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Solvability and Stability of a Coupled System of Multi-Term Fractional Differential Equations

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Abstract. In this work, we carry out a detailed qualitative analysis of a coupled system of multi-term fractional differential equations. More specifically, we investigate whether solutions exist and whether they are unique. The system under study includes coupled nonlocal boundary conditions. To obtain our results, we employ appropriate fixed-point methods that are well-suited to fractional problems. Beyond existence and uniqueness, we also conduct a thorough evaluation of the model's stability properties. Specifically, we examine Ulam-Hyers-type stability and its generalized version. These stability analyses confirm that the proposed system is reliable under small perturbations.

1. Introduction

Fractional calculus (FC) is currently a useful tool for describing processes that involve memory effects. Unlike traditional integer-order models, fractional operators allow researchers to capture the underlying dynamics of physical and biological systems with greater accuracy. As a result, interest in this area has grown substantially, leading to many studies on nonlocal fractional differential equations. Many boundary constraints, ranging from nonlinear multi-term to integral multi-strip conditions, have been explored by Ahmad et al. [1–3], and Mani et al. [4] . For more studies on this topic, see [5–7]. Collectively, these investigations provide the essential groundwork for current research on nonlocal fractional models. The importance of such models is evident from the wide range of applications where they appear. The description of memory effects in financial markets [8] relies heavily on fractional Langevin equations. Fractional derivatives have proven essential in the study of anomalous transport [9] and in formulating fractional advection-dispersion

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equations in hydrology [10]. For more applications, see [11–15]. These practical examples highlight why fractional operators are so valuable: they excel at modeling systems where past behavior continues to influence present dynamics.

While several researchers have studied systems of coupled FDEs with various BCs, the present study introduces several new elements that distinguish it from existing literature.

First, as far as we know, this is the first work to consider a system of coupled multi-term Caputo fractional differential equations where *both* equations contain sums of fractional derivatives with distinct orders, along with a class of *coupled nonlocal BCs* of a general form. Specifically, the boundary conditions at $r = 1$ for u depend on a weighted sum of v values at interior points η_i, and similarly the condition for v depends on u at those same points. This coupling in the boundary conditions adds significant mathematical complexity and has not been fully addressed in previous works.

Second, we derive an explicit integral form of the solution for the linearized version of the coupled equations. This representation is more general than those found in earlier studies because it handles multi-term derivatives on both equations simultaneously.

Third, we not only prove the existence of solutions using Krasnoselskii's fixed-point theorem and the uniqueness, but we also conduct a *complete stability analysis* covering four different stability types: Ulam-Hyers (UH), Ulam-Hyers-Rassias (UHR), and their generalization.Many existing studies treat only one or two of these stability types; we provide all four.

Fourth, the theoretical results are supported by a concrete numerical example with explicit parameter values, demonstrating that the conditions of our main theorems are verifiable in practice.

Motivated by the above considerations, in this paper we focus on a system of coupled multi-term Caputo FDEs given by:

$$\begin{cases} \sum_{l=1}^{N} \lambda_l {}^cD^{\hat{\varrho}_l}\varphi(\kappa) = h_1(\kappa, \varphi(\kappa), \chi(\kappa)), \\ \sum_{j=1}^{N} \mu_j {}^cD^{\hat{\beta}_j}\chi(\kappa) = h_2(\kappa, \varphi(\kappa), \chi(\kappa)), \end{cases} \tag{1.1}$$

$$\hat{\varrho}_\sigma \in (N-1, N], \hat{\beta}_\sigma \in (N-1, N], \sigma \neq l, \sigma \neq j$$

$$0 < \hat{\varrho}_l \leq 1, 0 < \hat{\beta}_j \leq 1$$

$$l = 1, 2, 3 \ldots N, j = 1, 2, 3, \ldots N$$

$$\lambda_l, \mu_j \in \mathbb{R}$$

equipped with nonlocal boundary conditions,

$$\varphi(0) = 0, \varphi'(0) = 0, \varphi^{N-2}(0) = 0, \varphi(1) = \sum_{i=1}^{k} a_i \chi(\eta_i).$$

$$\chi(0) = 0, \chi'(0) = 0, \chi^{N-2}(0) = 0, \chi(1) = \sum_{i=1}^{k} b_i \varphi(\eta_i), \quad \eta_i \in (0, 1).$$

In this formulation, ${}^cD^{\hat{\varrho}_l}$ and ${}^cD^{\hat{\beta}_j}$ stand for the Caputo fractional derivatives corresponding to the orders $\hat{\varrho}_l$ and $\hat{\beta}_j$. The functions h_1 and h_2 are assumed to be continuous.

This paper is organized into multiple sections: The foundational concepts are presented in Section 2. The core findings of this study are presented in the third section, where we utilize Krasnoselskii's fixed-point theorem to establish the existence of solutions to problem (1.1). The stability analysis of the solutions obtained in the previous section is examined in the final section.

2. Preliminaries

Before moving to the main results, we review a few definitions and Lemmas from (FC). which will be employed throughout this paper.

Definition 2.1. *[16] Suppose an integrable function y , with $-\infty \leq a \leq \kappa \leq b \leq \infty$. The $\hat{\varrho}$ order Riemann-Liouville fractional integral $I_a^{\hat{\varrho}}$ is given by:*

$$I_a^{\hat{\varrho}} = (y * K_{\hat{\varrho}}) = \int_a^{\kappa} \frac{(\kappa - z)^{\hat{\varrho}-1}}{\Gamma(\hat{\varrho})} y(z)dz, \tag{2.1}$$

Here, we use Γ to represent the Euler gamma function, and the kernel is $K_{\hat{\varrho}} = \frac{\kappa^{\hat{\varrho}-1}}{\Gamma(\hat{\varrho})}$.

Definition 2.2. *[16] Now consider a function $y : [a, \infty) \longrightarrow \mathbb{R}$ that is absolutely continuous and has $(N-1)$ derivatives. its Caputo fractional derivative of order $\hat{\varrho}$ is expressed as:*

$$^{c}D_a^{\hat{\varrho}} y(\kappa) = \int_a^{\kappa} \frac{(\kappa - z)^{N-\hat{\varrho}-1}}{\Gamma(N-\hat{\varrho})} y^{(N)}(z)dz, \tag{2.2}$$

In this definition, we require $N-1 < \hat{\varrho} \leq N$, with $N = [\hat{\varrho}] + 1$. The notation $[\hat{\varrho}]$ stands for the integer part of $\hat{\varrho}$.

Lemma 2.1. *[16] The general solution of $^{c}D_a^{\hat{\varrho}} y(\kappa) = 0$,where $\kappa \in [a, b], N-1 < \hat{\varrho} < N$, can be stated as:*

$$y(\kappa) = E_0 + E_1(\kappa - a) + E_2(\kappa - a)^2 + ... + E_{N-1}(\kappa - a)^{N-1},$$

the coefficients E_i are arbitrary real numbers for $i = 0, 1, .., N-1$. Moreover,

$$I^{\hat{\varrho}}(^{c}D_a^{\hat{\varrho}})y(\kappa) = y(\kappa) + \sum_{i=1}^{N-1} E_i(\kappa - a)^i.$$

Lemma 2.2. *Let $w, v \in \mathbf{C}([0,1], \mathbb{R})$ be given continuous functions. Then the solution to the following problem*

$$\begin{cases} \sum_{l=1}^{N} \lambda_l {}^{c}D^{\hat{\varrho}_l} \varphi(\kappa) = w(\kappa), \\ \sum_{j=1}^{N} \mu_j {}^{c}D^{\hat{\beta}_j} \chi(\kappa) = v(\kappa), \end{cases} \quad \kappa \in [0,1] \tag{2.3}$$

$$\varphi(0) = 0, \varphi'(0) = 0, \varphi^{N-2}(0) = 0, \varphi(1) = \sum_{i=1}^{k} a_i \chi(\eta_i).$$

$$\chi(0) = 0, \chi'(0) = 0, \chi^{N-2}(0) = 0, \chi(1) = \sum_{i=1}^{k} b_i \varphi(\eta_i).$$

$$0 < \hat{\varrho}_l \leq 1, 0 < \hat{\beta}_j \leq 1$$

$$l = 1, 2, 3 \ldots N, l \neq \sigma, j = 1, 2, 3 \ldots N, j \neq \sigma$$

can be written explicitly as follows:

$$\begin{aligned}
\varphi(\kappa) = \frac{\kappa^{N-1}}{\gamma}\Bigg[& \sum_{i=1}^{k} \frac{a_i}{\mu_\sigma} \int_0^{\eta_i} \frac{(\eta_i - z)^{\hat{\beta}_\sigma - 1}}{\Gamma(\hat{\beta}_\sigma)} v(z)\, dz \\
& - \sum_{i=1}^{k} \sum_{j=1, j\neq\sigma}^{N} a_i \frac{\mu_j}{\mu_\sigma} \int_0^{\eta_i} \frac{(\eta_i - z)^{\hat{\beta}_\sigma - \hat{\beta}_j - 1}}{\Gamma(\hat{\beta}_\sigma - \hat{\beta}_j)} \chi(z)\, dz \\
& - \frac{1}{\lambda_\sigma} \int_0^{1} \frac{(1 - z)^{\hat{\varrho}_\sigma - 1}}{\Gamma(\hat{\varrho}_\sigma)} w(z)\, dz \\
& + \sum_{l=1, l\neq\sigma}^{N} \frac{\lambda_l}{\lambda_\sigma} \int_0^{1} \frac{(1 - z)^{\hat{\varrho}_\sigma - \hat{\varrho}_l - 1}}{\Gamma(\hat{\varrho}_\sigma - \hat{\varrho}_l)} \varphi(z)\, dz \\
& + A_1\Bigg(\sum_{i=1}^{k} \frac{b_i}{\lambda_\sigma} \int_0^{\eta_i} \frac{(\eta_i - z)^{\hat{\varrho}_\sigma - 1}}{\Gamma(\hat{\varrho}_\sigma)} w(z)\, dz \\
& - \sum_{i=1}^{k} \sum_{l=1, l\neq\sigma}^{N} \frac{\lambda_l b_i}{\lambda_\sigma} \int_0^{\eta_i} \frac{(\eta_i - z)^{\hat{\varrho}_\sigma - \hat{\varrho}_l - 1}}{\Gamma(\hat{\varrho}_\sigma - \hat{\varrho}_l)} \varphi(z)\, dz \\
& - \frac{1}{\mu_\sigma} \int_0^{1} \frac{(1 - z)^{\hat{\beta}_\sigma - 1}}{\Gamma(\hat{\beta}_\sigma)} v(z)\, dz \\
& + \sum_{j=1, j\neq\sigma}^{N} \frac{\mu_j}{\mu_\sigma} \int_0^{1} \frac{(1 - z)^{\hat{\beta}_\sigma - \hat{\beta}_j - 1}}{\Gamma(\hat{\beta}_\sigma - \hat{\beta}_j)} \chi(z)\, dz \Bigg) \Bigg] \\
& + \frac{1}{\lambda_\sigma} \int_0^{\kappa} \frac{(\kappa - z)^{\hat{\varrho}_\sigma - 1}}{\Gamma(\hat{\varrho}_\sigma)} w(z) dz \\
& - \sum_{l=1, l\neq\sigma}^{N} \frac{\lambda_l}{\lambda_\sigma} \int_0^{\kappa} \frac{(\kappa - z)^{\hat{\varrho}_\sigma - \hat{\varrho}_l - 1}}{\Gamma(\hat{\varrho}_\sigma - \hat{\varrho}_l)} \varphi(z) dz,
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
\chi(\kappa) = \frac{\kappa^{N-1}}{\gamma}\Bigg[& \sum_{i=1}^{k} \frac{b_i}{\lambda_\sigma} \int_0^{\eta_i} \frac{(\eta_i - z)^{\hat{\varrho}_\sigma - 1}}{\Gamma(\hat{\varrho}_\sigma)} w(z)\, dz \\
& - \sum_{i=1}^{k} \sum_{l=1, l\neq\sigma}^{N} \frac{\lambda_l b_i}{\lambda_\sigma} \int_0^{\eta_i} \frac{(\eta_i - z)^{\hat{\varrho}_\sigma - \hat{\varrho}_l - 1}}{\Gamma(\hat{\varrho}_\sigma - \hat{\varrho}_l)} \varphi(z)\, dz \\
& - \frac{1}{\mu_\sigma} \int_0^{1} \frac{(1 - z)^{\hat{\beta}_\sigma - 1}}{\Gamma(\hat{\beta}_\sigma)} v(z)\, dz \\
& + \sum_{j=1, j\neq\sigma}^{N} \frac{\mu_j}{\mu_\sigma} \int_0^{1} \frac{(1 - z)^{\hat{\beta}_\sigma - \hat{\beta}_j - 1}}{\Gamma(\hat{\beta}_\sigma - \hat{\beta}_j)} \chi(z)\, dz \\
& + A_2\Bigg(\sum_{i=1}^{k} \frac{a_i}{\mu_\sigma} \int_0^{\eta_i} \frac{(\eta_i - z)^{\hat{\beta}_\sigma - 1}}{\Gamma(\hat{\beta}_\sigma)} v(z)\, dz
\end{aligned}$$

$$-\sum_{i=1}^{k}\sum_{j=1,j\neq\sigma}^{N} a_i\frac{\mu_j}{\mu_\sigma}\int_0^{\eta_i}\frac{(\eta_i-z)^{\hat{\beta}_\sigma-\hat{\beta}_j-1}}{\Gamma(\hat{\beta}_\sigma-\hat{\beta}_j)}\chi(z)\,dz$$
$$-\frac{1}{\lambda_\sigma}\int_0^1\frac{(1-z)^{\hat{\varrho}_\sigma-1}}{\Gamma(\hat{\varrho}_\sigma)}w(z)\,dz$$
$$+\sum_{l=1,l\neq\sigma}^{N}\frac{\lambda_l}{\lambda_\sigma}\int_0^1\frac{(1-z)^{\hat{\varrho}_\sigma-\hat{\varrho}_l-1}}{\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l)}\varphi(z)\,dz\Bigg)\Bigg]$$
$$+\frac{1}{\mu_\sigma}\int_0^\kappa\frac{(\kappa-z)^{\hat{\beta}_\sigma-1}}{\Gamma(\hat{\beta}_\sigma)}v(z)dz$$
$$-\sum_{j=1,j\neq\sigma}^{N}\frac{\mu_j}{\mu_\sigma}\int_0^\kappa\frac{(\kappa-z)^{\hat{\beta}_\sigma-\hat{\beta}_j-1}}{\Gamma(\hat{\beta}_\sigma-\hat{\beta}_j)}\chi(z)dz, \tag{2.5}$$

In the expressions above, the constant γ is given by $\gamma = 1 - A_1A_2$, which must be nonzero for the solution to be well-defined.

Proof. We begin by applying fractional integration of order $\hat{\varrho}_\sigma$ to the first equation in (2.3) and similarly of order $\hat{\beta}_\sigma$ to the second equation. We will have ;

$$\begin{aligned}\varphi(\kappa) &= E_1 + E_2\kappa + \cdots + E_N\kappa^{N-1} + \frac{1}{\lambda_\sigma}\int_0^\kappa\frac{(\kappa-z)^{\hat{\varrho}_\sigma-1}}{\Gamma(\hat{\varrho}_\sigma)}w(z)\,dz\\ &-\sum_{\substack{l=1\\ l\neq\sigma}}^{N}\frac{\lambda_l}{\lambda_\sigma}\int_0^\kappa\frac{(\kappa-z)^{\hat{\varrho}_\sigma-\hat{\varrho}_l-1}}{\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l)}\varphi(z)\,dz,\end{aligned} \tag{2.6}$$

$$\begin{aligned}\chi(\kappa) &= G_1 + G_2\kappa + \cdots + G_N\kappa^{N-1} + \frac{1}{\mu_\sigma}\int_0^\kappa\frac{(\kappa-z)^{\hat{\beta}_\sigma-1}}{\Gamma(\hat{\beta}_\sigma)}v(z)dz\\ &-\sum_{j=1,j\neq\sigma}^{N}\frac{\mu_j}{\mu_\sigma}\int_0^\kappa\frac{(\kappa-z)^{\hat{\beta}_\sigma-\hat{\beta}_j-1}}{\Gamma(\hat{\beta}_\sigma-\hat{\beta}_j)}\chi(z)dz,\end{aligned} \tag{2.7}$$

The symbols $E_1, E_2, ..., E_N$ and $G_1, G_2, ..., G_N$ represent arbitrary constants that arise from integration. Utilizing (2.6),(2.7) with the condition $\varphi(0) = 0, \varphi'(0) = 0, \varphi^{N-2}(0) = 0$,$\chi(0) = 0, \chi'(0) = 0, \chi^{N-2}(0) = 0$ we can find that $E_1 = 0$,E_2=0,$E_{N-1} = 0$,for i=1,2,..,N-1 $E_i = 0$,$G_1 = 0$,G_2=0,$G_{N-1} = 0$,for $i = 1, 2, .., N-1$, $G_i = 0$ Thus, (2.7),(2.6) as follows:

$$\begin{aligned}\varphi(\kappa) &= E_N\kappa^{N-1} + \frac{1}{\lambda_\sigma}\int_0^\kappa\frac{(\kappa-z)^{\hat{\varrho}_\sigma-1}}{\Gamma(\hat{\varrho}_\sigma)}w(z)\,dz\\ &-\sum_{\substack{l=1\\ l\neq\sigma}}^{N}\frac{\lambda_l}{\lambda_\sigma}\int_0^\kappa\frac{(\kappa-z)^{\hat{\varrho}_\sigma-\hat{\varrho}_l-1}}{\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l)}\varphi(z)\,dz,\end{aligned} \tag{2.8}$$

$$\chi(\kappa) = G_N \kappa^{N-1} + \frac{1}{\mu_\sigma} \int_0^\kappa \frac{(\kappa - z)^{\hat{\beta}_\sigma - 1}}{\Gamma(\hat{\beta}_\sigma)} v(z) dz - \sum_{j=1, j \neq \sigma}^{N} \frac{\mu_j}{\mu_\sigma} \int_0^\kappa \frac{(\kappa - z)^{\hat{\beta}_\sigma - \hat{\beta}_j - 1}}{\Gamma(\hat{\beta}_\sigma - \hat{\beta}_j)} \chi(z) dz, \tag{2.9}$$

Utilizing $\varphi(1) = \sum_{i=1}^{k} a_i \chi(\eta_i)$, $\chi(1) = \sum_{i=1}^{k} b_i \varphi(\eta_i)$ we get this system:

$$E_N - A_1 G_N = T_1, G_N - A_2 E_N = T_2 \tag{2.10}$$

where $A_1 = \sum_{i=1}^{k} a_i \eta_i^{N-1}$, $A_2 = \sum_{i=1}^{k} b_i \eta_i^{N-1}$,

$$\begin{aligned}
T_1 = & \sum_{i=1}^{k} \frac{a_i}{\mu_\sigma} \int_0^{\eta_i} \frac{(\eta_i - z)^{\hat{\beta}_\sigma - 1}}{\Gamma(\hat{\beta}_\sigma)} v(z)\, dz \\
& - \sum_{i=1}^{k} \sum_{j=1, j \neq \sigma}^{N} a_i \frac{\mu_j}{\mu_\sigma} \int_0^{\eta_i} \frac{(\eta_i - z)^{\hat{\beta}_\sigma - \hat{\beta}_j - 1}}{\Gamma(\hat{\beta}_\sigma - \hat{\beta}_j)} \chi(z)\, dz \\
& - \frac{1}{\lambda_\sigma} \int_0^1 \frac{(1 - z)^{\hat{\varrho}_\sigma - 1}}{\Gamma(\hat{\varrho}_\sigma)} w(z)\, dz \\
& + \sum_{l=1, l \neq \sigma}^{N} \frac{\lambda_l}{\lambda_\sigma} \int_0^1 \frac{(1 - z)^{\hat{\varrho}_\sigma - \hat{\varrho}_l - 1}}{\Gamma(\hat{\varrho}_\sigma - \hat{\varrho}_l)} \varphi(z)\, dz, \\
T_2 = & \sum_{i=1}^{k} \frac{b_i}{\lambda_\sigma} \int_0^{\eta_i} \frac{(\eta_i - z)^{\hat{\varrho}_\sigma - 1}}{\Gamma(\hat{\varrho}_\sigma)} w(z)\, dz \\
& - \sum_{i=1}^{k} \sum_{l=1, l \neq \sigma}^{N} \frac{\lambda_l b_i}{\lambda_\sigma} \int_0^{\eta_i} \frac{(\eta_i - z)^{\hat{\varrho}_\sigma - \hat{\varrho}_l - 1}}{\Gamma(\hat{\varrho}_\sigma - \hat{\varrho}_l)} \varphi(z)\, dz \\
& - \frac{1}{\mu_\sigma} \int_0^1 \frac{(1 - z)^{\hat{\beta}_\sigma - 1}}{\Gamma(\hat{\beta}_\sigma)} v(z)\, dz \\
& + \sum_{j=1, j \neq \sigma}^{N} \frac{\mu_j}{\mu_\sigma} \int_0^1 \frac{(1 - z)^{\hat{\beta}_\sigma - \hat{\beta}_j - 1}}{\Gamma(\hat{\beta}_\sigma - \hat{\beta}_j)} \chi(z)\, dz,
\end{aligned}$$

Evaluating the system(2.10) yields:

$$E_N = \frac{T_1 + A_1 T_2}{1 - A_1 A_2}, G_N = \frac{T_2 + A_2 T_1}{1 - A_1 A_2}$$

By inserting the expressions of E_N and G_N into (2.8),(2.9) the solutions are found to be (2.5),(2.4). □

Theorem 2.1. *[17] (Krasnoselskii's fixed-point theorem) Let $W \subset X$ be a nonempty, closed, bounded, and convex subset, and X be a Banach space. Suppose that we have two operators $C, D : W \to X$satisfying the following conditions:*

(A1) $\mathbf{C}x + \mathbf{D}z \in W$ *for all $x, z \in W$;*

(A2) $\mathbf{D}$ *satisfies the contraction mapping principle;*

(A3) $\mathbf{C}$ *is both continuous and compact.*

Under these assumptions, at least one fixed point $x^* \in W$ exists such that $x^* = \mathbf{C}x^* + \mathbf{D}x^*$.

Theorem 2.2. *[18] (ArzelàAscoli theorem)*
A subset of the space $X = C([0,1],\mathbb{R})$ is called relatively compact if and only if it is bounded and equicontinuous.

3. Main results

Consider the Banach space $X = C([0,1],\mathbb{R})$ endowed with the norm $\|f\| = \sup_{\kappa\in[0,1]} |f|$. Consequently, the space $(X \times X, \|(f_1, f_2)\|)$ is also a Banach space equipped with the norm $\|(f_1, f_2)\| = \|f_1\| + \|f_2\|$. We now define an operator $\mathcal{U} : X \times X \to X \times X$ using Lemma (2.2) , which will help us reframe the system (1.1) into a fixed-point problem by:

$$\mathcal{U}(\varphi(\kappa),\chi(\kappa)) = (\mathcal{U}_1(\varphi(\kappa),\chi(\kappa)),\mathcal{U}_2(\varphi(\kappa),\chi(\kappa)))$$

where,

$$\begin{aligned}
\mathcal{U}_1(\varphi(\kappa),\chi(\kappa)) = \frac{\kappa^{N-1}}{\gamma}\Bigg[& \sum_{i=1}^{k} \frac{a_i}{\mu_\sigma} \int_0^{\eta_i} \frac{(\eta_i - z)^{\hat{\beta}_\sigma - 1}}{\Gamma(\hat{\beta}_\sigma)} h_2(z,\varphi(z),\chi(z))\, dz \\
& - \sum_{i=1}^{k} \sum_{j=1, j\neq\sigma}^{N} a_i \frac{\mu_j}{\mu_\sigma} \int_0^{\eta_i} \frac{(\eta_i - z)^{\hat{\beta}_\sigma - \hat{\beta}_j - 1}}{\Gamma(\hat{\beta}_\sigma - \hat{\beta}_j)} \chi(z)\, dz \\
& - \frac{1}{\lambda_\sigma} \int_0^{1} \frac{(1-z)^{\hat{\varrho}_\sigma - 1}}{\Gamma(\hat{\varrho}_\sigma)} h_1(z,\varphi(z),\chi(z))\, dz \\
& + \sum_{l=1, l\neq\sigma}^{N} \frac{\lambda_l}{\lambda_\sigma} \int_0^{1} \frac{(1-z)^{\hat{\varrho}_\sigma - \hat{\varrho}_l - 1}}{\Gamma(\hat{\varrho}_\sigma - \hat{\varrho}_l)} \varphi(z)\, dz \\
& + A_1\Bigg(\sum_{i=1}^{k} \frac{b_i}{\lambda_\sigma} \int_0^{\eta_i} \frac{(\eta_i - z)^{\hat{\varrho}_\sigma - 1}}{\Gamma(\hat{\varrho}_\sigma)} h_1(z,\varphi(z),\chi(z))\, dz \\
& - \sum_{i=1}^{k} \sum_{l=1, l\neq\sigma}^{N} \frac{\lambda_l b_i}{\lambda_\sigma} \int_0^{\eta_i} \frac{(\eta_i - z)^{\hat{\varrho}_\sigma - \hat{\varrho}_l - 1}}{\Gamma(\hat{\varrho}_\sigma - \hat{\varrho}_l)} \varphi(z)\, dz \\
& - \frac{1}{\mu_\sigma} \int_0^{1} \frac{(1-z)^{\hat{\beta}_\sigma - 1}}{\Gamma(\hat{\beta}_\sigma)} h_2(z,\varphi(z),\chi(z))\, dz \\
& + \sum_{j=1, j\neq\sigma}^{N} \frac{\mu_j}{\mu_\sigma} \int_0^{1} \frac{(1-z)^{\hat{\beta}_\sigma - \hat{\beta}_j - 1}}{\Gamma(\hat{\beta}_\sigma - \hat{\beta}_j)} \chi(z)\, dz \Bigg) \Bigg] \\
& + \frac{1}{\lambda_\sigma} \int_0^{\kappa} \frac{(\kappa - z)^{\hat{\varrho}_\sigma - 1}}{\Gamma(\hat{\varrho}_\sigma)} h_1(z,\varphi(z),\chi(z)) dz \\
& - \sum_{l=1, l\neq\sigma}^{N} \frac{\lambda_l}{\lambda_\sigma} \int_0^{\kappa} \frac{(\kappa - z)^{\hat{\varrho}_\sigma - \hat{\varrho}_l - 1}}{\Gamma(\hat{\varrho}_\sigma - \hat{\varrho}_l)} \varphi(z) dz,
\end{aligned}$$

$$
\begin{aligned}
\mathcal{U}_2(\varphi(\kappa),\chi(\kappa)) = \frac{\kappa^{N-1}}{\gamma}\Bigg[&\sum_{i=1}^{k}\frac{b_i}{\lambda_\sigma}\int_0^{\eta_i}\frac{(\eta_i-z)^{\hat{\varrho}_\sigma-1}}{\Gamma(\hat{\varrho}_\sigma)}h_1(z,\varphi(z),\chi(z))\,dz\\
&-\sum_{i=1}^{k}\sum_{l=1,l\neq\sigma}^{N}\frac{\lambda_l b_i}{\lambda_\sigma}\int_0^{\eta_i}\frac{(\eta_i-z)^{\hat{\varrho}_\sigma-\hat{\varrho}_l-1}}{\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l)}\varphi(z)\,dz\\
&-\frac{1}{\mu_\sigma}\int_0^{1}\frac{(1-z)^{\hat{\beta}_\sigma-1}}{\Gamma(\hat{\beta}_\sigma)}h_2(z,\varphi(z),\chi(z))\,dz\\
&+\sum_{j=1,j\neq\sigma}^{N}\frac{\mu_j}{\mu_\sigma}\int_0^{1}\frac{(1-z)^{\hat{\beta}_\sigma-\hat{\beta}_j-1}}{\Gamma(\hat{\beta}_\sigma-\hat{\beta}_j)}\chi(z)\,dz\\
&+A_2\Bigg(\sum_{i=1}^{k}\frac{a_i}{\mu_\sigma}\int_0^{\eta_i}\frac{(\eta_i-z)^{\hat{\beta}_\sigma-1}}{\Gamma(\hat{\beta}_\sigma)}h_2(z,\varphi(z),\chi(z))\,dz\\
&-\sum_{i=1}^{k}\sum_{j=1,j\neq\sigma}^{N}a_i\frac{\mu_j}{\mu_\sigma}\int_0^{\eta_i}\frac{(\eta_i-z)^{\hat{\beta}_\sigma-\hat{\beta}_j-1}}{\Gamma(\hat{\beta}_\sigma-\hat{\beta}_j)}\chi(z)\,dz\\
&-\frac{1}{\lambda_\sigma}\int_0^{1}\frac{(1-z)^{\hat{\varrho}_\sigma-1}}{\Gamma(\hat{\varrho}_\sigma)}h_1(z,\varphi(z),\chi(z))\,dz\\
&+\sum_{l=1,l\neq\sigma}^{N}\frac{\lambda_l}{\lambda_\sigma}\int_0^{1}\frac{(1-z)^{\hat{\varrho}_\sigma-\hat{\varrho}_l-1}}{\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l)}\varphi(z)\,dz\Bigg)\Bigg]\\
&+\frac{1}{\mu_\sigma}\int_0^{\kappa}\frac{(\kappa-z)^{\hat{\beta}_\sigma-1}}{\Gamma(\hat{\beta}_\sigma)}h_2(z,\varphi(z),\chi(z))dz\\
&-\sum_{j=1,j\neq\sigma}^{N}\frac{\mu_j}{\mu_\sigma}\int_0^{\kappa}\frac{(\kappa-z)^{\hat{\beta}_\sigma-\hat{\beta}_j-1}}{\Gamma(\hat{\beta}_\sigma-\hat{\beta}_j)}\chi(z)dz,
\end{aligned}
$$

Where $\gamma = 1 - A_1A_2 \neq 0$.

4. Hypotheses

The following assumptions will be needed to support our main results. They concern the nonlinear functions h_1 and h_2.

(B_1) For all $\kappa \in [0,1]$ and $\varphi, \varphi^*, \chi, \chi^* \in \mathbb{R}$, there exists $L_1 > 0$ such that

$$|h_1(\kappa,\varphi(\kappa),\chi(\kappa)) - h_1(\kappa,\varphi^*(\kappa),\chi^*(\kappa))| \leq L_1(|\varphi-\varphi^*| + |\chi-\chi^*|)$$

(B_2) For all $\kappa \in [0,1]$ and $\varphi, \varphi^*, \chi, \chi^* \in \mathbb{R}$, there exists $L_2 > 0$ such that

$$|h_2(\kappa,\varphi(\kappa),\chi(\kappa)) - h_2(\kappa,\varphi^*(\kappa),\chi^*(\kappa))| \leq L_2(|\varphi-\varphi^*| + |\chi-\chi^*|)$$

(B_3) For constants $C_1, C_2, D_1, D_2, M_1, M_2 > 0$,

$$|h_1(\kappa,\varphi(\kappa),\chi(\kappa))| \leq C_1|\varphi| + D_1|\chi| + M_1$$

$$|h_2(\kappa,\varphi(\kappa),\chi(\kappa))| \le C_2|\varphi| + D_2|\chi| + M_2$$

Remark 4.1. *To keep the computations manageable, we introduce the following shorthand notation.*

$$\begin{aligned}
K_1 = & \frac{1}{|\gamma|}\Bigg[\sum_{i=1}^{k}\frac{|a_i|}{|\mu_\sigma|}\frac{\eta_i^{\hat{\beta}_\sigma}L_2}{\Gamma(\hat{\beta}_\sigma+1)} + \sum_{i=1}^{k}\sum_{j=1,j\neq\sigma}^{N}\frac{|a_i||\mu_j|}{|\mu_\sigma|}\frac{\eta_i^{\hat{\beta}_\sigma-\hat{\beta}_j}}{\Gamma(\hat{\beta}_\sigma-\hat{\beta}_j+1)} \\
& + \frac{L_1}{|\lambda_\sigma|\Gamma(\hat{\varrho}_\sigma+1)} + \sum_{l=1,l\neq\sigma}^{N}\frac{|\lambda_l|}{|\lambda_\sigma|\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l+1)} + |A_1|\Bigg(\sum_{i=1}^{k}\frac{L_1|b_i|\eta_i^{\hat{\varrho}_\sigma}}{|\lambda_\sigma|\Gamma(\hat{\varrho}_\sigma+1)} \\
& + \sum_{i=1}^{k}\sum_{l=1,l\neq\sigma}^{N}\frac{|\lambda_l||b_i|\eta_i^{\hat{\varrho}_\sigma-\hat{\varrho}_l}}{|\lambda_\sigma|\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l+1)} + \frac{L_2}{|\mu_\sigma|\Gamma(\hat{\beta}_\sigma+1)} + \sum_{j=1,j\neq\sigma}^{N}\frac{|\mu_j|}{|\mu_\sigma|\Gamma(\hat{\beta}_\sigma-\hat{\beta}_j+1)}\Bigg)\Bigg] \\
& + \frac{L_1}{|\lambda_\sigma|\Gamma(\hat{\varrho}_\sigma+1)} + \sum_{l=1,l\neq\sigma}^{N}\frac{|\lambda_l|}{|\lambda_\sigma|\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l+1)}, \\
K_2 = & \frac{1}{|\gamma|}\Bigg[\sum_{i=1}^{k}\frac{|b_i|}{|\lambda_\sigma|}\frac{\eta_i^{\hat{\varrho}_\sigma}L_1}{\Gamma(\hat{\varrho}_\sigma+1)} + \sum_{i=1}^{k}\sum_{l=1,l\neq\sigma}^{N}\frac{|b_i||\lambda_l|}{|\lambda_\sigma|}\frac{\eta_i^{\hat{\varrho}_\sigma-\hat{\varrho}_l}}{\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l+1)} \\
& + \frac{L_2}{|\mu_\sigma|\Gamma(\hat{\beta}_\sigma+1)} + \sum_{j=1,j\neq\sigma}^{N}\frac{|\mu_j|}{|\mu_\sigma|\Gamma(\hat{\beta}_\sigma-\hat{\beta}_j+1)} + |A_2|\Bigg(\sum_{i=1}^{k}\frac{L_2|a_i|\eta_i^{\hat{\beta}_\sigma}}{|\mu_\sigma|\Gamma(\hat{\beta}_\sigma+1)} \\
& + \sum_{i=1}^{k}\sum_{j=1,j\neq\sigma}^{N}\frac{|\mu_j||a_i|\eta_i^{\hat{\beta}_\sigma-\hat{\beta}_j}}{|\mu_\sigma|\Gamma(\hat{\beta}_\sigma-\hat{\beta}_j+1)} + \frac{L_1}{|\lambda_\sigma|\Gamma(\hat{\varrho}_\sigma+1)} + \sum_{l=1,l\neq\sigma}^{N}\frac{|\lambda_l|}{|\lambda_\sigma|\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l+1)}\Bigg)\Bigg] \\
& + \frac{L_2}{|\mu_\sigma|\Gamma(\hat{\beta}_\sigma+1)} + \sum_{j=1,j\neq\sigma}^{N}\frac{|\mu_j|}{|\mu_\sigma|\Gamma(\hat{\beta}_\sigma-\hat{\beta}_j+1)}.
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
& \frac{\rho}{2}\Bigg(1-\frac{1}{|\gamma|}\Bigg[\sum_{i=1}^{k}\frac{|a_i|}{|\mu_\sigma|}\frac{\eta_i^{\hat{\beta}_\sigma}}{\Gamma(\hat{\beta}_\sigma+1)}(C_2+D_2) + \sum_{i=1}^{k}\sum_{j=1,j\neq\sigma}^{N}\frac{|a_i||\mu_j|}{|\mu_\sigma|}\frac{\eta_i^{\hat{\beta}_\sigma-\hat{\beta}_j}}{\Gamma(\hat{\beta}_\sigma-\hat{\beta}_j+1)} \\
& + \frac{1}{|\lambda_\sigma|}\frac{1}{\Gamma(\hat{\varrho}_\sigma+1)}(C_1+D_1) + \sum_{l=1,l\neq\sigma}^{N}\frac{|\lambda_l|}{|\lambda_\sigma|}\frac{1}{\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l+1)} \\
& + |A_1|\Bigg(\sum_{i=1}^{k}\frac{|b_i|}{|\lambda_\sigma|}\frac{\eta_i^{\hat{\varrho}_\sigma}}{\Gamma(\hat{\varrho}_\sigma+1)}(C_1+D_1) + \frac{1}{|\mu_\sigma|}\frac{1}{\Gamma(\hat{\beta}_\sigma+1)}(C_2+D_2) \\
& + \sum_{i=1}^{k}\sum_{l=1,l\neq\sigma}^{N}\frac{|\lambda_l||b_i|}{|\lambda_\sigma|}\frac{\eta_i^{\hat{\varrho}_\sigma-\hat{\varrho}_l}}{\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l+1)} + \sum_{j=1,j\neq\sigma}^{N}\frac{|\mu_j|}{|\mu_\sigma|}\frac{1}{\Gamma(\hat{\beta}_\sigma-\hat{\beta}_j+1)}\Bigg)\Bigg] \\
& + \frac{1}{|\lambda_\sigma|}\frac{1}{\Gamma(\hat{\varrho}_\sigma+1)}(C_1+D_1) + \sum_{l=1,l\neq\sigma}^{N}\frac{|\lambda_l|}{|\lambda_\sigma|}\frac{1}{\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l+1)}\Bigg) > \frac{1}{|\gamma|}\Bigg[\sum_{i=1}^{k}\frac{|a_i|}{|\mu_\sigma|}\frac{\eta_i^{\hat{\beta}_\sigma}}{\Gamma(\hat{\beta}_\sigma+1)}M_2 \\
& + \frac{1}{|\lambda_\sigma|}\frac{1}{\Gamma(\hat{\varrho}_\sigma+1)}M_1 + |A_1|\Bigg(\sum_{i=1}^{k}\frac{|b_i|}{|\lambda_\sigma|}\frac{\eta_i^{\hat{\varrho}_\sigma}}{\Gamma(\hat{\varrho}_\sigma+1)}M_1 + \frac{1}{|\mu_\sigma|}\frac{1}{\Gamma(\hat{\beta}_\sigma+1)}M_2\Bigg) + \frac{|\gamma|}{|\lambda_\sigma|}\frac{1}{\Gamma(\hat{\varrho}_\sigma+1)}M_1\Bigg],
\end{aligned} \tag{4.2}$$

$$\frac{\rho}{2}\Bigg(1-\frac{1}{|\gamma|}\Bigg[\sum_{i=1}^{k}\frac{|b_i|}{|\lambda_\sigma|}\frac{\eta_i^{\hat{\varrho}_\sigma}}{\Gamma(\hat{\varrho}_\sigma+1)}(C_1+D_1)+\sum_{i=1}^{k}\sum_{l=1,l\neq\sigma}^{N}\frac{|\lambda_l||b_i|}{|\lambda_\sigma|}\frac{\eta_i^{\hat{\varrho}_\sigma-\hat{\varrho}_l}}{\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l+1)}$$
$$+\frac{1}{|\mu_\sigma|}\frac{1}{\Gamma(\hat{\beta}_\sigma+1)}(C_2+D_2)+\sum_{j=1,j\neq\sigma}^{N}\frac{|\mu_j|}{|\mu_\sigma|}\frac{1}{\Gamma(\hat{\beta}_\sigma-\hat{\beta}_j+1)}$$
$$+|A_2|\Bigg(\sum_{i=1}^{k}\frac{|a_i|}{|\mu_\sigma|}\frac{\eta_i^{\hat{\beta}_\sigma}}{\Gamma(\hat{\beta}_\sigma+1)}(C_2+D_2)+\frac{1}{|\lambda_\sigma|}\frac{1}{\Gamma(\hat{\varrho}_\sigma+1)}(C_1+D_1)$$
$$+\sum_{i=1}^{k}\sum_{j=1,j\neq\sigma}^{N}|a_i|\frac{|\mu_j|}{|\mu_\sigma|}\frac{\eta_i^{\hat{\beta}_\sigma-\hat{\beta}_j}}{\Gamma(\hat{\beta}_\sigma-\hat{\beta}_j+1)}+\sum_{l=1,l\neq\sigma}^{N}\frac{|\lambda_l|}{|\lambda_\sigma|}\frac{1}{\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l+1)}\Bigg)\Bigg]$$
$$+\frac{1}{|\mu_\sigma|}\frac{1}{\Gamma(\hat{\beta}_\sigma+1)}(C_2+D_2)+\sum_{j=1,j\neq\sigma}^{N}\frac{|\mu_j|}{|\mu_\sigma|}\frac{1}{\Gamma(\hat{\beta}_\sigma-\hat{\beta}_j+1)}\Bigg)>\frac{1}{|\gamma|}\Bigg[\sum_{i=1}^{k}\frac{|b_i|}{|\lambda_\sigma|}\frac{\eta_i^{\hat{\varrho}_\sigma}}{\Gamma(\hat{\varrho}_\sigma+1)}M_1$$
$$+\frac{1}{\mu_\sigma}\frac{1}{\Gamma(\hat{\beta}_\sigma+1)}M_2+|A_2|\Bigg(\sum_{i=1}^{k}\frac{|a_i|}{|\mu_\sigma|}\frac{\eta_i^{\hat{\beta}_\sigma}}{\Gamma(\hat{\beta}_\sigma+1)}M_2+\frac{1}{|\lambda_\sigma|}\frac{1}{\Gamma(\hat{\varrho}_\sigma+1)}M_1\Bigg)+\frac{1}{|\mu_\sigma|}\frac{|\gamma|}{\Gamma(\hat{\beta}_\sigma+1)}M_2\Bigg].\tag{4.3}$$

Theorem 4.1. *The system* (1.1) *admits a unique solution if the assumptions* (B_1)*and* (B_2) *, together with equations(4.1) hold ,and* $K<1$*, where* $K=max\{K_1,K_2\}$

Proof. Consider $\varphi(\kappa),\varphi^*(\kappa),\chi(\kappa),\chi^*(\kappa)\in X$

$$\begin{aligned}
&|\boldsymbol{\mathcal{U}}_1(\varphi(\kappa),\chi(\kappa))-\boldsymbol{\mathcal{U}}_1(\varphi^*(\kappa),\chi^*(\kappa))|\\
&\leq\frac{1}{|\gamma|}\Bigg[\sum_{i=1}^{k}\frac{|a_i|}{|\mu_\sigma|}\int_0^{\eta_i}\frac{(\eta_i-z)^{\hat{\beta}_\sigma-1}}{\Gamma(\hat{\beta}_\sigma)}|h_2(z,\varphi(z),\chi(z))-h_2(z,\varphi^*(z),\chi^*(z))\,dz\\
&+\sum_{i=1}^{k}\sum_{j=1,j\neq\sigma}^{N}|a_i|\frac{|\mu_j|}{|\mu_\sigma|}\int_0^{\eta_i}\frac{(\eta_i-z)^{\hat{\beta}_\sigma-\hat{\beta}_j-1}}{\Gamma(\hat{\beta}_\sigma-\hat{\beta}_j)}|\chi(z)-\chi^*(z)|\,dz\\
&+\frac{1}{|\lambda_\sigma|}\int_0^{1}\frac{(1-z)^{\hat{\varrho}_\sigma-1}}{\Gamma(\hat{\varrho}_\sigma)}|h_1(z,\varphi(z),\chi(z))-h_1(z,\varphi^*(z),\chi^*(z))\,dz\\
&+\sum_{l=1,l\neq\sigma}^{N}\frac{|\lambda_l|}{|\lambda_\sigma|}\int_0^{1}\frac{(1-z)^{\hat{\varrho}_\sigma-\hat{\varrho}_l-1}}{\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l)}|\varphi(z)-\varphi^*(z)|\,dz\\
&+|A_1|\Bigg(\sum_{i=1}^{k}\frac{|b_i|}{|\lambda_\sigma|}\int_0^{\eta_i}\frac{(\eta_i-z)^{\hat{\varrho}_\sigma-1}}{\Gamma(\hat{\varrho}_\sigma)}|h_1(z,\varphi(z),\chi(z))-h_1(z,\varphi^*(z),\chi^*(z))|\,dz\\
&+\sum_{i=1}^{k}\sum_{l=1,l\neq\sigma}^{N}\frac{|\lambda_l||b_i|}{|\lambda_\sigma|}\int_0^{\eta_i}\frac{(\eta_i-z)^{\hat{\varrho}_\sigma-\hat{\varrho}_l-1}}{\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l)}|\varphi(z)-\varphi^*(z)|\,dz\\
&+\frac{1}{|\mu_\sigma|}\int_0^{1}\frac{(1-z)^{\hat{\beta}_\sigma-1}}{\Gamma(\hat{\beta}_\sigma)}|h_2(z,\varphi(z),\chi(z))-h_2(z,\varphi^*(z),\chi^*(z)|\,dz
\end{aligned}$$

$$+ \sum_{j=1, j\neq\sigma}^{N} \frac{|\mu_j|}{|\mu_\sigma|} \int_0^1 \frac{(1-z)^{\hat{\beta}_\sigma-\hat{\beta}_j-1}}{\Gamma(\hat{\beta}_\sigma-\hat{\beta}_j)} |\chi(z)-\chi^*(z)|\, dz \Bigg)\Bigg] \tag{4.4}$$
$$+ \frac{1}{|\lambda_\sigma|} \int_0^\kappa \frac{(\kappa-z)^{\hat{\varrho}_\sigma-1}}{\Gamma(\hat{\varrho}_\sigma)} |h_1(z,\varphi(z),\chi(z)) - h_1(z,\varphi^*(z),\chi^*(z))| dz$$
$$+ \sum_{l=1, l\neq\sigma}^{N} \frac{|\lambda_l|}{|\lambda_\sigma|} \int_0^\kappa \frac{(\kappa-z)^{\hat{\varrho}_\sigma-\hat{\varrho}_l-1}}{\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l)} |\varphi(z)-\varphi^*(z)| dz,$$

Now, use (B_1), (B_2) in (4.4)

$$|\boldsymbol{\mathcal{U}}_1(\varphi(\kappa),\chi(\kappa)) - \boldsymbol{\mathcal{U}}_1(\varphi^*(\kappa),\chi^*(\kappa))|$$
$$\leq \frac{1}{|\gamma|}\Bigg[\sum_{i=1}^{k} \frac{|a_i|}{|\mu_\sigma|} \int_0^{\eta_i} \frac{(\eta_i-z)^{\hat{\beta}_\sigma-1}}{\Gamma(\hat{\beta}_\sigma)} L_2(|\varphi-\varphi^*| + |\chi-\chi^*|)\, dz$$
$$+ \sum_{i=1}^{k} \sum_{j=1, j\neq\sigma}^{N} |a_i| \frac{|\mu_j|}{|\mu_\sigma|} \int_0^{\eta_i} \frac{(\eta_i-z)^{\hat{\beta}_\sigma-\hat{\beta}_j-1}}{\Gamma(\hat{\beta}_\sigma-\hat{\beta}_j)} |\chi-\chi^*|\, dz$$
$$+ \frac{1}{|\lambda_\sigma|} \int_0^1 \frac{(1-z)^{\hat{\varrho}_\sigma-1}}{\Gamma(\hat{\varrho}_\sigma)} L_1(|\varphi-\varphi^*| + |\chi-\chi^*|)\, dz$$
$$+ \sum_{l=1, l\neq\sigma}^{N} \frac{|\lambda_l|}{|\lambda_\sigma|} \int_0^1 \frac{(1-z)^{\hat{\varrho}_\sigma-\hat{\varrho}_l-1}}{\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l)} |\varphi-\varphi^*|\, dz$$
$$+ |A_1| \Bigg(\sum_{i=1}^{k} \frac{|b_i|}{|\lambda_\sigma|} \int_0^{\eta_i} \frac{(\eta_i-z)^{\hat{\varrho}_\sigma-1}}{\Gamma(\hat{\varrho}_\sigma)} L_1(|\varphi-\varphi^*| + |\chi-\chi^*|)\, dz$$
$$+ \sum_{i=1}^{k} \sum_{l=1, l\neq\sigma}^{N} \frac{|\lambda_l||b_i|}{|\lambda_\sigma|} \int_0^{\eta_i} \frac{(\eta_i-z)^{\hat{\varrho}_\sigma-\hat{\varrho}_l-1}}{\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l)} |\varphi-\varphi^*|\, dz$$
$$+ \frac{1}{|\mu_\sigma|} \int_0^1 \frac{(1-z)^{\hat{\beta}_\sigma-1}}{\Gamma(\hat{\beta}_\sigma)} L_2(|\varphi-\varphi^*| + |\chi-\chi^*|)\, dz$$
$$+ \sum_{j=1, j\neq\sigma}^{N} \frac{|\mu_j|}{|\mu_\sigma|} \int_0^1 \frac{(1-z)^{\hat{\beta}_\sigma-\hat{\beta}_j-1}}{\Gamma(\hat{\beta}_\sigma-\hat{\beta}_j)} |\chi-\chi^*|\, dz \Bigg)\Bigg]$$
$$+ \frac{1}{|\lambda_\sigma|} \int_0^\kappa \frac{(\kappa-z)^{\hat{\varrho}_\sigma-1}}{\Gamma(\hat{\varrho}_\sigma)} L_1(|\varphi-\varphi^*| + |\chi-\chi^*|) dz$$
$$+ \sum_{l=1, l\neq\sigma}^{N} \frac{|\lambda_l|}{|\lambda_\sigma|} \int_0^\kappa \frac{(\kappa-z)^{\hat{\varrho}_\sigma-\hat{\varrho}_l-1}}{\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l)} |\varphi-\varphi^*| dz$$
$$= \frac{\kappa^{N-1}}{|\gamma|}\Bigg[\sum_{i=1}^{k} \frac{|a_i|}{|\mu_\sigma|} \frac{\eta_i^{\hat{\beta}_\sigma}}{\Gamma(\hat{\beta}_\sigma+1)} L_2(|\varphi-\varphi^*| + |\chi-\chi^*|)$$
$$+ \sum_{i=1}^{k} \sum_{j=1, j\neq\sigma}^{N} |a_i| \frac{|\mu_j|}{|\mu_\sigma|} \frac{\eta_i^{\hat{\beta}_\sigma-\hat{\beta}_j}}{\Gamma(\hat{\beta}_\sigma-\hat{\beta}_j+1)} |\chi-\chi^*|$$

$$
\begin{aligned}
&+\frac{1}{|\lambda_\sigma|}\frac{1}{\Gamma(\hat{\varrho}_\sigma+1)}L_1(|\varphi-\varphi^*|+|\chi-\chi^*|)\\
&+\sum_{l=1,l\neq\sigma}^{N}\frac{|\lambda_l|}{|\lambda_\sigma|}\frac{1}{\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l+1)}|\varphi-\varphi^*|\\
&+|A_1|\Bigg(\sum_{i=1}^{k}\frac{|b_i|}{|\lambda_\sigma|}\frac{\eta_i^{\hat{\varrho}_\sigma}}{\Gamma(\hat{\varrho}_\sigma+1)}L_1(|\varphi-\varphi^*|+|\chi-\chi^*|)\\
&+\sum_{i=1}^{k}\sum_{l=1,l\neq\sigma}^{N}\frac{|\lambda_l||b_i|}{|\lambda_\sigma|}\frac{\eta_i^{\hat{\varrho}_\sigma-\hat{\varrho}_l}}{\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l+1)}|\varphi-\varphi^*|\\
&+\frac{1}{|\mu_\sigma|}\frac{1}{\Gamma(\hat{\beta}_\sigma+1)}L_2(|\varphi-\varphi^*|+|\chi-\chi^*|)\\
&+\sum_{j=1,j\neq\sigma}^{N}\frac{|\mu_j|}{|\mu_\sigma|}\frac{1}{\Gamma(\hat{\beta}_\sigma-\hat{\beta}_j+1)}|\chi-\chi^*|\Bigg)\Bigg]\\
&+\frac{\kappa^{\hat{\varrho}_\sigma}}{|\lambda_\sigma|\Gamma(\hat{\varrho}_\sigma+1)}L_1(|\varphi-\varphi^*|+|\chi-\chi^*|)\\
&+\sum_{l=1,l\neq\sigma}^{N}\frac{|\lambda_l|}{|\lambda_\sigma|}\frac{\kappa^{\hat{\varrho}_\sigma-\hat{\varrho}_l}}{\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l+1)}|\varphi-\varphi^*|.
\end{aligned}
$$

$$
\begin{aligned}
&\|\boldsymbol{\mathcal{U}}_1(\varphi(\kappa),\chi(\kappa))-\boldsymbol{\mathcal{U}}_1(\varphi^*(\kappa),\chi^*(\kappa))\|\\
&\le\Bigg(\frac{1}{|\gamma|}\Bigg[\sum_{i=1}^{k}\frac{|a_i|}{|\mu_\sigma|}\frac{\eta_i^{\hat{\beta}_\sigma}L_2}{\Gamma(\hat{\beta}_\sigma+1)}+\sum_{i=1}^{k}\sum_{j=1,j\neq\sigma}^{N}\frac{|a_i||\mu_j|}{|\mu_\sigma|}\frac{\eta_i^{\hat{\beta}_\sigma-\hat{\beta}_j}}{\Gamma(\hat{\beta}_\sigma-\hat{\beta}_j+1)}\\
&+\frac{L_1}{|\lambda_\sigma|\Gamma(\hat{\varrho}_\sigma+1)}+\sum_{l=1,l\neq\sigma}^{N}\frac{|\lambda_l|}{|\lambda_\sigma|\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l+1)}\\
&+|A_1|\Bigg(\sum_{i=1}^{k}\frac{L_1|b_i|\eta_i^{\hat{\varrho}_\sigma}}{|\lambda_\sigma|\Gamma(\hat{\varrho}_\sigma+1)}+\sum_{i=1}^{k}\sum_{l=1,l\neq\sigma}^{N}\frac{|\lambda_l||b_i|\eta_i^{\hat{\varrho}_\sigma-\hat{\varrho}_l}}{|\lambda_\sigma|\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l+1)}\\
&+\frac{L_2}{|\mu_\sigma|\Gamma(\hat{\beta}_\sigma+1)}+\sum_{j=1,j\neq\sigma}^{N}\frac{|\mu_j|}{|\mu_\sigma|\Gamma(\hat{\beta}_\sigma-\hat{\beta}_j+1)}\Bigg)\Bigg]\\
&+\frac{L_1}{|\lambda_\sigma|\Gamma(\hat{\varrho}_\sigma+1)}+\sum_{l=1,l\neq\sigma}^{N}\frac{|\lambda_l|}{|\lambda_\sigma|\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l+1)}\Bigg)\|(\varphi,\chi)-(\varphi^*,\chi^*)\|.
\end{aligned}
$$

using (4.1)

$$\|\boldsymbol{\mathcal{U}}_1(\varphi(\kappa),\chi(\kappa))-\boldsymbol{\mathcal{U}}_1(\varphi^*(\kappa),\chi^*(\kappa))\|\le K_1\|(\varphi,\chi)-(\varphi^*,\chi^*)\|. \tag{4.5}$$

Applying the same approach as previously, we get:

$$\|\boldsymbol{\mathcal{U}}_2(\varphi(\kappa),\chi(\kappa))-\boldsymbol{\mathcal{U}}_2(\varphi^*(\kappa),\chi^*(\kappa))\|\le K_2\|(\varphi,\chi)-(\varphi^*,\chi^*)\|. \tag{4.6}$$

Consequently , by utilizing the relation (4.5) and (4.6), we obtain

$$\begin{aligned}
\|\mathcal{U}(\varphi(\kappa),\chi(\kappa))-\mathcal{U}(\varphi^*(\kappa),\chi^*(\kappa))\| &= \|\mathcal{U}_1(\varphi(\kappa),\chi(\kappa))-\mathcal{U}_1(\varphi^*(\kappa),\chi^*(\kappa))\| \\
&\quad + \|\mathcal{U}_2(\varphi(\kappa),\chi(\kappa))-\mathcal{U}_2(\varphi^*(\kappa),\chi^*(\kappa))\| \\
&\leq K_1\|(\varphi,\chi)-(\varphi^*,\chi^*)\| + K_2\|(\varphi,\chi)-(\varphi^*,\chi^*)\| \\
&\leq K\|(\varphi,\chi)-(\varphi^*,\chi^*)\|,
\end{aligned} \tag{4.7}$$

Therefore , the system (1.1) admits a unique solution,since the operator $\mathcal{U}$ possesses a unique fixed point . □

Theorem 4.2. *The system (1.1) admits at least one solution if the assumptions* $(B_1)-(B_3)$ *hold.*

Proof. Let us divide the operator $\mathcal{U}$ into two operators S and P defined as:

$$\begin{aligned}
S(\varphi(\kappa),\chi(\kappa)) &= \Big(S_1(\varphi(\kappa),\chi(\kappa)),\, S_2(\varphi(\kappa),\chi(\kappa))\Big). \\
P(\varphi(\kappa),\chi(\kappa)) &= \Big(P_1(\varphi(\kappa),\chi(\kappa)),\, P_2(\varphi(\kappa),\chi(\kappa))\Big).
\end{aligned}$$

$$\begin{aligned}
S_1(\varphi(\kappa),\chi(\kappa)) = \frac{\kappa^{N-1}}{\gamma}\Bigg[&\sum_{i=1}^{k}\frac{a_i}{\mu_\sigma}\int_0^{\eta_i}\frac{(\eta_i-z)^{\hat{\beta}_\sigma-1}}{\Gamma(\hat{\beta}_\sigma)}h_2(z,\varphi(z),\chi(z))\,dz \\
&-\sum_{i=1}^{k}\sum_{j=1,j\neq\sigma}^{N}a_i\frac{\mu_j}{\mu_\sigma}\int_0^{\eta_i}\frac{(\eta_i-z)^{\hat{\beta}_\sigma-\hat{\beta}_j-1}}{\Gamma(\hat{\beta}_\sigma-\hat{\beta}_j)}\chi(z)\,dz \\
&-\frac{1}{\lambda_\sigma}\int_0^{1}\frac{(1-z)^{\hat{\varrho}_\sigma-1}}{\Gamma(\hat{\varrho}_\sigma)}h_1(z,\varphi(z),\chi(z))\,dz \\
&+\sum_{l=1,l\neq\sigma}^{N}\frac{\lambda_l}{\lambda_\sigma}\int_0^{1}\frac{(1-z)^{\hat{\varrho}_\sigma-\hat{\varrho}_l-1}}{\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l)}\varphi(z)\,dz \\
&+A_1\Bigg(\sum_{i=1}^{k}\frac{b_i}{\lambda_\sigma}\int_0^{\eta_i}\frac{(\eta_i-z)^{\hat{\varrho}_\sigma-1}}{\Gamma(\hat{\varrho}_\sigma)}h_1(z,\varphi(z),\chi(z))\,dz \\
&-\sum_{i=1}^{k}\sum_{l=1,l\neq\sigma}^{N}\frac{\lambda_l b_i}{\lambda_\sigma}\int_0^{\eta_i}\frac{(\eta_i-z)^{\hat{\varrho}_\sigma-\hat{\varrho}_l-1}}{\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l)}\varphi(z)\,dz \\
&-\frac{1}{\mu_\sigma}\int_0^{1}\frac{(1-z)^{\hat{\beta}_\sigma-1}}{\Gamma(\hat{\beta}_\sigma)}h_2(z,\varphi(z),\chi(z))\,dz \\
&+\sum_{j=1,j\neq\sigma}^{N}\frac{\mu_j}{\mu_\sigma}\int_0^{1}\frac{(1-z)^{\hat{\beta}_\sigma-\hat{\beta}_j-1}}{\Gamma(\hat{\beta}_\sigma-\hat{\beta}_j)}\chi(z)\,dz\Bigg)\Bigg],
\end{aligned}$$

$$\begin{aligned}
S_2(\varphi(\kappa),\chi(\kappa)) = \frac{\kappa^{N-1}}{\gamma}\Bigg[&\sum_{i=1}^{k}\frac{b_i}{\lambda_\sigma}\int_0^{\eta_i}\frac{(\eta_i-z)^{\hat{\varrho}_\sigma-1}}{\Gamma(\hat{\varrho}_\sigma)}h_1(z,\varphi(z),\chi(z))\,dz \\
&-\sum_{i=1}^{k}\sum_{l=1,l\neq\sigma}^{N}\frac{\lambda_l b_i}{\lambda_\sigma}\int_0^{\eta_i}\frac{(\eta_i-z)^{\hat{\varrho}_\sigma-\hat{\varrho}_l-1}}{\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l)}\varphi(z)\,dz
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{\mu_\sigma} \int_0^1 \frac{(1-z)^{\hat{\beta}_\sigma - 1}}{\Gamma(\hat{\beta}_\sigma)} h_2(z, \varphi(z), \chi(z))\, dz \\
&+ \sum_{j=1, j\neq\sigma}^{N} \frac{\mu_j}{\mu_\sigma} \int_0^1 \frac{(1-z)^{\hat{\beta}_\sigma - \hat{\beta}_j - 1}}{\Gamma(\hat{\beta}_\sigma - \hat{\beta}_j)} \chi(z)\, dz \\
&+ A_2\Bigg(\sum_{i=1}^{k} \frac{a_i}{\mu_\sigma} \int_0^{\eta_i} \frac{(\eta_i - z)^{\hat{\beta}_\sigma - 1}}{\Gamma(\hat{\beta}_\sigma)} h_2(z, \varphi(z), \chi(z))\, dz \\
&- \sum_{i=1}^{k} \sum_{j=1, j\neq\sigma}^{N} a_i \frac{\mu_j}{\mu_\sigma} \int_0^{\eta_i} \frac{(\eta_i - z)^{\hat{\beta}_\sigma - \hat{\beta}_j - 1}}{\Gamma(\hat{\beta}_\sigma - \hat{\beta}_j)} \chi(z)\, dz \\
&- \frac{1}{\lambda_\sigma} \int_0^1 \frac{(1-z)^{\hat{\varrho}_\sigma - 1}}{\Gamma(\hat{\varrho}_\sigma)} h_1(z, \varphi(z), \chi(z))\, dz \\
&+ \sum_{l=1, l\neq\sigma}^{N} \frac{\lambda_l}{\lambda_\sigma} \int_0^1 \frac{(1-z)^{\hat{\varrho}_\sigma - \hat{\varrho}_l - 1}}{\Gamma(\hat{\varrho}_\sigma - \hat{\varrho}_l)} \varphi(z)\, dz \Bigg) \Bigg].
\end{aligned}
$$

$$
\begin{aligned}
P_1(\varphi(\kappa), \chi(\kappa)) &= \frac{1}{\lambda_\sigma} \int_0^{\kappa} \frac{(\kappa - z)^{\hat{\varrho}_\sigma - 1}}{\Gamma(\hat{\varrho}_\sigma)} h_1(z, \varphi(z), \chi(z)) dz \\
&\quad - \sum_{l=1, l\neq\sigma}^{N} \frac{\lambda_l}{\lambda_\sigma} \int_0^{\kappa} \frac{(\kappa - z)^{\hat{\varrho}_\sigma - \hat{\varrho}_l - 1}}{\Gamma(\hat{\varrho}_\sigma - \hat{\varrho}_l)} \varphi(z) dz, \\
P_2(\varphi(\kappa), \chi(\kappa)) &= \frac{1}{\mu_\sigma} \int_0^{\kappa} \frac{(\kappa - z)^{\hat{\beta}_\sigma - 1}}{\Gamma(\hat{\beta}_\sigma)} h_2(z, \varphi(z), \chi(z)) dz \\
&\quad - \sum_{j=1, j\neq\sigma}^{N} \frac{\mu_j}{\mu_\sigma} \int_0^{\kappa} \frac{(\kappa - z)^{\hat{\beta}_\sigma - \hat{\beta}_j - 1}}{\Gamma(\hat{\beta}_\sigma - \hat{\beta}_j)} \chi(z) dz.
\end{aligned}
$$

Furthermore, we Define a ball $\Omega_\rho = \{(\varphi, \chi) \in X \times X : \|(\varphi, \chi)\| \leq \rho, \|\varphi\| \leq \frac{\rho}{2}, \|\chi\| \leq \frac{\rho}{2}\}$, with positive radius $\rho > max\{\rho_1, \rho_2\}$, where ρ_1 ,ρ_2 are defined by equations (4.2) and (4.3). For simplicity, the proof is carried out through several steps .

Step 1 We claim that

$$S(\varphi(\kappa), \chi(\kappa)) + P(\varphi^*(\kappa), \chi^*(\kappa)) \in \Omega_\rho \subset X \times X$$

for every

$$(\varphi(\kappa), \chi(\kappa), \varphi^*(\kappa), \chi^*(\kappa)) \in \Omega_\rho.$$

For this

$$
\begin{aligned}
|S_1(\varphi(\kappa), \chi(\kappa)) + P_1(\varphi^*(\kappa), \chi^*(\kappa))| &\leq \frac{1}{|\gamma|} \Bigg[\sum_{i=1}^{k} \frac{|a_i|}{|\mu_\sigma|} \int_0^{\eta_i} \frac{(\eta_i - z)^{\hat{\beta}_\sigma - 1}}{\Gamma(\hat{\beta}_\sigma)} |h_2(z, \varphi(z), \chi(z))|\, dz \\
&\quad + \sum_{i=1}^{k} \sum_{j=1, j\neq\sigma}^{N} |a_i| \frac{|\mu_j|}{|\mu_\sigma|} \int_0^{\eta_i} \frac{(\eta_i - z)^{\hat{\beta}_\sigma - \hat{\beta}_j - 1}}{\Gamma(\hat{\beta}_\sigma - \hat{\beta}_j)} |\chi(z)|\, dz \\
&\quad + \frac{1}{|\lambda_\sigma|} \int_0^1 \frac{(1-z)^{\hat{\varrho}_\sigma - 1}}{\Gamma(\hat{\varrho}_\sigma)} |h_1(z, \varphi(z), \chi(z))|\, dz
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{l=1, l\neq\sigma}^{N} \frac{|\lambda_l|}{|\lambda_\sigma|} \int_0^1 \frac{(1-z)^{\hat{\varrho}_\sigma-\hat{\varrho}_l-1}}{\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l)} |\varphi(z)|\, dz \\
&+ |A_1| \Bigg(\sum_{i=1}^{k} \frac{|b_i|}{|\lambda_\sigma|} \int_0^{\eta_i} \frac{(\eta_i-z)^{\hat{\varrho}_\sigma-1}}{\Gamma(\hat{\varrho}_\sigma)} |h_1(z,\varphi(z),\chi(z))|\, dz \\
&+ \sum_{i=1}^{k} \sum_{l=1, l\neq\sigma}^{N} \frac{|\lambda_l||b_i|}{|\lambda_\sigma|} \int_0^{\eta_i} \frac{(\eta_i-z)^{\hat{\varrho}_\sigma-\hat{\varrho}_l-1}}{\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l)} |\varphi(z)|\, dz \\
&+ \frac{1}{|\mu_\sigma|} \int_0^1 \frac{(1-z)^{\hat{\beta}_\sigma-1}}{\Gamma(\hat{\beta}_\sigma)} |h_2(z,\varphi(z),\chi(z))|\, dz \\
&+ \sum_{j=1, j\neq\sigma}^{N} \frac{|\mu_j|}{|\mu_\sigma|} \int_0^1 \frac{(1-z)^{\hat{\beta}_\sigma-\hat{\beta}_j-1}}{\Gamma(\hat{\beta}_\sigma-\hat{\beta}_j)} |\chi(z)|\, dz \Bigg) \Bigg] \\
&+ \frac{1}{|\lambda_\sigma|} \int_0^{\kappa} \frac{(\kappa-z)^{\hat{\varrho}_\sigma-1}}{\Gamma(\hat{\varrho}_\sigma)} |h_1(z,\varphi(z),\chi(z))| dz \\
&+ \sum_{l=1, l\neq\sigma}^{N} \frac{|\lambda_l|}{|\lambda_\sigma|} \int_0^{\kappa} \frac{(\kappa-z)^{\hat{\varrho}_\sigma-\hat{\varrho}_l-1}}{\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l)} |\varphi(z)| dz,
\end{aligned}
\tag{4.8}
$$

Apply B_3 in (4.8)

$$
\begin{aligned}
\|S_1(\varphi(\kappa),\chi(\kappa)) + P_1(\varphi^*(\kappa),\chi^*(\kappa))\| \leq{}& \frac{1}{|\gamma|} \Bigg[\sum_{i=1}^{k} \frac{|a_i|}{|\mu_\sigma|} \frac{\eta_i^{\hat{\beta}_\sigma}}{\Gamma(\hat{\beta}_\sigma+1)} \Big(C_2\frac{\rho}{2} + D_2\frac{\rho}{2} + M_2\Big) \\
&+ \sum_{i=1}^{k} \sum_{j=1, j\neq\sigma}^{N} |a_i| \frac{|\mu_j|}{|\mu_\sigma|} \frac{\eta_i^{\hat{\beta}_\sigma-\hat{\beta}_j}}{\Gamma(\hat{\beta}_\sigma-\hat{\beta}_j+1)} \frac{\rho}{2} \\
&+ \frac{1}{|\lambda_\sigma|} \frac{1}{\Gamma(\hat{\varrho}_\sigma+1)} \Big(C_1\frac{\rho}{2} + D_1\frac{\rho}{2} + M_1\Big) \\
&+ \sum_{l=1, l\neq\sigma}^{N} \frac{|\lambda_l|}{|\lambda_\sigma|} \frac{1}{\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l+1)} \frac{\rho}{2} \\
&+ |A_1| \Bigg(\sum_{i=1}^{k} \frac{|b_i|}{|\lambda_\sigma|} \frac{\eta_i^{\hat{\varrho}_\sigma}}{\Gamma(\hat{\varrho}_\sigma+1)} \Big(C_1\frac{\rho}{2} + D_1\frac{\rho}{2} + M_1\Big) \\
&+ \sum_{i=1}^{k} \sum_{l=1, l\neq\sigma}^{N} \frac{|\lambda_l||b_i|}{|\lambda_\sigma|} \frac{\eta_i^{\hat{\varrho}_\sigma-\hat{\varrho}_l}}{\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l+1)} \frac{\rho}{2} \\
&+ \frac{1}{|\mu_\sigma|} \frac{1}{\Gamma(\hat{\beta}_\sigma+1)} \Big(C_2\frac{\rho}{2} + D_2\frac{\rho}{2} + M_2\Big) \\
&+ \sum_{j=1, j\neq\sigma}^{N} \frac{|\mu_j|}{|\mu_\sigma|} \frac{1}{\Gamma(\hat{\beta}_\sigma-\hat{\beta}_j+1)} \frac{\rho}{2} \Bigg) \Bigg] \\
&+ \frac{1}{|\lambda_\sigma|} \frac{1}{\Gamma(\hat{\varrho}_\sigma+1)} \Big(C_1\frac{\rho}{2} + D_1\frac{\rho}{2} + M_1\Big)
\end{aligned}
$$

$$+\sum_{l=1,l\neq\sigma}^{N}\frac{|\lambda_l|}{|\lambda_\sigma|}\frac{1}{\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l+1)}\frac{\rho}{2}.$$

by use (4.2)

$$\|S_1(\varphi(\kappa),\chi(\kappa))+P_1(\varphi^*(\kappa),\chi^*(\kappa))\|\leq\frac{\rho}{2},\tag{4.9}$$

Similarly

$$\|S_2(\varphi(\kappa),\chi(\kappa))+P_2(\varphi^*(\kappa),\chi^*(\kappa))\|\leq\frac{\rho}{2},\tag{4.10}$$

Hence, from (4.9) and (4.10), we have

$$S(\varphi(\kappa),\chi(\kappa))+P(\varphi^*(\kappa),\chi^*(\kappa))\in\Omega_\rho.$$

Step 2 To assert the boundedness of S. Take any $\varphi,\chi\in\Omega_\rho$ yielding

$$\begin{aligned}|S_1(\varphi(\kappa),\chi(\kappa))|\leq\frac{1}{|\gamma|}\Bigg[&\sum_{i=1}^{k}\frac{|a_i|}{|\mu_\sigma|}\int_0^{\eta_i}\frac{(\eta_i-z)^{\hat{\beta}_\sigma-1}}{\Gamma(\hat{\beta}_\sigma)}|h_2(z,\varphi(z),\chi(z))|\,dz\\
&+\sum_{i=1}^{k}\sum_{j=1,j\neq\sigma}^{N}|a_i|\frac{|\mu_j|}{|\mu_\sigma|}\int_0^{\eta_i}\frac{(\eta_i-z)^{\hat{\beta}_\sigma-\hat{\beta}_j-1}}{\Gamma(\hat{\beta}_\sigma-\hat{\beta}_j)}|\chi(z)|\,dz\\
&+\frac{1}{|\lambda_\sigma|}\int_0^1\frac{(1-z)^{\hat{\varrho}_\sigma-1}}{\Gamma(\hat{\varrho}_\sigma)}|h_1(z,\varphi(z),\chi(z))|\,dz\\
&+\sum_{l=1,l\neq\sigma}^{N}\frac{|\lambda_l|}{|\lambda_\sigma|}\int_0^1\frac{(1-z)^{\hat{\varrho}_\sigma-\hat{\varrho}_l-1}}{\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l)}|\varphi(z)|\,dz\\
&+|A_1|\Bigg(\sum_{i=1}^{k}\frac{|b_i|}{|\lambda_\sigma|}\int_0^{\eta_i}\frac{(\eta_i-z)^{\hat{\varrho}_\sigma-1}}{\Gamma(\hat{\varrho}_\sigma)}|h_1(z,\varphi(z),\chi(z))|\,dz\\
&+\sum_{i=1}^{k}\sum_{l=1,l\neq\sigma}^{N}\frac{|\lambda_l||b_i|}{|\lambda_\sigma|}\int_0^{\eta_i}\frac{(\eta_i-z)^{\hat{\varrho}_\sigma-\hat{\varrho}_l-1}}{\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l)}|\varphi(z)|\,dz\\
&+\frac{1}{|\mu_\sigma|}\int_0^1\frac{(1-z)^{\hat{\beta}_\sigma-1}}{\Gamma(\hat{\beta}_\sigma)}|h_2(z,\varphi(z),\chi(z))|\,dz\\
&+\sum_{j=1,j\neq\sigma}^{N}\frac{|\mu_j|}{|\mu_\sigma|}\int_0^1\frac{(1-z)^{\hat{\beta}_\sigma-\hat{\beta}_j-1}}{\Gamma(\hat{\beta}_\sigma-\hat{\beta}_j)}|\chi(z)|\,dz\Bigg)\Bigg].\end{aligned}\tag{4.11}$$

use (B_3) in(4.11)

$$\begin{aligned}\|S_1(\varphi(\kappa),\chi(\kappa))\|\leq\frac{1}{|\gamma|}\Bigg[&\sum_{i=1}^{k}\frac{|a_i|}{|\mu_\sigma|}\frac{\eta_i^{\hat{\beta}_\sigma}}{\Gamma(\hat{\beta}_\sigma+1)}(C_2\frac{\rho}{2}+D_2\frac{\rho}{2}+M_2)\\
&+\sum_{i=1}^{k}\sum_{j=1,j\neq\sigma}^{N}|a_i|\frac{|\mu_j|}{|\mu_\sigma|}\frac{\eta_i^{\hat{\beta}_\sigma-\hat{\beta}_j}}{\Gamma(\hat{\beta}_\sigma-\hat{\beta}_j+1)}\frac{\rho}{2}\\
&+\frac{1}{|\lambda_\sigma|}\frac{1}{\Gamma(\hat{\varrho}_\sigma+1)}(C_1\frac{\rho}{2}+D_1\frac{\rho}{2}+M_1)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{l=1,l\neq\sigma}^{N}\frac{|\lambda_l|}{|\lambda_\sigma|}\frac{1}{\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l+1)}\frac{\rho}{2}\\
&+|A_1|\Bigg(\sum_{i=1}^{k}\frac{|b_i|}{|\lambda_\sigma|}\frac{\eta_i^{\hat{\varrho}_\sigma}}{\Gamma(\hat{\varrho}_\sigma+1)}(C_1\frac{\rho}{2}+D_1\frac{\rho}{2}+M_1)\\
&+\sum_{i=1}^{k}\sum_{l=1,l\neq\sigma}^{N}\frac{|\lambda_l||b_i|}{|\lambda_\sigma|}\frac{\eta_i^{\hat{\varrho}_\sigma-\hat{\varrho}_l}}{\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l+1)}\frac{\rho}{2}\\
&+\frac{1}{|\mu_\sigma|}\frac{1}{\Gamma(\hat{\beta}_\sigma+1)}(C_2\frac{\rho}{2}+D_2\frac{\rho}{2}+M_2)\\
&+\sum_{j=1,j\neq\sigma}^{N}\frac{|\mu_j|}{|\mu_\sigma|}\frac{1}{\Gamma(\hat{\beta}_\sigma-\hat{\beta}_j+1)}\frac{\rho}{2}\Bigg)\Bigg].
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\|S_2(\varphi(\kappa),\chi(\kappa))\| \le \frac{1}{|\gamma|}\Bigg[&\sum_{i=1}^{k}\frac{|b_i|}{|\lambda_\sigma|}\frac{\eta_i^{\hat{\varrho}_\sigma}}{\Gamma(\hat{\varrho}_\sigma+1)}(C_1\frac{\rho}{2}+D_1\frac{\rho}{2}+M_1)\\
&+\sum_{i=1}^{k}\sum_{l=1,l\neq\sigma}^{N}\frac{|\lambda_l||b_i|}{|\lambda_\sigma|}\frac{\eta_i^{\hat{\varrho}_\sigma-\hat{\varrho}_l}}{\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l+1)}\frac{\rho}{2}\\
&+\frac{1}{|\mu_\sigma|}\frac{1}{\Gamma(\hat{\beta}_\sigma+1)}(C_2\frac{\rho}{2}+D_2\frac{\rho}{2}+M_2)\\
&+\sum_{j=1,j\neq\sigma}^{N}\frac{|\mu_j|}{|\mu_\sigma|}\frac{1}{\Gamma(\hat{\beta}_\sigma-\hat{\beta}_j+1)}\frac{\rho}{2}\\
&+|A_2|\Bigg(\sum_{i=1}^{k}\frac{|a_i|}{|\mu_\sigma|}\frac{\eta_i^{\hat{\beta}_\sigma}}{\Gamma(\hat{\beta}_\sigma+1)}(C_2\frac{\rho}{2}+D_2\frac{\rho}{2}+M_2)\\
&+\sum_{i=1}^{k}\sum_{j=1,j\neq\sigma}^{N}a_i\frac{|\mu_j|}{|\mu_\sigma|}\frac{\eta_i^{\hat{\beta}_\sigma-\hat{\beta}_j}}{\Gamma(\hat{\beta}_\sigma-\hat{\beta}_j+1)}\frac{\rho}{2} \qquad (4.12)\\
&+\frac{1}{|\lambda_\sigma|}\frac{1}{\Gamma(\hat{\varrho}_\sigma+1)}(C_1\frac{\rho}{2}+D_1\frac{\rho}{2}+M_1)\\
&+\sum_{l=1,l\neq\sigma}^{N}\frac{|\lambda_l|}{|\lambda_\sigma|}\frac{1}{\Gamma(\hat{\varrho}_\sigma-\hat{\varrho}_l+1)}\frac{\rho}{2}\Bigg)\Bigg].
\end{aligned}
$$

Thus, S_1 and S_2 are uniformly bounded; it follows that S is uniformly bounded.

Step 3 Now, our next goal is to prove the continuity of the operator S . To this end, let a sequence $(\varphi_n,\chi_n)\in\Omega_\rho$ which converges to (φ,χ). Then

$$
\begin{aligned}
&|S_1(\varphi_n(\kappa),\chi_n(\kappa))-S_1(\varphi(\kappa),\chi(\kappa))|\\
&\le\frac{1}{|\gamma|}\Bigg[\sum_{i=1}^{k}\frac{|a_i|}{|\mu_\sigma|}\int_0^{\eta_i}\frac{(\eta_i-z)^{\beta_\sigma-1}}{\Gamma(\beta_\sigma)}|h_2(z,\varphi_n(z),\chi_n(z))-h_2(z,\varphi(z),\chi(z))|\,dz
\end{aligned}
$$

$$+ \sum_{i=1}^{k} \sum_{j=1, j\neq\sigma}^{N} |a_i| \frac{|\mu_j|}{|\mu_\sigma|} \int_0^{\eta_i} \frac{(\eta_i - z)^{\beta_\sigma - \beta_j - 1}}{\Gamma(\beta_\sigma - \beta_j)} |\chi_n(z) - \chi(z)|\, dz$$
$$+ \frac{1}{|\lambda_\sigma|} \int_0^1 \frac{(1-z)^{\varrho_\sigma - 1}}{\Gamma(\varrho_\sigma)} |h_1(z, \varphi_n(z), \chi_n(z)) - h_1(z, \varphi(z), \chi(z))|\, dz$$
$$+ \sum_{l=1, l\neq\sigma}^{N} \frac{|\lambda_l|}{|\lambda_\sigma|} \int_0^1 \frac{(1-z)^{\varrho_\sigma - \varrho_l - 1}}{\Gamma(\varrho_\sigma - \varrho_l)} |\varphi_n(z) - \varphi(z)|\, dz$$
$$+ |A_1| \Bigg(\sum_{i=1}^{k} \frac{|b_i|}{|\lambda_\sigma|} \int_0^{\eta_i} \frac{(\eta_i - z)^{\varrho_\sigma - 1}}{\Gamma(\varrho_\sigma)} |h_1(z, \varphi_n(z), \chi_n(z)) - h_1(z, \varphi(z), \chi(z))|\, dz$$
$$+ \sum_{i=1}^{k} \sum_{l=1, l\neq\sigma}^{N} \frac{|\lambda_l||b_i|}{|\lambda_\sigma|} \int_0^{\eta_i} \frac{(\eta_i - z)^{\varrho_\sigma - \varrho_l - 1}}{\Gamma(\varrho_\sigma - \varrho_l)} |\varphi_n(z) - \varphi(z)|\, dz$$
$$+ \frac{1}{|\mu_\sigma|} \int_0^1 \frac{(1-z)^{\beta_\sigma - 1}}{\Gamma(\beta_\sigma)} |h_2(z, \varphi_n(z), \chi_n(z)) - h_2(z, \varphi(z), \chi(z))|\, dz$$
$$+ \sum_{j=1, j\neq\sigma}^{N} \frac{|\mu_j|}{|\mu_\sigma|} \int_0^1 \frac{(1-z)^{\beta_\sigma - \beta_j - 1}}{\Gamma(\beta_\sigma - \beta_j)} |\chi_n(z) - \chi(z)|\, dz \Bigg) \Bigg].$$

Applying the Lebesgue Dominated Convergence Theorem, we obtain,

$$\|S_1(\varphi_n(\kappa), \chi_n(\kappa)) - S_1(\varphi(\kappa), \chi(\kappa))\| \to 0$$

as $n \to \infty$. Likewise, the continuity of S_2 can be derived. Therefore, S is continuous.

Step 4 We will show the equicontinuity of the operator S.Let $\kappa_1, \kappa_2 \in [0,1], \kappa_1 < \kappa_2$, $(S_1, S_2 \in X \times X)$

$$|S_1(\varphi(\kappa_2), \chi(\kappa_2)) - S_1(\varphi(\kappa_1), \chi(\kappa_1))| \leq \frac{|\kappa_2^{N-1} - \kappa_1^{N-1}|}{|\gamma|} \Bigg[\sum_{i=1}^{k} \frac{|a_i|}{|\mu_\sigma|} \int_0^{\eta_i} \frac{(\eta_i - z)^{\beta_\sigma - 1}}{\Gamma(\beta_\sigma)} |h_2(z, \varphi(z), \chi(z))|\, dz$$
$$+ \sum_{i=1}^{k} \sum_{j=1, j\neq\sigma}^{N} |a_i| \frac{|\mu_j|}{|\mu_\sigma|} \int_0^{\eta_i} \frac{(\eta_i - z)^{\beta_\sigma - \beta_j - 1}}{\Gamma(\beta_\sigma - \beta_j)} |\chi(z)|\, dz$$
$$+ \frac{1}{|\lambda_\sigma|} \int_0^1 \frac{(1-z)^{\varrho_\sigma - 1}}{\Gamma(\varrho_\sigma)} |h_1(z, \varphi(z), \chi(z))| dz$$
$$+ \sum_{l=1, l\neq\sigma}^{N} \frac{|\lambda_l|}{|\lambda_\sigma|} \int_0^1 \frac{(1-z)^{\varrho_\sigma - \varrho_l - 1}}{\Gamma(\varrho_\sigma - \varrho_l)} |\varphi(z)|\, dz$$
$$+ |A_1| \Bigg(\sum_{i=1}^{k} \frac{|b_i|}{|\lambda_\sigma|} \int_0^{\eta_i} \frac{(\eta_i - z)^{\varrho_\sigma - 1}}{\Gamma(\varrho_\sigma)} |h_1(z, \varphi(z), \chi(z))| dz$$
$$+ \sum_{i=1}^{k} \sum_{l=1, l\neq\sigma}^{N} \frac{|\lambda_l||b_i|}{|\lambda_\sigma|} \int_0^{\eta_i} \frac{(\eta_i - z)^{\varrho_\sigma - \varrho_l - 1}}{\Gamma(\varrho_\sigma - \varrho_l)} |\varphi(z)|\, dz$$
$$+ \frac{1}{|\mu_\sigma|} \int_0^1 \frac{(1-z)^{\beta_\sigma - 1}}{\Gamma(\beta_\sigma)} |h_2(z, \varphi(z), \chi(z))|\, dz$$

$$
\begin{aligned}
&+\sum_{j=1,j\neq\sigma}^{N}\frac{|\mu_j|}{|\mu_\sigma|}\int_0^1\frac{(1-z)^{\beta_\sigma-\beta_j-1}}{\Gamma(\beta_\sigma-\beta_j)}|\chi(z)|\,dz\Bigg)\Bigg]\\
&\leq\frac{|\kappa_2^{N-1}-\kappa_1^{N-1}|}{|\gamma|}\Bigg[\sum_{i=1}^{k}\frac{|a_i|}{|\mu_\sigma|}\frac{\eta_i^{\beta_\sigma}}{\Gamma(\beta_\sigma+1)}(C_2\frac{\rho}{2}+D_2\frac{\rho}{2}+M_2)\\
&+\sum_{i=1}^{k}\sum_{j=1,j\neq\sigma}^{N}|a_i|\frac{|\mu_j|}{|\mu_\sigma|}\frac{\eta_i^{\beta_\sigma-\beta_j}}{\Gamma(\beta_\sigma-\beta_j+1)}\frac{\rho}{2}\\
&+\frac{1}{|\lambda_\sigma|}\frac{1}{\Gamma(\varrho_\sigma+1)}(C_1\frac{\rho}{2}+D_1\frac{\rho}{2}+M_1)\\
&+\sum_{l=1,l\neq\sigma}^{N}\frac{|\lambda_l|}{|\lambda_\sigma|}\frac{1}{\Gamma(\varrho_\sigma-\varrho_l+1)}\frac{\rho}{2}\\
&+|A_1|\Bigg(\sum_{i=1}^{k}\frac{|b_i|}{|\lambda_\sigma|}\frac{\eta_i^{\varrho_\sigma}}{\Gamma(\varrho_\sigma+1)}(C_1\frac{\rho}{2}+D_1\frac{\rho}{2}+M_1)\\
&+\sum_{i=1}^{k}\sum_{l=1,l\neq\sigma}^{N}\frac{|\lambda_l||b_i|}{|\lambda_\sigma|}\frac{\eta_i^{\varrho_\sigma-\varrho_l}}{\Gamma(\varrho_\sigma-\varrho_l+1)}\frac{\rho}{2}\\
&+\frac{1}{|\mu_\sigma|}\frac{1}{\Gamma(\beta_\sigma+1)}(C_2\frac{\rho}{2}+D_2\frac{\rho}{2}+M_2)\\
&+\sum_{j=1,j\neq\sigma}^{N}\frac{|\mu_j|}{|\mu_\sigma|}\frac{1}{\Gamma(\beta_\sigma-\beta_j+1)}\frac{\rho}{2}\Bigg)\Bigg],
\end{aligned}
$$

Evidently, $\|S_1(\varphi(\kappa_2),\chi(\kappa_2))-S_1(\varphi(\kappa_1),\chi(\kappa_1))\|\to 0$ as $\kappa_1\to\kappa_2$ Similarly, we have $\|S_2(\varphi(\kappa_2),\chi(\kappa_2))-S_2(\varphi(\kappa_1),\chi(\kappa_1))\|\to 0$ as $\kappa_1\to\kappa_2$. Therefore, S is equicontinuous. Consequently, by virtue of theorem (2.2) , S is a completely continuous operator on Ω_ρ .

Step 5 Our objective is to establish the contraction P. Consider that $\varphi,\chi,\varphi^*,\chi^*\in X$

$$
\begin{aligned}
|P_1(\varphi(\kappa),\chi(\kappa))-P_1(\varphi^*(\kappa),\chi^*(\kappa))|&\leq\frac{1}{|\lambda_\sigma|}\int_0^\kappa\frac{(\kappa-z)^{\varrho_\sigma-1}}{\Gamma(\varrho_\sigma)}|h_1(z,\varphi(z),\chi(z))-h_1(z,\varphi^*(z),\chi^*(z))|dz\\
&+\sum_{l=1,l\neq\sigma}^{N}\frac{|\lambda_l|}{|\lambda_\sigma|}\int_0^\kappa\frac{(\kappa-z)^{\varrho_\sigma-\varrho_l-1}}{\Gamma(\varrho_\sigma-\varrho_l)}|\varphi(z)-\varphi^*(z)|dz\\
&\leq\frac{1}{|\lambda_\sigma|}\int_0^\kappa\frac{(\kappa-z)^{\varrho_\sigma-1}}{\Gamma(\varrho_\sigma)}L_1(|\varphi-\varphi^*|+|\chi-\chi^*|)dz\\
&+\sum_{l=1,l\neq\sigma}^{N}\frac{|\lambda_l|}{|\lambda_\sigma|}\int_0^\kappa\frac{(\kappa-z)^{\varrho_\sigma-\varrho_l-1}}{\Gamma(\varrho_\sigma-\varrho_l)}(|\varphi(z)-\varphi^*(z)|)dz\\
&\leq\frac{1}{|\lambda_\sigma|}\frac{L_1}{\Gamma(\varrho_\sigma+1)}(|\varphi-\varphi^*|+|\chi-\chi^*|)\\
&+\sum_{l=1,l\neq\sigma}^{N}\frac{|\lambda_l|}{|\lambda_\sigma|}\frac{1}{\Gamma(\varrho_\sigma-\varrho_l+1)}(|\varphi(z)-\varphi^*(z)|),
\end{aligned}
$$

$$\|P_1(\varphi(\kappa),\chi(\kappa)) - P_1(\varphi^*(\kappa),\chi^*(\kappa))\| \leq \left(\frac{1}{|\lambda_\sigma|}\frac{L_1}{\Gamma(\varrho_\sigma+1)} + \sum_{l=1,l\neq\sigma}^{N}\frac{|\lambda_l|}{|\lambda_\sigma|}\frac{1}{\Gamma(\varrho_\sigma-\varrho_l+1)}\right)\|(\varphi,\chi)-(\varphi^*,\chi^*)\|,$$

$$\|P_1(\varphi(\kappa),\chi(\kappa)) - P_1(\varphi^*(\kappa),\chi^*(\kappa))\| \leq K_1'\|(\varphi,\chi)-(\varphi^*,\chi^*)\|. \tag{4.13}$$

Similarly

$$\|P_2(\varphi(\kappa),\chi(\kappa)) - P_2(\varphi^*(\kappa),\chi^*(\kappa))\| \leq \left(\frac{1}{|\mu_\sigma|}\frac{L_2}{\Gamma(\beta_\sigma+1)} + \sum_{j=1,j\neq\sigma}^{N}\frac{|\mu_j|}{|\mu_\sigma|}\frac{1}{\Gamma(\beta_\sigma-\beta_j+1)}\right)\|(\varphi,\chi)-(\varphi^*,\chi^*)\|, \tag{4.14}$$

$$\|P_2(\varphi(\kappa),\chi(\kappa)) - P_2(\varphi^*(\kappa),\chi^*(\kappa))\| \leq K_2'\|(\varphi,\chi)-(\varphi^*,\chi^*)\|.$$

let $K' = max\{K_1', K_2'\}$ then:

$$\|P(\varphi(\kappa),\chi(\kappa)) - P(\varphi^*(\kappa),\chi^*(\kappa))\| \leq K'\|(\varphi,\chi)-(\varphi^*,\chi^*)\|$$

Evidently, P satisfies the contraction condition. Thus,in view of theorem (2.1), the considered coupled multi-term Caputo FDEs possess at least one solution.

□

5. Analysis of Stability

Now we focus on the stability analysis of solutions to the system (1.1). Specifically, we examine four types of stability: Ulam-Hyers (UH), Ulam-Hyers-Rassias (UHR), and their generalization (GUH) and (GUHR).

Definition 5.1. *Let (φ^*,χ^*) be the unique solution of the system(1.1),and the solution (φ,χ) satisfies the following inequalities*

$$\begin{aligned} &|\sum_{l=1}^{N}\lambda_l{}^cD^{\hat{\varrho}_l}\varphi(\kappa) - h_1(\kappa,\varphi(\kappa),\chi(\kappa))| \leq \varepsilon_1,\\ &|\sum_{j=1}^{N}\mu_j{}^cD^{\hat{\beta}_j}\chi(\kappa) - h_2(\kappa,\varphi(\kappa),\chi(\kappa))| \leq \varepsilon_2,\\ &\textit{where},\quad \kappa\in[0.1],\lambda_l,\mu_j\in\mathbb{R}\end{aligned} \tag{5.1}$$

for some $\epsilon = max\{\varepsilon_1,\varepsilon_2\} > 0$. If there exists a constants $\mathcal{H} = max\{\mathcal{H}_1,\mathcal{H}_2\}$(positive) , such that $|(\varphi,\chi)-(\varphi^*,\chi^*)| \leq \mathcal{H}\epsilon$, then the solution (φ,χ) of the system (1.1) is termed as (UH) stable . In the same manner, the solution (φ,χ) of the system (1.1) is said to be generalized Ulam-Hyers (GUH) stable. if we can find Non-negative function $\mathcal{K}:\mathbb{R}^+\to\mathbb{R}^+$ with $\mathcal{K}(0)=0$,such that $|(\varphi,\chi)-(\varphi^*,\chi^*)| \leq \mathcal{H}\mathcal{K}(\epsilon)$.

Definition 5.2. *Let (φ^*,χ^*) be the unique solution of the system(1.1),and the solution (φ,χ) satisfies the following inequalities ,*

$$\begin{aligned}
&|\sum_{l=1}^{N}\lambda_l{}^{c}D^{\hat{\varrho}_l}\varphi(\kappa)-h_1(\kappa,\varphi(\kappa),\chi(\kappa))|\leq \mathrm{v}_1(\kappa)\varepsilon_1,\\
&|\sum_{j=1}^{N}\mu_j{}^{c}D^{\hat{\beta}_j}\chi(\kappa)-h_2(\kappa,\varphi(\kappa),\chi(\kappa))|\leq \mathrm{v}_2(\kappa)\varepsilon_2,\\
&\textit{where,}\quad \kappa\in[0.1],\lambda_l,\mu_j\in\mathbb{R}
\end{aligned}\tag{5.2}$$

for some $\epsilon = max\{\varepsilon_1,\varepsilon_2\} > 0$. If there exists a constant $\mathcal{H} = max\{\mathcal{H}_1,\mathcal{H}_2\}$(positive) , such that $|(\varphi,\chi)-(\varphi^*,\chi^*)|\leq \mathcal{H}\mathrm{v}(\kappa)\epsilon$,then the solution (φ,χ) of the system (1.1) is said to be (UHR) stable for a continuous function $\mathrm{v} = max\{\mathrm{v}_1,\mathrm{v}_2\}\in X\times X$.

Definition 5.3. *Let (φ^*,χ^*) be the unique solution of the system(1.1),and the solution (φ,χ) satisfies the following inequalities ,*

$$\begin{aligned}
&|\sum_{l=1}^{N}\lambda_l{}^{c}D^{\hat{\varrho}_l}\varphi(\kappa)-h_1(\kappa,\varphi(\kappa),\chi(\kappa))|\leq \mathrm{v}_1(\kappa),\\
&|\sum_{j=1}^{N}\mu_j{}^{c}D^{\hat{\beta}_j}\chi(\kappa)-h_2(\kappa,\varphi(\kappa),\chi(\kappa))|\leq \mathrm{v}_2(\kappa),\\
&\textit{where}\quad \kappa\in[0.1],\lambda_l,\mu_j\in\mathbb{R}
\end{aligned}\tag{5.3}$$

for some $\epsilon = max\{\varepsilon_1,\varepsilon_2\} > 0$.If there exists a constant $\mathcal{H} = max\{\mathcal{H}_1,\mathcal{H}_2\}$(positive),such that $|(\varphi,\chi)-(\varphi^*,\chi^*)|\leq \mathcal{H}\mathrm{v}(\kappa)$, then the solution (φ,χ) of the system(1.1) is said to be (GUHR) stable for a continuous function $\mathrm{v} = max\{\mathrm{v}_1,\mathrm{v}_2\}\in X\times X$.

Remark 5.1. *Let $(\varphi,\chi)\in X\times X$ be a solution of the inequality (5.1) if and only if we can find a functions $\psi_1,\psi_2\in X$, depending on φ and χ, such that:*

(1) $\epsilon_1\geq|\psi_1(\kappa)|,\epsilon_2\geq|\psi_2(\kappa)|$,
(2)

$$\begin{aligned}
&\sum_{l=1}^{N}\lambda_l{}^{c}D^{\hat{\varrho}_l}\varphi(\kappa)-h_1(\kappa,\varphi(\kappa),\chi(\kappa))-\psi_1(\kappa)=0,\\
&\sum_{j=1}^{N}\mu_j{}^{c}D^{\hat{\beta}_j}\chi(\kappa)-h_2(\kappa,\varphi(\kappa),\chi(\kappa))-\psi_2(\kappa)=0,
\end{aligned}\tag{5.4}$$

where $\kappa\in[0.1],\lambda_l,\mu_j\in\mathbb{R}$.

Remark 5.2. *Let $(\varphi,\chi)\in X\times X$ be a solution of the inequality (5.2), if and only if we can find functions $\psi_1,\psi_2\in X$, depending on φ and χ, such that:*

(1) $|\psi_1(\kappa)|\leq \mathrm{v}_1(\kappa)\epsilon_1,|\psi_2(\kappa)|\leq \mathrm{v}_2(\kappa)\epsilon_2$, $\quad\kappa\in[0.1]$,

(2)

$$\sum_{l=1}^{N} \lambda_l {}^c D^{\hat{\varrho}_l} \varphi(\kappa) - h_1(\kappa, \varphi(\kappa), \chi(\kappa)) - \psi_1(\kappa) = 0,$$
$$\sum_{j=1}^{N} \mu_j {}^c D^{\hat{\beta}_j} \chi(\kappa) - h_2(\kappa, \varphi(\kappa), \chi(\kappa)) - \psi_2(\kappa) = 0, \tag{5.5}$$

where $\kappa \in [0.1], \lambda_l, \mu_j \in \mathbb{R}$.

Remark 5.3. *$(\varphi, \chi) \in X \times X$ is a solution of the inequality (5.3) if and only if one can find a functions $\psi_1, \psi_2 \in X$, depending on φ and χ, such that:*

(1) $|\psi_1(\kappa)| \leq \mathrm{v}_1(\kappa), |\psi_2(\kappa)| \leq \mathrm{v}_2(\kappa), \quad \kappa \in [0.1]$,

(2)

$$\sum_{l=1}^{N} \lambda_l {}^c D^{\hat{\varrho}_l} \varphi(\kappa) - h_1(\kappa, \varphi(\kappa), \chi(\kappa)) - \psi_1(\kappa) = 0,$$
$$\sum_{j=1}^{N} \mu_j {}^c D^{\hat{\beta}_j} \chi(\kappa) - h_2(\kappa, \varphi(\kappa), \chi(\kappa)) - \psi_2(\kappa) = 0, \tag{5.6}$$

where $\kappa \in [0.1], \lambda_l, \mu_j \in \mathbb{R}$.

Lemma 5.1. *Let $(\varphi, \chi) \in X \times X$ be a solution to the following system:*

$$\begin{cases} \sum_{l=1}^{N} \lambda_l {}^c D^{\hat{\varrho}_l} \varphi(\kappa) = h_1(\kappa, \varphi(\kappa), \chi(\kappa) + \psi_1(\kappa), \\ \sum_{j=1}^{N} \mu_j {}^c D_a^{\hat{\beta}_j} \chi(\kappa) = h_2(\kappa, \varphi(\kappa), \chi(\kappa)) + \psi_2(\kappa), \end{cases} \quad \kappa \in [0,1] \tag{5.7}$$

$$\varphi(0) = 0, \varphi'(0) = 0, \varphi^{N-2}(0) = 0, \varphi(1) = \sum_{i=1}^{k} a_i \chi(\eta_i),$$

$$\chi(0) = 0, \chi'(0) = 0, \chi^{N-2}(0) = 0, \chi(1) = \sum_{i=1}^{k} b_i \varphi(\eta_i),$$

satisfying the following relations:

$$\|\varphi(\kappa) - \mathcal{U}_1(\varphi, \chi)\| \leq \frac{\epsilon_1}{|\gamma|} \left(\sum_{i=1}^{k} \frac{|A_1||b_i|}{|\lambda_\sigma|} \frac{\eta_i^{\hat{\varrho}_\sigma}}{\Gamma(\hat{\varrho}_\sigma + 1)} + \frac{1}{|\lambda_\sigma| \Gamma(\hat{\varrho}_\sigma + 1)} + \frac{|\gamma|}{|\lambda_\sigma| \Gamma(\hat{\varrho}_\sigma + 1)} \right) + \frac{\epsilon_2}{|\gamma|} \left(\sum_{i=1}^{k} \frac{|a_i|}{|\mu_\sigma|} \frac{\eta_i^{\hat{\beta}_\sigma}}{\Gamma(\hat{\beta}_\sigma + 1)} + \frac{|A_1|}{|\mu_\sigma| \Gamma(\hat{\beta}_\sigma + 1)} \right),$$

$$\|\chi(\kappa) - \mathcal{U}_2(\varphi, \chi)\| \leq \frac{\epsilon_2}{|\gamma|} \left(\sum_{i=1}^{k} \frac{|A_2||a_i|}{|\mu_\sigma|} \frac{\eta_i^{\hat{\beta}_\sigma}}{\Gamma(\hat{\beta}_\sigma + 1)} + \frac{1}{|\mu_\sigma| \Gamma(\hat{\beta}_\sigma + 1)} + \frac{|\gamma|}{|\mu_\sigma| \Gamma(\hat{\beta}_\sigma + 1)} \right) + \frac{\epsilon_1}{|\gamma|} \left(\sum_{i=1}^{k} \frac{|b_i|}{|\lambda_\sigma|} \frac{\eta_i^{\hat{\varrho}_\sigma}}{\Gamma(\hat{\varrho}_\sigma + 1)} + \frac{|A_2|}{|\lambda_\sigma| \Gamma(\hat{\varrho}_\sigma + 1)} \right),$$

Proof. For any solution $(\varphi,\chi)\in X\times X$ of (1.1), we have

$$\begin{aligned}\varphi(\kappa)=\mathcal{U}_1(\varphi,\chi)+\frac{\kappa^{N-1}}{\gamma}\Bigg[&\sum_{i=1}^{k}\frac{a_i}{\mu_\sigma}\int_0^{\eta_i}\frac{(\eta_i-z)^{\hat{\beta}_\sigma-1}}{\Gamma(\hat{\beta}_\sigma)}\psi_2(z)\,dz-\frac{A_1}{\mu_\sigma}\int_0^1\frac{(1-z)^{\hat{\beta}_\sigma-1}}{\Gamma(\hat{\beta}_\sigma)}\psi_2(z)\,dz\\&+\sum_{i=1}^{k}\frac{A_1 b_i}{\lambda_\sigma}\int_0^{\eta_i}\frac{(\eta_i-z)^{\hat{\varrho}_\sigma-1}}{\Gamma(\hat{\varrho}_\sigma)}\psi_1(z)\,dz-\frac{1}{\lambda_\sigma}\int_0^1\frac{(1-z)^{\hat{\varrho}_\sigma-1}}{\Gamma(\hat{\varrho}_\sigma)}\psi_1(z)\,dz\Bigg]\\&+\frac{1}{\lambda_\sigma}\int_0^{\kappa}\frac{(\kappa-z)^{\hat{\varrho}_\sigma-1}}{\Gamma(\hat{\varrho}_\sigma)}\psi_1(z)dz.\end{aligned}$$

Using Remark 5.1, we get

$$\begin{aligned}|\varphi(\kappa)-\mathcal{U}_1(\varphi,\chi)|\le\frac{1}{|\gamma|}\Bigg[&\sum_{i=1}^{k}\frac{|a_i|}{|\mu_\sigma|}\int_0^{\eta_i}\frac{(\eta_i-z)^{\hat{\beta}_\sigma-1}}{\Gamma(\hat{\beta}_\sigma)}|\psi_2(z)|\,dz+\frac{|A_1|}{|\mu_\sigma|}\int_0^1\frac{(1-z)^{\hat{\beta}_\sigma-1}}{\Gamma(\hat{\beta}_\sigma)}|\psi_2(z)|\,dz\\&+\sum_{i=1}^{k}\frac{|A_1||b_i|}{|\lambda_\sigma|}\int_0^{\eta_i}\frac{(\eta_i-z)^{\hat{\varrho}_\sigma-1}}{\Gamma(\hat{\varrho}_\sigma)}|\psi_1(z)|dz+\frac{1}{|\lambda_\sigma|}\int_0^1\frac{(1-z)^{\hat{\varrho}_\sigma-1}}{\Gamma(\hat{\varrho}_\sigma)}|\psi_1(z)|\,dz\Bigg]\\&+\frac{1}{|\lambda_\sigma|}\int_0^{\kappa}\frac{(\kappa-z)^{\hat{\varrho}_\sigma-1}}{\Gamma(\hat{\varrho}_\sigma)}|\psi_1(z)|dz\\&\le\frac{\epsilon_1}{|\gamma|}\left(\sum_{i=1}^{k}\frac{|A_1||b_i|}{|\lambda_\sigma|}\frac{\eta_i^{\hat{\varrho}_\sigma}}{\Gamma(\hat{\varrho}_\sigma+1)}+\frac{1}{|\lambda_\sigma|\Gamma(\hat{\varrho}_\sigma+1)}+\frac{|\gamma|}{|\lambda_\sigma|\Gamma(\hat{\varrho}_\sigma+1)}\right)\\&+\frac{\epsilon_2}{|\gamma|}\left(\sum_{i=1}^{k}\frac{|a_i|}{|\mu_\sigma|}\frac{\eta_i^{\hat{\beta}_\sigma}}{\Gamma(\hat{\beta}_\sigma+1)}+\frac{|A_1|}{|\mu_\sigma|\Gamma(\hat{\beta}_\sigma+1)}\right).\end{aligned}$$

Similarly, we can find:

$$\begin{aligned}\chi(\kappa)=\mathcal{U}_2(\varphi,\chi)+\frac{\kappa^{N-1}}{\gamma}\Bigg[&\sum_{i=1}^{k}\frac{b_i}{\lambda_\sigma}\int_0^{\eta_i}\frac{(\eta_i-z)^{\varrho_\sigma-1}}{\Gamma(\varrho_\sigma)}\psi_1(z)\,dz-\frac{A_2}{\lambda_\sigma}\int_0^1\frac{(1-z)^{\varrho_\sigma-1}}{\Gamma(\varrho_\sigma)}\psi_1(z)\,dz\\&+\sum_{i=1}^{k}\frac{A_2 a_i}{\mu_\sigma}\int_0^{\eta_i}\frac{(\eta_i-z)^{\beta_\sigma-1}}{\Gamma(\beta_\sigma)}\psi_2(z)\,dz-\frac{1}{\mu_\sigma}\int_0^1\frac{(1-z)^{\beta_\sigma-1}}{\Gamma(\beta_\sigma)}\psi_2(z)\,dz\Bigg]\\&+\frac{1}{\mu_\sigma}\int_0^{\kappa}\frac{(\kappa-z)^{\beta_\sigma-1}}{\Gamma(\beta_\sigma)}\psi_2(z)dz.\end{aligned}$$

$$\begin{aligned}|\chi(\kappa)-\mathcal{U}_2(\varphi,\chi)|\le\frac{1}{|\gamma|}\Bigg[&\sum_{i=1}^{k}\frac{|b_i|}{|\lambda_\sigma|}\int_0^{\eta_i}\frac{(\eta_i-z)^{\varrho_\sigma-1}}{\Gamma(\varrho_\sigma)}|\psi_1(z)|\,dz+\frac{|A_2|}{|\lambda_\sigma|}\int_0^1\frac{(1-z)^{\varrho_\sigma-1}}{\Gamma(\varrho_\sigma)}|\psi_1(z)|\,dz\\&+\sum_{i=1}^{k}\frac{|A_2||a_i|}{|\mu_\sigma|}\int_0^{\eta_i}\frac{(\eta_i-z)^{\beta_\sigma-1}}{\Gamma(\beta_\sigma)}|\psi_2(z)|\,dz+\frac{1}{|\mu_\sigma|}\int_0^1\frac{(1-z)^{\beta_\sigma-1}}{\Gamma(\beta_\sigma)}|\psi_2(z)|\,dz\Bigg]\\&+\frac{1}{|\mu_\sigma|}\int_0^{\kappa}\frac{(\kappa-z)^{\beta_\sigma-1}}{\Gamma(\beta_\sigma)}|\psi_2(z)|dz,\\&\le\frac{\epsilon_2}{|\gamma|}\left(\sum_{i=1}^{k}\frac{|A_2||a_i|}{|\mu_\sigma|}\frac{\eta_i^{\beta_\sigma}}{\Gamma(\beta_\sigma+1)}+\frac{1}{|\mu_\sigma|\Gamma(\beta_\sigma+1)}+\frac{|\gamma|}{|\mu_\sigma|\Gamma(\beta_\sigma+1)}\right)\end{aligned}$$

$$+\frac{\epsilon_1}{|\gamma|}\bigg(\sum_{i=1}^{k}\frac{|b_i|}{|\lambda_\sigma|}\frac{\eta_i^{\varrho_\sigma}}{\Gamma(\varrho_\sigma+1)}+\frac{|A_2|}{|\lambda_\sigma|\Gamma(\varrho_\sigma+1)}\bigg).$$

Therefore

$$\begin{aligned}\|\varphi(\kappa)-\mathcal{U}_1(\varphi,\chi)\| &\le \frac{\epsilon_1}{|\gamma|}\bigg(\sum_{i=1}^{k}\frac{|A_1||b_i|}{|\lambda_\sigma|}\frac{\eta_i^{\varrho_\sigma}}{\Gamma(\varrho_\sigma+1)}+\frac{1}{|\lambda_\sigma|\Gamma(\varrho_\sigma+1)}+\frac{|\gamma|}{|\lambda_\sigma|\Gamma(\varrho_\sigma+1)}\bigg)\\ &+\frac{\epsilon_2}{|\gamma|}\bigg(\sum_{i=1}^{k}\frac{|a_i|}{|\mu_\sigma|}\frac{\eta_i^{\beta_\sigma}}{\Gamma(\beta_\sigma+1)}+\frac{|A_1|}{|\mu_\sigma|\Gamma(\beta_\sigma+1)}\bigg),\\ \|\chi(\kappa)-\mathcal{U}_2(\varphi,\chi)\| &\le \frac{\epsilon_2}{|\gamma|}\bigg(\sum_{i=1}^{k}\frac{|A_2||a_i|}{|\mu_\sigma|}\frac{\eta_i^{\beta_\sigma}}{\Gamma(\beta_\sigma+1)}+\frac{1}{|\mu_\sigma|\Gamma(\beta_\sigma+1)}+\frac{|\gamma|}{|\mu_\sigma|\Gamma(\beta_\sigma+1)}\bigg)\\ &+\frac{\epsilon_1}{|\gamma|}\bigg(\sum_{i=1}^{k}\frac{|b_i|}{|\lambda_\sigma|}\frac{\eta_i^{\varrho_\sigma}}{\Gamma(\varrho_\sigma+1)}+\frac{|A_2|}{|\lambda_\sigma|\Gamma(\varrho_\sigma+1)}\bigg).\end{aligned}$$

□

Theorem 5.1. *Under the hypotheses $(B_1),(B_2)$ and assuming Equations(4.1) hold with $K<1$, where $K=max\{K_1,K_2\}$. Then, the problem (1.1) is UH and GUHstable.*

Proof. for any solution $(\varphi,\chi)\in X\times X$ and (φ^*,χ^*) unique solution of problem (1.1), we derive:

$$\begin{aligned}\|(\varphi,\chi)-(\varphi^*,\chi^*)\| &= \|(\varphi,\chi)-\mathcal{U}(\varphi^*,\chi^*)\|\\ &\le \|\varphi-\mathcal{U}_1(\varphi,\chi)\|+\|\chi-\mathcal{U}_2(\varphi,\chi)\|\\ &+\|\mathcal{U}_1(\varphi,\chi)-\mathcal{U}_1(\varphi^*,\chi^*)\|+\|\mathcal{U}_2(\varphi,\chi)-\mathcal{U}_2(\varphi^*,\chi^*)\|.\end{aligned}$$

Applying lemma(5.1) and theorem(4.1) we have:

$$\begin{aligned}\|(\varphi,\chi)-(\varphi^*,\chi^*)\| &\le \frac{\epsilon_1}{|\gamma|}\bigg(\sum_{i=1}^{k}\frac{|A_1||b_i|}{|\lambda_\sigma|}\frac{\eta_i^{\hat\varrho_\sigma}}{\Gamma(\hat\varrho_\sigma+1)}+\frac{1}{|\lambda_\sigma|\Gamma(\hat\varrho_\sigma+1)}+\frac{|\gamma|}{|\lambda_\sigma|\Gamma(\hat\varrho_\sigma+1)}\bigg)\\ &+\frac{\epsilon_2}{|\gamma|}\bigg(\sum_{i=1}^{k}\frac{|a_i|}{|\mu_\sigma|}\frac{\eta_i^{\hat\beta_\sigma}}{\Gamma(\hat\beta_\sigma+1)}+\frac{|A_1|}{|\mu_\sigma|\Gamma(\hat\beta_\sigma+1)}\bigg)\\ &+\frac{\epsilon_2}{|\gamma|}\bigg(\sum_{i=1}^{k}\frac{|A_2||a_i|}{|\mu_\sigma|}\frac{\eta_i^{\hat\beta_\sigma}}{\Gamma(\hat\beta_\sigma+1)}+\frac{1}{|\mu_\sigma|\Gamma(\hat\beta_\sigma+1)}+\frac{|\gamma|}{|\mu_\sigma|\Gamma(\hat\beta_\sigma+1)}\bigg)\\ &+\frac{\epsilon_1}{|\gamma|}\bigg(\sum_{i=1}^{k}\frac{|b_i|}{|\lambda_\sigma|}\frac{\eta_i^{\hat\varrho_\sigma}}{\Gamma(\hat\varrho_\sigma+1)}+\frac{|A_2|}{|\lambda_\sigma|\Gamma(\hat\varrho_\sigma+1)}\bigg)+K\|(\varphi,\chi)-(\varphi^*,\chi^*\|\\ &\le \frac{1}{1-K}\bigg[\frac{\epsilon_1}{|\gamma|}\bigg(\sum_{i=1}^{k}\frac{|A_1||b_i|}{|\lambda_\sigma|}\frac{\eta_i^{\hat\varrho_\sigma}}{\Gamma(\hat\varrho_\sigma+1)}+\frac{1}{|\lambda_\sigma|\Gamma(\hat\varrho_\sigma+1)}\\ &+\frac{|\gamma|}{|\lambda_\sigma|\Gamma(\hat\varrho_\sigma+1)}+\sum_{i=1}^{k}\frac{|b_i|}{|\lambda_\sigma|}\frac{\eta_i^{\hat\varrho_\sigma}}{\Gamma(\hat\varrho_\sigma+1)}+\frac{|A_2|}{|\lambda_\sigma|\Gamma(\hat\varrho_\sigma+1)}\bigg)\end{aligned}$$

$$
\begin{aligned}
&+\frac{\epsilon_2}{|\gamma|}\Big(\sum_{i=1}^{k}\frac{|A_2||a_i|}{|\mu_\sigma|}\frac{\eta_i^{\hat{\beta}_\sigma}}{\Gamma(\hat{\beta}_\sigma+1)}+\frac{1}{|\mu_\sigma|\Gamma(\hat{\beta}_\sigma+1)}+\frac{|\gamma|}{|\mu_\sigma|\Gamma(\hat{\beta}_\sigma+1)}\\
&+\sum_{i=1}^{k}\frac{|a_i|}{|\mu_\sigma|}\frac{\eta_i^{\hat{\beta}_\sigma}}{\Gamma(\hat{\beta}_\sigma+1)}+\frac{|A_1|}{|\mu_\sigma|\Gamma(\hat{\beta}_\sigma+1)}\Big)\Big]\\
&\leq \mathcal{H}\epsilon,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{H}=\max\Big\{&\frac{1}{(1-K)|\gamma|}\Big(\sum_{i=1}^{k}\frac{|A_1||b_i|}{|\lambda_\sigma|}\frac{\eta_i^{\varrho_\sigma}}{\Gamma(\varrho_\sigma+1)}+\frac{1}{|\lambda_\sigma|\Gamma(\varrho_\sigma+1)}+\frac{|\gamma|}{|\lambda_\sigma|\Gamma(\varrho_\sigma+1)}+\sum_{i=1}^{k}\frac{|b_i|}{|\lambda_\sigma|}\frac{\eta_i^{\varrho_\sigma}}{\Gamma(\varrho_\sigma+1)}\\
&+\frac{|A_2|}{|\lambda_\sigma|\Gamma(\varrho_\sigma+1)}\Big),\frac{1}{(1-K)|\gamma|}\Big(\sum_{i=1}^{k}\frac{|A_2||a_i|}{|\mu_\sigma|}\frac{\eta_i^{\beta_\sigma}}{\Gamma(\beta_\sigma+1)}+\frac{1}{|\mu_\sigma|\Gamma(\beta_\sigma+1)}+\frac{|\gamma|}{|\mu_\sigma|\Gamma(\beta_\sigma+1)}\\
&+\sum_{i=1}^{k}\frac{|a_i|}{|\mu_\sigma|}\frac{\eta_i^{\beta_\sigma}}{\Gamma(\beta_\sigma+1)}+\frac{|A_1|}{|\mu_\sigma|\Gamma(\beta_\sigma+1)}\Big)\Big\},
\end{aligned}
$$

Consequently, it is established that the system solution (1.1) is (UH) stable. Moreover, the system (1.1) becomes (GUH) stable when $\mathcal{K}(\epsilon)$ is taken as $\mathcal{K}(\epsilon)=\epsilon$. □

Lemma 5.2. *Let the assumptions in Remark (5.2) be fulfilled. Then , any solution of (5.7), satisfies the following relation:*

$$
\begin{aligned}
\|\varphi(\kappa)-\mathcal{U}_1(\varphi,\chi)\|\leq{}&\frac{\mathrm{v}_1(\kappa)\epsilon_1}{|\gamma|}\Big(\sum_{i=1}^{k}\frac{|A_1||b_i|}{|\lambda_\sigma|}\frac{\eta_i^{\hat{\varrho}_\sigma}}{\Gamma(\hat{\varrho}_\sigma+1)}+\frac{1}{|\lambda_\sigma|\Gamma(\hat{\varrho}_\sigma+1)}+\frac{|\gamma|}{|\lambda_\sigma|\Gamma(\hat{\varrho}_\sigma+1)}\Big)\\
&+\frac{\mathrm{v}_2(\kappa)\epsilon_2}{|\gamma|}\Big(\sum_{i=1}^{k}\frac{|a_i|}{|\mu_\sigma|}\frac{\eta_i^{\hat{\beta}_\sigma}}{\Gamma(\hat{\beta}_\sigma+1)}+\frac{|A_1|}{|\mu_\sigma|\Gamma(\hat{\beta}_\sigma+1)}\Big),\\
\|\chi(\kappa)-\mathcal{U}_2(\varphi,\chi)\|\leq{}&\frac{\mathrm{v}_2(\kappa)\epsilon_2}{|\gamma|}\Big(\sum_{i=1}^{k}\frac{|A_2||a_i|}{|\mu_\sigma|}\frac{\eta_i^{\hat{\beta}_\sigma}}{\Gamma(\hat{\beta}_\sigma+1)}+\frac{1}{|\mu_\sigma|\Gamma(\hat{\beta}_\sigma+1)}+\frac{|\gamma|}{|\mu_\sigma|\Gamma(\hat{\beta}_\sigma+1)}\Big)\\
&+\frac{\mathrm{v}_1(\kappa)\epsilon_1}{|\gamma|}\Big(\sum_{i=1}^{k}\frac{|b_i|}{|\lambda_\sigma|}\frac{\eta_i^{\hat{\varrho}_\sigma}}{\Gamma(\hat{\varrho}_\sigma+1)}+\frac{|A_2|}{|\lambda_\sigma|\Gamma(\hat{\varrho}_\sigma+1)}\Big),
\end{aligned}
$$

Proof. Assuming that $(\varphi,\chi)\in X\times X$ is the solution to (1.1), then

$$
\begin{aligned}
\varphi(\kappa)=\mathcal{U}_1(\varphi,\chi)+\frac{\kappa^{N-1}}{\gamma}\Big[&\sum_{i=1}^{k}\frac{a_i}{\mu_\sigma}\int_0^{\eta_i}\frac{(\eta_i-z)^{\beta_\sigma-1}}{\Gamma(\beta_\sigma)}\psi_2(z)\,dz-\frac{A_1}{\mu_\sigma}\int_0^1\frac{(1-z)^{\beta_\sigma-1}}{\Gamma(\beta_\sigma)}\psi_2(z)\,dz\\
&+\sum_{i=1}^{k}\frac{A_1b_i}{\lambda_\sigma}\int_0^{\eta_i}\frac{(\eta_i-z)^{\varrho_\sigma-1}}{\Gamma(\varrho_\sigma)}\psi_1(z)\,dz-\frac{1}{\lambda_\sigma}\int_0^1\frac{(1-z)^{\varrho_\sigma-1}}{\Gamma(\varrho_\sigma)}\psi_1(z)\,dz\Big]\\
&+\frac{1}{\lambda_\sigma}\int_0^{\kappa}\frac{(\kappa-z)^{\varrho_\sigma-1}}{\Gamma(\varrho_\sigma)}\psi_1(z)dz,
\end{aligned}
\tag{5.8}
$$

Using Remark 5.2, we get

$$\begin{aligned}
|\varphi(\kappa)-\mathcal{U}_1(\varphi,\chi)| \leq{}& \frac{1}{|\gamma|}\Bigg[\sum_{i=1}^{k}\frac{|a_i|}{|\mu_\sigma|}\int_0^{\eta_i}\frac{(\eta_i-z)^{\beta_\sigma-1}}{\Gamma(\beta_\sigma)}|\psi_2(z)|\,dz+\frac{|A_1|}{|\mu_\sigma|}\int_0^1\frac{(1-z)^{\beta_\sigma-1}}{\Gamma(\beta_\sigma)}|\psi_2(z)|\,dz\\
&+\sum_{i=1}^{k}\frac{|A_1||b_i|}{|\lambda_\sigma|}\int_0^{\eta_i}\frac{(\eta_i-z)^{\varrho_\sigma-1}}{\Gamma(\varrho_\sigma)}|\psi_1(z)|dz+\frac{1}{|\lambda_\sigma|}\int_0^1\frac{(1-z)^{\varrho_\sigma-1}}{\Gamma(\varrho_\sigma)}|\psi_1(z)|\,dz\Bigg]\\
&+\frac{1}{|\lambda_\sigma|}\int_0^{\kappa}\frac{(\kappa-z)^{\varrho_\sigma-1}}{\Gamma(\varrho_\sigma)}|\psi_1(z)|dz\\
\leq{}& \frac{\mathrm{v}_1(\kappa)\epsilon_1}{|\gamma|}\left(\sum_{i=1}^{k}\frac{|A_1||b_i|}{|\lambda_\sigma|}\frac{\eta_i^{\varrho_\sigma}}{\Gamma(\varrho_\sigma+1)}+\frac{1}{|\lambda_\sigma|\Gamma(\varrho_\sigma+1)}+\frac{|\gamma|}{|\lambda_\sigma|\Gamma(\varrho_\sigma+1)}\right),\\
&+\frac{\mathrm{v}_2(\kappa)\epsilon_2}{|\gamma|}\left(\sum_{i=1}^{k}\frac{|a_i|}{|\mu_\sigma|}\frac{\eta_i^{\beta_\sigma}}{\Gamma(\beta_\sigma+1)}+\frac{|A_1|}{|\mu_\sigma|\Gamma(\beta_\sigma+1)}\right).\frac{\eta_i^{\varrho_\sigma}}{\Gamma(\varrho_\sigma+1)}+\frac{|A_2|}{|\lambda_\sigma|\Gamma(\varrho_\sigma+1)}\Bigg).
\end{aligned}$$

Similarly we can find:

$$\begin{aligned}
\chi(\kappa)={}&\mathcal{U}_2(\varphi,\chi)+\frac{\kappa^{N-1}}{\gamma}\Bigg[\sum_{i=1}^{k}\frac{b_i}{\lambda_\sigma}\int_0^{\eta_i}\frac{(\eta_i-z)^{\varrho_\sigma-1}}{\Gamma(\varrho_\sigma)}\psi_1(z)\,dz-\frac{A_2}{\lambda_\sigma}\int_0^1\frac{(1-z)^{\varrho_\sigma-1}}{\Gamma(\varrho_\sigma)}\psi_1(z)\,dz\\
&+\sum_{i=1}^{k}\frac{A_2a_i}{\mu_\sigma}\int_0^{\eta_i}\frac{(\eta_i-z)^{\beta_\sigma-1}}{\Gamma(\beta_\sigma)}\psi_2(z)\,dz-\frac{1}{\mu_\sigma}\int_0^1\frac{(1-z)^{\beta_\sigma-1}}{\Gamma(\beta_\sigma)}\psi_2(z)\,dz\Bigg]\\
&+\frac{1}{\mu_\sigma}\int_0^{\kappa}\frac{(\kappa-z)^{\beta_\sigma-1}}{\Gamma(\beta_\sigma)}\psi_2(z)dz,\\
|\chi(\kappa)-\mathcal{U}_2(\varphi,\chi)|\leq{}&\frac{1}{|\gamma|}\Bigg[\sum_{i=1}^{k}\frac{|b_i|}{|\lambda_\sigma|}\int_0^{\eta_i}\frac{(\eta_i-z)^{\varrho_\sigma-1}}{\Gamma(\varrho_\sigma)}|\psi_1(z)|\,dz+\frac{|A_2|}{|\lambda_\sigma|}\int_0^1\frac{(1-z)^{\varrho_\sigma-1}}{\Gamma(\varrho_\sigma)}|\psi_1(z)|\,dz\\
&+\sum_{i=1}^{k}\frac{|A_2||a_i|}{|\mu_\sigma|}\int_0^{\eta_i}\frac{(\eta_i-z)^{\beta_\sigma-1}}{\Gamma(\beta_\sigma)}|\psi_2(z)|\,dz+\frac{1}{|\mu_\sigma|}\int_0^1\frac{(1-z)^{\beta_\sigma-1}}{\Gamma(\beta_\sigma)}|\psi_2(z)|\,dz\Bigg]\\
&+\frac{1}{|\mu_\sigma|}\int_0^{\kappa}\frac{(\kappa-z)^{\beta_\sigma-1}}{\Gamma(\beta_\sigma)}|\psi_2(z)|dz,\\
\leq{}&\frac{\mathrm{v}_2(\kappa)\epsilon_2}{|\gamma|}\left(\sum_{i=1}^{k}\frac{|A_2||a_i|}{|\mu_\sigma|}\frac{\eta_i^{\beta_\sigma}}{\Gamma(\beta_\sigma+1)}+\frac{1}{|\mu_\sigma|\Gamma(\beta_\sigma+1)}+\frac{|\gamma|}{|\mu_\sigma|\Gamma(\beta_\sigma+1)}\right)\\
&+\frac{\mathrm{v}_1(\kappa)\epsilon_1}{|\gamma|}\left(\sum_{i=1}^{k}\frac{|b_i|}{|\lambda_\sigma|}\frac{\eta_i^{\varrho_\sigma}}{\Gamma(\varrho_\sigma+1)}+\frac{|A_2|}{|\lambda_\sigma|\Gamma(\varrho_\sigma+1)}\right),\\
\|\varphi(\kappa)-\mathcal{U}_1(\varphi,\chi)\|\leq{}&\frac{\mathrm{v}_1(\kappa)\epsilon_1}{|\gamma|}\left(\sum_{i=1}^{k}\frac{|A_1||b_i|}{|\lambda_\sigma|}\frac{\eta_i^{\varrho_\sigma}}{\Gamma(\varrho_\sigma+1)}+\frac{1}{|\lambda_\sigma|\Gamma(\varrho_\sigma+1)}+\frac{|\gamma|}{|\lambda_\sigma|\Gamma(\varrho_\sigma+1)}\right)\\
&+\frac{\mathrm{v}_2(\kappa)\epsilon_2}{|\gamma|}\left(\sum_{i=1}^{k}\frac{|a_i|}{|\mu_\sigma|}\frac{\eta_i^{\beta_\sigma}}{\Gamma(\beta_\sigma+1)}+\frac{|A_1|}{|\mu_\sigma|\Gamma(\beta_\sigma+1)}\right),
\end{aligned}$$

$$\|\chi(\kappa)-\mathcal{U}_2(\varphi,\chi)\| \leq \frac{\mathrm{v}_2(\kappa)\epsilon_2}{|\gamma|}\Big(\sum_{i=1}^{k}\frac{|A_2||a_i|}{|\mu_\sigma|}\frac{\eta_i^{\beta_\sigma}}{\Gamma(\beta_\sigma+1)}+\frac{1}{|\mu_\sigma|\Gamma(\beta_\sigma+1)}+\frac{|\gamma|}{|\mu_\sigma|\Gamma(\beta_\sigma+1)}\Big)$$
$$+\frac{\mathrm{v}_1(\kappa)\epsilon_1}{|\gamma|}\Big(\sum_{i=1}^{k}\frac{|b_i|}{|\lambda_\sigma|}\frac{\eta_i^{\varrho_\sigma}}{\Gamma(\varrho_\sigma+1)}+\frac{|A_2|}{|\lambda_\sigma|\Gamma(\varrho_\sigma+1)}\Big).$$

□

Theorem 5.2. *Under the hypotheses $(B_1),(B_2)$ and assuming Equations (4.1) hold with $K<1$, where $K=max\{K_1,K_2\}$. Then, the problem (1.1) is both (UHR) and (GUHR) stable.*

Proof. Given any solution $(\varphi,\chi)\in X\times X$ and the unique solution (φ^*,χ^*) of problem(1.1), we obtain

$$\|(\varphi,\chi)-(\varphi^*,\chi^*)\| = \|(\varphi,\chi)-\mathcal{U}(\varphi^*,\chi^*)\|$$
$$\leq \|\varphi-\mathcal{U}_1(\varphi,\chi)\|+\|\chi-\mathcal{U}_2(\varphi,\chi)\|+\|\mathcal{U}_1(\varphi,\chi)-\mathcal{U}_1(\varphi^*,\chi^*)\|+\|\mathcal{U}_2(\varphi,\chi)-\mathcal{U}_2(\varphi^*,\chi^*)\|$$

by use lemma(5.2) and theorem(4.1) we have:

$$\begin{aligned}
\|(\varphi,\chi)-(\varphi^*,\chi^*)\| \leq{}& \frac{\mathrm{v}_1(\kappa)\epsilon_1}{|\gamma|}\Bigg(\sum_{i=1}^{k}\frac{|A_1||b_i|}{|\lambda_\sigma|}\frac{\eta_i^{\hat{\varrho}_\sigma}}{\Gamma(\hat{\varrho}_\sigma+1)}+\frac{1}{|\lambda_\sigma|\Gamma(\hat{\varrho}_\sigma+1)}+\frac{|\gamma|}{|\lambda_\sigma|\Gamma(\hat{\varrho}_\sigma+1)}\Bigg)\\
&+\frac{\mathrm{v}_2(\kappa)\epsilon_2}{|\gamma|}\Bigg(\sum_{i=1}^{k}\frac{|a_i|}{|\mu_\sigma|}\frac{\eta_i^{\hat{\beta}_\sigma}}{\Gamma(\hat{\beta}_\sigma+1)}+\frac{|A_1|}{|\mu_\sigma|\Gamma(\hat{\beta}_\sigma+1)}\Bigg)\\
&+\frac{\mathrm{v}_2(\kappa)\epsilon_2}{|\gamma|}\Bigg(\sum_{i=1}^{k}\frac{|A_2||a_i|}{|\mu_\sigma|}\frac{\eta_i^{\hat{\beta}_\sigma}}{\Gamma(\hat{\beta}_\sigma+1)}+\frac{1}{|\mu_\sigma|\Gamma(\hat{\beta}_\sigma+1)}+\frac{|\gamma|}{|\mu_\sigma|\Gamma(\hat{\beta}_\sigma+1)}\Bigg)\\
&+\frac{\mathrm{v}_1(\kappa)\epsilon_1}{|\gamma|}\Bigg(\sum_{i=1}^{k}\frac{|b_i|}{|\lambda_\sigma|}\frac{\eta_i^{\hat{\varrho}_\sigma}}{\Gamma(\hat{\varrho}_\sigma+1)}+\frac{|A_2|}{|\lambda_\sigma|\Gamma(\hat{\varrho}_\sigma+1)}\Bigg),\\
&+K\|(\varphi,\chi)-(\varphi^*,\chi^*\|\\
\leq{}& \frac{1}{1-K}\Bigg[\frac{\mathrm{v}_1(\kappa)\epsilon_1}{|\gamma|}\Bigg(\sum_{i=1}^{k}\frac{|A_1||b_i|}{|\lambda_\sigma|}\frac{\eta_i^{\hat{\varrho}_\sigma}}{\Gamma(\hat{\varrho}_\sigma+1)}+\frac{1}{|\lambda_\sigma|\Gamma(\hat{\varrho}_\sigma+1)}\\
&+\frac{|\gamma|}{|\lambda_\sigma|\Gamma(\hat{\varrho}_\sigma+1)}+\sum_{i=1}^{k}\frac{|b_i|}{|\lambda_\sigma|}\frac{\eta_i^{\hat{\varrho}_\sigma}}{\Gamma(\hat{\varrho}_\sigma+1)}+\frac{|A_2|}{|\lambda_\sigma|\Gamma(\hat{\varrho}_\sigma+1)}\Bigg)\\
&+\frac{\mathrm{v}_2(\kappa)\epsilon_2}{|\gamma|}\Bigg(\sum_{i=1}^{k}\frac{|A_2||a_i|}{|\mu_\sigma|}\frac{\eta_i^{\hat{\beta}_\sigma}}{\Gamma(\hat{\beta}_\sigma+1)}+\frac{1}{|\mu_\sigma|\Gamma(\hat{\beta}_\sigma+1)}+\frac{|\gamma|}{|\mu_\sigma|\Gamma(\hat{\beta}_\sigma+1)}\\
&+\sum_{i=1}^{k}\frac{|a_i|}{|\mu_\sigma|}\frac{\eta_i^{\hat{\beta}_\sigma}}{\Gamma(\hat{\beta}_\sigma+1)}+\frac{|A_1|}{|\mu_\sigma|\Gamma(\hat{\beta}_\sigma+1)}\Bigg)\Bigg]\\
\leq{}& \mathcal{H}\mathrm{v}(\kappa)\epsilon,
\end{aligned}$$

where

$$\mathcal{H}=max\Bigg\{\frac{1}{(1-K)|\gamma|}\Bigg(\sum_{i=1}^{k}\frac{|A_1||b_i|}{|\lambda_\sigma|}\frac{\eta_i^{\hat{\varrho}_\sigma}}{\Gamma(\hat{\varrho}_\sigma+1)}+\frac{1}{|\lambda_\sigma|\Gamma(\hat{\varrho}_\sigma+1)}+\frac{|\gamma|}{|\lambda_\sigma|\Gamma(\hat{\varrho}_\sigma+1)}+\sum_{i=1}^{k}\frac{|b_i|}{|\lambda_\sigma|}\frac{\eta_i^{\hat{\varrho}_\sigma}}{\Gamma(\hat{\varrho}_\sigma+1)}$$

$$+\frac{|A_2|}{|\lambda_\sigma|\Gamma(\hat{\varrho}_\sigma+1)}\Bigg),\frac{1}{(1-K)|\gamma|}\Bigg(\sum_{i=1}^{k}\frac{|A_2||a_i|}{|\mu_\sigma|}\frac{\eta_i^{\hat{\beta}_\sigma}}{\Gamma(\hat{\beta}_\sigma+1)}+\frac{1}{|\mu_\sigma|\Gamma(\hat{\beta}_\sigma+1)}+\frac{|\gamma|}{|\mu_\sigma|\Gamma(\hat{\beta}_\sigma+1)}$$

$$+\sum_{i=1}^{k}\frac{|a_i|}{|\mu_\sigma|}\frac{\eta_i^{\hat{\beta}_\sigma}}{\Gamma(\hat{\beta}_\sigma+1)}+\frac{|A_1|}{|\mu_\sigma|\Gamma(\hat{\beta}_\sigma+1)}\Bigg)\Bigg\},$$

Hence, the system (1.1) enjoys the (UHR) stability properties. Also, the (GUHR) stability of system (1.1) follows by setting $\epsilon=1$. □

6. Illustrative Example

Example 6.1. *We consider a coupled system of multi-term fractional differential equations with nonlocal boundary conditions:*

$$\begin{cases}\sum_{l=1}^{4}\lambda_l{}^cD^{\hat{\varrho}_l}\varphi(\kappa)=h_1(\kappa,\varphi(\kappa),\chi(\kappa)),\\ \sum_{j=1}^{4}\mu_j{}^cD^{\hat{\beta}_j}\chi(\kappa)=h_2(\kappa,\varphi(\kappa),\chi(\kappa)),\end{cases}\tag{6.1}$$

$$\varphi(0)=0,\varphi'(0)=0,\varphi^{N-2}(0)=0,\varphi(1)=\sum_{i=1}^{3}a_i\chi(\eta_i).$$

$$\chi(0)=0,\chi'(0)=0,\chi^{N-2}(0)=0,\chi(1)=\sum_{i=1}^{3}b_i\varphi(\eta_i).$$

where

$$\hat{\varrho}_1=\frac{1}{2},\quad \hat{\varrho}_2=\frac{1}{4},\quad \hat{\varrho}_3=\frac{1}{8},\quad \hat{\varrho}_4=\frac{63}{16}=\hat{\varrho}_\sigma,\quad \hat{\beta}_1=\frac{1}{3},\quad \hat{\beta}_2=\frac{1}{9},\quad \hat{\beta}_3=\frac{1}{27},\quad \hat{\beta}_4=\frac{323}{81}=\hat{\beta}_\sigma$$

$$\mu_1=2,\quad \mu_2=-\frac{1}{15},\quad \mu_3=\frac{1}{25},\quad \mu_4=-2,\quad \lambda_1=3,\quad \lambda_2=\frac{-1}{10},\quad \lambda_3=\frac{3}{50},\quad \lambda_4=-3$$

$$a_1=\frac{-1}{5},\quad a_2=\frac{1}{10},\quad a_3=\frac{3}{10},\quad b_1=\frac{1}{50},\quad b_2=\frac{3}{100},\quad b_3=\frac{3}{50},\quad \eta_1=\frac{1}{2},\quad \eta_2=\frac{1}{4},\quad \eta_3=\frac{1}{8},$$

$$h_1(\kappa,\varphi(\kappa),\chi(\kappa))=\frac{1}{\sqrt{\kappa^2+4}}\Big(\cos\varphi(\kappa)+|\chi(\kappa)|+\tan^{-1}\kappa\Big)$$
$$h_2(\kappa,\varphi(\kappa),\chi(\kappa))=\frac{|\varphi(\kappa)|+\sin\chi(\kappa)}{\kappa^3+4}+\frac{\kappa^2}{8}$$

we obtain

$$\begin{aligned}|h_1(\kappa,\varphi(\kappa),\chi(\kappa))-h_1(\kappa,\varphi^*(\kappa),\chi^*(\kappa))|&\leq\frac{1}{2}\Big(|\varphi(\kappa)-\varphi^*(\kappa)|+|\chi(\kappa)-\chi^*(\kappa)|\Big),\\ |h_2(\kappa,\varphi(\kappa),\chi(\kappa))-h_2(\kappa,\varphi^*(\kappa),\chi^*(\kappa))|&\leq\frac{1}{4}\Big(|\varphi(\kappa)-\varphi^*(\kappa)|+|\chi(\kappa)-\chi^*(\kappa)|\Big),\end{aligned}\tag{6.2}$$

with $L_1 = 1/2, L_2 = 1/4$ *and* $K = 0.212314121903 < 1$ *,and we have* $\gamma = 1.000070518494 \neq 0$. *Accordingly, since the conditions required by Theorem (4.1) hold, problem (1.1) admits a unique solution on* $[0, 1]$.

7. Conclusion

This paper investigated a system of coupled fractional differential equations and a specific class of boundary value problems involving nonlocal conditions. We first derived the integral operator that corresponds to the proposed system. Then, to prove the existence and uniqueness of solutions, we utilized Krasnoselskii's fixed-point theorem and the Banach contraction principle. Finally, we developed several stability frameworks, including Ulam-Hyers, Ulam-Hyers-Rassias, and their generalization. These results give new contributions to the field of boundary value problems in fractional calculus.

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