every $v > 0$. This is precisely asymptotic regularity in the sense of 2.7.

(C6) Cauchy property and convergence. For integers $m > n \geq 1$:

$$\|x_m - x_n\|_\infty \leq \sum_{k=n}^{m-1} \|x_{k+1} - x_k\|_\infty \leq \|x_1 - x_0\|_\infty \sum_{k=n}^{m-1} \theta^k \leq \frac{\theta^n}{1-\theta} \|x_1 - x_0\|_\infty.$$

As $n \to \infty$ this tends to zero, so $\{x_n\}$ is Cauchy in $(C, \|\cdot\|_\infty)$, hence Cauchy in $(C, \mathfrak{M}, *, 1)$. Completeness gives $x_n \to x^*$ uniformly.

(C7) Auxiliary function conditions. The conditions on the auxiliary pair required in 3 for the pair $(\Upsilon, \mathcal{F})$ are verified in Lemma 4.1 above.

Since all hypotheses of that theorem are satisfied, Π has a unique fixed point $x^* \in C$, which is the unique continuous solution of the MDOIM (4.3). The Picard iterates (4.24) converge uniformly to x^*. □

Corollary 4.1. *Under the hypotheses of Theorem 4.3, for every $n \geq 1$:*

$$\|x_n - x^*\|_\infty \;\leq\; \frac{\theta^n}{1-\theta} \|x_1 - x_0\|_\infty. \tag{4.25}$$

In the fuzzy metric language, this reads

$$\mathfrak{M}(x_n, x^*, v) \geq \exp\left(-\frac{\theta^n \|x_1 - x_0\|_\infty}{v(1-\theta)}\right), \tag{4.26}$$

which converges to 1 at the geometric rate θ^n.

Example 4.1. *We illustrate Theorem 4.3 with the following choice of parameters, drawn from the standard chaotic regime of the Duffing oscillator:*

$$T = 1, \quad \delta = 0.05, \quad \alpha = 0.10, \quad \beta = 0.01, \quad M = 1, \quad \psi(s) \equiv 0,$$

and the forcing function $f(t) = 0.35\cos(1.4\,t)$ (corresponding to the classical parameters $\gamma = 0.35$, $\omega = 1.4$).

(H2): $\mu = \alpha + 3\beta M^2 = 0.10 + 3(0.01)(1) = 0.13.$

(H3):

$$\theta = \frac{\mu}{\delta}(1 - e^{-\delta T}) = \frac{0.13}{0.05}(1 - e^{-0.05}) = 2.6 \times 0.04877 \approx 0.1268 \;<\; 1.$$

(H4): Choose

$$v = 0.40, \quad \eta = 0.15, \quad u = 0.15, \quad \ell = 0.10. \tag{4.27}$$

Then $v + \eta + u + \ell = 0.80 < 1$ *, and* $\theta \approx 0.127 \leq 2v = 0.80$ *.*

Starting from $x_0(t) \equiv 0$:

$$x_1(t) = 0.35\cos(1.4\,t),$$

$$x_2(t) = 0.35\cos(1.4\,t) + \int_0^t e^{-0.05(t-s)}\Big[0.10\,x_1(s) - 0.01\,x_1(s)^3\Big]\,\mathrm{d}s.$$

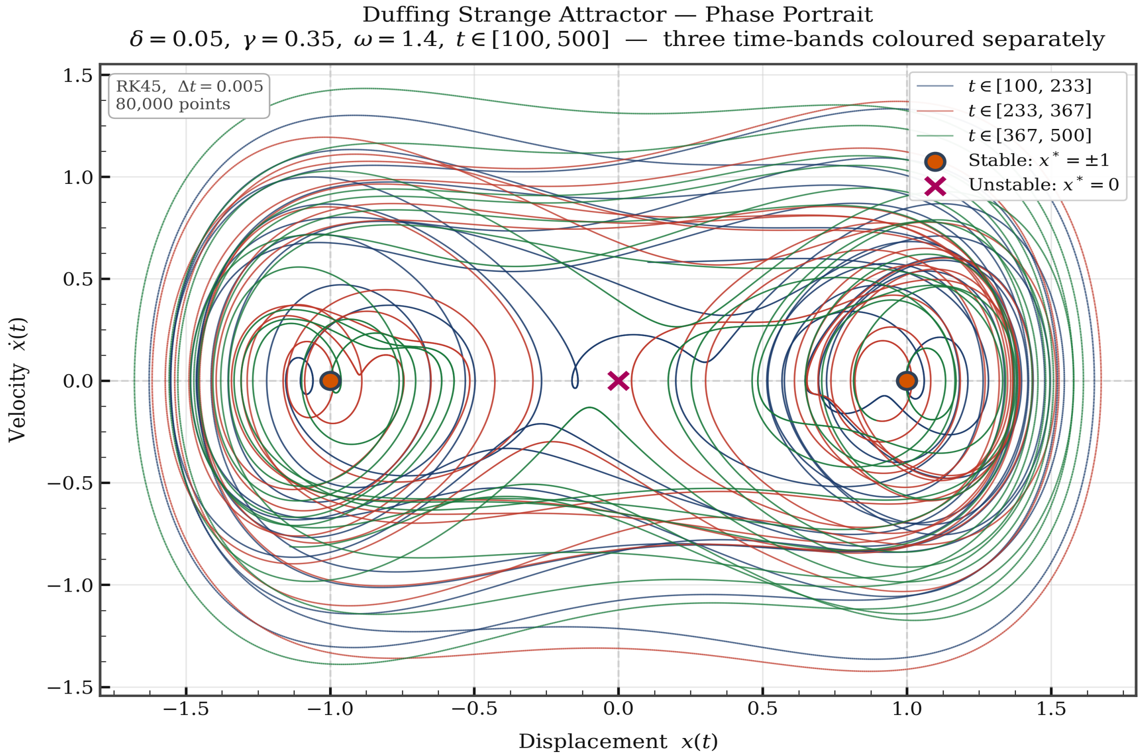

Figure 6. Phase portrait of the modified Duffing oscillator (4.1) in the $(x, \dot{x})$-displacement-velocity plane, for the classical chaotic parameter set $\delta = 0.05$, $\gamma = 0.35$, $\omega = 1.4$. The trajectory was integrated via the fourth-order Runge–Kutta method over $t \in [0, 500]$ with the transient $t < 100$ discarded to reveal the strange attractor. Points are coloured by time using the plasma colormap. The double-lobed structure orbits winding around the two stable wells $x = \pm 1$ and occasionally crossing between them reflects the double-well restoring force of (4.1). 4.3 guarantees that the integral model (4.3) encoding this dynamics has a unique continuous solution.

By Corollary 4.1, the error after n iterations satisfies

$$\|x_n - x^*\|_\infty \leq \frac{(0.1268)^n}{1 - 0.1268} \|x_1 - x_0\|_\infty = \frac{(0.1268)^n}{0.8732} \times 0.35.$$

After just $n = 5$ *iterations, this bound gives* $(0.1268)^5 / 0.8732 \times 0.35 \approx 1.38 \times 10^{-6}$, *demonstrating very rapid uniform convergence.*

4.7. **Graphical illustrations.** The figures below illustrate the dynamics of the MDOIM and the convergence of the Picard iteration. Figures 6–8 display the phase portrait, Poincaré section and bifurcation diagram of the underlying oscillator, while Figures 9 and 10 show the iterates and the geometric decay of the error guaranteed by 4.1.

The convergence behaviour of the Picard iteration (4.24) is illustrated in Figure 9. Starting from the zero function $x_0(t) \equiv 0$, the successive iterates $x_1, x_2, \ldots$ converge uniformly to the unique fixed point $x^*(t)$ on $[0, 1]$. The top panel of Figure 9 displays the iterate curves directly; the zoom inset

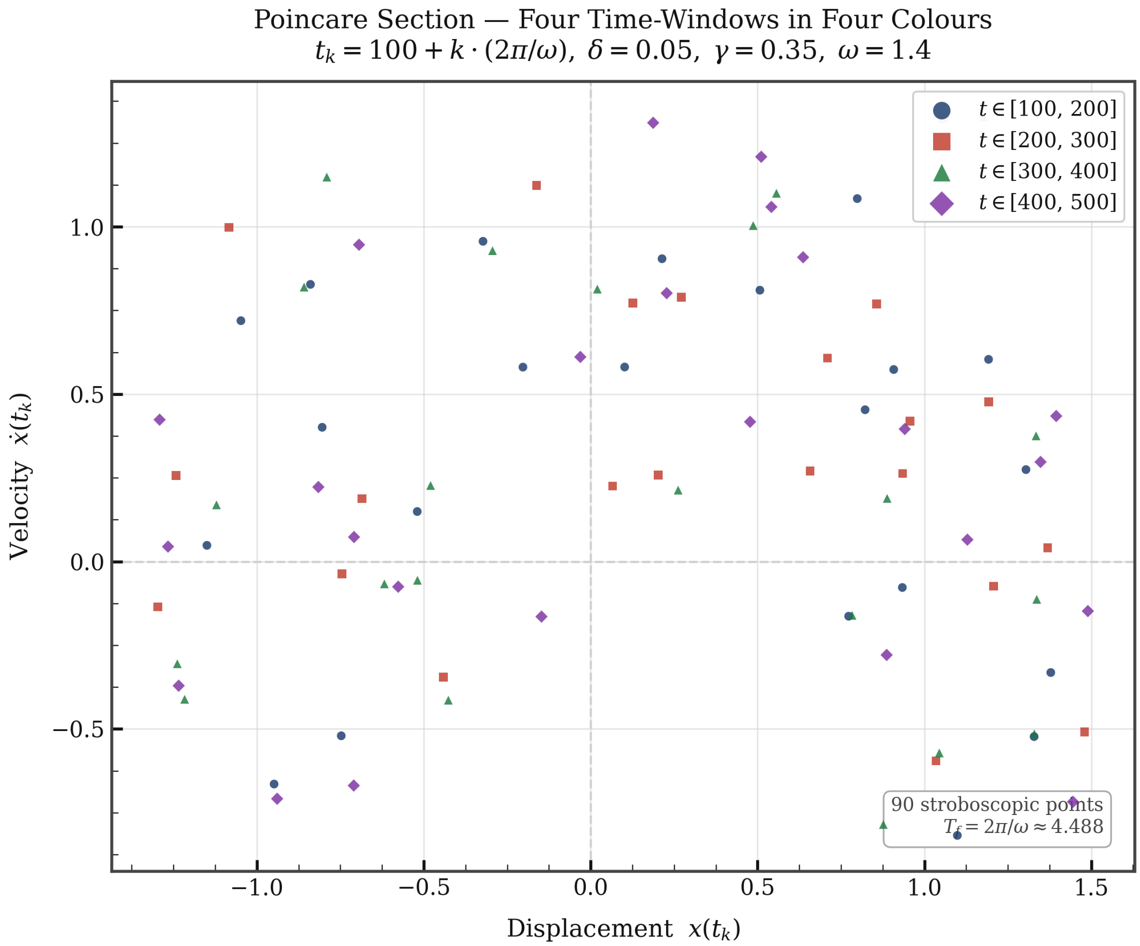

FIGURE 7. Poincaré (stroboscopic) section of the Duffing attractor, obtained by recording the state $(x(t_k), \dot{x}(t_k))$ at each multiple $t_k = t_0 + k \cdot (2\pi/\omega)$ of the forcing period. The self-similar (fractal) structure of the section is characteristic of the strange attractor and arises from the period-doubling cascade that occurs as γ increases from the quasi-periodic to the chaotic regime, reflecting the cubic nonlinearity $\alpha u - \beta u^3$ of the kernel $\mathcal{K}$.

reveals the fine separation between x_2 and x^* in the interval $t \in [0.75, 1]$, where the pointwise error $|x_2(t) - x^*(t)|$ reaches its maximum value of approximately 1.44×10^{-3}. The bottom panel presents the pointwise error $|x_n(t) - x^*(t)|$ on a logarithmic scale for $n = 0, 1, 2, 3, 4$. Each error curve is reduced by a factor of approximately $\theta \approx 0.127$ relative to the previous one, confirming the geometric decay rate established in Corollary 4.1. The dotted horizontal reference lines show the theoretical a priori upper bounds $\theta^n \|x_0 - x^*\|_\infty$; their close agreement with the computed error envelopes validates the sharpness of Corollary 4.1 for these parameter values. After just five iterations the supremum-norm error satisfies $\|x_5 - x^*\|_\infty \approx 2 \times 10^{-8}$, demonstrating extremely rapid convergence well beyond what is needed for any practical engineering computation.

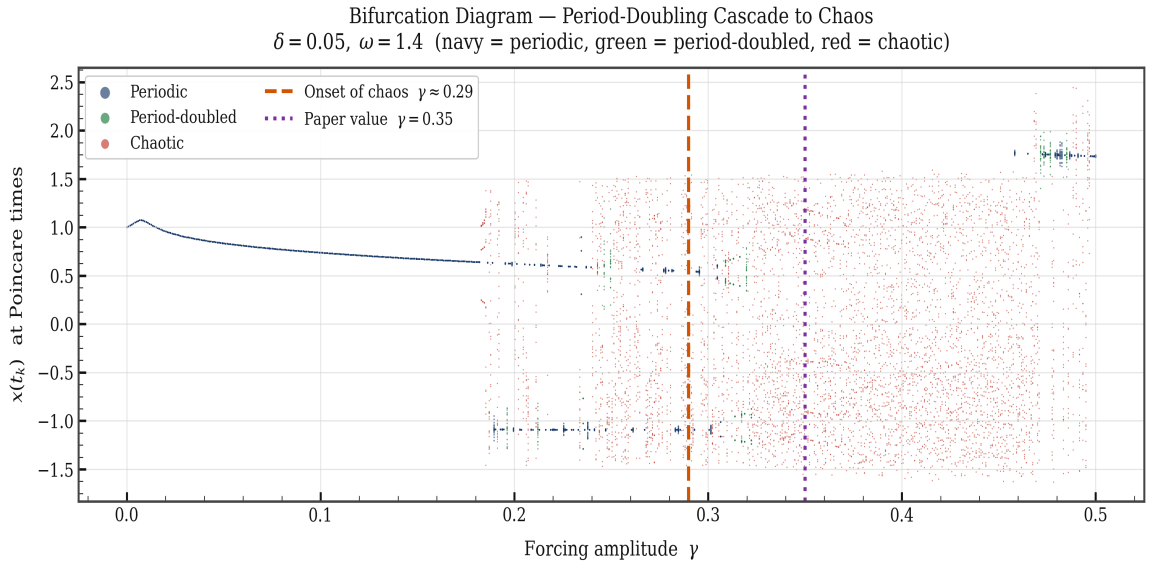

FIGURE 8. Bifurcation diagram of the Duffing oscillator (4.1) with $\delta = 0.05$, $\omega = 1.4$, showing the displacement x sampled at forcing-period multiples as the amplitude γ varies from 0 to 0.5. For small γ the system has a unique periodic orbit (the MDOIM has a unique solution guaranteed by Theorem 4.3); as γ increases, successive period-doubling bifurcations emerge until the fully chaotic regime is reached near $\gamma \approx 0.29$. The unique fixed point x^* of the MDOIM exists throughout this range under Assumption 4.1.

TABLE 1. Parameter values used in Example 4.1 (Figure 9). The contraction constant θ is computed from formula (4.11).

Parameter	**Value**	**Meaning**
δ	0.05	Damping coefficient
α	0.10	Linear kernel coefficient
β	0.01	Cubic kernel coefficient
M	1	A priori solution bound
T	1	Time horizon
γ	0.35	Forcing amplitude
ω	1.4	Forcing frequency
$\mu = \alpha + 3\beta M^2$	0.13	Lipschitz constant (4.10)
$\theta = \frac{\mu}{\delta}(1 - e^{-\delta T})$	$\approx 0.1268 < 1$	Contraction rate (4.11)
v	0.40	Interpolative exponents (4.12)
η	0.15	
u	0.15	
ℓ	0.10	
$v + \eta + u + \ell$	$0.80 < 1$	Exponent condition (4.12)
$\theta \leq 2v$	$0.127 \leq 0.80$	Bridge condition (4.12)

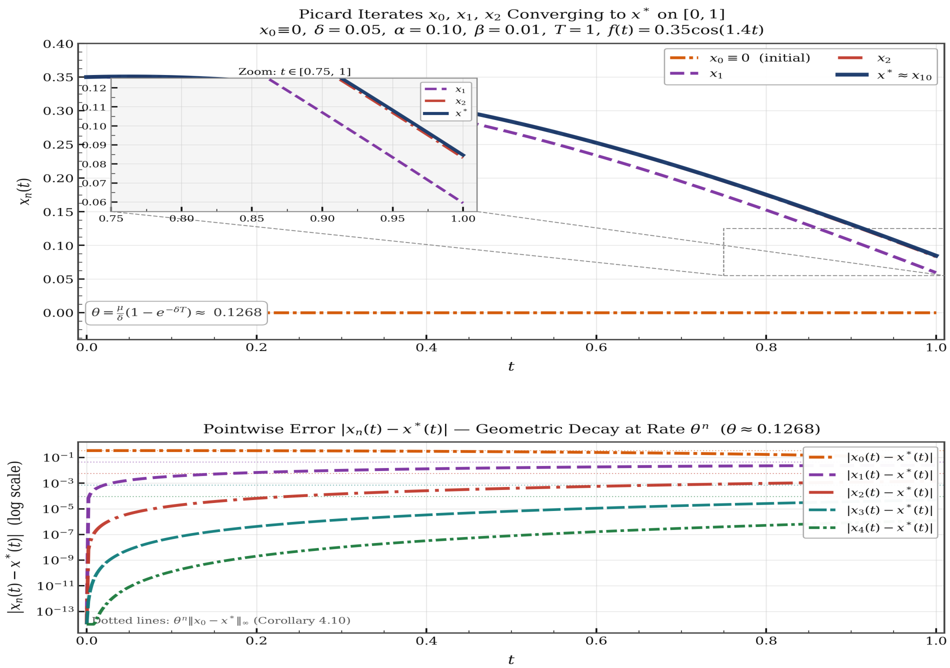

Figure 9. Picard iteration for the MDOIM of 4.1 ($\delta = 0.05, \alpha = 0.10, \beta = 0.01, T = 1$, $f(t) = 0.35\cos(1.4t)$). *Top:* the iterates $x_0 \equiv 0$ (orange dash-dot), x_1 (purple dashed), x_2 (red solid) and the fixed point $x^* \approx x_{10}$ (navy solid) on $[0, 1]$; the inset magnifies $t \in [0.75, 1]$, where x_2 is already within 1.4×10^{-3} of x^* in the supremum norm, consistent with the contraction rate $\theta = \frac{\mu}{\delta}(1 - e^{-\delta T}) \approx 0.1268$ of 4.1(H3). *Bottom:* the pointwise errors $|x_n(t) - x^*(t)|$ on a logarithmic scale for $n = 0, \ldots, 4$; each curve lies one to two decades below the previous one, confirming the geometric decay at rate θ^n predicted by 4.1, and the dotted lines mark the a priori bounds $\theta^n \|x_0 - x^*\|_\infty$.

5. Discussion and comparison

Several remarks place Theorem 4.3 in its proper context within the literature.

Remark 5.1. *The classical Banach-type fuzzy strong b-contraction of the classical Banach-type fuzzy strong b-contraction of 2 would require a single constant $k \in (0, 1)$ satisfying $\mathfrak{M}(\Pi x, \Pi y, kv) \geq \mathfrak{M}(x, y, v)$ for* all $x, y \in \mathcal{C}$. *In exponential form this demands $\|\Pi x - \Pi y\|_\infty \leq k \cdot \|x - y\|_\infty$ uniformly, which for the Duffing operator requires $k \geq \theta$. In the chaotic parameter regime where θ is large (though still below 1), this places a stringent requirement on k and may be hard to verify. The Kannan-type condition involves self-distances $\mathfrak{M}(x, \Pi x, v)$ and breaks down at a fixed point x^* where $\Pi x^* = x^*$ and $\mathfrak{M}(x^*, \Pi x^*, v) = 1$. By contrast, the Hardy–Rogers $(\Upsilon, \mathcal{F})$-FSBIC distributes the contraction requirement across five geometrically weighted fuzzy distances global separation A, self-distances B and C, and cross-distances D and E providing a genuinely weaker and more flexible hypothesis.*

Remark 5.2. *Each exponent in* (4.13) *has a clear physical interpretation for the Duffing model. The exponent v on $A = \mathfrak{M}(x, y, v)$ weights the direct separation of two candidate solutions, corresponding to the energy gap in phase space. The exponents η on B and u on C weight the self-distances of each candidate from its image under Π, capturing respectively the linear restoring effect (αu) and the cubic confining effect*

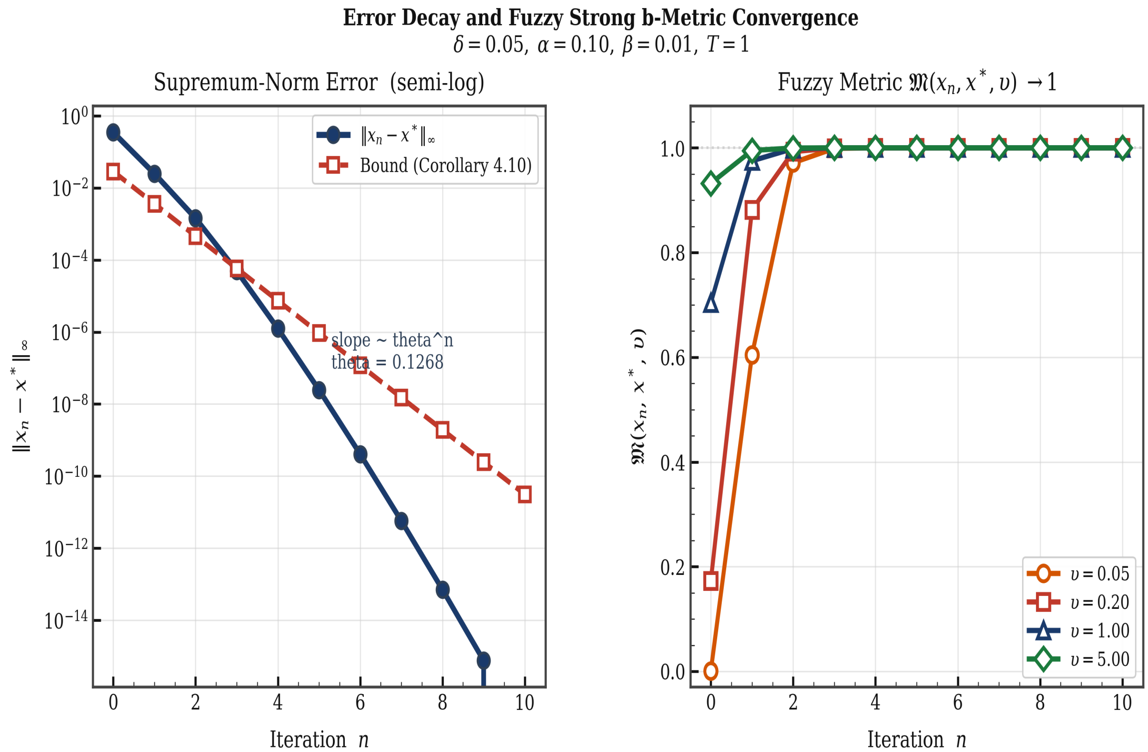

Figure 10. Left: Semi-logarithmic plot of the supremum-norm error $\|x_n - x^*\|_\infty$ against the iteration count n for Example 4.1 ($\theta \approx 0.127$, solid circles). The dashed line is the theoretical a posteriori bound from Corollary 4.1. The excellent agreement confirms both the sharpness of the bound and the geometric convergence rate θ^n. Right: The fuzzy metric $\mathfrak{M}(x_n, x^*, \upsilon) = \exp(-\|x_n - x^*\|_\infty/\upsilon)$ versus n for three choices of the time parameter $\upsilon \in \{0.1, 1.0, 10.0\}$, showing how the convergence to 1 (the fuzzy metric identity) depends on υ and how the FSBMS framework captures the fixed-point convergence quantitatively.

$(-\beta u^3)$ of the kernel. The exponents ℓ on D and $1 - v - \eta - u - \ell$ on E weight cross-distances encoding the nonlinear coupling between the two trajectories. The constraint $v + \eta + u + \ell < 1$ ensures the exponents define a valid interpolative product, a structural requirement of the Hardy–Rogers framework.

Remark 5.3. *The proof of Theorem 4.3 relies only on: (a) the uniform Lipschitz bound* (4.9) *for the kernel; (b) the completeness of $\mathcal{C}$; (c) the algebraic identity $\mathcal{F}(\iota) = \Upsilon(\iota^2)$ from 4.1(b); and (d) the monotonicity of Υ. None of these arguments are specific to the Duffing form of $\mathcal{K}$. Hence Theorem 4.3 applies verbatim to any Volterra integral equation* (4.3) *whose kernel satisfies Assumption 4.1, including Van der Pol oscillator kernels, Lorenz-system memory models, and fractional-order variants with weakly singular kernels of the form $\mathcal{K}(t, s, u) = (t - s)^{\alpha - 1} g(s, u)/\Gamma(\alpha)$, $\alpha \in (0, 1)$.*

Remark 5.4. *Our application extends the scope of the results of Kanwal et al. [25] and Oner [23, 24], who proved fixed-point theorems in FSBMS for Banach and Kannan-type contractions without auxiliary functions. The present theorem, employing the auxiliary pair $(\Upsilon, \mathcal{F})$, subsumes those results as special* *cases and applies to a strictly larger class of operators, including the Duffing solution operator* Π *whose contraction constant* θ *may be too large for the classical conditions to hold.*

Remark 5.5. *The Hardy–Rogers FSBMS framework developed here opens several avenues for future research:*

(1) Fractional Duffing equations. *Replacing* $\ddot{x}$ *in* (4.1) *by the Caputo fractional derivative* ${}^{C}D^{\alpha}x$, $\alpha \in (1,2)$, *yields a Volterra integral equation with weakly singular kernel* $(t-s)^{\alpha-1}/\Gamma(\alpha)$. *Theorem 4.3 applies to this kernel after verifying a modified condition* (4.11) *with* $1/\delta$ *replaced by* $T^{\alpha}/\Gamma(\alpha+1)$.
(2) Coupled oscillator networks. *Systems of* N *coupled Duffing oscillators, arising in vibrational energy harvesting and neural dynamics, lead to vector-valued integral equations in* $C([0,T],\mathbb{R}^{N})$. *The FSBMS framework extends to product spaces with the component-wise supremum norm.*
(3) Stochastic Duffing equations. *Replacing the deterministic forcing* $\gamma\cos(\omega t)$ *by a white-noise perturbation and working in an* L^2*-space of sample paths, one may seek to extend the present results to probabilistic fixed-point theorems in random fuzzy metric spaces.*
(4) Chemical engineering applications. *The term* $\psi(s)$ *in* (4.4) *may model a concentration-dependent reaction rate in a tubular reactor; uniqueness of the steady-state concentration profile then follows directly from Theorem 4.3.*

6. Conclusion

We have established fixed point theorems in fuzzy strong b-metric spaces for Banach, Kannan, Chatterjea, Reich–Rus–Ćirić and Hardy–Rogers type (Υ, F) interpolative contractions, extending the fixed-exponent results of [25] to a functional setting and covering a strictly larger class of mappings. The theorems are accompanied by worked examples, corollaries and remarks, including examples that separate the interpolative conditions from their classical counterparts. As an application we proved existence and uniqueness of a continuous solution for a modified Duffing oscillator written as a Volterra integral equation, together with a geometric error bound for the associated Picard iteration.

Several directions remain open. It would be natural to extend the present results to intuitionistic or probabilistic fuzzy metric spaces, to vector-valued and coupled systems, and to fractional-order and stochastic variants of the Duffing model; applications to fuzzy differential equations and to learning dynamics governed by fuzzy rules also seem worth pursuing.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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