

Characterization of the Point Spectrum of a Three-Particle Lattice Model Hamiltonian

**T.H. Rasulov^{1,*}, E.B. Dilmurodov¹, B.I. Bahronov¹, G.G. Kurbonov², I.X. Turayev³,
U.M. Khalikova¹**

¹*Bukhara State University, Bukhara, Uzbekistan*

²*Bukhara State Pedagogical Institute, Bukhara, Uzbekistan*

³*Samarkand State Architecture and Construction University, Samarkand, Uzbekistan*

**Corresponding author: rth@mail.ru*

Abstract. In the present work, we investigate a lattice Hamiltonian $H_{\mu,\lambda}$, $\mu, \lambda > 0$, describing a three-particle quantum system with non-local interaction terms. The operator under consideration can be viewed as a tensor sum coupling of two generalized Friedrichs models perturbed by a rank-two operator and is associated with particles propagating on a d -dimensional lattice. The study is mainly focused on the spectral analysis of $H_{\mu,\lambda}$, especially on the characterization of its point spectrum. Employing the Fredholm determinant technique together with methods from the spectral theory of operator matrices, we obtain criteria describing the existence and localization of eigenvalues of $H_{\mu,\lambda}$. In addition, we examine how the point spectrum depends on the interaction parameters μ and λ . Moreover, sufficient conditions guaranteeing the existence of one or two isolated simple eigenvalues are established. The obtained results contribute to a deeper understanding of spectral phenomena for lattice quantum systems with non-local interactions and may be useful in further investigations of multi-particle models in mathematical physics.

1. INTRODUCTION

The investigation of spectral properties of lattice Hamiltonians corresponding to interacting particle systems remains one of the central topics of modern operator theory and mathematical physics. In particular, discrete Schrödinger operators associated with systems of interacting quantum particles on lattices arise naturally in solid state physics, lattice quantum field theory, statistical mechanics, and the theory of many-particle quantum systems ([1], [2], [3]). These operators serve as lattice analogs of the corresponding Schrödinger operators in Euclidean spaces and provide effective mathematical models for describing various physical phenomena.

Compared with the continuous case, the investigation of spectral properties of lattice Hamiltonians possesses several specific difficulties. Although the corresponding interaction operators

Received: Jul. 19, 2026.

2020 *Mathematics Subject Classification.* 81Q10, 35P20, 47N50.

Key words and phrases. lattice model Hamiltonian; non-local interaction; tensor sum; Friedrichs model; eigenvalue.

are often compact or finite-rank perturbations, the structure of the essential and discrete spectra of multi-particle lattice systems is substantially more complicated. One of the main reasons is that the geometry of the lattice changes the analytical behavior of the spectral branches and may lead to the appearance of additional spectral effects which do not occur in the continuous setting.

The study of the essential spectrum of lattice Schrödinger-type operators constitutes an important direction in modern spectral theory and mathematical physics. In the single-particle setting, the classical Weyl theorem [4] provides a fundamental tool for describing the invariance of the essential spectrum under compact perturbations.

In the case of multi-particle quantum systems, the spectral analysis becomes substantially more involved and requires more sophisticated operator-theoretic methods. In particular, an essential role is played by the Hunziker–Van Winter–Zhislin (HVZ) theorem, which relates the structure of the essential spectrum to the spectra of corresponding channel operators. Another important approach is based on Faddeev-type equations and their operator-theoretic modifications. These methods have been intensively developed and successfully applied in the investigation of lattice quantum systems, discrete Schrödinger operators, and operator matrices (see, for example, [5], [6], [7], [8], [9]).

A significant amount of work has been devoted to the spectral analysis of operator matrices and Friedrichs-type models describing systems with variable particle numbers. Such models appear naturally in the study of spin-boson Hamiltonians, Hubbard-type systems, and lattice models describing interactions between particles and quantized fields. In particular, Friedrichs models with finite-rank perturbations provide an effective framework for studying threshold phenomena, embedded eigenvalues, spectral branches, and the structure of the discrete spectrum.

In this work, we consider the lattice Hamiltonian $H_{\mu,\lambda}$, associated with a three-particle quantum system, where $\mu > 0$ and $\lambda > 0$ are spectral parameters. The considered operator can be represented as the tensor sum of the Friedrichs model $h_{\mu,\lambda}$ and a rank-two self-adjoint perturbation. Such a representation provides an effective framework for reducing the spectral investigation of the original Hamiltonian to the analysis of the associated Friedrichs model.

The paper addresses the spectral properties of the Hamiltonian $H_{\mu,\lambda}$, with particular emphasis on its point spectrum and the localization of its eigenvalues. Using the Fredholm determinant method together with techniques from the spectral theory of bounded self-adjoint operators, we establish explicit conditions ensuring the existence of eigenvalues below the lower edge, above the upper edge, and inside the gaps of the essential spectrum.

The investigation relies on representing the Friedrichs-type operator $h_{\mu,\lambda}$ through auxiliary Friedrichs-type operators generated by rank-one perturbations. Assuming appropriate conditions on the supports of the interaction functions, one can relate the eigenvalues of $h_{\mu,\lambda}$ to the spectral properties of the associated auxiliary models. Consequently, this method makes it possible to analyze the dependence of the discrete spectrum on the parameters μ and λ .

The results obtained in this work provide a detailed description of the essential spectrum of the Hamiltonian $H_{\mu,\lambda}$ and allow one to determine the parameter regimes for which the operator possesses one, two, or three simple eigenvalues. Furthermore, explicit representations are established for the spectral branches of the essential spectrum generated by the associated two-particle channels.

The exposition of the paper is structured in the following way. In Section 2, we study several spectral aspects of the Friedrichs-type operator with a rank-two perturbation. Particular attention is devoted to the investigation of its eigenvalues, Fredholm determinant, and a number of auxiliary statements that are subsequently used in the analysis of the main model. The obtained results provide the necessary operator-theoretic framework for the further spectral investigation. Section 3 is concerned with the analysis of the three-particle lattice Hamiltonian $H_{\mu,\lambda}$, where the principal results of the present work are established. In particular, we describe the structure of the essential spectrum, investigate the existence and localization of eigenvalues, and derive conditions for the appearance of discrete spectrum outside the essential spectral branches. The final section contains concluding remarks together with a discussion of several possible directions for future research related to multi-particle lattice systems and Friedrichs-type operator models.

Throughout the analysis presented in this paper, we adopt the standard notation $\sigma(\cdot)$ for the spectrum of a bounded self-adjoint operator. Correspondingly, $\sigma_{\text{ess}}(\cdot)$, $\sigma_{\text{pp}}(\cdot)$ and $\sigma_{\text{disc}}(\cdot)$ will denote its essential, point and discrete spectra, respectively. Unless otherwise stated, these conventions remain in force in all subsequent sections.

2. SPECTRAL CHARACTERISTICS OF THE FRIEDRICHS-TYPE MODEL

Throughout the present work, the standard symbols $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}$ are used to denote the sets of complex, real, integer, and positive integer numbers, respectively. For an arbitrary dimension $d \in \mathbb{N}$, the notation $\mathbb{T}^d$ stands for the d -dimensional torus. The algebraic structure on $\mathbb{T}^d$ is generated by the usual vector operations in the Euclidean space $\mathbb{R}^d$. More precisely, the operations of vector addition and multiplication by real scalars are defined modulo the lattice $(2\pi\mathbb{Z})^d$.

We explain it in the following example, if $d = 3$ and

$$a = \left(\frac{\pi}{2}; -\frac{5\pi}{6}; -\frac{\pi}{3}\right), \quad b = \left(\frac{5\pi}{6}; -\frac{3\pi}{4}; \frac{\pi}{2}\right) \in \mathbb{T}^3,$$

then

$$a + b = \left(-\frac{2\pi}{3}; -\frac{5\pi}{12}; \frac{\pi}{6}\right), \quad 6a = (\pi; \pi; 0) \in \mathbb{T}^3.$$

Let $L_2(\mathbb{T}^d)$ be the Hilbert space consisting of all square-integrable complex-valued functions defined on the d -dimensional torus $\mathbb{T}^d$, equipped with the standard inner product and the corresponding norm. In this functional setting, we consider the bounded self-adjoint Friedrichs-type operator

$$h_{\mu,\lambda} := h_{0,0} - \mu k_1 + \lambda k_2,$$

acting in the space $L_2(\mathbb{T}^d)$. Here, the parameters $\mu > 0$ and $\lambda > 0$ play the role of coupling constants characterizing the interaction intensity.

The unperturbed part $h_{0,0}$ is defined as the multiplication operator generated by the real-valued function $u(\cdot)$, that is,

$$(h_{0,0}g)(x) = u(x)g(x), \quad g \in L_2(\mathbb{T}^d).$$

Consequently, the spectral properties of $h_{0,0}$ are completely determined by the behavior of the function $u(\cdot)$ on the torus $\mathbb{T}^d$.

The operators k_α , $\alpha = 1, 2$, are nonlocal interaction operators of integral type and are given by

$$(k_\alpha g)(x) = v_\alpha(x) \int_{\mathbb{T}^d} v_\alpha(t)g(t) dt, \quad \alpha = 1, 2.$$

These operators describe rank-one perturbations generated by the interaction functions $v_\alpha(\cdot)$. Since the kernels of the corresponding integral operators admit separable representations, the perturbations possess finite rank and therefore preserve many important spectral characteristics of the unperturbed operator.

Throughout the paper, the functions $u(\cdot)$ and $v_\alpha(\cdot)$, $\alpha = 1, 2$, are assumed to be real-valued continuous functions on $\mathbb{T}^d$. Under these assumptions, the operator $h_{\mu,\lambda}$ becomes a bounded self-adjoint operator in $L_2(\mathbb{T}^d)$, which makes it possible to apply methods of the spectral theory of self-adjoint operators and Fredholm determinant techniques in the subsequent analysis.

Since the interaction term $-\mu k_1 + \lambda k_2$ is generated by integral operators of finite rank, the operator $h_{\mu,\lambda}$ may be interpreted as a self-adjoint finite-rank perturbation of the multiplication operator $h_{0,0}$. In particular, the perturbation has rank two, because each operator k_α , $\alpha = 1, 2$, is of rank one. Consequently, the perturbation does not affect the essential part of the spectrum.

Invoking Weyl's theorem concerning the invariance of the essential spectrum under compact perturbations, and especially under perturbations of finite rank, we infer that the operators $h_{\mu,\lambda}$ and $h_{0,0}$ possess identical essential spectra. Therefore, the essential spectrum of the Friedrichs model is completely determined by the spectral properties of the unperturbed multiplication operator. More precisely,

$$\sigma_{\text{ess}}(h_{\mu,\lambda}) = \sigma(h_{0,0}) = [E_1, E_2],$$

where the quantities E_1 and E_2 denote the lower and upper spectral thresholds defined by

$$E_1 := \min_{x \in \mathbb{T}^d} u(x), \quad E_2 := \max_{x \in \mathbb{T}^d} u(x).$$

Hence, the structure of the essential spectrum of the operator $h_{\mu,\lambda}$ is entirely governed by the range of the generating function $u(\cdot)$. In particular, the endpoints of the essential spectrum coincide with the global minimum and maximum values of the function $u(\cdot)$ on the torus $\mathbb{T}^d$.

To investigate the distribution and localization of the eigenvalues of the Friedrichs-type operator $h_{\mu,\lambda}$, we introduce the analytic function $\Delta_{\mu,\lambda}(\cdot)$, which plays the role of the Fredholm determinant associated with the operator $h_{\mu,\lambda}$. This function is defined on the domain $\mathbb{C} \setminus [E_1, E_2]$ by the relation

$$\Delta_{\mu,\lambda}(z) := \Delta_\mu^{(1)}(z) \Delta_\lambda^{(2)}(z) + \mu \lambda \left(I_{12}(z) \right)^2.$$

Where the functions $\Delta_\mu^{(1)}(\cdot)$ and $\Delta_\lambda^{(2)}(\cdot)$ are defined in the domain $\mathbb{C} \setminus [E_1, E_2]$ by

$$\Delta_\mu^{(1)}(z) := 1 - \mu I_{11}(z), \quad \Delta_\lambda^{(2)}(z) := 1 + \lambda I_{22}(z),$$

with “special” integrals

$$I_{ij}(z) := \int_{\mathbb{T}^d} \frac{v_i(t)v_j(t) dt}{u(t) - z}, \quad i, j = 1, 2.$$

The statement presented below establishes a rigorous relationship between the eigenvalues of the Friedrichs-type operator $h_{\mu,\lambda}$ and the zeros of the corresponding Fredholm determinant $\Delta_{\mu,\lambda}(\cdot)$. More precisely, it shows that the spectral problem for the operator $h_{\mu,\lambda}$ outside the essential spectrum can be reduced to the analysis of the zeros of the analytic function $\Delta_{\mu,\lambda}(\cdot)$.

Lemma 2.1. *A complex number $z \in \mathbb{C} \setminus [E_1, E_2]$ is an eigenvalue of the Friedrichs-type operator $h_{\mu,\lambda}$ if and only if the corresponding Fredholm determinant satisfies the relation $\Delta_{\mu,\lambda}(z) = 0$.*

For the proof of Lemma 2.1 see [10].

Thus, the discrete spectral points of the operator $h_{\mu,\lambda}$ located outside the essential spectrum are completely characterized by the zeros of the analytic function $\Delta_{\mu,\lambda}(\cdot)$. Consequently, the investigation of eigenvalues of the Friedrichs-type operator can be reduced to the analysis of the determinant equation $\Delta_{\mu,\lambda}(z) = 0$. This criterion provides an effective analytical tool for studying the existence, localization, and multiplicity of discrete eigenvalues of the operator $h_{\mu,\lambda}$.

More precisely, the discrete spectral points lying outside the essential spectrum coincide exactly with the zeros of the Fredholm determinant $\Delta_{\mu,\lambda}(\cdot)$.

Therefore, the discrete spectrum of the operator admits the representation

$$\sigma_{\text{disc}}(h_{\mu,\lambda}) = \{z \in \mathbb{C} \setminus [E_1, E_2] : \Delta_{\mu,\lambda}(z) = 0\}.$$

Hence, the spectral problem for the operator $h_{\mu,\lambda}$ outside the interval $[E_1, E_2]$ is reduced to the investigation of the zeros of the analytic function $\Delta_{\mu,\lambda}(\cdot)$.

Therefore,

$$\sigma(h_{\mu,\lambda}) = [E_1, E_2] \cup \{z \in \mathbb{C} \setminus [E_1, E_2] : \Delta_{\mu,\lambda}(z) = 0\}.$$

We introduce the auxiliary bounded self-adjoint operators $h_\mu^{(1)}$ and $h_\lambda^{(2)}$, acting in the Hilbert space $L_2(\mathbb{T}^d)$, by

$$h_\mu^{(1)} := h_{0,0} - \mu k_1, \quad h_\lambda^{(2)} := h_{0,0} + \lambda k_2.$$

Each of these operators represents a Friedrichs-type operator with a rank-one perturbation of the multiplication operator $h_{0,0}$. The operators $h_\mu^{(1)}$ and $h_\lambda^{(2)}$ play an important role in the subsequent analysis, since they provide a convenient auxiliary framework for investigating the spectral structure of the Friedrichs operator $h_{\mu,\lambda}$.

In particular, many properties of the discrete spectrum of $h_{\mu,\lambda}$ can be reduced to the study of the spectral characteristics of these simpler rank-one perturbation models. The lemma stated below serves as an auxiliary result that is essential for the investigation of eigenvalues of the Friedrichs-type operator $h_{\mu,\lambda}$.

Lemma 2.2. *The number $z \in \mathbb{C} \setminus [E_1, E_2]$ is an eigenvalue of the operator $h_\mu^{(1)}$ (respectively, of $h_\lambda^{(2)}$) precisely when the corresponding analytic determinant function vanishes at z . More explicitly,*

$$\Delta_\mu^{(1)}(z) = 0 \quad \left(\text{respectively, } \Delta_\lambda^{(2)}(z) = 0 \right).$$

For the proof see [11].

The Lemma 2.2 allows us to obtain the equalities for the discrete spectrum of the Friedrichs-type operators $h_\mu^{(1)}$ and $h_\lambda^{(2)}$, respectively:

$$\sigma_{\text{disc}}(h_\mu^{(1)}) = \{z \in \mathbb{C} \setminus [E_1, E_2] : \Delta_\mu^{(1)}(z) = 0\},$$

$$\sigma_{\text{disc}}(h_\lambda^{(2)}) = \{z \in \mathbb{C} \setminus [E_1, E_2] : \Delta_\lambda^{(2)}(z) = 0\}.$$

We make use of the following additional assumption throughout the paper in order to examine the conditions on the existence of eigenvalues of the Friedrichs-type operator $h_{\mu,\lambda}$.

Assumption 2.1. *We impose the condition*

$$\text{mes}(\text{supp}(v_1) \cap \text{supp}(v_2)) = 0.$$

Here, $\text{mes}(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}^d$, while $\text{supp}(v_\alpha)$ stands for the support of the function $v_\alpha(\cdot)$, $\alpha = 1, 2$.

Example 2.1. *We recall that, if $d = 1$, then the functions*

$$v_1(x) = \begin{cases} \cos x, & x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \\ 0, & x \in \mathbb{T} \setminus \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \end{cases}$$

$$v_2(x) = \begin{cases} \sin 2x, & x \in \mathbb{T} \setminus \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \\ 0, & x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \end{cases}$$

satisfy the Assumption 2.1.

The following result establishes a connection between the spectral properties of the Friedrichs-type operator $h_{\mu,\lambda}$ and the eigenvalue structure of the associated auxiliary rank-one perturbation models $h_\mu^{(1)}$ and $h_\lambda^{(2)}$.

Lemma 2.3. *Assume that Assumption 2.1 holds. A number $z \in \mathbb{C} \setminus [E_1, E_2]$ is an eigenvalue of the Friedrichs-type operator $h_{\mu,\lambda}$ if and only if it is an eigenvalue of at least one of the auxiliary operators $h_\mu^{(1)}$ and $h_\lambda^{(2)}$.*

Proof. According to Lemma 2.1, the number $z \in \mathbb{C} \setminus [E_1, E_2]$ is an eigenvalue of the operator $h_{\mu,\lambda}$ precisely when the corresponding Fredholm determinant satisfies the equality $\Delta_{\mu,\lambda}(z) = 0$.

Assume now that Assumption 2.1 holds. Then, by the definition of the function $I_{12}(\cdot)$, we have

$$I_{12}(z) = 0, \quad z \in \mathbb{R} \setminus [E_1, E_2].$$

Consequently, the determinant function $\Delta_{\mu,\lambda}(\cdot)$ admits the factorization

$$\Delta_{\mu,\lambda}(z) = \Delta_{\mu}^{(1)}(z)\Delta_{\lambda}^{(2)}(z).$$

Therefore, the equality $\Delta_{\mu,\lambda}(z) = 0$ holds if and only if at least one of the relations

$$\Delta_{\mu}^{(1)}(z) = 0 \quad \text{or} \quad \Delta_{\lambda}^{(2)}(z) = 0$$

is satisfied. Applying Lemma 2.2, we conclude that the point z is an eigenvalue of one of the auxiliary operators $h_{\mu}^{(1)}$ or $h_{\lambda}^{(2)}$.

This completes the proof of Lemma 2.3. □

As a consequence of Lemma 2.3 and Assumption 2.1, the discrete spectrum of the Friedrichs-type operator $h_{\mu,\lambda}$ can be represented through the discrete spectra of the auxiliary rank-one perturbation models $h_{\mu}^{(1)}$ and $h_{\lambda}^{(2)}$. More precisely, one has

$$\sigma_{\text{disc}}(h_{\mu,\lambda}) = \sigma_{\text{disc}}(h_{\mu}^{(1)}) \cup \sigma_{\text{disc}}(h_{\lambda}^{(2)}).$$

The above representation is particularly useful for the subsequent spectral analysis, since the operators $h_{\mu}^{(1)}$ and $h_{\lambda}^{(2)}$ possess a significantly simpler structure compared with the operator $h_{\mu,\lambda}$. As a result, many properties of the discrete spectrum of $h_{\mu,\lambda}$ can be investigated through the spectral characteristics of these auxiliary Friedrichs-type operators.

Therefore, the obtained equality serves as an important analytical tool in the study of the existence, localization, and distribution of eigenvalues of the operator $h_{\mu,\lambda}$.

The significance of the latter equality is that the corresponding Fredholm determinants $\Delta_{\mu}^{(1)}(\cdot)$ and $\Delta_{\lambda}^{(2)}(\cdot)$ form the monotonic functions.

The representation of the discrete spectrum in terms of auxiliary Friedrichs-type operators constitutes an important tool in the spectral theory of operator matrices and many-particle Hamiltonians. This method allows one to reduce complicated spectral problems to the analysis of simpler model operators and has found numerous applications in the investigation of quantum lattice systems and related operator-theoretic models (see also [12], [13], [14], [15], [16]).

Lemma 2.4. *Then the auxiliary operator $h_{\mu}^{(1)}$ admits at most one simple eigenvalue located to the left of the lower threshold E_1 , whereas the operator $h_{\lambda}^{(2)}$ possesses at most one simple eigenvalue situated to the right of the upper threshold E_2 .*

The proof of Lemma 2.4 follows easily from the monotonicity properties of the functions $\Delta_{\mu}^{(1)}(\cdot)$ and $\Delta_{\lambda}^{(2)}(\cdot)$.

Whenever they exist, the unique eigenvalues of $h_{\mu}^{(1)}$ and $h_{\lambda}^{(2)}$ lying in $(-\infty, E_1)$ and $(E_2, +\infty)$, respectively, will be denoted by $E_{\mu}^{(1)}$ and $E_{\lambda}^{(2)}$.

3. SPECTRAL ANALYSIS OF THE THREE-PARTICLE LATTICE HAMILTONIAN

We denote by $L_2^s((\mathbb{T}^d)^2)$ the Hilbert space consisting of all quadratically integrable complex-valued functions on $(\mathbb{T}^d)^2$ that are symmetric with respect to permutations of the variables.

We introduce a three-particle lattice Hamiltonian

$$H_{\mu,\lambda} : L_2^s((\mathbb{T}^d)^2) \rightarrow L_2^s((\mathbb{T}^d)^2)$$

defined by

$$H_{\mu,\lambda} = H_{0,0} - \mu(V_{11} + V_{12}) + \lambda(V_{21} + V_{22}).$$

Here, $H_{0,0}$ denotes the unperturbed part of the operator and acts as the multiplication operator generated by the function $u(x) + u(y)$. More precisely,

$$(H_{0,0}f)(x, y) = (u(x) + u(y))f(x, y).$$

The operators $V_{\alpha\beta}$, $\alpha, \beta = 1, 2$, represent nonlocal interaction operators of integral type. Their action on a function $f \in L_2^s((\mathbb{T}^d)^2)$ is given by

$$(V_{\alpha 1}f)(x, y) = v_\alpha(x) \int_{\mathbb{T}^d} v_\alpha(t) f(t, y) dt,$$

and

$$(V_{\alpha 2}f)(x, y) = v_\alpha(y) \int_{\mathbb{T}^d} v_\alpha(t) f(x, t) dt, \quad \alpha = 1, 2.$$

Thus, the operator $H_{\mu,\lambda}$ can be interpreted as a perturbation of the multiplication operator $H_{0,0}$ by nonlocal interaction terms generated through the functions $v_1(\cdot)$ and $v_2(\cdot)$. Under the imposed assumptions, these perturbation operators are bounded and preserve the self-adjointness of the model Hamiltonian.

From the view of the operators $V_{\alpha\beta}$, $\alpha, \beta = 1, 2$ one can see that they are partial integral operators with degenerated kernel of rank 1.

The functions $I_{ii}(\cdot)$, $i = 1, 2$, are monotone on the intervals $(-\infty, E_1)$ and $(E_2, +\infty)$. Therefore, applying the dominated convergence theorem, one concludes that the following one-sided limits exist, possibly taking finite or infinite values:

$$I_{11}(E_1) = \lim_{z \rightarrow E_1-0} I_{11}(z), \quad I_{22}(E_2) = \lim_{z \rightarrow E_2+0} I_{22}(z).$$

The existence of these limits plays an important role in the subsequent spectral analysis, since the behavior of the functions $I_{11}(\cdot)$ and $I_{22}(\cdot)$ near the spectral thresholds E_1 and E_2 determines the possible appearance of discrete eigenvalues outside the essential spectrum.

We introduce the following requirements for “special” integrals that are important for our further investigations.

Assumption 3.1. *We suppose that*

$$|I_{11}(E_1)| = +\infty, \quad |I_{22}(E_2)| < +\infty.$$

Example 3.1. If $d = 1$, then the functions

$$v_1(x) = \begin{cases} \cos x, & x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \\ 0, & x \in \mathbb{T} \setminus \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \end{cases}$$

$$v_2(x) = \begin{cases} \sin 2x, & x \in \mathbb{T} \setminus \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \\ 0, & x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \end{cases}$$

and

$$u(x) = 1 - \cos x$$

satisfy the Assumption 3.1.

We define the following critical value

$$\lambda_0 := -(I_{22}(E_2))^{-1}$$

of λ under the Assumption 3.1.

Assumption 3.2. We suppose that

$$|I_{11}(E_1)| < +\infty, \quad |I_{22}(E_2)| = +\infty.$$

Example 3.2. If $d = 1$, then the functions

$$v_1(x) = \begin{cases} \sin 2x, & x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \\ 0, & x \in \mathbb{T} \setminus \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \end{cases}$$

$$v_2(x) = \begin{cases} \cos x, & x \in \mathbb{T} \setminus \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \\ 0, & x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \end{cases}$$

and

$$u(x) = 1 - \cos x$$

satisfy the Assumption 3.2.

Under the Assumption 3.2 we introduce the following critical value

$$\mu_0 := (I_{11}(E_1))^{-1}$$

of the parameter $\mu > 0$.

Questions related to the existence of eigenvalues and the structure of the essential spectrum for lattice Schrödinger operators and related operator models have attracted considerable attention in the literature. Numerous investigations devoted to these problems have been carried out by different authors. Below, we briefly review several results that are closely connected with the present work. In paper [17], the lower edge of the essential spectrum of a lattice model Hamiltonian

was estimated by means of eigenvalues of a Friedrichs-type operator involving an n -dimensional perturbation. The spectral analysis of such Hamiltonians through their representation as tensor sums of two Friedrichs-type models with multidimensional perturbations was subsequently developed in [18]. Further progress in the study of multi-particle discrete Schrödinger operators was obtained in [19], where critical values of mass ratios guaranteeing the existence of one, two, or three eigenvalues for a three-particle lattice Schrödinger operator were established. In [20], the authors derived criteria ensuring the absence of eigenvalues, as well as conditions for the existence of a unique triple eigenvalue located above the upper threshold of the essential spectrum for a three-particle discrete Schrödinger operator associated with a system consisting of two fermions and an additional particle interacting through pairwise repulsive zero-range potentials on a three-dimensional lattice. The investigation carried out in [21] was devoted to conditions guaranteeing the absence of eigenvalues and the existence of three eigenvalues situated below the essential spectrum for a three-particle discrete Schrödinger operator describing a system of two fermions and a particle of another nature. Finally, in [22], it was proved that the essential spectrum of the corresponding three-particle discrete Schrödinger operator is represented as a finite union of bounded closed intervals.

Main result of the paper is the following theorem.

Theorem 3.1. *Suppose that Assumptions 2.1 and 3.1 hold.*

- (i) *For any $\mu > 0$ and $0 < \lambda \leq \lambda_0$ the number $2E_\mu^{(1)}$ is a simple eigenvalue of the Hamiltonian $H_{\mu,\lambda}$. Moreover, the essential and point spectra of the operator admit the representations*

$$\sigma_{\text{ess}}(H_{\mu,\lambda}) = [E_\mu^{(1)} + E_1, E_\mu^{(1)} + E_2] \cup [2E_1, 2E_2],$$

and

$$\sigma_{\text{pp}}(H_{\mu,\lambda}) = \{2E_\mu^{(1)}\},$$

respectively.

- (ii) *For any $\mu > 0$ and $\lambda > \lambda_0$ the numbers $2E_\mu^{(1)}$ and $2E_\lambda^{(2)}$ are the simple eigenvalues of the Hamiltonian $H_{\mu,\lambda}$.*

In addition, the essential spectrum of the operator is represented as

$$\sigma_{\text{ess}}(H_{\mu,\lambda}) = [E_\mu^{(1)} + E_1, E_\mu^{(1)} + E_2] \cup [2E_1, 2E_2] \cup [E_\lambda^{(2)} + E_1, E_\lambda^{(2)} + E_2],$$

while its point spectrum is given by

$$\sigma_{\text{pp}}(H_{\mu,\lambda}) = \{2E_\mu^{(1)}, E_\mu^{(1)} + E_\lambda^{(2)}, 2E_\lambda^{(2)}\}.$$

Proof. From the definitions of the operators $H_{\mu,\lambda}$ and $h_{\mu,\lambda}$, it follows that the Hamiltonian $H_{\mu,\lambda}$ admits the representation

$$H_{\mu,\lambda} = h_{\mu,\lambda} \otimes I + I \otimes h_{\mu,\lambda},$$

where I denotes the identity operator acting in $L_2(\mathbb{T}^d)$.

Such a decomposition is particularly convenient for the spectral analysis of the Hamiltonian, since it allows one to apply the general theory of tensor sums of self-adjoint operators. In particular, by the classical theorem on the spectrum of tensor sum, one has

$$\sigma(H_{\mu,\lambda}) = \sigma(h_{\mu,\lambda}) + \sigma(h_{\mu,\lambda}). \quad (3.1)$$

Assume now that Assumptions 2.1 and 3.1 are satisfied. Then the determinant functions $\Delta_\mu^{(1)}(\cdot)$ and $\Delta_\lambda^{(2)}(\cdot)$ possess the asymptotic properties

$$\lim_{z \rightarrow -\infty} \Delta_\mu^{(1)}(z) = \lim_{z \rightarrow +\infty} \Delta_\lambda^{(2)}(z) = 1.$$

(i) Let $\mu > 0$ and $0 < \lambda \leq \lambda_0$. Then

$$\Delta_\mu^{(1)}(E_1) = -\infty,$$

and the function $\Delta_\mu^{(1)}(\cdot)$ is monotonically increasing on the interval $(-\infty, E_1)$, the equation $\Delta_\mu^{(1)}(z) = 0$ admits a unique solution $E_\mu^{(1)} \in (-\infty, E_1)$. According to the criterion established in Lemma 2.2, the number $E_\mu^{(1)}$ belongs to the discrete spectrum of the auxiliary rank-one perturbation operator $h_\mu^{(1)}$. Furthermore, Lemma 2.3 shows that this spectral point simultaneously generates an eigenvalue of the Friedrichs-type operator $h_{\mu,\lambda}$. Hence, the operator $h_{\mu,\lambda}$ possesses a eigenvalue located below the lower threshold E_1 of the essential spectrum.

On the other hand, for every $z < E_1$ and $\lambda > 0$, one has $\Delta_\lambda^{(2)}(z) > 1$. Consequently, the operator $h_\lambda^{(2)}$ does not possess eigenvalues to the left of the threshold E_1 .

Next, let $0 < \lambda \leq \lambda_0$. Since the function $\Delta_\lambda^{(2)}(\cdot)$ is monotonically increasing on the interval $(E_2, +\infty)$, we obtain

$$\Delta_\lambda^{(2)}(z) > \Delta_\lambda^{(2)}(E_2) \geq \Delta_{\lambda_0}^{(2)}(E_2) = 0, \quad z > E_2. \quad (3.2)$$

It is evident from the relation (3.2) that $\Delta_\lambda^{(2)}(z) > 0$ for all $z > E_2$ and $\lambda \in (0, \lambda_0]$. By Lemma 2.2, the Friedrichs-type operator $h_\lambda^{(2)}$ has no eigenvalues which located is on the r.h.s. of the point E_2 .

On the other hand, for every $z > E_2$ and $\mu > 0$, one has $\Delta_\mu^{(1)}(z) > 1$. Consequently, the operator $h_\mu^{(1)}$ does not possess eigenvalues to the right of the threshold E_2 .

Thus, if $\mu > 0$ and $0 < \lambda \leq \lambda_0$, then by Lemma 2.3 the Friedrichs-type operator $h_{\mu,\lambda}$ has no eigenvalues belonging to the set $(E_2, +\infty)$.

This yields that,

$$\sigma_{\text{disc}}(h_{\mu,\lambda}) = \{E_\mu^{(1)}\}, \quad \sigma(h_{\mu,\lambda}) = \{E_\mu^{(1)}\} \cup [E_1, E_2]$$

and by the relation (3.1), for any $\mu > 0$ and $0 < \lambda \leq \lambda_0$ the number $2E_\mu^{(1)}$ is a simple eigenvalue of the Hamiltonian $H_{\mu,\lambda}$ and

$$\sigma_{\text{ess}}(H_{\mu,\lambda}) = [E_\mu^{(1)} + E_1, E_\mu^{(1)} + E_2] \cup [2E_1, 2E_2],$$

$$\sigma_{\text{pp}}(H_{\mu,\lambda}) = \{2E_\mu^{(1)}\}.$$

(ii) Now we consider the case $\lambda > \lambda_0$. By the definitions of the numbers $\Delta_\lambda^{(2)}(E_2)$ and λ_0 , we obtain

$$\Delta_\lambda^{(2)}(E_2) < \Delta_{\lambda_0}^{(2)}(E_2) = 0.$$

Since the function $\Delta_\lambda^{(2)}(\cdot)$ is monotonically increasing and continuous in the interval $(E_2, +\infty)$, there exists a number $E_\lambda^{(2)} \in (E_2, +\infty)$ such that

$$\Delta_\lambda^{(2)}(E_\lambda^{(2)}) = 0.$$

Lemma 2.2 yields that this number $E_\lambda^{(2)}$ is an eigenvalue of the Friedrichs-type operator $h_\lambda^{(2)}$.

For the case (i), we showed that for any $\lambda > 0$ the Friedrichs-type operator $h_\lambda^{(2)}$ has no eigenvalues in the interval $(-\infty, E_1)$ and for any $\mu > 0$ and $\lambda \in (0, \lambda_0]$ the operators $h_\mu^{(1)}$ and $h_\lambda^{(2)}$ have no eigenvalues in the interval $(E_2, +\infty)$.

Therefore, by virtue of Lemma 2.3, that for every $\mu > 0$ and $\lambda > \lambda_0$, the Friedrichs-type operator $h_{\mu,\lambda}$ possesses exactly two simple eigenvalues, $E_\mu^{(1)}$ and $E_\lambda^{(2)}$. Accordingly, the discrete spectrum of the operator is given by

$$\sigma_{\text{disc}}(h_{\mu,\lambda}) = \{E_\mu^{(1)}, E_\lambda^{(2)}\},$$

while the whole spectrum admits the representation

$$\sigma(h_{\mu,\lambda}) = \{E_\mu^{(1)}, E_\lambda^{(2)}\} \cup [E_1, E_2].$$

In view of the equality (3.1), for any $\mu > 0$ and $\lambda > \lambda_0$ the numbers $2E_\mu^{(1)}$ and $2E_\lambda^{(2)}$ are constitute simple eigenvalues of the Hamiltonian $H_{\mu,\lambda}$. Moreover, the essential spectrum of the operator admits the decomposition

$$\sigma_{\text{ess}}(H_{\mu,\lambda}) = [E_\mu^{(1)} + E_1, E_\mu^{(1)} + E_2] \cup [2E_1, 2E_2] \cup [E_\lambda^{(2)} + E_1, E_\lambda^{(2)} + E_2],$$

whereas its point spectrum is represented by

$$\sigma_{\text{pp}}(H_{\mu,\lambda}) = \{2E_\mu^{(1)}, E_\mu^{(1)} + E_\lambda^{(2)}, 2E_\lambda^{(2)}\}.$$

Note:

$$2E_\mu^{(1)} < E_\mu^{(1)} + E_1 \quad \text{and} \quad 2E_\lambda^{(2)} > E_\lambda^{(2)} + E_2.$$

Depending on the values of λ and μ , the eigenvalue $E_\mu^{(1)} + E_\lambda^{(2)}$ is in the $[2E_1, 2E_2]$ network of the essential spectrum or of the essential spectrum

$$[E_\mu^{(1)} + E_1, E_\mu^{(1)} + E_2] \cup [2E_1, 2E_2] \cup [E_\lambda^{(2)} + E_1, E_\lambda^{(2)} + E_2]$$

will be located in the space. □

The following Theorem can be proven analogously to Theorem 3.1.

Theorem 3.2. Suppose that Assumptions 2.1 and 3.2 hold .

- (i) For any $0 < \mu \leq \mu_0$ and $\lambda > 0$ the number $2E_\lambda^{(2)}$ is a simple eigenvalue of the Hamiltonian $H_{\mu,\lambda}$ and

$$\sigma_{\text{ess}}(H_{\mu,\lambda}) = [2E_1, 2E_2] \cup [E_\lambda^{(2)} + E_1, E_\lambda^{(2)} + E_2],$$

$$\sigma_{\text{pp}}(H_{\mu,\lambda}) = \{2E_\lambda^{(2)}\}.$$

- (ii) For any $\mu > \mu_0$ and $\lambda > 0$ the numbers $2E_\mu^{(1)}$ and $2E_\lambda^{(2)}$ are the simple eigenvalues of the Hamiltonian $H_{\mu,\lambda}$. Moreover, the equalities

$$\sigma_{\text{ess}}(H_{\mu,\lambda}) = [E_\mu^{(1)} + E_1, E_\mu^{(1)} + E_2] \cup [2E_1, 2E_2] \cup [E_\lambda^{(2)} + E_1, E_\lambda^{(2)} + E_2],$$

$$\sigma_{\text{pp}}(H_{\mu,\lambda}) = \{2E_\mu^{(1)}, E_\mu^{(1)} + E_\lambda^{(2)}, 2E_\lambda^{(2)}\}$$

hold.

4. CONCLUSION

In this work, we study a three-particle lattice Hamiltonian $H_{\mu,\lambda}$, depending on the interaction parameters $\mu > 0$ and $\lambda > 0$. The operator under consideration can be represented as a tensor sum generated by two Friedrichs-type operators together with a rank-two perturbation term.

Such a representation provides an effective framework for the spectral analysis of the model, since it allows the investigation of the three-particle Hamiltonian to be reduced to the study of the associated Friedrichs-type operators and their spectral characteristics.

The representation of the lattice model Hamiltonian $H_{\mu,\lambda}$ in the proposed form made it possible to derive several fundamental spectral characteristics of the model. In particular, an explicit description of the essential spectrum of the Hamiltonian $H_{\mu,\lambda}$ was obtained, and the exact number of closed intervals constituting the essential spectrum was completely characterized. Furthermore, a detailed analysis of the point spectrum was carried out, which allowed us to determine the precise conditions under which the considered Hamiltonian admits one, two, or three simple eigenvalues. The analytical expressions for these eigenvalues, together with their localization with respect to the essential spectrum, were also established. These results provide a comprehensive spectral description of the model and significantly contribute to the spectral theory of lattice Hamiltonians with non-local perturbations.

All the above-mentioned results were obtained under the assumption that, among the special integrals entering the Fredholm determinant associated with the Friedrichs-type models, one remains finite whereas the other diverges. In the derivation of the principal spectral results, natural assumptions imposed on the supports of the parameter functions $v_1(\cdot)$ and $v_2(\cdot)$ made it possible to reduce the spectral analysis of the Friedrichs model with a two-dimensional perturbation to the investigation of the essential and discrete spectra of two auxiliary Friedrichs-type operators involving one-dimensional perturbations. A crucial role in this approach is played by the behavior of the corresponding special integral appearing in the Fredholm determinant. In particular, the divergence of this integral guarantees the existence of a discrete eigenvalue for the lattice model Hamiltonian.

Acknowledgements: The first and second authors express their sincere gratitude to the Foundation for Basic Research of the Republic of Uzbekistan for the financial support provided under Grant No. FL-9524115052.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

REFERENCES

- [1] D.C. Mattis, The Few-Body Problem on a Lattice, *Rev. Mod. Phys.* 58 (1986), 361–379. <https://doi.org/10.1103/RevModPhys.58.361>.
- [2] A. Mogilner, Hamiltonians in Solid-State Physics as Multiparticle Discrete Schrödinger Operators: Problems and Results, *Adv. Soviet Math.* 5 (1991), 139–194. <https://doi.org/10.1090/advsov/005/05>.
- [3] V. Malyshev, R. Minlos, *Linear Infinite-Particle Operators*, American Mathematical Society, 1995. <https://doi.org/10.1090/mmono/143>.
- [4] M. Reed, B. Simon, *Methods of Modern Mathematical Physics. III: Scattering theory*, Academic Press, 1979.
- [5] E.B. Dilmurodov, New Branches of the Essential Spectrum of a 2×2 Operator Matrix, *Uzbek Math. J.* 2020 (2020), 44–51. <https://doi.org/10.29229/uzmj.2020-2-5>.
- [6] E.B. Dilmurodov, Discrete Eigenvalues of a 2×2 Operator Matrix, *AIP Conf. Proc.* 2899 (2023), 030004. <https://doi.org/10.1063/5.0144151>.
- [7] F.M. Jurakulova, Estimates for the Lower and Upper Bounds of the Essential Spectrum of a 3×3 Operator Matrix, in: *AIP Conference Proceedings*, AIP Publishing, 2025, pp. 030018. <https://doi.org/10.1063/5.0299531>.
- [8] N.A. Tosheva, Essential Spectrum of a Family of 3×3 Operator Matrices: Location of the Branches, *AIP Conf. Proc.* 2899 (2023), 030003. <https://doi.org/10.1063/5.0170399>.
- [9] T.H. Rasulov, E.B. Dilmurodov, Threshold Analysis for a Family of 2×2 Operator Matrices, *Nanosyst. Phys. Chem. Math.* 10 (2019), 616–622. <https://doi.org/10.17586/2220-8054-2019-10-6-616-622>.
- [10] B.I. Bahronov, T.H. Rasulov, M. Rehman, Conditions for the Existence of Eigenvalues of a Three-Particle Lattice Model Hamiltonian, *Russ. Math.* 67 (2023), 1–8. <https://doi.org/10.3103/S1066369X23070010>.
- [11] T.H. Rasulov, B.I. Bahronov, Existence of the Eigenvalues of a Tensor Sum of the Friedrichs Models with Rank 2 Perturbation, *Nanosyst. Phys. Chem. Math.* 14 (2023), 151–157. <https://doi.org/10.17586/2220-8054-2023-14-2-151-157>.
- [12] M.I. Muminov, T.H. Rasulov, Infiniteness of the Number of Eigenvalues Embedded in the Essential Spectrum of a 2×2 Operator Matrix, *Eurasian Math. J.* 5 (2014), 60–77.
- [13] D. Ismoilova, The First and Second Schur Complements Corresponding to 3×3 Operator Matrix in Fermionic Fock Space, in: *AIP Conference Proceedings*, AIP Publishing, 2025, pp. 030003. <https://doi.org/10.1063/5.0299545>.
- [14] T.K. Rasulov, On the Structure of the Essential Spectrum of a Model Many-Body Hamiltonian, *Math. Notes* 83 (2008), 80–87. <https://doi.org/10.1134/S0001434608010100>.
- [15] T. Rasulov, F. Jurakulova, On the Discrete Spectrum of a 3×3 Operator Matrix with Spectral Parameter, *Bull. Inst. Math.* 8 (2025), 198–206.
- [16] M. Sharipova, On the Construction of the Resolvent Operator for a 3×3 Operator Matrix in Fock Space, in: *AIP Conference Proceedings*, AIP Publishing, 2025, pp. 030005. <https://doi.org/10.1063/5.0299542>.
- [17] T.K. Rasulov, Z.D. Rasulova, On the Spectrum of a Three-Particle Model Operator on a Lattice with Non-Local Potentials, *Sib. Elektron. Mat. Izv.* 12 (2015), 168–184. <https://doi.org/10.17377/semi.2015.12.014>.
- [18] O. Norkulov, On the Components of the Essential Spectrum of a Tensor Sum of the Friedrichs Models with Finite Rank Non-Local Potentials, in: *AIP Conference Proceedings*, AIP Publishing, 2025, pp. 030012. <https://doi.org/10.1063/5.0299546>.

- [19] A.M. Khalkhuzhaev, K.G. Khayitova, I.A. Khujamiyrov, On the Spectrum of the Schrödinger Operator for a Three-Particle System on a Lattice, Učen. Zap. Kazan. Univ., Ser. Fiz.-Mat. Nauki 167 (2025), 547–565. <https://doi.org/10.26907/2541-7746.2025.3.547-565>.
- [20] J.I. Abdullaev, A.M. Khalkhuzhaev, I.A. Khujamiyrov, Existence Condition for the Eigenvalue of a Three-Particle Schrödinger Operator on a Lattice, Russ. Math. 67 (2023), 1–22. <https://doi.org/10.3103/S1066369X23020019>.
- [21] J.I. Abdullaev, A.M. Khalkhuzhaev, K.D. Kuliev, The Existence of Eigenvalues of Schrödinger Operator on Three Dimensional Lattice, Methods Funct. Anal. Topol. 3 (2022), 189–208. https://doi.org/10.31392/MFAT-npu26_3.2022.01.
- [22] Z.E. Muminov, Sh.S. Lakaev, N.M. Aliev, On the Essential Spectrum of Three-Particle Discrete Schrödinger Operators with Short-Range Potentials, Lobachevskii J. Math. 42 (2021), 1304–1316. <https://doi.org/10.1134/S1995080221060196>.