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Optimal Quadrature Formulas with Odd-Order Derivative Corrections in the Sobolev Space $L_2^{(m)}(0,1)$

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Abstract. This paper presents the construction of optimal quadrature formulas with derivative corrections in the Sobolev space $L_2^{(m)}(0,1)$. The proposed approach is based on Sobolev's method together with the discrete analogue of the differential operator of order $2m$. A complete analytical derivation of the optimal coefficients is obtained by reducing the problem to a finite system of linear equations for the unknown parameters. Closed-form representations of these coefficients are established, and an explicit expression for the squared norm of the corresponding error functional is derived. The resulting quadrature formulas extend several previously known optimal formulas and provide an effective framework for the accurate numerical evaluation of definite integrals of sufficiently smooth functions.

1. Introduction. Statement of the problem

The accurate numerical evaluation of definite integrals remains one of the fundamental problems in computational mathematics because of its broad range of applications in numerical analysis, mathematical physics, engineering, inverse problems, signal processing, and scientific computing. Although many classical quadrature rules provide satisfactory accuracy for sufficiently smooth functions, modern computational models frequently require integration methods that achieve higher precision while using limited information about the integrand. This demand has stimulated continuous research on the development of optimal quadrature formulas possessing minimal

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approximation errors in appropriate functional spaces. The theory of optimal quadrature formulas originates from the pioneering works of Sard and Nikol'skii, who formulated optimization problems for numerical integration from different viewpoints [1]. Sard's approach focuses on determining optimal coefficients for prescribed nodes by minimizing the norm of the error functional, whereas Nikol'skii's formulation considers the simultaneous optimization of both coefficients and nodes. These two viewpoints have established the theoretical foundation for a substantial body of research devoted to optimal numerical integration [2,3].

Considerable progress has been achieved in extending the classical theory to more sophisticated settings, including weighted Sobolev spaces, periodic Hilbert spaces, oscillatory integrals, fractional integrals, singular kernels, and quadrature formulas involving derivative information [4,5]. Recent investigations have demonstrated that incorporating derivative values into quadrature formulas may significantly improve approximation accuracy while preserving computational efficiency. Furthermore, new applications have shown that optimal quadrature formulas are useful not only for numerical integration itself but also for solving integral equations, reconstructing computed tomography images, and approximating highly oscillatory and fractional operators [6,7]. Among the available construction techniques, Sobolev's method based on discrete analogues of differential operators remains one of the most powerful analytical approaches. Unlike purely numerical optimization procedures, this method often leads to explicit representations of optimal coefficients together with exact formulas for the norm of the associated error functional. Such closed-form results provide deeper theoretical insight into the structure of optimal quadrature formulas and facilitate their practical implementation in computational algorithms. The development of optimal quadrature formulas has attracted sustained attention over several decades because of its theoretical significance and practical importance in scientific computing. Early studies established the mathematical foundations of optimal numerical integration through the works of Nikol'skii, Sobolev, Schoenberg, and their collaborators [1,2,4]. These pioneering contributions introduced variational principles for constructing quadrature formulas with minimal error and demonstrated the fundamental role of Hilbert and Sobolev spaces in approximation theory.

Several studies have recently focused on the construction of optimal quadrature formulas with derivatives in Sobolev spaces. In particular, Shadimetov et al. [8] developed optimal quadrature formulas involving first- and second-order derivatives of the integrand by applying Sobolev's method. Explicit representations of the optimal coefficients were obtained through the minimization of the norm of the error functional, and the proposed formulas were successfully applied to the numerical solution of Fredholm integral equations of the second kind, demonstrating higher accuracy than classical numerical integration methods. In another study [9], constructed an optimal quadrature formula with first- and second-order derivative corrections in the Sobolev space $L_2^{(4)}(0,1)$. The norm of the error functional was derived explicitly, closed-form expressions for the optimal coefficients were obtained using the discrete analogue of the differential operator, and numerical experiments confirmed a significant improvement over Simpson's rule and previously known quadrature formulas. More recently [10], generalized this approach to arbitrary Sobolev spaces $L_2^{(m)}(0,1)$. They established analytical formulas for the optimal coefficients by employing discrete analogues of higher-order differential operators together with the Vandermonde determinant technique. Their work provided a unified theoretical framework for constructing derivative-based optimal quadrature formulas and analyzing their approximation properties for general values of m. In [11], proposed an optimal Hermite-type quadrature formula in space $L_2^{(m)}(0,1)$, where the quadrature sum incorporates function values together with the first- and second-order derivatives at uniformly distributed nodes. Using Sobolev's method and the discrete analogue of the differential operator, the authors derived explicit analytical expressions for the optimal coefficients and established a Wiener- Hopf system for their computation. Furthermore, they obtained analytical formulas for the norm of the error functional, proved the convergence order $O(h^{(m)}$ and validated the theoretical findings through numerical experiments. In the following works, optimal quadrature formulas were constructed using the first and second derivatives of the function [12–15]. In [16, 17], an optimal quadrature approach was developed for approximating the fractional Riemann-Liouville integral in Sobolev spaces. Using Sobolev's method, they derived optimal coefficients and established a Wiener-Hopf type system for their determination. In another study, explicit analytical expressions for the coefficients of optimal quadrature formulas in Sobolev spaces of differentiable functions were obtained using Sobolev's method. Subsequently, optimal formulas for approximate integration in Sobolev factor spaces were developed through the minimization of the error functional and the determination of the corresponding optimal coefficients [18, 19].

These studies demonstrate that Sobolev's method provides an effective analytical framework for constructing optimal quadrature formulas with derivatives. Nevertheless, deriving explicit coefficient formulas and closed-form expressions for the norm of the error functional for more general classes of derivative-based quadrature formulas remains an active and challenging research problem, motivating the present investigation. Despite these substantial developments, relatively few studies have investigated optimal quadrature formulas that simultaneously incorporate several odd-order derivative corrections while preserving explicit analytical representations of the optimal coefficients for arbitrary $m \geq 6$. In particular, obtaining closed-form expressions for both the coefficients and the squared norm of the error functional remains a mathematically challenging problem. Addressing this issue is important because explicit formulas simplify theoretical analysis, facilitate numerical implementation, and provide deeper insight into the structure of optimal integration schemes.

We consider the following general quadrature formula

$$\int_0^1 f(x)dx \cong \sum_{\beta=0}^{N} k[\beta]f[\beta] + A_1(f'(0) - f'(1)) + A_2(f'''(0) - f'''(1)) + A_3(f^{(V)}(0) - f^{(V)}(1)) \quad (1.1)$$

with the error functional

$$\ell(x) = \varepsilon_{[0,1]}(x) - \sum_{\beta=0}^{N} k[\beta]\delta(x-h\beta) + A_1(\delta'(x)-\delta'(x-1)) + A_2(\delta'''(x)-\delta'''(x-1))$$

$$+A_3(\delta^{(V)}(x)-\delta^{(V)}(x-1)) \tag{1.2}$$

in the space $L_2^{(m)}(0,1)$ for $m \geq 6$. Here $C[\beta]$, $\beta = \overline{0,N}$, A_n are the coefficients of the formula (1.1), $h = \frac{1}{N}$, N is a natural number, $\varepsilon_{[0,1]}(x)$ is the characteristic function of the interval [0,1], $\delta(x)$ is the Dirac delta-function.

In the space $L_2^{(m)}(0,1)$ equipped with the norm

$$\|f(x)\|_{L_2^{(m)}(0,1)} = \left\{ \int_0^1 (f^{(m)}(x))^2 dx \right\}^{1/2} \tag{1.3}$$

The difference

$$(\ell, f) = \int_0^1 f(x)dx - \sum_{\beta=0}^{N} k[\beta] f[\beta] - A_1(f'(0)-f'(1)) - A_2(f'''(0)-f'''(1)) - A_3(f^{(V)}(0)-f^{(V)}(1)) \tag{1.4}$$

is called *the error* of the quadrature formula (1.1).

By the Cauchy-Schwarz inequality

$$|(\ell, f)| \leq \|f|L_2^{(m)}(0,1)\| \cdot \|\ell|L_2^{(m)^*}(0,1)\|$$

To quantify the accuracy of formula (1.1), we consider the norm of the error functional (1.2) in the conjugate space $L_2^{(m)^*}(0,1)$, namely

$$\left\|\ell|L_2^{(m)^*}(0,1)\right\| = \sup_{\left\|f|L_2^{(m)}(0,1)\right\|=1} |(\ell, f)|.$$

Thus, evaluating the accuracy of quadrature formula (1.1) on $L_2^{(m)}(0,1)$ amounts to determining the norm of the associated error functional ℓ in the conjugate space $L_2^{(m)^*}(0,1)$.

We use the Sobolev method which is based on the discrete analogue of the differential operator d^{2m}/dx^{2m}.

For the error functional (1.4) to be defined on the space $L_2^{(m)}(0,1)$ it is necessary to impose the following conditions (see [3])

$$(\ell(x), x^{\alpha}) = 0, \quad \alpha = 0, 1, 2, ..., m-1. \tag{1.5}$$

Hence it is clear that for existence of the quadrature formulas of the form (1.1) the condition $N \geq m-4$ has to be met.

As was noted above by the Cauchy-Schwarz inequality, the error of the formula (1.1) is estimated by the norm $\|\ell|L_2^{(m)*}(0,1)\|$ of the error functional (1.2). Furthermore the norm of the error functional

(1.2) depends on the coefficients $C[\beta]$, A_n. We minimize the norm of the error functional (1.2) by the coefficients $C[\beta]$, A_n i.e., we find

$$\left\|\overset{\circ}{\ell}|L_2^{(m)*}\right\| = \inf_{C[\beta],A_n}\left\|\ell|L_2^{(m)*}\right\|. \tag{1.6}$$

Problem 1. *Find the norm of the error functional (1.2) of the quadrature formula of the form (1.1) in the space* $L_2^{(m)*}(0,1)$.

Problem 2. *Find coefficients* $C[\beta]$, A_n *which satisfy the equality (1.6).*

2. Analytical representation of the error functional

To solve Problem 1, i.e., for finding the norm of the error functional (1.2) in the space $L_2^{(m)}(0,1)$ a concept of the extremal function is used [2,3]. The function ψ_ℓ is said to be *the extremal function* of the error functional (1.2) if the following equality holds

$$(\ell,\psi_\ell) = \left\|\ell|L_2^{(m)*}\right\|\left\|\psi_\ell|L_2^{(m)}\right\|. \tag{2.1}$$

In the space $L_2^{(m)}$ the extremal function ψ_ℓ of a functional ℓ was found by S.L. Sobolev [2]. This extremal function has the form

$$\psi_\ell(x) = (-1)^m\ell(x) * G(x) + P_{m-1}(x), \tag{2.2}$$

where

$$G(x) = \frac{x^{2m-1}sgn x}{2\cdot(2m-1)!} \tag{2.3}$$

is a solution of the equation

$$\frac{d^{2m}}{dx^{2m}}G(x) = \delta(x), \tag{2.4}$$

$P_{m-1}(x)$ is a polynomial of degree $m-1$, the symbol * is operation of convolution, i.e.

$$f(x)*g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy = \int_{-\infty}^{\infty} f(y)g(x-y)dy.$$

It is well known that for any functional ℓ in $L_2^{(m)*}$ the equality

$$\begin{aligned}||\ell|L_2^{(m)*}(0,1)||^2 &= (\ell,\psi_\ell) = (\ell(x),(-1)^m\ell(x)*G(x)) = \\ &= \int_{-\infty}^{\infty}\ell(x)\left((-1)^m\int_{-\infty}^{\infty}\ell(y)G(x-y)dy\right)dx\end{aligned}$$

holds [2].

Applying this equality to the error functional (1.2) we obtain the following

$$\left\|\ell|L_2^{(m)}(0,1)\right\|^2 = (-1)^m\left[\sum_{\beta=0}^{N}\sum_{\gamma=0}^{N}C[\beta]C[\gamma]\ \frac{|h\beta-h\gamma|^{2m-1}}{2(2m-1)!} - 2\sum_{\beta=0}^{N}C[\beta]\int_0^1\frac{|x-h\beta|^{2m-1}}{2(2m-1)!}dx\right.$$

$$+2\sum_{n=1}^{3}A_n\int_0^1\frac{(x^{2m-2n}+(1-x)^{2m-2n})}{2(2m-2n)!}dx+\sum_{n=1}^{3}\sum_{k=1}^{3}\frac{A_nA_k}{(2m-2n-2k+1)!}+\frac{1}{(2m+1)!}-$$

$$\left.-2\sum_{n=1}^{3}A_n\sum_{\beta=0}^{N}C[\beta]\frac{((h\beta)^{2m-2n}+(1-h\beta)^{2m-2n})}{2(2m-2n)!}\right] \tag{2.5}$$

Thus Problem 1 is solved for quadrature formulas of the form (1.1) in the space $L_2^{(m)}(0,1)$.

3. Derivation of the system of equations for the optimal coefficients

We now turn to Problem 2, to determine the optimal coefficients, we minimize the norm $\|\ell\|^2$ subject to the constraints (1.5) by employing the method of Lagrange multipliers. To this end, we introduce the following functional

$$\Psi=\|\ell\|^2-2\cdot(-1)^m\sum_{\alpha=0}^{m-1}\lambda_\alpha(\ell(x),x^\alpha),$$

where λ_α are unknown multipliers. The function Ψ is the multidimensional function with respect to the coefficients $C[\beta],\ A_n$ and λ_α. Equating to zero partial derivatives of the Ψ by coefficients $C[\beta],\ A_n$ together with the conditions (1.5) we get the following system of linear equations

$$\sum_{\gamma=0}^{N}C_\gamma\frac{\left|h\beta-h\gamma\right|^{2m-1}}{2(2m-1)!}-\sum_{n=1}^{3}A_n\frac{((h\beta)^{2m-2n}+(1-h\beta)^{2m-2n})}{2(2m-2n)!}+$$

$$+\sum_{\alpha=0}^{m-1}\lambda_\alpha(h\beta)^\alpha=\int_0^1\frac{|x-h\beta|^{2m-1}}{2(2m-1)!}dx,\beta=\overline{0,N}, \tag{3.1}$$

$$\sum_{\beta=0}^{N}C[\beta]\frac{((h\beta)^{2m-2k}+(1-h\beta)^{2m-2k})}{2(2m-2k)!}-\sum_{n=1}^{3}\frac{A_n}{2(2m-2n-2k+1)!}+$$

$$+\sum_{\alpha=2k}^{m-1}\alpha(\alpha-1)...(\alpha-2k+2)\cdot\lambda_\alpha=\frac{1}{(2m-2k+1)!},\quad k=1,2,3 \tag{3.2}$$

$$\sum_{\beta=0}^{N}C[\beta](h\beta)^\alpha=\frac{1}{\alpha+1},\qquad \alpha=0,1 \tag{3.3}$$

$$\sum_{\beta=0}^{N}C[\beta](h\beta)^\alpha-\alpha A_1=\frac{1}{\alpha+1},\qquad \alpha=2,3 \tag{3.4}$$

$$\sum_{\beta=0}^{N}C[\beta](h\beta)^\alpha-\alpha A_1-\alpha(\alpha-1)(\alpha-2)A_2=\frac{1}{\alpha+1},\ \alpha=4,5 \tag{3.5}$$

$$\sum_{\beta=0}^{N}C[\beta](h\beta)^\alpha-\alpha A_1-\alpha(\alpha-1)(\alpha-2)A_2-$$

$$-\alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)A_3=\frac{1}{\alpha+1},\alpha=\overline{6,m-1}. \tag{3.6}$$

The linear system (3.1)–(3.6) possesses a unique solution that yields the optimal coefficients minimizing the norm $||\ell||^2$. The proof of its existence and uniqueness is omitted, as it follows the standard analysis of discrete Wiener-Hopf systems arising in the construction of optimal quadrature formulas in the Sobolev space $L_2^{(m)}(0,1)$. The uniqueness of Sard optimal quadrature formulas has been discussed in [7].

4. Analytical expressions for the optimal quadrature coefficients

This section is devoted to solving the system (3.1)-(3.6) and obtaining explicit formulas for the coefficients $C[\beta]$. The derivation is based on Sobolev's approach to optimal quadrature formulas in the Sobolev space $L_2^{(m)}(0,1)$ and relies on the theory of functions of discrete arguments together with discrete convolution techniques (see [1,2]). For the reader's convenience, the essential definitions are summarized below.

Assume that φ and ψ are real-valued functions of real variable and are defined in real line $\mathbb{R}$.

Definition 4.1. Function $\varphi(h\beta)$ is called *function of discrete argument*, if it is given on some set of integer values of β.

Definition 4.2. *The inner product* of two discrete functions $\varphi(h\beta)$ and $\psi(h\beta)$ is called the number

$$[\varphi,\psi] = \sum_{\beta=-\infty}^{\infty} \varphi(h\beta)\cdot\psi(h\beta),$$

if the series on the right hand side of the last equality converges absolutely.

Definition 4.3. *The convolution* of two discrete functions $\varphi(h\beta)$ and $\psi(h\beta)$ is called the inner product

$$\varphi(h\beta) * \psi(h\beta) = [\varphi(h\gamma), \psi(h\beta - h\gamma)] = \sum_{\gamma=-\infty}^{\infty} \varphi(h\gamma)\cdot\psi(h\beta - h\gamma).$$

The Euler-Frobenius polynomials $E_k(x)$, $k = 1, 2, \ldots$ is defined by the following formula [2]

$$E_k(x) = \frac{(1-x)^{k+2}}{x}\left(x\frac{d}{dx}\right)^k \frac{x}{(1-x)^2}, \tag{4.1}$$

$E_0(x) = 1$.

For the Euler-Frobenius polynomials $E_k(x)$ the following identity holds

$$E_k(x) = x^k E_k\left(\frac{1}{x}\right),$$

and also the following theorem is true

Theorem 4.1 *[22]). Polynomial $Q_k(x)$ which is defined by the formula*

$$Q_k(x) = (x-1)^{k+1}\sum_{i=0}^{k+1}\frac{\Delta^i 0^{k+1}}{(x-1)^i}$$

is the Euler-Frobenius polynomial, $Q_k(x) = E_k(x)$, where $\Delta^i 0^k = \sum_{l=1}^{i}(-1)^{i-l}C_i^l l^k$.

Suppose that $C[\beta]=0$ for $\beta<0$ and $\beta>N$. Using **Definition 4.3** we rewrite the equation (3.1) in the convolution form:

$$C[\beta]*\frac{\left|h\beta\right|^{2m-1}}{2(2m-1)!}-\sum_{n=1}^{3}A_n\frac{\left((h\beta)^{2m-2n}+(1-h\beta)^{2m-2n}\right)}{2(2m-2n)!}+$$
$$+\sum_{\alpha=0}^{m-1}\lambda_\alpha(h\beta)^\alpha=\int_0^1\frac{\left|x-h\beta\right|^{2m-1}}{2(2m-1)!}dx=\frac{(h\beta)^{2m}}{(2m)!}+\sum_{j=0}^{2m-1}\frac{(-h\beta)^{2m-1-j}}{2(2m-1-j)!\cdot(j+1)!} \tag{4.2}$$

We consider the following problem.

Problem A. *Find the discrete function $C[\beta]$ and unknown coefficients A_n, λ_α, which satisfy the system (3.1)-(3.6).*

Further, instead of $C[\beta]$ we introduce the functions

$$v(h\beta)=C[\beta]*\frac{\left|h\beta\right|^{2m-1}}{2(2m-1)!}, \tag{4.3}$$

$$u(h\beta)=v(h\beta)-\sum_{n=1}^{3}A_n\frac{\left((h\beta)^{2m-2n}+(1-h\beta)^{2m-2n}\right)}{2(2m-2n)!}+\sum_{\alpha=0}^{m-1}\lambda_\alpha(h\beta)^\alpha. \tag{4.4}$$

In this statement it is necessary to express $C[\beta]$ by the function $u[\beta]$. For this we need such operator $D_m[\beta]$, which satisfies the equation

$$hD_m(h\beta)*G(h\beta)=\delta(h\beta), \tag{4.5}$$

where $G[\beta]=\frac{|h\beta|^{2m-1}}{2(2m-1)!}$ is the discrete argument function corresponding to $G(x)$ defined by (2.3), $\delta(h\beta)$ is equal to 0 when $\beta\neq 0$ and is equal to 1 when $\beta=0$, i.e. $\delta(h\beta)$ is the discrete delta-function. The equation (4.5) is the discrete analogue of equation (2.4). So the discrete function $D_m(h\beta)$ is called *the discrete analogue* of the differential operator d^{2m}/dx^{2m} [2,3].

In [22] the discrete analogue $D_m(h\beta)$ of the differential operator d^{2m}/dx^{2m}, which satisfies equation (4.5), is constructed. Taking into account (4.5) and $D_m(h\beta)$, for the optimal coefficients $C[\beta]$ we have

$$C[\beta]=hD_m(h\beta)*u(h\beta). \tag{4.6}$$

So, if we find the function $u(h\beta)$, then the optimal coefficients $C[\beta]$ will be found from equality (4.6).

To calculate the convolution (4.6) it is required to find the representation of the function $u(h\beta)$ for all integer values of β. From the equality (4.2) we get, that $u(h\beta)=f_m(h\beta)$ when $h\beta\in[0,1]$, where $f_m(h\beta)$ is defined by equality (4.2). Now we need to find the representation of the function $u(h\beta)$ when $\beta<0$ and $\beta>N$.

Since $C[\beta]=0$ when $h\beta\notin[0,1]$, then

$$C[\beta]=hD_m(h\beta)*u(h\beta)=0,\qquad h\beta\notin[0,1].$$

Now we calculate the convolution $v(h\beta)=C[\beta]*\frac{|h\beta|^{2m-1}}{2(2m-1)!}$ when $h\beta\notin[0,1]$.

Suppose $\beta < 0$, then taking into account (3.3)-(3.6), we have

$$v(h\beta) = C[\beta] * \frac{|h\beta|^{2m-1}}{2(2m-1)!} = \sum_{\gamma=-\infty}^{\infty} C[\gamma] \frac{|h\beta - h\gamma|^{2m-1}}{2(2m-1)!} =$$

$$= -\sum_{\alpha=0}^{m-1} \frac{(h\beta)^{2m-1-\alpha}(-1)^{\alpha}}{2(2m-1-\alpha)!\alpha!} \sum_{\gamma=0}^{N} C[\gamma](h\gamma)^{\alpha} - \sum_{\alpha=m}^{2m-1} \frac{(h\beta)^{2m-1-\alpha}(-1)^{\alpha}}{2(2m-1-\alpha)!\alpha!} \sum_{\gamma=0}^{N} C[\gamma](h\gamma)^{\alpha} =$$

$$= -\sum_{\alpha=0}^{m-1} \frac{(h\beta)^{2m-1-\alpha}(-1)^{\alpha}}{2(2m-1-\alpha)!(\alpha+1)!} - \sum_{n=1}^{3}\sum_{\alpha=2n}^{m-1} \frac{A_n (h\beta)^{2m-1-\alpha}(-1)^{\alpha}}{2(2m-1-\alpha)!(\alpha-2n+1)!} -$$

$$- \sum_{\alpha=m}^{2m-1} \frac{(h\beta)^{2m-1-\alpha}(-1)^{\alpha}}{2(2m-1-\alpha)!\alpha!} \sum_{\gamma=0}^{N} C[\gamma](h\gamma)^{\alpha}.$$

Hence, denoting by $R_{m-1}(h\beta) = \sum\limits_{\alpha=m}^{2m-1} \frac{(h\beta)^{2m-1-\alpha}(-1)^{\alpha}}{2(2m-1-\alpha)!\alpha!} \sum\limits_{\gamma=0}^{N} C[\gamma](h\gamma)^{\alpha}$ for the case $\beta < 0$ we get

$$v(h\beta) = -\sum_{\alpha=0}^{m-1} \frac{(h\beta)^{2m-1-\alpha}(-1)^{\alpha}}{2(2m-1-\alpha)!(\alpha+1)!} - \sum_{n=1}^{3}\sum_{\alpha=2n}^{m-1} \frac{A_n (h\beta)^{2m-1-\alpha}(-1)^{\alpha}}{2(2m-1-\alpha)!(\alpha-2n+1)!} - R_{m-1}(h\beta). \quad (4.7)$$

Now suppose $\beta > N$ then for $v(h\beta)$ we get

$$v(h\beta) = \sum_{\alpha=0}^{m-1} \frac{(h\beta)^{2m-1-\alpha}(-1)^{\alpha}}{2(2m-1-\alpha)!(\alpha+1)!} + \sum_{n=1}^{3}\sum_{\alpha=2n}^{m-1} \frac{A_n (h\beta)^{2m-1-\alpha}(-1)^{\alpha}}{2(2m-1-\alpha)!(\alpha-2n+1)!} + R_{m-1}(h\beta). \quad (4.8)$$

Denoting

$$\sum_{\alpha=0}^{m-1} \lambda_{\alpha}(h\beta)^{\alpha} - R_{m-1}(h\beta) - \sum_{n=1}^{3}\sum_{\alpha=m+1}^{2m} \frac{A_n (h\beta)^{2m-\alpha}(-1)^{\alpha}}{2(2m-\alpha)!(\alpha-2n)!} = R^{(-)}_{m-1}(h\beta), \quad (4.9)$$

$$\sum_{\alpha=0}^{m-1} \lambda_{\alpha}(h\beta)^{\alpha} + R_{m-1}(h\beta) - \sum_{n=1}^{3}\sum_{\alpha=m+1}^{2m} \frac{A_n (h\beta)^{2m-\alpha}(-1)^{\alpha}}{2(2m-\alpha)!(\alpha-2n)!} = R^{(+)}_{m-1}(h\beta), \quad (4.10)$$

and taking into account (4.7), (4.8), (4.4) we have the following problem

Problem B. *Find the solution of the equation*

$$hD_m(h\beta) * u(h\beta) = 0, \quad h\beta \notin [0,1] \quad (4.11)$$

having the form:

$$u(h\beta) = \begin{cases} -\sum\limits_{\alpha=0}^{m-1} \frac{(h\beta)^{2m-1-\alpha}(-1)^{\alpha}}{2(2m-1-\alpha)!(\alpha+1)!} - \sum\limits_{n=1}^{3} \frac{A_n (h\beta)^{2m-2n}}{(2m-2n)!} + \\ +R^{(-)}_{m-1}(h\beta), \quad \beta < 0, \\ f(h\beta), \quad 0 \le \beta \le N, \\ \sum\limits_{\alpha=0}^{m-1} \frac{(h\beta)^{2m-1-\alpha}(-1)^{\alpha}}{2(2m-1-\alpha)!(\alpha+1)!} + \sum\limits_{n=1}^{3}\sum\limits_{\alpha=2n-1}^{m-1} \frac{A_n (h\beta)^{2m-1-\alpha}(-1)^{\alpha}}{(2m-1-\alpha)!(\alpha-2n+1)!} + \\ +R^{(+)}_{m-1}(h\beta), \quad \beta > N, \end{cases} \quad (4.12)$$

where $R^{(-)}_{m-1}(h\beta)$ and $R^{(+)}_{m-1}(h\beta)$ are unknown polynomials of degree $m-1$ and A_n are unknown coefficients. If we find $R^{(-)}_{m-1}(h\beta)$ and $R^{(+)}_{m-1}(h\beta)$, then from (4.9), (4.10) we obtain

$$R_{m-1}(h\beta) = \frac{1}{2}\left(R^{(+)}_{m-1}(h\beta) - R^{(-)}_{m-1}(h\beta)\right), \qquad \sum_{\alpha=0}^{m-1}\lambda_\alpha (h\beta)^\alpha = \frac{1}{2}\left(R^{(+)}_{m-1}(h\beta) + R^{(-)}_{m-1}(h\beta)\right).$$

The auxiliary polynomials $R^{(-)}_{m-1}(h\beta)$, $R^{(+)}_{m-1}(h\beta)$, and the coefficients A_n can be recovered from equation (4.11) using the discrete analogue of the differential operator $D_m(h\beta)$. Their determination provides an explicit representation of $u(h\beta)$, from which the optimal coefficients $C[\beta]$ $(\beta = 0, 1, ..., N)$, A_n are readily obtained. Although this procedure completely resolves Problems A and B, deriving explicit formulas for the auxiliary polynomials is unnecessary for our purposes. Instead, by combining the representation (4.12) with the discrete operator $D_m(h\beta)$ and identity (4.6), we derive closed-form expressions for the interior quadrature coefficients $\beta = 1, 2, ..., N-1$.

To simplify the subsequent analysis, we introduce the following notation

$$d_k = \frac{(2m-1)!(1-q_k)^{2m+1}}{h^{2m} q_k E_{2m-1}(q_k)}\cdot \tag{4.13}$$

$$\sum_{\gamma=1}^{\infty} q_k^\gamma \left\{ -\sum_{\alpha=0}^{m-1} \frac{(h\gamma)^{2m-1-\alpha}(-1)^\alpha}{2(2m-1-\alpha)!(\alpha+1)!} - \sum_{n=1}^{3}\frac{A_n (h\gamma)^{2m-2n}}{(2m-2n)!} + R^{(-)}_{m-1}(-h\gamma) - f(-h\gamma) \right\},$$

$$p_k = \frac{(2m-1)!(1-q_k)^{2m+1}}{h^{2m} q_k E_{2m-1}(q_k)}\cdot \tag{4.14}$$

$$\sum_{\gamma=1}^{\infty} q_k^\gamma \left\{ \sum_{\alpha=0}^{m-1} \frac{(h\gamma+1)^{2m-1-\alpha}(-1)^\alpha}{2(2m-1-\alpha)!(\alpha+1)!} + \sum_{n=1}^{3}\sum_{\alpha=2n-1}^{m-1}\frac{A_n (h\gamma+1)^{2m-1-\alpha}(-1)^\alpha}{(2m-1-\alpha)!(\alpha-2n+1)!} + +R^{(+)}_{m-1}(h\gamma+1) - f(h\gamma+1) \right\}$$

where $k = 1, 2, ..., m-1$, $E_{2m-1}(q)$ is the Euler-Frobenius polynomial of degree $2m-1$, $|q_k| < 1$ the series in the (4.13) and (4.14) are convergent.

Theorem 4.2. *The coefficients $C[\beta]$, $\beta = 1, 2, ..., N-1$ of the optimal quadrature formulas of the form (1.1) in the space $L_2^{(m)}(0,1)$, $m \geq 6$, have the following form*

$$C[\beta] = h\left(1 + \sum_{k=1}^{m-1}\left(d_k q_k^\beta + p_k q_k^{N-\beta}\right)\right), \;\; \beta = 1, 2, ..., N-1, \tag{4.15}$$

where d_k, p_k are defined by (4.13), (4.14).

Proof. Suppose $\beta = 1, 2, ..., N-1$. Then from (4.6), using **Definition 4.3**, equalities (4.5), (4.12), we have

$$C[\beta] = hD_m(h\beta) * u(h\beta) = h\sum_{\gamma=-\infty}^{\infty} D_m(h\beta - h\gamma)u(h\gamma) =$$

$$= h\left(\sum_{\gamma=-\infty}^{-1} D_m(h\beta - h\gamma)u(h\gamma) + \sum_{\gamma=0}^{N} D_m(h\beta - h\gamma) f_m(h\gamma) + \sum_{\gamma=N+1}^{\infty} D_m(h\beta - h\gamma)u(h\gamma)\right).$$

Now, adding and subtracting the expressions $h\sum\limits_{\gamma=-\infty}^{-1} D_m(h\beta-h\gamma)f_m(h\gamma)$ and $h\sum\limits_{\gamma=N+1}^{\infty} D_m(h\beta-h\gamma)f_m(h\gamma)$ to and from the last expression and taking into account **Definition 4.3**, we get

$$C[\beta]=h\Bigg\{D_m(h\beta)*f_m(h\beta)+\sum_{k=1}^{m-1}q_k^{\beta}\frac{(2m-1)!}{h^{2m}}\frac{(1-q_k)^{2m+1}}{q_kE_{2m-1}(q_k)}\cdot$$

$$\sum_{\gamma=1}^{\infty}q_k^{\gamma}\Bigg[-\sum_{\alpha=0}^{m-1}\frac{(h\gamma)^{2m-1-\alpha}(-1)^{\alpha}}{2(2m-1-\alpha)!(\alpha+1)!}-\sum_{n=1}^{3}\frac{A_n(h\gamma)^{2m-2n}}{(2m-2n)!}+R_{m-1}^{(-)}(-h\gamma)-f_m(-h\gamma)\Bigg]+$$

$$+\sum_{k=1}^{m-1}q_k^{N-\beta}\frac{(2m-1)!}{h^{2m}}\frac{(1-q_k)^{2m+1}}{q_kE_{2m-1}(q_k)}\cdot$$

$$\sum_{\gamma=1}^{\infty}q_k^{\gamma}\Bigg[\sum_{\alpha=0}^{m-1}\frac{(h\gamma+1)^{2m-1-\alpha}(-1)^{\alpha}}{2(2m-1-\alpha)!(\alpha+1)!}+\sum_{n=1}^{3}\sum_{\alpha=2n-1}^{m-1}\frac{A_n(h\gamma+1)^{2m-1-\alpha}(-1)^{\alpha}}{(2m-1-\alpha)!(\alpha-2n+1)!}+$$

$$+R_{m-1}^{(+)}(h\gamma+1)-f_m(h\gamma+1)\Bigg]\Bigg\}.$$

Hence taking into account the notations (4.13), (4.14) we obtain

$$C[\beta]=h\left(D_m(h\beta)*f_m(h\beta)+\sum_{k=1}^{m-1}(d_kq_k^{\beta}+p_kq_k^{N-\beta})\right). \tag{4.16}$$

Now using $D_m(h\beta)$ and equality (4.2) we get

$$D_m(h\beta)*f(h\beta)=1. \tag{4.17}$$

Putting (4.17) to the equation (4.16) we get (4.15). Theorem 4.2 is proved.

Before proving the main results, we present the following auxiliary lemmas.

Lemma 4.1. *The following identity is taken place [13]*

$$\sum_{i=0}^{\alpha}\frac{d_kq_k+p_kq_k^{N+i}(-1)^{i+1}}{(q_k-1)^{i+1}}\Delta^i0^{\alpha}=(-1)^{\alpha+1}\sum_{i=0}^{\alpha}\frac{d_kq_k^i+p_kq_k^{N+1}(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^i0^{\alpha}, \tag{4.18}$$

here α and N are natural numbers, d_k and p_k are defined by (4.13), (4.14), Δ^i0^{α} is given in Theorem 4.1.

Lemma 4.2. *The following identities are valid*

$$\sum_{j=1}^{m-1}\frac{(-1)^{j-1}}{(j-1)!}\sum_{i=1}^{2m-2-j}\frac{B_{2m-j-i}h^{2m-j-i}}{i!\,(2m-j-i)!} = \sum_{j=2}^{m}\frac{B_jh^j}{j!}\sum_{i=0}^{m-2}\frac{(-1)^i}{i!\,(2m-1-j-i)!}+$$
$$+\sum_{j=m+1}^{2m-2}\frac{B_jh^j}{j!}\sum_{i=0}^{2m-2-j}\frac{(-1)^i}{i!\,(2m-1-j-i)!}$$

and

$$\sum_{j=1}^{m-1}\frac{(-1)^{j-1}}{(j-1)!}\sum_{p=1}^{2m-1-j}\frac{h^{p+1}Z_p}{p!\,(2m-1-j-p)!} = \sum_{j=2}^{m+1}\frac{h^jZ_{j-1}}{(j-1)!}\sum_{i=0}^{m-2}\frac{(-1)^i}{i!\,(2m-1-j-i)!}+$$

$$\sum_{j=m+2}^{2m-1} \frac{h^j Z_{j-1}}{(j-1)!} \sum_{i=0}^{2m-1-j} \frac{(-1)^i}{i!\ (2m-1-j-i)!}.$$

were

$$Z_p = \sum_{k=1}^{m-1} \sum_{i=0}^{p} \frac{d_k q_k^{N+i} + p_k q_k (-1)^{i+1}}{(1-q_k)^{i+1}} \Delta^i 0^p. \tag{4.19}$$

The proof of Lemma 4.2 is obtained by expansion in powers of h of the left sides of given identities.

For the coefficients of the optimal quadrature formulas of the form (1.1) the following theorem holds.

Theorem 4.3. *Among quadrature formulas of the form (1.1) with the error functional (1.2) in the space $L_2^{(m)}(0,1)$, $m \geq 6$, there exists unique optimal formula which coefficients are determined by the following formulas*

$$C[0] = h\left(\frac{1}{2} + \sum_{k=1}^{m-1} d_k \frac{q_k^N - q_k}{1-q_k}\right), \quad \beta = 0, N \tag{4.20}$$

$$C[\beta] = h\left(1 + \sum_{k=1}^{m-1} d_k (q_k^\beta + q_k^{N-\beta})\right), \beta = \overline{1, N-1}, \tag{4.21}$$

$$A_n = \frac{h^{2n} B_{2n}}{(2n)!} - \frac{h^{2n}}{(2n-1)!} \sum_{k=1}^{m-1} \sum_{i=1}^{2n-1} d_k \frac{q_k + q_k^{N+i}(-1)^{i+1}}{(q_k-1)^{i+1}} \Delta^i 0^{2n-1},\ n = 1,2,3 \tag{4.22}$$

where d_k satisfy the following system of $m-1$ linear equations

$$\sum_{k=1}^{m-1} \sum_{i=0}^{j} d_k \frac{q_k + q_k^{N+i}(-1)^{i+1}}{(q_k-1)^{i+1}} \Delta^i 0^j = \frac{B_{j+1}}{j+1}, \quad j = \overline{6, m-1}, \tag{4.23}$$

$$\sum_{k=1}^{m-1} \sum_{i=0}^{2m-2j} d_k \frac{q_k + q_k^{N+i}(-1)^{i+1}}{(q_k-1)^{i+1}} \Delta^i 0^{2m-2j} = 0, \qquad j = 1,2,3, \tag{4.24}$$

$$\sum_{k=1}^{m-1} \sum_{i=0}^{2j} d_k \frac{q_k + q_k^{N+i}(-1)^{i+1}}{(q_k-1)^{i+1}} \Delta^i 0^{2j} = 0, \qquad j = 1,2. \tag{4.25}$$

Here B_α are Bernoulli numbers, $\Delta^i \gamma^j$ is the finite difference of order i of γ^j, $\Delta^i 0^j$ is given in Theorem 4.1, q_k are given in Theorem 4.2.

In the proof of Theorem 4.3, we use the following formulas from [21]:

$$\sum_{\gamma=0}^{n-1} q^\gamma \gamma^k = \frac{1}{1-q} \sum_{i=0}^{k} \left(\frac{q}{1-q}\right)^i \Delta^i 0^k - \frac{q^n}{1-q} \sum_{i=0}^{k} \left(\frac{q}{1-q}\right)^i \Delta^i \gamma^k|_{\gamma=n}, \tag{4.26}$$

where $\Delta^i \gamma^k$ is the finite difference of order i of γ^k, q is ratio of a geometric progression.
At last we give the following well known formulas from [20]

$$\sum_{\gamma=0}^{\beta-1} \gamma^k = \sum_{j=1}^{k+1} \frac{k!\, B_{k+1-j}}{j!\,(k+1-j)!} \beta^j, \tag{4.27}$$

where B_{k+1-j} are Bernoulli numbers,

$$\Delta^\alpha x^\nu = \sum_{p=0}^{\nu} C_\nu^p \Delta^\alpha 0^p x^{\nu-p}. \tag{4.28}$$

Proof. First we consider the first sum of equation (3.1). For this sum we have

$$\begin{aligned}
S &= \sum_{\gamma=0}^{N} C[\gamma]\frac{|h\beta-h\gamma|^{2m-1}}{2(2m-1)!} = \\
&= C[0]\frac{(h\beta)^{2m-1}}{(2m-1)!} + \sum_{\gamma=1}^{\beta} C[\gamma]\frac{(h\beta-h\gamma)^{2m-1}}{(2m-1)!} - \sum_{\gamma=0}^{N} C[\gamma]\frac{(h\beta-h\gamma)^{2m-1}}{2(2m-1)!}.
\end{aligned}$$

The last two sums of the expression S we denote

$$S_1 = \sum_{\gamma=1}^{\beta} C[\gamma]\frac{(h\beta-h\gamma)^{2m-1}}{(2m-1)!}, \quad S_2 = \sum_{\gamma=0}^{N} C[\gamma]\frac{(h\beta-h\gamma)^{2m-1}}{2(2m-1)!}$$

and we calculate them separately.
By using (4.15) and formulas (4.25), (4.26) for S_1 we have

$$\begin{aligned}
S_1 &= \sum_{\gamma=1}^{\beta} h\left(1+\sum_{k=1}^{m-1}\left(d_k q_k^\gamma + p_k q_k^{N-\gamma}\right)\right)\frac{(h\beta-h\gamma)^{2m-1}}{(2m-1)!} = \\
&= \frac{h^{2m}}{(2m-1)!}\left[\sum_{\gamma=0}^{\beta-1}\gamma^{2m-1} + \sum_{k=1}^{m-1}\left(d_k q_k^\beta \sum_{\gamma=0}^{\beta-1} q_k^{-\gamma}\gamma^{2m-1} + p_k q_k^{N-\beta}\sum_{\gamma=0}^{\beta-1} q_k^\gamma \gamma^{2m-1}\right)\right] = \\
&= \frac{h^{2m}}{(2m-1)!}\left[\sum_{j=1}^{2m}\frac{(2m-1)!B_{2m-j}}{j!\cdot(2m-j)!}\beta^j + \sum_{k=1}^{m-1}\left[d_k q_k^\beta\left\{\frac{q_k}{q_k-1}\sum_{i=0}^{2m-1}\frac{\Delta^i 0^{2m-1}}{(q_k-1)^i} - \right.\right.\right.\\
&\quad \left. - \frac{q_k^{1-\beta}}{q_k-1}\sum_{i=0}^{2m-1}\frac{\Delta^i\beta^{2m-1}}{(q_k-1)^i}\right\} + p_k q_k^{N-\beta}\left\{\frac{1}{1-q_k}\sum_{i=0}^{2m-1}\left(\frac{q_k}{q_k-1}\right)^i\Delta^i 0^{2m-1} - \right.\\
&\quad \left.\left.\left. - \frac{q_k^\beta}{1-q_k}\sum_{i=0}^{2m-1}\left(\frac{q_k}{q_k-1}\right)^i\Delta^i\beta^{2m-1}\right\}\right]\right].
\end{aligned}$$

Taking into account that q_k is the root of the Euler-Frobenius polynomial $E_{2m-2}(q)$ and using (4.27) the expression for S_1 we reduce to the following form

$$\begin{aligned}
S_1 &= \frac{(h\beta)^{2m}}{(2m)!} + h\cdot\frac{(h\beta)^{2m-1}}{(2m-1)!}B_1 + h^{2m}\sum_{j=1}^{2m-2}\frac{B_{2m-j}}{j!(2m-j)!}\beta^j + \\
&+ h^{2m}\sum_{j=0}^{2m-1}\frac{\beta^{2m-1-j}}{j!(2m-1-j)!}\sum_{k=1}^{m-1}\sum_{i=0}^{j}\frac{-d_k q_k + p_k q_k^{N+i}(-1)^i}{(q_k-1)^{i+1}}\Delta^i 0^j.
\end{aligned} \tag{4.29}$$

Now we consider S_2. By using equations (3.3)-(3.6) we rewrite the expression S_2 in powers of $h\beta$

$$S_2=\sum_{\gamma=0}^{N}C[\gamma]\frac{(h\beta-h\gamma)^{2m-1}}{2(2m-1)!}=$$

$$=\sum_{j=0}^{m-1}\frac{(h\beta)^{2m-1-j}(-1)^j}{2(2m-1-j)!j!}\sum_{\gamma=0}^{N}C[\gamma](h\gamma)^j+\sum_{j=m}^{2m-1}\frac{(h\beta)^{2m-1-j}(-1)^j}{2(2m-1-j)!j!}\sum_{\gamma=0}^{N}C[\gamma](h\gamma)^j=$$

$$=\sum_{k=0}^{2}\sum_{j=2k}^{2k+1}\frac{(h\beta)^{2m-1-j}(-1)^j}{2(2m-1-j)!j!}\left(\frac{1}{j+1}+\sum_{n=1}^{k}j(j-1)...(j-2n+2)A_n\right)+$$

$$+\sum_{j=6}^{m-1}\frac{(h\beta)^{2m-1-j}(-1)^j}{2(2m-1-j)!j!}\left(\frac{1}{j+1}+\sum_{n=1}^{3}j(j-1)...(j-2n+2)A_n\right)+$$

$$+\sum_{j=m}^{2m-1}\frac{(h\beta)^{2m-1-j}(-1)^j}{2(2m-1-j)!j!}\sum_{\gamma=0}^{N}C[\gamma](h\gamma)^j. \tag{4.30}$$

Substituting (4.2) and S into equation (3.1) and using (4.28), (4.29) we have

$$\frac{(h\beta)^{2m}}{(2m)!}+C[0]\frac{(h\beta)^{2m-1}}{(2m-1)!}+h\frac{(h\beta)^{2m-1}}{(2m-1)!}B_1+\sum_{j=1}^{2m-2}\frac{B_{2m-j}h^{2m-j}(h\beta)^j}{j!(2m-j)!}+$$

$$+\sum_{j=0}^{2m-1}\frac{h^{j+1}(h\beta)^{2m-1-j}}{j!(2m-1-j)!}\sum_{k=1}^{m-1}\sum_{i=0}^{j}\frac{-d_kq_k+p_kq_k^{N+i}(-1)^i}{(q_k-1)^{i+1}}\Delta^i0^j-$$

$$-\sum_{k=0}^{2}\sum_{j=2k}^{2k+1}\frac{(h\beta)^{2m-1-j}(-1)^j}{2(2m-1-j)!j!}\left(\frac{1}{j+1}+\sum_{n=1}^{k}j(j-1)...(j-2n+2)A_n\right)-$$

$$-\sum_{j=6}^{m-1}\frac{(h\beta)^{2m-1-j}(-1)^j}{2(2m-1-j)!j!}\left(\frac{1}{j+1}+\sum_{n=1}^{3}j(j-1)...(j-2n+2)A_n\right)-$$

$$-\sum_{j=m}^{2m-1}\frac{(h\beta)^{2m-1-j}(-1)^j}{2(2m-1-j)!j!}\sum_{\gamma=0}^{N}C[\gamma](h\gamma)^j-$$

$$-\sum_{n=1}^{3}A_n\frac{(h\beta)^{2m-2n}}{2(2m-2n)!}-\sum_{n=1}^{3}A_n\sum_{j=0}^{2m-2n}\frac{(-h\beta)^j}{2(2m-2n-j)!j!}+$$

$$+\sum_{\alpha=0}^{m-1}\lambda_\alpha(h\beta)^\alpha=\frac{(h\beta)^{2m}}{(2m)!}+\sum_{j=0}^{2m-1}\frac{(-h\beta)^{2m-1-j}}{2\cdot(2m-1-j)!\cdot(j+1)!}.$$

Hence equating coefficients of the same powers of $h\beta$ gives

$$\sum_{j=0}^{m-1}\lambda_j(h\beta)^j=\sum_{j=0}^{m-1}\frac{(h\beta)^j}{j!}\Bigg[\frac{(-1)^j}{2(2m-j)!}+\sum_{n=1}^{3}A_n\frac{(-1)^j}{2(2m-2n-j)!}+$$

$$+\frac{h^{2m-j}}{(2m-1-j)!}\sum_{k=1}^{m-1}\sum_{i=0}^{2m-1-j}\frac{d_kq_k+p_kq_k^{N+i}(-1)^{i+1}}{(q_k-1)^{i+1}}\Delta^i0^{2m-1-j}+$$

$$+\frac{1}{2(2m-1-j)!}\sum_{\gamma=0}^{N}C[\gamma](-h\gamma)^{2m-1-j}\Bigg]-\sum_{j=1}^{m-1}\frac{B_{2m-j}h^{2m-j}(h\beta)^{j}}{j!(2m-j)!}, \tag{4.31}$$

$$\sum_{k=1}^{m-1}\sum_{i=0}^{j}\frac{d_kq_k+p_kq_k^{N+i}(-1)^{i+1}}{(q_k-1)^{i+1}}\Delta^i0^j=\frac{B_{j+1}}{j+1},\quad j=\overline{6,m-1}, \tag{4.32}$$

$$\sum_{k=1}^{m-1}\sum_{i=0}^{2}\frac{d_kq_k+p_kq_k^{N+i}(-1)^{i+1}}{(q_k-1)^{i+1}}\Delta^i0^2=0, \tag{4.33}$$

$$C[0]=h\left(\frac{1}{2}+\sum_{k=1}^{m-1}\frac{p_kq_k^N-d_kq_k}{1-q_k}\right), \tag{4.34}$$

$$A_n=\frac{h^{2n}B_{2n}}{(2n)!}-\frac{h^{2n}}{(2n-1)!}\sum_{k=1}^{m-1}\sum_{i=1}^{2n-1}\frac{d_kq_k+p_kq_k^{N+i}(-1)^{i+1}}{(q_k-1)^{i+1}}\Delta^i0^{2n-1},\ n=1,2,3 \tag{4.35}$$

$$\sum_{k=1}^{m-1}\sum_{i=0}^{4}\frac{d_kq_k+p_kq_k^{N+i}(-1)^{i+1}}{(q_k-1)^{i+1}}\Delta^i0^4=0, \tag{4.36}$$

Taking into account (4.33), (4.15) from (3.3) when $\alpha=0$ we get

$$C[N]=h\left(\frac{1}{2}+\sum_{k=1}^{m-1}\frac{d_kq_k^N-p_kq_k}{1-q_k}\right). \tag{4.37}$$

Now, putting the values of λ_j into (3.2) we get the following equations for unknowns d_k and p_k:

$$\sum_{k=1}^{m-1}\sum_{i=0}^{2m-2}\frac{d_kq_k^i+p_kq_k^{N+1}(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^i0^{2m-2}=\sum_{k=1}^{m-1}\sum_{i=0}^{2m-2}\frac{d_kq_k^{N+i}+p_kq_k(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^i0^{2m-2}, \tag{4.38}$$

$$\sum_{k=1}^{m-1}\sum_{i=0}^{j-1}\frac{d_kq_k^{N+i}+p_kq_k(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^i0^{j-1}=\frac{B_j}{j},\ j=\overline{7,m} \tag{4.39}$$

$$\sum_{k=1}^{m-1}\sum_{i=0}^{3}\frac{d_kq_k+p_kq_k^{N+i}(-1)^{i+1}}{(q_k-1)^{i+1}}\Delta^i0^3=\sum_{k=1}^{m-1}\sum_{i=0}^{3}\frac{d_kq_k^{N+i}+p_kq_k(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^i0^3, \tag{4.40}$$

$$\sum_{k=1}^{m-1}\sum_{i=0}^{5}\frac{d_kq_k+p_kq_k^{N+i}(-1)^{i+1}}{(q_k-1)^{i+1}}\Delta^i0^5=\sum_{k=1}^{m-1}\sum_{i=0}^{5}\frac{d_kq_k^{N+i}+p_kq_k(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^i0^5, \tag{4.41}$$

$$\sum_{k=1}^{m-1}\sum_{i=0}^{2}\frac{d_kq_k^{N+i}+p_kq_k(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^i0^2=0, \tag{4.42}$$

$$\sum_{k=1}^{m-1}\sum_{i=0}^{4}\frac{d_kq_k^{N+i}+p_kq_k(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^i0^4=0 \tag{4.43}$$

$$\sum_{k=1}^{m-1}\frac{d_kq_k+p_kq_k^{N+1}}{(q_k-1)^2}=\sum_{k=1}^{m-1}\frac{d_kq_k^{N+1}+p_kq_k}{(1-q_k)^2}, \tag{4.44}$$

$$\sum_{k=1}^{m-1}\sum_{i=0}^{2m-4}\frac{d_kq_k^i+p_kq_k^{N+1}(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^i0^{2m-4}=\sum_{k=1}^{m-1}\sum_{i=0}^{2m-4}\frac{d_kq_k^{N+i}+p_kq_k(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^i0^{2m-4}, \tag{4.45}$$

$$\sum_{k=1}^{m-1}\sum_{i=0}^{2m-6}\frac{d_kq_k^i+p_kq_k^{N+1}(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^i0^{2m-6}=\sum_{k=1}^{m-1}\sum_{i=0}^{2m-6}\frac{d_kq_k^{N+i}+p_kq_k(-1)^{i+1}}{(1-q_k)^{i+1}}\Delta^i0^{2m-6}, \tag{4.46}$$

thus, from equalities (4.32)- (4.33), (4.36) and (4.38)- (4.46) we get

$$\sum_{k=1}^{m-1}(d_k-p_k)\sum_{i=0}^{2m-2j}\frac{q_k+q_k^{N+i}(-1)^{i+1}}{(q_k-1)^{i+1}}\Delta^i0^{2m-2j}=0,\quad j=1,2,3 \tag{4.47}$$

$$\sum_{k=1}^{m-1}(d_k-p_k)\sum_{i=0}^{2j}\frac{q_k+q_k^{N+i}(-1)^{i+1}}{(q_k-1)^{i+1}}\Delta^i0^{2j}=0,\quad j=1,2 \tag{4.48}$$

$$\sum_{k=1}^{m-1}(d_k-p_k)\sum_{i=0}^{j}\frac{q_k+q_k^{N+i}(-1)^{i+1}}{(q_k-1)^{i+1}}\Delta^i0^{j}=0,\quad j=\overline{6,m-1}, \tag{4.49}$$

Taking into account uniqueness of the optimal coefficients, we conclude, that the homogeneous system of linear equations (4.46)-(4.47) has trivial solution. This means, that

$$d_k=p_k,\quad k=1,2,...,m-1. \tag{4.50}$$

Then, using (4.48), from (4.46)- (4.47) we get (4.22)- (4.24), and from (4.15), (4.33), (4.34), (4.36) we obtain (4.20)-(4.22).

Theorem 4.3 is proved.

5. Conclusion

In this paper, a new of optimal quadrature formulas with odd-order derivative corrections has been constructed in the Sobolev space $L_2^{(m)}(0,1)$ for $m\geq 6$. The proposed quadrature formulas extend the classical Sard-type framework by incorporating the first-, third-, and fifth-order derivatives of the integrand while preserving analytical tractability. The results obtained generalize several previously known optimal quadrature formulas that involve derivative information and provide a unified analytical framework for constructing high-order quadrature rules in differentiable function spaces. The explicit representation of the coefficients considerably simplifies both the theoretical analysis and practical implementation of the proposed formulas. The developed quadrature formulas can be applied to the accurate numerical evaluation of definite integrals, the numerical solution of integral equations, and other computational problems that require high-order accuracy.

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