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Weakly Regular Semigroups Characterized in Terms of Neutrosophic Bipolar-Valued Fuzzy Ideals

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Abstract. In this paper, we introduce several new concepts of neutrosophic bipolar valued fuzzy structures in semigroups, specifically focusing on neutrosophic bipolar valued fuzzy bi-ideals, generalized bi-ideals, ideals, and interior ideals. We investigate the fundamental properties of these notions in detail. Furthermore, we establish several characterizations of weakly regular semigroups in terms of these newly defined fuzzy ideals. Our results extend existing studies on bipolar fuzzy sets and neutrosophic structures and provide a more robust framework for managing uncertainty within algebraic systems.

1. Introduction

Many real-world problems involve uncertainty, incomplete information, and vagueness. To address these challenges, fuzzy set theory provides an effective mathematical framework for modeling such situations. This theory was first proposed by Zadeh in 1965 to represent imprecise information through membership functions. Subsequently, Atanassov [1] introduced intuitionistic fuzzy sets, which generalize classical fuzzy sets by incorporating both membership and non-membership degrees, thereby providing a more flexible way to describe the uncertainty.

To further extend these concepts, Smarandache [17] introduced the cencepts of neutrosophic sets in 1999. In this model, the degrees of truth, indeterminacy, and falsity are treated independently, making it uniquely suitable for modeling scenarious where uncertainty and inconsistency coexist. These ideas have been widely applied to various algebraic structures such as fields, rings, vector spaces, groups, and semigroups [4, 7–10, 16, 18, 19]. In particular, Kuroki [14] initiated the study of fuzzy sets in semigroups in 1979 by introducing the concept of fuzzy ideals and related structures.

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Another significant development is the notion of bipolar fuzzy sets, introduced by Zhang [21] in 1994. This concept allows the simultaneous representation of positive and negative information, which is useful in decision-making processes and uncertainty analysis. Since then, bipolar fuzzy structures have been extensively studied in various algebraic contexts.

In recent years, Gaketem and Khamrot [5] studied bipolar fuzzy weakly interior ideals in semigroups and examined their relationships with other fuzzy ideals. Subsequently, Gaketem et al. [6] introduced bipolar fuzzy implicative UP-filters in UP-algebras. Further research has continued to develop the theory and applications of bipolar fuzzy sets [2,12,13].

More recently, Deetae and Khamrot [3] investigated neutrosophic structures on bipolar valued fuzzy sets by incorporating positive and negative components of truth, indeterminacy, and falsity. They also studied the fundamental properties of bipolar valued fuzzy subsemigroups. Later Khamrot et al. [11] characterized quasi-regular semigroups in terms neutrosophic bipolar valued fuzzy ideals.

Motivated by these developments, this paper introduces several new classes of neutrosophic bipolar valued fuzzy ideals in semigroups. We investigate their fundamental properties and establish characterizations of weakly regular semigroups based on these notions. The paper is organized as follows. Section 2 reviews basic definitions related to semigroups and fuzzy structures, while the subsequent sections present the main results and their characterizations.

2. Preliminaries

This section presents the fundamental definitions and notations related to semigroups and various fuzzy set models that will be used throughout this paper.

Let $\emptyset \neq \mathfrak{W} \subseteq \mathfrak{G}$, where G is a semigroup (SG). Then we define the following

(1) $\mathfrak{W}$ is called a *subsemigroup* (SSG) of $\mathfrak{G}$ if $\mathfrak{W}\mathfrak{W}^2 \subseteq \mathfrak{W}$.
(2) $\mathfrak{W}$ is called a *left ideal* (LID) [right ideal (RID)] of $\mathfrak{G}$ if $\mathfrak{G}\mathfrak{W} \subseteq \mathfrak{W}[\mathfrak{W}\mathfrak{G} \subseteq \mathfrak{W}]$.
(3) $\mathfrak{W}$ is called an *ideal* (ID) of $\mathfrak{G}$ if it is an LID and a RID of $\mathfrak{G}$.
(4) $\mathfrak{W}$ is called a *generalized bi-ideal* (GBID) of $\mathfrak{G}$ if $\mathfrak{W}\mathfrak{S}\mathfrak{W} \subseteq \mathfrak{W}$.
(5) $\mathfrak{W}$ is called a *bi-ideal* (BID) of $\mathfrak{G}$ if $\mathfrak{W}$ is an SSG and $\mathfrak{W}$ is a GIBd of $\mathfrak{G}$.
(6) $\mathfrak{W}$ is called an *interior ideal* (INID) of $\mathfrak{G}$ if $\mathfrak{W}$ is an SSG and $\mathfrak{G}\mathfrak{W}\mathfrak{G} \subseteq \mathfrak{W}$.
(7) $\mathfrak{W}$ is called a *quasi-ideal* (QID) of $\mathfrak{G}$ if $\mathfrak{G}\mathfrak{W} \cap \mathfrak{W}\mathfrak{G} \subseteq \mathfrak{W}$.

An SG $\mathfrak{G}$ is called *weakly regular* (WR) if, for every $\mathfrak{v} \in \mathfrak{G}$, there exists $\mathfrak{x}, \mathfrak{y} \in \mathfrak{G}$ such that $\mathfrak{v} = \mathfrak{v}\mathfrak{x}\mathfrak{v}\mathfrak{y}$.

A *fuzzy set* (FS) ω of a non-empty set $\mathfrak{V}$ is defined as a mapping from $\mathfrak{V}$ into the unit interval $[0,1]$.

Definition 2.1. *[1] An **intuitionistic fuzzy set** (IF set) $\mathfrak{E} \neq \emptyset$ in set $\mathfrak{V}$ is defined by*

$$\mathfrak{E} := \{(\mathfrak{z}, \rho_{\mathfrak{E}}(\mathfrak{z}), \upsilon_{\mathfrak{E}}(\mathfrak{z})) \mid \mathfrak{z} \in \mathfrak{V}\},$$

where $\rho_{\mathfrak{E}} : \mathfrak{V} \to [0,1]$ and $\upsilon_{\mathfrak{E}} : \mathfrak{V} \to [0,1]$ denote the grade of membership and non-membership functions, respectively, satisfying $0 \leq \rho_{\mathfrak{E}}(\mathfrak{z}) + \upsilon_{\mathfrak{E}}(\mathfrak{z}) \leq 1$ for all $\mathfrak{z} \in \mathfrak{V}$.

Definition 2.2. *[21] A* bipolar fuzzy set *(shortly, BF set) ω in $\mathfrak{V}$ is defined as*

$$\omega := \{(\mathfrak{z}, \omega^+(\mathfrak{z}), \omega^-(\mathfrak{z})) \mid \mathfrak{z} \in \mathfrak{V}\},$$

where $\omega^+ : \mathfrak{V} \to [0,1]$ represents the positive membership degree and $\omega^- : \mathfrak{V} \to [-1,0]$ represents the negative membership degree.

Definition 2.3. *[17] A neutrosophic set (NS) $\mathfrak{E}$ in a non-empty set $\mathfrak{V}$ is defined as*

$$\mathfrak{E} = \{\langle \mathfrak{z}, \mathfrak{T}_{\mathfrak{E}}(\mathfrak{z}), \mathfrak{I}_{\mathfrak{E}}(\mathfrak{z}), \mathfrak{F}_{\mathfrak{E}}(\mathfrak{z})\rangle : \mathfrak{z} \in \mathfrak{V}\},$$

where $\mathfrak{T}_{\mathfrak{E}}, \mathfrak{I}_{\mathfrak{E}}, \mathfrak{F}_{\mathfrak{E}} : \mathfrak{V} \to [0,1]$ denote the truth-membership, the indeterminacy-membership, the falsity-membership functions, respectively.

Next, we recall the notion of neutrosophic bipolar valued fuzzy sets.

Definition 2.4. *[17] Let $\emptyset \neq \mathfrak{V}$. A neutrosophic bipolar valued fuzzy set (NSBF) $\mathfrak{E}$ in $\mathfrak{V}$ is an object of the form $\mathfrak{E} = \{\langle \mathfrak{z}, \mathfrak{T}^+_{\mathfrak{E}}(\mathfrak{z}), \mathfrak{I}^+_{\mathfrak{E}}(\mathfrak{z}), \mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{z}), \mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{v}), \mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{z}), \mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{z})\rangle : \mathfrak{z} \in \mathfrak{V}\}$,*
where $\mathfrak{T}^+_{\mathfrak{E}}, \mathfrak{I}^+_{\mathfrak{E}}, \mathfrak{F}^+_{\mathfrak{E}} : \mathfrak{V} \to [0,1]$ and $\mathfrak{T}^-_{\mathfrak{E}}, \mathfrak{I}^-_{\mathfrak{E}}, \mathfrak{F}^-_{\mathfrak{E}} : \mathfrak{V} \to [-1,0]$.

For convenience, we denote $\mathfrak{E} = (\mathfrak{E}^-, \mathfrak{E}^+)$ by the NSBF

$$\mathfrak{E} = \{\langle \mathfrak{z}, \mathfrak{T}^+_{\mathfrak{E}}(\mathfrak{z}), \mathfrak{I}^+_{\mathfrak{E}}(\mathfrak{z}), \mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{z}), \mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{z}), \mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{z}), \mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{z})\rangle : \mathfrak{z} \in \mathfrak{V}\}.$$

Definition 2.5. *[3] An NSBF set $\mathfrak{E} = (\mathfrak{E}^-, \mathfrak{E}^+)$ in an SG $\mathfrak{G}$ is called an NSBF subsemigroup (NSBF SSG) if it satisfies:*

$$(\forall \mathfrak{z}, \mathfrak{k} \in \mathfrak{S}) \begin{pmatrix} \mathfrak{T}^+_{\mathfrak{E}}(\mathfrak{z}\mathfrak{k}) \geq \mathfrak{T}^+_{\mathfrak{E}}(\mathfrak{z}) \wedge \mathfrak{T}^+_{\mathfrak{E}}(\mathfrak{k}), \\ \mathfrak{I}^+_{\mathfrak{E}}(\mathfrak{z}\mathfrak{k}) \leq \mathfrak{I}^+_{\mathfrak{E}}(\mathfrak{z}) \vee \mathfrak{I}^+_{\mathfrak{E}}(\mathfrak{k}), \\ \mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{z}\mathfrak{k}) \geq \mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{z}) \wedge \mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{k}), \\ \mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{z}\mathfrak{k}) \leq \mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{z}) \vee \mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{k}), \\ \mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{z}\mathfrak{k}) \geq \mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{z}) \wedge \mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{k}), \\ \mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{z}\mathfrak{k}) \leq \mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{z}) \vee \mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{k}). \end{pmatrix}.$$

Example 2.1. *Consider an SG $\mathfrak{S} = \{\check{\delta}_1, \check{\delta}_2, \check{\delta}_3\}$ with the following Cayley table:*

$\blacktriangleright$	$\check{\delta}_1$	$\check{\delta}_2$	$\check{\delta}_3$
$\check{\delta}_1$	$\check{\delta}_3$	$\check{\delta}_3$	$\check{\delta}_3$
$\check{\delta}_2$	$\check{\delta}_3$	$\check{\delta}_3$	$\check{\delta}_1$
$\check{\delta}_3$	$\check{\delta}_3$	$\check{\delta}_2$	$\check{\delta}_3$

Define an NSBF set $\mathfrak{E} = (\mathfrak{E}^-, \mathfrak{E}^+)$ in $\mathfrak{S}$ as follows:

$\mathfrak{S}$	$\mathfrak{T}^+_{\mathfrak{E}}$	$\mathfrak{I}^+_{\mathfrak{E}}$	$\mathfrak{F}^+_{\mathfrak{E}}$	$\mathfrak{T}^-_{\mathfrak{E}}$	$\mathfrak{I}^-_{\mathfrak{E}}$	$\mathfrak{F}^-_{\mathfrak{E}}$
$\check{\delta}_1$	0.3	0.5	0.6	−0.4	−0.6	−0.8
$\check{\delta}_2$	0.2	0.3	0.8	−0.6	−0.7	−0.6
$\check{\delta}_3$	0.7	0.8	0.5	−0.2	−0.3	−0.9

Then $\mathfrak{E} = (\mathfrak{E}^-, \mathfrak{E}^+)$ *is an NSBF SSG of* $\mathfrak{S}$.

Definition 2.6. *[3] Let* $\mathfrak{G}$ *be an SG. An NSBF set* $\mathfrak{E} = (\mathfrak{E}^-, \mathfrak{E}^+)$ *in* $\mathfrak{G}$ *is called an NSBF right ideal (NSBF RID) if the following assertions are valid:*

$$(\forall \mathfrak{z}, \mathfrak{k} \in \mathfrak{G}) \begin{pmatrix} \mathfrak{T}^+_{\mathfrak{E}}(\mathfrak{z}\mathfrak{k}) \geq \mathfrak{T}^+_{\mathfrak{E}}(\mathfrak{z}), \\ \mathfrak{I}^+_{\mathfrak{E}}(\mathfrak{z}\mathfrak{k}) \leq \mathfrak{I}^+_{\mathfrak{E}}(\mathfrak{z}), \\ \mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{z}\mathfrak{k}) \geq \mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{z}), \\ \mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{z}\mathfrak{k}) \leq \mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{z}), \\ \mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{z}\mathfrak{k}) \geq \mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{z}), \\ \mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{z}\mathfrak{k}) \leq \mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{z}). \end{pmatrix}$$

Definition 2.7. *[3] Let* $\mathfrak{G}$ *be an SG. An NSBF set* $\mathfrak{E} = (\mathfrak{E}^-, \mathfrak{E}^+)$ *in* $\mathfrak{G}$ *is called an NSBF left ideal (NSBF LID) if the following assertions are valid:*

$$(\forall \mathfrak{z}, \mathfrak{k} \in \mathfrak{G}) \begin{pmatrix} \mathfrak{T}^+_{\mathfrak{E}}(\mathfrak{z}\mathfrak{k}) \geq \mathfrak{T}^+_{\mathfrak{E}}(\mathfrak{k}), \\ \mathfrak{I}^+_{\mathfrak{E}}(\mathfrak{z}\mathfrak{k}) \leq \mathfrak{I}^+_{\mathfrak{E}}(\mathfrak{k}), \\ \mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{z}\mathfrak{k}) \geq \mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{k}), \\ \mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{z}\mathfrak{k}) \leq \mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{k}), \\ \mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{z}\mathfrak{k}) \geq \mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{k}), \\ \mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{z}\mathfrak{k}) \leq \mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{k}). \end{pmatrix}$$

Definition 2.8. *An NSBF set* $\mathfrak{E} = (\mathfrak{E}^-, \mathfrak{E}^+)$ *in an SG* $\mathfrak{G}$ *is called an NSBF ideal (NSBF ID) if it is both NSBF LID and NSBF RID of* $\mathfrak{G}$.

Example 2.2. *Consider an SG* $\mathfrak{G} = \{\check{\delta}_1, \check{\delta}_2, \check{\delta}_3\}$ *with the ensuing Cayley table:*

$\blacktriangleright$	$\check{\delta}_1$	$\check{\delta}_2$	$\check{\delta}_3$
$\check{\delta}_1$	$\check{\delta}_1$	$\check{\delta}_1$	$\check{\delta}_1$
$\check{\delta}_2$	$\check{\delta}_1$	$\check{\delta}_1$	$\check{\delta}_1$
$\check{\delta}_3$	$\check{\delta}_1$	$\check{\delta}_1$	$\check{\delta}_3$

Define an NSBF $\mathfrak{E} = (\mathfrak{E}^-, \mathfrak{E}^+)$ in $\mathfrak{G}$ as follows:

$\mathfrak{S}$	$\mathfrak{T}^+_{\mathfrak{E}}$	$\mathfrak{I}^+_{\mathfrak{E}}$	$\mathfrak{F}^+_{\mathfrak{E}}$	$\mathfrak{T}^-_{\mathfrak{E}}$	$\mathfrak{I}^-_{\mathfrak{E}}$	$\mathfrak{F}^-_{\mathfrak{E}}$
$\check{\delta}_1$	0.7	0.8	0.1	−0.2	−0.3	−0.9
$\check{\delta}_2$	0.2	0.3	0.2	−0.6	−0.7	−0.7
$\check{\delta}_3$	0.1	0.5	0.2	−0.7	−0.5	−0.8

It is evident that $\mathfrak{E} = (\mathfrak{E}^-, \mathfrak{E}^+)$ is an NSBF ID of $\mathfrak{G}$. Every NSBF RID (resp., NSBF LID) is an NSBF SSG. However, the converse does not necessarily hold, as illustrated by the following example.

Example 2.3. *Consider an SG* $\mathfrak{S} = \{\check{\delta}_1, \check{\mathfrak{z}}_2, \check{\delta}_3, z_4\}$ *with the following Cayley table:*

$\blacktriangleright$	$\check{\delta}_1$	$\check{\delta}_2$	$\check{\delta}_3$	$\check{\delta}_4$
$\check{\delta}_1$	$\check{\delta}_1$	$\check{\delta}_1$	$\check{\delta}_1$	$\check{\delta}_1$
$\check{\delta}_2$	$\check{\delta}_1$	$\check{\delta}_1$	$\check{\delta}_1$	$\check{\delta}_1$
$\check{\delta}_3$	$\check{\delta}_1$	$\check{\delta}_1$	$\check{\delta}_1$	$\check{\delta}_2$
$\check{\delta}_4$	$\check{\delta}_1$	$\check{\delta}_1$	$\check{\delta}_2$	$\check{\delta}_3$

Define an NSBF $\mathfrak{E} = (\mathfrak{E}^-, \mathfrak{E}^+)$ in $\mathfrak{G}$ as follows:

$\mathfrak{S}$	$\mathfrak{T}^+_{\mathfrak{E}}$	$\mathfrak{I}^+_{\mathfrak{E}}$	$\mathfrak{F}^+_{\mathfrak{E}}$	$\mathfrak{T}^-_{\mathfrak{E}}$	$\mathfrak{I}^-_{\mathfrak{E}}$	$\mathfrak{F}^-_{\mathfrak{E}}$
$\check{\delta}_1$	0.5	0.7	0.1	−0.2	−0.1	−0.3
$\check{\delta}_2$	0.3	0.4	0.3	−0.6	−0.7	−0.4
$\check{\delta}_3$	0.5	0.5	0.2	−0.4	−0.5	−0.4
$\check{\delta}_4$	0.2	0.2	0.5	−0.6	−0.8	−0.5

It is evident that $\mathfrak{E} = (\mathfrak{E}^-, \mathfrak{E}^+)$ is an NSBF SSG of $\mathfrak{G}$, but it is not a left NSBF ideal of $\mathfrak{G}$, since $\mathfrak{T}^+_{\mathfrak{E}}(\check{\delta}_4\check{\delta}_3) = \mathfrak{T}^+_{\mathfrak{E}}(\check{\delta}_2) = 0.3 < 0.5 = \mathfrak{T}^+_{\mathfrak{E}}(\check{\delta}_3)$.

Definition 2.9. *[3] Let $\mathfrak{G}$ be an SG. An NSBF SSG $\mathfrak{E} = (\mathfrak{E}^-, \mathfrak{E}^+)$ in $\mathfrak{G}$ is an NSBF interior ideal (NSBF IN ID) in $\mathfrak{G}$ if the below assertions are valid:*

$$(\forall \mathfrak{x}, \mathfrak{z}, \mathfrak{k} \in \mathfrak{S}) \begin{pmatrix} \mathfrak{T}^+_{\mathfrak{E}}(\mathfrak{x}\mathfrak{z}\mathfrak{k}) \geq \mathfrak{T}^+_{\mathfrak{E}}(\mathfrak{z}), \\ \mathfrak{I}^+_{\mathfrak{E}}(\mathfrak{x}\mathfrak{z}\mathfrak{k}) \leq \mathfrak{I}^+_{\mathfrak{E}}(\mathfrak{z}), \\ \mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{x}\mathfrak{z}\mathfrak{k}) \geq \mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{z}), \\ \mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{x}\mathfrak{z}\mathfrak{k}) \leq \mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{z}), \\ \mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{x}\mathfrak{z}\mathfrak{k}) \geq \mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{z}), \\ \mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{x}\mathfrak{z}\mathfrak{k}) \leq \mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{z}). \end{pmatrix}.$$

Remark 2.1. *Every NSBF IDs of an SG $\mathfrak{G}$ is an NSBF In IDs of $\mathfrak{G}$.*

Definition 2.10. *[3] Let NSBF $\mathfrak{E} = (\mathfrak{E}^-, \mathfrak{E}^+)$ in an SG $\mathfrak{G}$ and $\check{\mu}_1, \check{\mu}_2, \check{\mu}_3 \in [0,1]$, $\check{\delta}_1, \check{\delta}_2, \check{\delta}_3 \in [-1,0]$, the sets*

$$(\mathfrak{T}^+_{\mathfrak{E}})^{\check{\mu}_1} = \{\mathfrak{z} \in \mathfrak{S} | \mathfrak{T}^+_{\mathfrak{E}}(\mathfrak{z}) \geq \check{\mu}_1\},$$

$$(\mathfrak{I}^+_{\mathfrak{E}})^{\check{\mu}_2} = \{\mathfrak{z} \in \mathfrak{S} | \mathfrak{I}^+_{\mathfrak{E}}(\mathfrak{z}) \leq \check{\mu}_2\},$$

$$(\mathfrak{F}^+_{\mathfrak{E}})^{\check{\mu}_3} = \{\mathfrak{z} \in \mathfrak{S} | \mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{z}) \geq \check{\mu}_3\}.$$

The set $\mathfrak{E}^+(\check{\mu}_1, \check{\mu}_2, \check{\mu}_3) := \{\mathfrak{z} \in \mathfrak{G} | \mathfrak{T}^+_{\mathfrak{E}}(\mathfrak{z}) \geq \check{\mu}_1, \mathfrak{I}^+_{\mathfrak{E}}(\mathfrak{z}) \leq \check{\mu}_2, \mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{z}) \geq \check{\mu}_3\}$ is called a positive $(\check{\mu}_1, \check{\mu}_2, \check{\mu}_3)$-level of $\mathfrak{E} = (\mathfrak{E}^-, \mathfrak{E}^+)$. It is evident that $P^+_{\mathfrak{E}}(\check{\mu}_1, \check{\mu}_2, \check{\mu}_3) = (\mathfrak{T}^+_{\mathfrak{E}})^{\check{\mu}_1} \cap (\mathfrak{I}^+_{\mathfrak{E}})^{\check{\mu}_2} \cap (\mathfrak{F}^+_{\mathfrak{E}})^{\check{\mu}_3}$, and

$$(\mathfrak{T}^-_{\mathfrak{E}})^{\check{\delta}_1} = \{\mathfrak{z} \in \mathfrak{G} | \mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{z}) \leq \check{\delta}_1\},$$

$$(\mathfrak{I}^-_{\mathfrak{E}})^{\check{\delta}_2} = \{\mathfrak{z} \in \mathfrak{G} | \mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{z}) \geq \check{\delta}_2\},$$

$$(\mathfrak{F}^-_{\mathfrak{E}})^{\check{\delta}_3} = \{\mathfrak{z} \in \mathfrak{G} | \mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{z}) \leq \check{\delta}_3\}.$$

The set $N^-_{\mathfrak{E}}(\check{\delta}_1, \check{\delta}_2, \check{\delta}_3) := \{\mathfrak{z} \in \mathfrak{G} | \mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{z}) \leq \check{\delta}_1, \mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{z}) \geq \check{\delta}_2, \mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{z}) \leq \check{\delta}_3\}$ is called a negative $(\check{\delta}_1, \check{\delta}_2, \check{\delta}_3)$-level of $\mathfrak{E} = (\mathfrak{E}^-, \mathfrak{E}^+)$. It is evident that $N^-_{\mathfrak{E}}(\check{\delta}_1, \check{\delta}_2, \check{\delta}_3) = (\mathfrak{T}^-_{\mathfrak{E}})^{\check{\delta}_1} \cap (\mathfrak{I}^-_{\mathfrak{E}})^{\check{\delta}_2} \cap (\mathfrak{F}^-_{\mathfrak{E}})^{\check{\delta}_3}$.

The set $C^{\pm}_{\mathfrak{E}}(\breve{\mu}_1, \breve{\mu}_2, \breve{\mu}_3, \breve{\delta}_1, \breve{\delta}_2, \breve{\delta}_3) = P\mathfrak{E}^{+}(\breve{\mu}_1, \breve{\mu}_2, \breve{\mu}_3) \cap N\mathfrak{E}^{-}(\breve{\delta}_1, \breve{\delta}_2, \breve{\delta}_3)$ *is called the bipolar* $(\breve{\mu}_1, \breve{\mu}_2, \breve{\mu}_3, \breve{\delta}_1, \breve{\delta}_2, \breve{\delta}_3)$*-level of* $\mathfrak{E} = (\mathfrak{E}^{-}, \mathfrak{E}^{+})$.

Definition 2.11. *[3] For any* $\emptyset \neq \mathfrak{W} \subseteq \mathfrak{V}$*, the characteristic NSBF function of* $\mathfrak{W}$ *in* $\mathfrak{V}$ *is defined to be a structure* $\chi_{\mathfrak{W}} = \{\langle x, \mathfrak{T}^{+}_{\chi_{\mathfrak{W}}}(\mathfrak{z}), \mathfrak{I}^{+}_{\chi_{\mathfrak{W}}}(\mathfrak{z}), \mathfrak{F}^{+}_{\chi_{\mathfrak{W}}}(\mathfrak{z}), \mathfrak{T}^{-}_{\chi_{\mathfrak{W}}}(\mathfrak{z}), \mathfrak{I}^{-}_{\chi_{\mathfrak{W}}}(\mathfrak{z}), \mathfrak{F}^{-}_{\chi_{\mathfrak{W}}}(\mathfrak{z})\rangle : \mathfrak{z} \in \mathfrak{V}\}$*, where*

$$\mathfrak{T}^{+}_{\chi_{\mathfrak{W}}} : \mathfrak{V} \to [0,1]; \mathfrak{z} \mapsto \mathfrak{T}^{+}_{\chi_{\mathfrak{W}}}(\mathfrak{z}) := \begin{cases} 1 & if \ \mathfrak{z} \in \mathfrak{W} \\ 0 & if \ \mathfrak{z} \notin \mathfrak{W}, \end{cases}$$

$$\mathfrak{I}^{+}_{\chi_{\mathfrak{W}}} : \mathfrak{V} \to [0,1]; \mathfrak{z} \mapsto \mathfrak{I}^{+}_{\chi_{\mathfrak{W}}}(\mathfrak{z}) := \begin{cases} 0 & if \ \mathfrak{z} \in \mathfrak{W} \\ 1 & if \ \mathfrak{z} \notin \mathfrak{W}, \end{cases}$$

$$\mathfrak{F}^{+}_{\chi_{\mathfrak{W}}} : \mathfrak{V} \to [0,1]; \mathfrak{z} \mapsto \mathfrak{F}^{+}_{\chi_{\mathfrak{W}}}(\mathfrak{z}) := \begin{cases} 1 & if \ \mathfrak{z} \in \mathfrak{W} \\ 0 & if \ \mathfrak{z} \notin \mathfrak{W}, \end{cases}$$

$$\mathfrak{T}^{-}_{\chi_{\mathfrak{W}}} : \mathfrak{V} \to [-1,0]; \mathfrak{z} \mapsto \mathfrak{T}^{-}_{\chi_{\mathfrak{W}}}(\mathfrak{z}) := \begin{cases} -1 & if \ \mathfrak{z} \in \mathfrak{W} \\ 0 & if \ \mathfrak{z} \notin \mathfrak{W}, \end{cases}$$

$$\mathfrak{I}^{-}_{\chi_{\mathfrak{W}}} : \mathfrak{V} \to [-1,0]; \mathfrak{z} \mapsto \mathfrak{I}^{-}_{\chi_{\mathfrak{W}}}(\mathfrak{z}) := \begin{cases} 0 & if \ \mathfrak{z} \in \mathfrak{W} \\ -1 & if \ \mathfrak{z} \notin \mathfrak{W}, \end{cases}$$

$$\mathfrak{F}^{-}_{\chi_{\mathfrak{W}}} : \mathfrak{V} \to [-1,0]; \mathfrak{z} \mapsto \mathfrak{F}^{-}_{\chi_{\mathfrak{W}}}(\mathfrak{z}) := \begin{cases} -1 & if \ \mathfrak{z} \in \mathfrak{W} \\ 0 & if \ \mathfrak{z} \notin \mathfrak{W}. \end{cases}$$

For convenience, we denote $\chi_{\mathfrak{W}} = (\chi^{+}_{\mathfrak{W}}, \chi^{-}_{\mathfrak{W}})$ by the characteristic NSBF (shortly, CNSBF) function $\chi_{\mathfrak{W}} = \{\langle \mathfrak{z}, \mathfrak{T}^{+}_{\chi_{\mathfrak{W}}}(\mathfrak{z}), \mathfrak{I}^{+}_{\chi_{\mathfrak{W}}}(\mathfrak{z}), \mathfrak{F}^{+}_{\chi_{\mathfrak{W}}}(\mathfrak{z}), \mathfrak{T}^{-}_{\chi_{\mathfrak{W}}}(\mathfrak{z}), \mathfrak{I}^{-}_{\chi_{\mathfrak{W}}}(\mathfrak{r}), \mathfrak{F}^{-}_{\chi_{\mathfrak{W}}}(\mathfrak{z})\rangle : \mathfrak{z} \in \mathfrak{V}\}$. The SG $\mathfrak{G}$ can be considered a fuzzy subset of itself, i.e., $\chi_{\mathfrak{G}}(\mathfrak{z}) = \langle 1, 0, 1, -1, 0, -1\rangle$ for all $\mathfrak{z} \in \mathfrak{G}$.

Definition 2.12. *Let* $\mathfrak{E} = (\mathfrak{E}^{-}, \mathfrak{E}^{+})$ *and* $\mathfrak{B} = (\mathfrak{B}^{-}, \mathfrak{B}^{+})$ *be NSBFs in a semigroup* $\mathfrak{S}$*. Then*

(1) $\mathfrak{E} = (\mathfrak{E}^{-}, \mathfrak{E}^{+})$ *is said to be an NSBF contained in* $\mathfrak{B} = (\mathfrak{B}^{-}, \mathfrak{B}^{+})$*, denoted by* $\mathfrak{E} \sqsubseteq \mathfrak{B} = (\mathfrak{E}^{+} \sqsubseteq \mathfrak{B}^{+}, \mathfrak{E}^{-} \sqsubseteq \mathfrak{B}^{-})$ *if*

$$\mathfrak{T}^{+}_{\mathfrak{E}}(\mathfrak{r}) \leq \mathfrak{T}^{+}_{\mathfrak{B}}(\mathfrak{r}), \quad \mathfrak{I}^{+}_{\mathfrak{E}}(\mathfrak{r}) \geq \mathfrak{I}^{+}_{\mathfrak{B}}(\mathfrak{r}), \quad \mathfrak{F}^{+}_{\mathfrak{E}}(\mathfrak{r}) \leq \mathfrak{F}^{+}_{\mathfrak{B}}(\mathfrak{r}),$$

$$\mathfrak{T}^{-}_{\mathfrak{E}}(\mathfrak{r}) \geq \mathfrak{T}^{-}_{\mathfrak{B}}(\mathfrak{r}), \quad \mathfrak{I}^{-}_{\mathfrak{E}}(\mathfrak{r}) \leq \mathfrak{I}^{-}_{\mathfrak{B}}(\mathfrak{r}), \quad \mathfrak{F}^{-}_{\mathfrak{E}}(\mathfrak{r}) \geq \mathfrak{F}^{-}_{\mathfrak{B}}(\mathfrak{r}),$$

for all $\mathfrak{r} \in \mathfrak{S}$*. If* $\mathfrak{E} \sqsubseteq \mathfrak{B}$ *and* $\mathfrak{B} \sqsubseteq \mathfrak{E}$*, then we say that* $\mathfrak{E} = \mathfrak{B}$*.*

(2) *The union of two NSBFs* $\mathfrak{E} = (\mathfrak{E}^{-}, \mathfrak{E}^{+})$ *and* $\mathfrak{B} = (\mathfrak{B}^{-}, \mathfrak{B}^{+})$ *is defined as*

$$\mathfrak{E} \sqcup \mathfrak{B} = (\mathfrak{E}^{+} \sqcup \mathfrak{B}^{+}, \mathfrak{E}^{-} \sqcup \mathfrak{B}^{-})$$

$$= \{\langle \mathfrak{r}, (\mathfrak{T}^{+}_{\mathfrak{E}} \cup \mathfrak{T}^{+}_{\mathfrak{B}})(\mathfrak{r}), (\mathfrak{I}^{+}_{\mathfrak{E}} \cup \mathfrak{I}^{+}_{\mathfrak{B}})(\mathfrak{r}), (\mathfrak{F}^{+}_{\mathfrak{E}} \cup \mathfrak{F}^{+}_{\mathfrak{B}})(\mathfrak{r}),$$

$$(\mathfrak{T}^-_{\mathfrak{E}} \cup \mathfrak{T}^-_{\mathfrak{B}})(\mathfrak{r}), (\mathfrak{I}^-_{\mathfrak{E}} \cup \mathfrak{I}^-_{\mathfrak{B}})(\mathfrak{r}), (\mathfrak{F}^-_{\mathfrak{E}} \cup \mathfrak{F}^-_{\mathfrak{B}})(\mathfrak{r})\rangle : \mathfrak{r} \in \mathfrak{S}\},$$

where, for all $\mathfrak{r} \in \mathfrak{S}$,

$$(\mathfrak{T}^+_{\mathfrak{E}} \cup \mathfrak{T}^+_{\mathfrak{B}})(\mathfrak{r}) = \mathfrak{T}^+_{\mathfrak{E}}(\mathfrak{r}) \vee \mathfrak{T}^+_{\mathfrak{B}}(\mathfrak{r}), (\mathfrak{I}^+_{\mathfrak{E}} \cup \mathfrak{I}^+_{\mathfrak{B}})(\mathfrak{r}) = \mathfrak{I}^+_{\mathfrak{E}}(\mathfrak{r}) \wedge \mathfrak{I}^+_{\mathfrak{B}}(\mathfrak{r}),$$

$$(\mathfrak{F}^+_{\mathfrak{E}} \cup \mathfrak{F}^+_{\mathfrak{B}})(\mathfrak{r}) = \mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{r}) \vee \mathfrak{F}^+_{\mathfrak{B}}(\mathfrak{r}),$$

$$(\mathfrak{T}^-_{\mathfrak{E}} \cup \mathfrak{T}^-_{\mathfrak{B}})(\mathfrak{r}) = \mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{r}) \wedge \mathfrak{T}^-_{\mathfrak{B}}(\mathfrak{r}),$$

$$(\mathfrak{I}^-_{\mathfrak{E}} \cup \mathfrak{I}^-_{\mathfrak{B}})(\mathfrak{r}) = \mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{r}) \vee \mathfrak{I}^-_{\mathfrak{B}}(\mathfrak{r}),$$

$$(\mathfrak{F}^-_{\mathfrak{E}} \cup \mathfrak{F}^-_{\mathfrak{B}})(\mathfrak{r}) = \mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{r}) \wedge \mathfrak{F}^-_{\mathfrak{B}}(\mathfrak{r}).$$

(3) *The intersection of two NSBFs* $\mathfrak{E} = (\mathfrak{E}^-, \mathfrak{E}^+)$ *and* $\mathfrak{B} = (\mathfrak{B}^-, \mathfrak{B}^+)$ *is defined as*

$$\mathfrak{E} \sqcap \mathfrak{B} = (\mathfrak{E}^+ \sqcap \mathfrak{B}^+, \mathfrak{E}^- \sqcap \mathfrak{B}^-)$$

$$= \{\langle \mathfrak{r}, (\mathfrak{T}^+_{\mathfrak{E}} \cap \mathfrak{T}^+_{\mathfrak{B}})(\mathfrak{r}), (\mathfrak{I}^+_{\mathfrak{E}} \cap \mathfrak{I}^+_{\mathfrak{B}})(\mathfrak{r}), (\mathfrak{F}^+_{\mathfrak{E}} \cap \mathfrak{F}^+_{\mathfrak{B}})(\mathfrak{r}),$$

$$(\mathfrak{T}^-_{\mathfrak{E}} \cap \mathfrak{T}^-_{\mathfrak{B}})(\mathfrak{r}), (\mathfrak{I}^-_{\mathfrak{E}} \cap \mathfrak{I}^-_{\mathfrak{B}})(\mathfrak{r}), (\mathfrak{F}^-_{\mathfrak{E}} \cap \mathfrak{F}^-_{\mathfrak{B}})(\mathfrak{r})\rangle : \mathfrak{r} \in \mathfrak{S}\},$$

where, for all $\mathfrak{r} \in \mathfrak{S}$,

$$(\mathfrak{T}^+_{\mathfrak{E}} \cap \mathfrak{T}^+_{\mathfrak{B}})(\mathfrak{r}) = \mathfrak{T}^+_{\mathfrak{E}}(\mathfrak{r}) \wedge \mathfrak{T}^+_{\mathfrak{B}}(\mathfrak{r}),$$

$$(\mathfrak{I}^+_{\mathfrak{E}} \cap \mathfrak{I}^+_{\mathfrak{B}})(\mathfrak{r}) = \mathfrak{I}^+_{\mathfrak{E}}(\mathfrak{r}) \vee \mathfrak{I}^+_{\mathfrak{B}}(\mathfrak{r}),$$

$$(\mathfrak{F}^+_{\mathfrak{E}} \cap \mathfrak{F}^+_{\mathfrak{B}})(\mathfrak{r}) = \mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{r}) \wedge \mathfrak{F}^+_{\mathfrak{B}}(\mathfrak{r}),$$

$$(\mathfrak{T}^-_{\mathfrak{E}} \cap \mathfrak{T}^-_{\mathfrak{B}})(\mathfrak{r}) = \mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{r}) \vee \mathfrak{T}^-_{\mathfrak{B}}(\mathfrak{r}),$$

$$(\mathfrak{I}^-_{\mathfrak{E}} \cap \mathfrak{I}^-_{\mathfrak{B}})(\mathfrak{r}) = \mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{r}) \wedge \mathfrak{I}^-_{\mathfrak{B}}(\mathfrak{r}),$$

$$(\mathfrak{F}^-_{\mathfrak{E}} \cap \mathfrak{F}^-_{\mathfrak{B}})(\mathfrak{r}) = \mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{r}) \vee \mathfrak{F}^-_{\mathfrak{B}}(\mathfrak{r}).$$

(4) *The product (composition) of two NSBFs is defined by*

$$\mathfrak{E} \mathbin{\bar{\circ}} \mathfrak{B} = (\mathfrak{E}^{+} \circ \mathfrak{B}^{+}, \mathfrak{E}^{-} \circ \mathfrak{B}^{-}).$$

For $\mathfrak{u} \in \mathfrak{S}$, *we define*

$$(\mathfrak{T}^{+}_{\mathfrak{E}} \circ \mathfrak{T}^{+}_{\mathfrak{B}})(\mathfrak{u}) = \begin{cases} \displaystyle\bigvee_{(\mathfrak{h},\mathfrak{r}) \in \mathcal{F}_{\mathfrak{u}}} \left\{ \mathfrak{T}^{+}_{\mathfrak{E}}(\mathfrak{h}) \wedge \mathfrak{T}^{+}_{\mathfrak{B}}(\mathfrak{r}) \right\}, & \text{if } \mathcal{F}_{\mathfrak{u}} \neq \emptyset, \\ 0, & \text{if } \mathcal{F}_{\mathfrak{u}} = \emptyset. \end{cases}$$

$$(\mathfrak{I}^{+}_{\mathfrak{E}} \circ \mathfrak{I}^{+}_{\mathfrak{B}})(\mathfrak{u}) = \begin{cases} \displaystyle\bigwedge_{(\mathfrak{h},\mathfrak{r}) \in \mathcal{F}_{\mathfrak{u}}} \left\{ \mathfrak{I}^{+}_{\mathfrak{E}}(\mathfrak{h}) \vee \mathfrak{I}^{+}_{\mathfrak{B}}(\mathfrak{r}) \right\}, & \text{if } \mathcal{F}_{\mathfrak{u}} \neq \emptyset, \\ 0, & \text{if } \mathcal{F}_{\mathfrak{u}} = \emptyset. \end{cases}$$

$$(\mathfrak{F}^{+}_{\mathfrak{E}} \circ \mathfrak{F}^{+}_{\mathfrak{B}})(\mathfrak{u}) = \begin{cases} \displaystyle\bigvee_{(\mathfrak{h},\mathfrak{r}) \in \mathcal{F}_{\mathfrak{u}}} \left\{ \mathfrak{F}^{+}_{\mathfrak{E}}(\mathfrak{h}) \wedge \mathfrak{F}^{+}_{\mathfrak{B}}(\mathfrak{r}) \right\}, & \text{if } \mathcal{F}_{\mathfrak{u}} \neq \emptyset, \\ 0, & \text{if } \mathcal{F}_{\mathfrak{u}} = \emptyset. \end{cases}$$

$$(\mathfrak{T}^{-}_{\mathfrak{E}} \circ \mathfrak{T}^{-}_{\mathfrak{B}})(\mathfrak{u}) = \begin{cases} \displaystyle\bigwedge_{(\mathfrak{h},\mathfrak{r}) \in \mathcal{F}_{\mathfrak{u}}} \left\{ \mathfrak{T}^{-}_{\mathfrak{E}}(\mathfrak{h}) \vee \mathfrak{T}^{-}_{\mathfrak{B}}(\mathfrak{r}) \right\}, & \text{if } \mathcal{F}_{\mathfrak{u}} \neq \emptyset, \\ 0, & \text{if } \mathcal{F}_{\mathfrak{u}} = \emptyset. \end{cases}$$

$$(\mathfrak{I}^{-}_{\mathfrak{E}} \circ \mathfrak{I}^{-}_{\mathfrak{B}})(\mathfrak{u}) = \begin{cases} \displaystyle\bigvee_{(\mathfrak{h},\mathfrak{r}) \in \mathcal{F}_{\mathfrak{u}}} \left\{ \mathfrak{I}^{-}_{\mathfrak{E}}(\mathfrak{h}) \wedge \mathfrak{I}^{-}_{\mathfrak{B}}(\mathfrak{r}) \right\}, & \text{if } \mathcal{F}_{\mathfrak{u}} \neq \emptyset, \\ 0, & \text{if } \mathcal{F}_{\mathfrak{u}} = \emptyset. \end{cases}$$

$$(\mathfrak{F}^{-}_{\mathfrak{E}} \circ \mathfrak{F}^{-}_{\mathfrak{B}})(\mathfrak{u}) = \begin{cases} \displaystyle\bigwedge_{(\mathfrak{h},\mathfrak{r}) \in \mathcal{F}_{\mathfrak{u}}} \left\{ \mathfrak{F}^{-}_{\mathfrak{E}}(\mathfrak{h}) \vee \mathfrak{F}^{-}_{\mathfrak{B}}(\mathfrak{r}) \right\}, & \text{if } \mathcal{F}_{\mathfrak{u}} \neq \emptyset, \\ 0, & \text{if } \mathcal{F}_{\mathfrak{u}} = \emptyset. \end{cases}$$

where $\mathcal{F}_{\mathfrak{u}} = \{(\mathfrak{h}, \mathfrak{r}) \in \mathfrak{S} \times \mathfrak{S} \mid \mathfrak{u} = \mathfrak{h}\mathfrak{r}\}$.

Theorem 2.1. *Let* $\emptyset \neq \mathfrak{K}$ *of an SG* $\mathfrak{S}$. *Then* $\mathfrak{K}$ *is an SSG (LID, RID, INID) of* $\mathfrak{S}$ *if and only if* $\chi_{\mathfrak{W}} = (\chi^{+}_{\mathfrak{W}}, \chi^{-}_{\mathfrak{W}})$ *is an NSBF SSG (NSBF LID, NSBF RID, NSBF INID) of* $\mathfrak{S}$.

Theorem 2.2. *Let* $\mathfrak{G}$ *be an SG. Then the arbitrary intersection (resp., union) of NSBF SSGs (NSBF LIDs, NSBF RIDs, NSBF INIDs) in* $\mathfrak{S}$ *is an NSBF SSG (NSBF LID, NSBF RID, NSBF INID) of* $\mathfrak{S}$.

3. Main Results

In this section, we provide the definitions of NSBF bi-ideals of an SG and examine the properties of NSBF bi-ideals and NSBF generalized bi-ideals.

Definition 3.1. *Let $\mathfrak{G}$ be an SG. An NSBF SSG $\mathfrak{E} = (\mathfrak{E}^-, \mathfrak{E}^+)$ in $\mathfrak{G}$ is an NSBF bi-ideal (NSBF B ID) in $\mathfrak{G}$ if the below assertions are valid:*

$$(\forall \mathfrak{x}, \mathfrak{z}, \mathfrak{k} \in \mathfrak{S}) \begin{pmatrix} \mathfrak{T}^+_{\mathfrak{E}}(\mathfrak{x}\mathfrak{z}\mathfrak{k}) \geq \mathfrak{T}^+_{\mathfrak{E}}(\mathfrak{x}) \wedge \mathfrak{T}^+_{\mathfrak{E}}(\mathfrak{k}), \\ \mathfrak{I}^+_{\mathfrak{E}}(\mathfrak{x}\mathfrak{z}\mathfrak{k}) \leq \mathfrak{I}^+_{\mathfrak{E}}(\mathfrak{x}) \vee \mathfrak{I}^+_{\mathfrak{E}}(\mathfrak{k}), \\ \mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{x}\mathfrak{z}\mathfrak{k}) \geq \mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{x}) \wedge \mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{k}), \\ \mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{x}\mathfrak{z}\mathfrak{k}) \leq \mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{x}) \vee \mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{k}), \\ \mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{x}\mathfrak{z}\mathfrak{k}) \geq \mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{x}) \wedge \mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{k}), \\ \mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{x}\mathfrak{z}\mathfrak{k}) \leq \mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{x}) \vee \mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{k}). \end{pmatrix}$$

Example 3.1. *Consider an SG $\mathfrak{G} = \{\check{\delta}_1, \check{\mathfrak{z}}_2, \check{\delta}_3, \check{\delta}_4\}$ with the following Cayley table:*

$\blacktriangleright$	$\check{\delta}_1$	$\check{\delta}_2$	$\check{\delta}_3$	$\check{\delta}_4$
$\check{\delta}_1$	$\check{\delta}_1$	$\check{\delta}_1$	$\check{\delta}_1$	$\check{\delta}_1$
$\check{\delta}_2$	$\check{\delta}_1$	$\check{\delta}_1$	$\check{\delta}_1$	$\check{\delta}_1$
$\check{\delta}_3$	$\check{\delta}_1$	$\check{\delta}_1$	$\check{\delta}_1$	$\check{\delta}_2$
$\check{\delta}_4$	$\check{\delta}_1$	$\check{\delta}_1$	$\check{\delta}_2$	$\check{\delta}_3$

Define an NSBF $\mathfrak{E} = (\mathfrak{E}^-, \mathfrak{E}^+)$ in $\mathfrak{G}$ as follows:

$\mathfrak{S}$	$\mathfrak{T}^+_{\mathfrak{E}}$	$\mathfrak{I}^+_{\mathfrak{E}}$	$\mathfrak{F}^+_{\mathfrak{E}}$	$\mathfrak{T}^-_{\mathfrak{E}}$	$\mathfrak{I}^-_{\mathfrak{E}}$	$\mathfrak{F}^-_{\mathfrak{E}}$
$\check{\delta}_1$	0.6	0.1	0.5	−0.1	−0.6	−0.1
$\check{\delta}_2$	0.3	0.3	0.2	−0.4	−0.3	−0.3
$\check{\delta}_3$	0.4	0.2	0.1	−0.3	−0.4	−0.2
$\check{\delta}_4$	0.1	0.4	0.1	−0.5	−0.1	−0.4

It is easy to examine that $\mathfrak{E} = (\mathfrak{E}^-, \mathfrak{E}^+)$ is an NSBF B ID of $\mathfrak{G}$, but it is not a left NSBF ID of $\mathfrak{G}$, since $\mathfrak{T}^+_{\mathfrak{E}}(\check{\delta}_4\check{\delta}_3) = \mathfrak{T}^+_{\mathfrak{E}}(\check{\delta}_2) = 0.3 < 0.4 = \mathfrak{T}^+_{\mathfrak{E}}(\check{\delta}_3)$.

Remark 3.1. *Every NSBF IDs of an SG $\mathfrak{G}$ is an NSBF B IDs of $\mathfrak{G}$.*

Definition 3.2. *Let $\mathfrak{G}$ be an SG. An NSBF SG $\mathfrak{E} = (\mathfrak{E}^-, \mathfrak{E}^+)$ in $\mathfrak{G}$ is an NSBF generalized bi-ideal (NSBF GB ID) in $\mathfrak{G}$ if the below assertions are valid:*

$$(\forall \mathfrak{x}, \mathfrak{z}, \mathfrak{k} \in \mathfrak{G}) \begin{pmatrix} \mathfrak{T}^+_{\mathfrak{E}}(\mathfrak{x}\mathfrak{z}\mathfrak{k}) \geq \mathfrak{T}^+_{\mathfrak{E}}(\mathfrak{x}) \wedge \mathfrak{T}^+_{\mathfrak{E}}(\mathfrak{k}), \\ \mathfrak{I}^+_{\mathfrak{E}}(\mathfrak{x}\mathfrak{z}\mathfrak{k}) \leq \mathfrak{I}^+_{\mathfrak{E}}(\mathfrak{x}) \vee \mathfrak{I}^+_{\mathfrak{E}}(\mathfrak{k}), \\ \mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{x}\mathfrak{z}\mathfrak{k}) \geq \mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{x}) \wedge \mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{k}), \\ \mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{x}\mathfrak{z}\mathfrak{k}) \leq \mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{x}) \vee \mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{k}), \\ \mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{x}\mathfrak{z}\mathfrak{k}) \geq \mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{x}) \wedge \mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{k}), \\ \mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{x}\mathfrak{z}\mathfrak{k}) \leq \mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{x}) \vee \mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{k}). \end{pmatrix}.$$

Remark 3.2. *Every NSBF BIDs of an SG $\mathfrak{G}$ is an NSBF GB IDs of an SG $\mathfrak{G}$.*

In order to establish the converses of Remarks 3.1 and 3.2, we need to strengthen the condition imposed on $\mathfrak{G}$.

Theorem 3.1. *Let $\mathfrak{G}$ be a WR semigroup. Then the following statements are holds.*

(1) *Every NSBF INIDs and the NSBF IDs coincide*
(2) *Every NSBF GB IDs and the NSBF B IDs coincide*

Proof. (1) Suppose that $\mathfrak{E} = (\mathfrak{E}^-, \mathfrak{E}^+)$ is an NSBF INID of $\mathfrak{G}$ and let $\mathfrak{z}, \mathfrak{y} \in \mathfrak{S}$. By the weak regularity of $\mathfrak{G}$, there exist $\mathfrak{t}, \mathfrak{n} \in \mathfrak{G}$ such that $\mathfrak{z} = \mathfrak{z}\mathfrak{t}\mathfrak{z}\mathfrak{n}$. Thus, $\mathfrak{T}^+_{\mathfrak{E}}(\mathfrak{z}\mathfrak{y}) = \mathfrak{T}^+_{\mathfrak{E}}(\mathfrak{z}\mathfrak{t}\mathfrak{z}\mathfrak{n}\mathfrak{y}) = \mathfrak{T}^+_{\mathfrak{E}}(\mathfrak{z}\mathfrak{t})\mathfrak{z}(\mathfrak{n}\mathfrak{y}) \geq \mathfrak{T}^+_{\mathfrak{E}}(\mathfrak{z})$, $\mathfrak{I}^+_{\mathfrak{E}}(\mathfrak{z}\mathfrak{y}) = \mathfrak{I}^+_{\mathfrak{E}}(\mathfrak{z}\mathfrak{t}\mathfrak{z}\mathfrak{n}\mathfrak{y}) = \mathfrak{I}^+_{\mathfrak{E}}(\mathfrak{z}\mathfrak{t})\mathfrak{z}(\mathfrak{n}\mathfrak{y}) \leq \mathfrak{I}^+_{\mathfrak{E}}(\mathfrak{z})$ and $\mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{z}\mathfrak{y}) = \mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{z}\mathfrak{t}\mathfrak{z}\mathfrak{n}\mathfrak{y}) = \mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{z}\mathfrak{t})\mathfrak{z}(\mathfrak{n}\mathfrak{y}) \geq \mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{z})$. And $\mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{z}\mathfrak{y}) = \mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{z}\mathfrak{t}\mathfrak{z}\mathfrak{n}\mathfrak{y}) = \mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{z}\mathfrak{t})\mathfrak{z}(\mathfrak{n}\mathfrak{y}) \leq \mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{z})$, $\mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{z}\mathfrak{y}) = \mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{z}\mathfrak{t}\mathfrak{z}\mathfrak{n}\mathfrak{y}) = \mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{z}\mathfrak{t})\mathfrak{z}(\mathfrak{n}\mathfrak{y}) \geq \mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{z})$ and $\mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{z}\mathfrak{y}) = \mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{z}\mathfrak{t}\mathfrak{z}\mathfrak{n}\mathfrak{y}) = \mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{z}\mathfrak{t})\mathfrak{z}(\mathfrak{n}\mathfrak{y}) \leq \mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{z})$. Hence $\mathfrak{E} = (\mathfrak{E}^-, \mathfrak{E}^+)$ is an NSBF RID of $\mathfrak{G}$. Similarly, we can show that $\mathfrak{E} = (\mathfrak{E}^-, \mathfrak{E}^+)$ is an NSBF LID of $\mathfrak{G}$. Thus, $\mathfrak{E} = (\mathfrak{E}^-, \mathfrak{E}^+)$ is a NSBF ID of $\mathfrak{G}$.

(2) Suppose that $\mathfrak{E} = (\mathfrak{E}^-, \mathfrak{E}^+)$ is an NSBF GB ID of $\mathfrak{G}$ and let $\mathfrak{z}, \mathfrak{y} \in \mathfrak{S}$. By the weak regularity of $\mathfrak{G}$, there exist $\mathfrak{k}, \mathfrak{n} \in \mathfrak{G}$ such that $\mathfrak{z} = \mathfrak{z}\mathfrak{k}\mathfrak{z}\mathfrak{n}$ Thus, $\mathfrak{T}^+_{\mathfrak{E}}(\mathfrak{z}\mathfrak{y}) = \mathfrak{T}^+_{\mathfrak{E}}(\mathfrak{z}(\mathfrak{k}\mathfrak{z}\mathfrak{n})\mathfrak{y}) \geq \mathfrak{T}^+_{\mathfrak{E}}(\mathfrak{z}) \wedge \mathfrak{T}^+_{\mathfrak{E}}(\mathfrak{y})$, $\mathfrak{I}^+_{\mathfrak{E}}(\mathfrak{z}\mathfrak{y}) = \mathfrak{I}^+_{\mathfrak{E}}(\mathfrak{z}(\mathfrak{k}\mathfrak{z}\mathfrak{n})\mathfrak{y}) \leq \mathfrak{I}^+_{\mathfrak{E}}(\mathfrak{z}) \vee \mathfrak{I}^+_{\mathfrak{E}}(\mathfrak{y})$ and $\mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{z}\mathfrak{y}) = \mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{z}(\mathfrak{k}\mathfrak{z}\mathfrak{n})\mathfrak{y}) \geq \mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{z}) \wedge \mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{y})$. And $\mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{z}\mathfrak{y}) = \mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{z}(\mathfrak{k}\mathfrak{z}\mathfrak{n})\mathfrak{y}) \leq \mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{z}) \vee \mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{y})$, $\mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{z}\mathfrak{y}) = \mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{z}(\mathfrak{k}\mathfrak{z}\mathfrak{n})\mathfrak{y}) \geq \mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{z}) \wedge \mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{y})$ and $\mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{z}\mathfrak{y}) = \mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{z}(\mathfrak{k}\mathfrak{z}\mathfrak{n})\mathfrak{y}) \leq \mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{z}) \vee \mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{y})$. Hence, $\mathfrak{E} = (\mathfrak{E}^-, \mathfrak{E}^+)$ is an NSBF SSG of $\mathfrak{G}$. Thus, $\mathfrak{E} = (\mathfrak{E}^-, \mathfrak{E}^+)$ is an NSBF B ID of $\mathfrak{G}$.

□

Theorem 3.2. *Let $\mathfrak{G}$ be an SG. Then, for any $\emptyset \neq \mathfrak{W} \subseteq \mathfrak{S}$, the given assertions are equivalent:*

(1) $\mathfrak{W}$ is a BID (GB ID, QID),
(2) $\chi_{\mathfrak{W}} = (\chi^+_{\mathfrak{W}}, \chi^-_{\mathfrak{W}})$ is an NSBF B ID (NSBF GB ID, NSBF QID).

Proof. (1⇒2) Suppose that $\mathfrak{W}$ is a BID of $\mathfrak{G}$ and $\mathfrak{z}, \mathfrak{y}, \mathfrak{E} \in \mathfrak{S}$. If $\mathfrak{z}, \mathfrak{y} \in \mathfrak{W}$, then $\mathfrak{x}\mathfrak{a}\mathfrak{y} \in \mathfrak{W}$. Thus, $\mathfrak{T}^+_{\chi_{\mathfrak{W}}}(\mathfrak{x}\mathfrak{a}\mathfrak{y}) = \mathfrak{T}^+_{\chi_{\mathfrak{W}}}(\mathfrak{x}) = \mathfrak{T}^+_{\chi_{\mathfrak{W}}}(\mathfrak{y}) = 1, \mathfrak{I}^+_{\chi_{\mathfrak{W}}}(\mathfrak{x}\mathfrak{a}\mathfrak{y}) = \mathfrak{I}^+_{\chi_{\mathfrak{W}}}(\mathfrak{x}) = \mathfrak{I}^+_{\chi_{\mathfrak{W}}}(\mathfrak{y}) = 0, \mathfrak{F}^+_{\chi_{\mathfrak{W}}}(\mathfrak{x}\mathfrak{a}\mathfrak{y}) = \mathfrak{F}^+_{\chi_{\mathfrak{W}}}(\mathfrak{x}) = \mathfrak{F}^+_{\chi_{\mathfrak{W}}}(\mathfrak{y}) = 1$ and $\mathfrak{T}^-_{\chi_{\mathfrak{W}}}(\mathfrak{x}\mathfrak{a}\mathfrak{y}) = \mathfrak{T}^-_{\chi_{\mathfrak{W}}}(\mathfrak{x}) = \mathfrak{T}^-_{\chi_{\mathfrak{W}}}(\mathfrak{y}) = 0, \mathfrak{I}^-_{\chi_{\mathfrak{W}}}(\mathfrak{x}\mathfrak{a}\mathfrak{y}) = \mathfrak{I}^-_{\chi_{\mathfrak{W}}}(\mathfrak{x}) = \mathfrak{I}^-_{\chi_{\mathfrak{W}}}(\mathfrak{y}) = 1, \mathfrak{F}^-_{\chi_{\mathfrak{W}}}(\mathfrak{x}\mathfrak{a}\mathfrak{y}) = \mathfrak{F}^-_{\chi_{\mathfrak{W}}}(\mathfrak{x}) = \mathfrak{F}^-_{\chi_{\mathfrak{W}}}(\mathfrak{y}) = 0$. Hence, $\mathfrak{T}^+_{\chi_{\mathfrak{W}}}(\mathfrak{x}\mathfrak{a}\mathfrak{y}) \geq \mathfrak{T}^-_{\chi_{\mathfrak{W}}}(\mathfrak{x}) \wedge \mathfrak{T}^+_{\chi_{\mathfrak{W}}}(\mathfrak{y}), \mathfrak{I}^+_{\chi_{\mathfrak{W}}}(\mathfrak{x}\mathfrak{a}\mathfrak{y}) \leq \mathfrak{I}^+_{\chi_{\mathfrak{W}}}(\mathfrak{x}) \vee \mathfrak{I}^+_{\chi_{\mathfrak{W}}}(\mathfrak{y}), \mathfrak{F}^+_{\chi_{\mathfrak{W}}}(\mathfrak{x}\mathfrak{a}\mathfrak{y}) \geq \mathfrak{F}^+_{\chi_{\mathfrak{W}}}(\mathfrak{x}) \wedge \mathfrak{F}^+_{\chi_{\mathfrak{W}}}(\mathfrak{y}), \mathfrak{T}^-_{\chi_{\mathfrak{W}}}(\mathfrak{x}\mathfrak{a}\mathfrak{y}) \leq \mathfrak{T}^-_{\chi_{\mathfrak{W}}}(\mathfrak{x}) \vee \mathfrak{T}^-_{\chi_{\mathfrak{W}}}(\mathfrak{y}), \mathfrak{I}^-_{\chi_{\mathfrak{W}}}(\mathfrak{x}\mathfrak{a}\mathfrak{y}) \geq \mathfrak{I}^-_{\chi_{\mathfrak{W}}}(\mathfrak{x}) \wedge \mathfrak{I}^-_{\chi_{\mathfrak{W}}}(\mathfrak{y}), \mathfrak{F}^-_{\chi_{\mathfrak{W}}}(\mathfrak{x}\mathfrak{a}\mathfrak{y}) \leq \mathfrak{F}^-_{\chi_{\mathfrak{W}}}(\mathfrak{x}) \vee \mathfrak{F}^-_{\chi_{\mathfrak{W}}}(\mathfrak{y})$. Therefore, $\chi_{\mathfrak{W}} = (\chi^+_{\mathfrak{W}}, \chi^-_{\mathfrak{W}})$ is an NSBF SSG. By Definition 2.11, $\chi_{\mathfrak{W}} = (\chi^+_{\mathfrak{W}}, \chi^-_{\mathfrak{W}})$ is an NSBF B ID.

(2⇒1) Suppose that $\chi_{\mathfrak{W}} = (\chi^+_{\mathfrak{W}}, \chi^-_{\mathfrak{W}})$ is an NSBF B ID. Then $\chi_{\mathfrak{W}} = (\chi^+_{\mathfrak{W}}, \chi^-_{\mathfrak{W}})$ is an NSBF SSG. Thus, $\mathfrak{W}$ is an SSG. Let $\mathfrak{z}, \mathfrak{y} \in \mathfrak{W}$ and $\mathfrak{a} \in \mathfrak{S}$. Then $\mathfrak{T}^+_{\chi_{\mathfrak{W}}}(\mathfrak{z}) = \mathfrak{T}^+_{\chi_{\mathfrak{W}}}(\mathfrak{y}) = 1, \mathfrak{I}^+_{\chi_{\mathfrak{W}}}(\mathfrak{z}) = \mathfrak{I}^+_{\chi_{\mathfrak{W}}}(\mathfrak{y}) = 0, \mathfrak{F}^+_{\chi_{\mathfrak{W}}}(\mathfrak{z}) = \mathfrak{F}^+_{\chi_{\mathfrak{W}}}(\mathfrak{y}) = 1$. By assumptions, which imply $\mathfrak{x}\mathfrak{a}\mathfrak{y} \in \mathfrak{W}$. Hence, by Definition 2.11, $\mathfrak{W}$ is a BID. □

4. Characterize Weakly Regular Semigroups in Terms of Neutrosophic Bipolar-Valued Fuzzy Ideals

In this section, we establish several characterizations of WR SGs using NSBF ideals.

Lemma 4.1. *Let $\mathfrak{W}, \mathfrak{L}(\neq \emptyset) \subseteq \mathfrak{G}$. Then the ensuing pronouncements are true*

(1) $(\chi_{\mathfrak{W}}) \wedge (\chi_{\mathfrak{L}}) = (\chi_{\mathfrak{W} \cap \mathfrak{L}})$.

(2) $(\chi_{\mathfrak{W}})\bar{\circ}(\chi_{\mathfrak{L}}) = (\chi_{\mathfrak{W}\mathfrak{L}})$.

The ensuing lemma will be used to examine in Theorem 4.1.

Lemma 4.2. *[15] For SG $\mathfrak{G}$ is WR if and only if $\mathfrak{K} \cap \mathfrak{L} \subseteq \mathfrak{K}\mathfrak{L}$ for every RID $\mathfrak{K}$ and every ID $\mathfrak{L}$ of $\mathfrak{G}$.*

Theorem 4.1. *Let $\mathfrak{G}$ be an SG. Then the ensuing are equivalent:*

(1) *$\mathfrak{G}$ is WR,*

(2) *$\mathfrak{E} \sqcap \mathfrak{B} \sqsubseteq \mathfrak{E}\bar{\circ}\mathfrak{B}$, for every NSBF RID $\mathfrak{E} = (\mathfrak{E}^-, \mathfrak{E}^+)$ and every NSBF ID $\mathfrak{B} = (\mathfrak{B}^-, \mathfrak{B}^+)$ of $\mathfrak{G}$.*

Proof. (1) $\Rightarrow$ (2) Let $\mathfrak{E} = (\mathfrak{E}^-, \mathfrak{E}^+)$ and $\mathfrak{B} = (\mathfrak{B}^-, \mathfrak{B}^+)$ be an NSBF RID and an NSBF ID of $\mathfrak{G}$ respectively. Let $\mathfrak{z} \in \mathfrak{G}$. By the weak regularity of $\mathfrak{G}$, there exist $\mathfrak{r}, \mathfrak{n} \in \mathfrak{G}$ such that $\mathfrak{z} = \mathfrak{z}\mathfrak{r}\mathfrak{z}\mathfrak{n}$. Thus,

$$\begin{aligned}
(\mathfrak{T}^+_{\mathfrak{E}} \circ \mathfrak{T}^+_{\mathfrak{B}})(\mathfrak{z}) &= \bigvee_{\mathfrak{z}=\mathfrak{m}\mathfrak{q}} \{\mathfrak{T}^+_{\mathfrak{E}}(\mathfrak{m}) \wedge \mathfrak{T}^+_{\mathfrak{B}}(\mathfrak{q})\} \\
&= \bigvee_{\mathfrak{z}\mathfrak{r}\mathfrak{z}\mathfrak{n}=\mathfrak{m}\mathfrak{q}} \{\mathfrak{T}^+_{\mathfrak{E}}(\mathfrak{m}) \wedge \mathfrak{T}^+_{\mathfrak{B}}(\mathfrak{q})\} \\
&\geq \mathfrak{T}^+_{\mathfrak{E}}(\mathfrak{z}\mathfrak{r}) \wedge \mathfrak{T}^+_{\mathfrak{B}}(\mathfrak{z}\mathfrak{n}) \\
&\geq \mathfrak{T}^+_{\mathfrak{E}}(\mathfrak{z}) \wedge \mathfrak{T}^+_{\mathfrak{B}}(\mathfrak{z}) \\
&= (\mathfrak{T}^+_{\mathfrak{E}} \wedge \mathfrak{T}^+_{\mathfrak{B}})(\mathfrak{z}),
\end{aligned}$$

$$\begin{aligned}
(\mathfrak{I}^+_{\mathfrak{E}} \circ \mathfrak{I}^+_{\mathfrak{B}})(\mathfrak{z}) &= \bigwedge_{\mathfrak{z}=\mathfrak{m}\mathfrak{q}} \{\mathfrak{I}^+_{\mathfrak{E}}(\mathfrak{m}) \vee \mathfrak{I}^+_{\mathfrak{B}}(\mathfrak{q})\} \\
&= \bigwedge_{\mathfrak{z}\mathfrak{r}\mathfrak{z}\mathfrak{n}=\mathfrak{m}\mathfrak{q}} \{\mathfrak{I}^+_{\mathfrak{E}}(\mathfrak{m}) \vee \mathfrak{I}^+_{\mathfrak{B}}(\mathfrak{q})\} \\
&\leq \mathfrak{I}^+_{\mathfrak{E}}(\mathfrak{z}\mathfrak{r}) \vee \mathfrak{I}^+_{\mathfrak{B}}(\mathfrak{z}\mathfrak{n}) \\
&\leq \mathfrak{I}^+_{\mathfrak{E}}(\mathfrak{z}) \vee \mathfrak{I}^+_{\mathfrak{B}}(\mathfrak{z}) \\
&= (\mathfrak{I}^+_{\mathfrak{E}} \vee \mathfrak{I}^+_{\mathfrak{B}})(\mathfrak{z}),
\end{aligned}$$

$$\begin{aligned}
(\mathfrak{F}^+_{\mathfrak{E}} \circ \mathfrak{F}^+_{\mathfrak{B}})(\mathfrak{z}) &= \bigvee_{\mathfrak{z}=\mathfrak{m}\mathfrak{q}} \{\mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{m}) \wedge \mathfrak{F}^+_{\mathfrak{B}}(\mathfrak{q})\} \\
&= \bigvee_{\mathfrak{z}\mathfrak{r}\mathfrak{z}\mathfrak{n}=\mathfrak{m}\mathfrak{q}} \{\mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{m}) \wedge \mathfrak{F}^+_{\mathfrak{B}}(\mathfrak{q})\} \\
&\geq \mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{z}\mathfrak{r}) \wedge \mathfrak{F}^+_{\mathfrak{B}}(\mathfrak{z}\mathfrak{n}) \\
&\geq \mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{z}) \wedge \mathfrak{F}^+_{\mathfrak{B}}(\mathfrak{z}) \\
&= (\mathfrak{F}^+_{\mathfrak{E}} \wedge \mathfrak{F}^+_{\mathfrak{B}})(\mathfrak{z}),
\end{aligned}$$

and

$$\begin{aligned}
(\mathfrak{T}^-_{\mathfrak{E}} \circ \mathfrak{T}^-_{\mathfrak{B}})(\mathfrak{z}) &= \bigwedge_{\mathfrak{z}=\mathfrak{m}\mathfrak{q}} \{\mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{m}) \vee \mathfrak{T}^-_{\mathfrak{B}}(\mathfrak{q})\} \\
&= \bigwedge_{\mathfrak{z}\mathfrak{r}\mathfrak{z}\mathfrak{n}=\mathfrak{m}\mathfrak{q}} \{\mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{m}) \vee \mathfrak{T}^-_{\mathfrak{B}}(\mathfrak{q})\} \\
&\leq \mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{z}\mathfrak{r}) \vee \mathfrak{T}^-_{\mathfrak{B}}(\mathfrak{z}\mathfrak{n}) \\
&\leq \mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{z}) \vee \mathfrak{T}^-_{\mathfrak{B}}(\mathfrak{z}) \\
&= (\mathfrak{T}^-_{\mathfrak{E}} \wedge \mathfrak{T}^-_{\mathfrak{B}})(\mathfrak{z}),
\end{aligned}$$

$$
\begin{aligned}
(\mathfrak{I}^-_{\mathfrak{E}} \circ \mathfrak{I}^-_{\mathfrak{B}})(\mathfrak{z}) &= \bigvee_{\mathfrak{z}=\mathfrak{m}\mathfrak{q}} \{\mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{m}) \wedge \mathfrak{I}^-_{\mathfrak{B}}(\mathfrak{q})\} \\
&= \bigvee_{\mathfrak{z}\mathfrak{r}\mathfrak{z}\mathfrak{n}=\mathfrak{m}\mathfrak{q}} \{\mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{m}) \wedge \mathfrak{I}^-_{\mathfrak{B}}(\mathfrak{q})\} \\
&\geq \mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{z}\mathfrak{r}) \wedge \mathfrak{I}^-_{\mathfrak{B}}(\mathfrak{z}\mathfrak{n}) \\
&\geq \mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{z}) \wedge \mathfrak{I}^+_{\mathfrak{B}}(\mathfrak{z}) \\
&= (\mathfrak{I}^-_{\mathfrak{E}} \wedge \mathfrak{I}^-_{\mathfrak{B}})(\mathfrak{z}),
\end{aligned}
$$

$$
\begin{aligned}
(\mathfrak{F}^-_{\mathfrak{E}} \circ \mathfrak{F}^-_{\mathfrak{B}})(\mathfrak{h}) &= \bigwedge_{\mathfrak{h}=\mathfrak{m}\mathfrak{q}} \{\mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{m}) \vee \mathfrak{F}^-_{\mathfrak{B}}(\mathfrak{q})\} \\
&= \bigwedge_{\mathfrak{h}\mathfrak{x}\mathfrak{h}=\mathfrak{m}\mathfrak{q}} \{\mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{m}) \vee \mathfrak{F}^-_{\mathfrak{B}}(\mathfrak{q})\} \\
&\leq \mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{h}) \vee \mathfrak{F}^-_{\mathfrak{B}}(\mathfrak{x}\mathfrak{h}) \\
&\leq \mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{h}) \vee \mathfrak{F}^-_{\mathfrak{B}}(\mathfrak{h}) \\
&= (\mathfrak{F}^-_{\mathfrak{E}} \vee \mathfrak{F}^-_{\mathfrak{B}})(\mathfrak{h}).
\end{aligned}
$$

So, $(\mathfrak{T}^+_{\mathfrak{E}} \circ \mathfrak{T}^+_{\mathfrak{B}})(\mathfrak{z}) \geq (\mathfrak{T}^+_{\mathfrak{E}} \wedge \mathfrak{T}^+_{\mathfrak{B}})(\mathfrak{z})$, $(\mathfrak{I}^+_{\mathfrak{E}} \circ \mathfrak{I}^+_{\mathfrak{B}})(\mathfrak{z}) \leq (\mathfrak{I}^+_{\mathfrak{E}} \vee \mathfrak{I}^+_{\mathfrak{B}})(\mathfrak{z})$, $(\mathfrak{F}^+_{\mathfrak{E}} \circ \mathfrak{F}^+_{\mathfrak{B}})(\mathfrak{z}) \geq (\mathfrak{F}^+_{\mathfrak{E}} \wedge \mathfrak{F}^+_{\mathfrak{B}})(\mathfrak{z})$ and $(\mathfrak{T}^-_{\mathfrak{E}} \circ \mathfrak{T}^-_{\mathfrak{B}})(\mathfrak{z}) \leq (\mathfrak{T}^-_{\mathfrak{E}} \vee \mathfrak{T}^-_{\mathfrak{B}})(\mathfrak{z})$, $(\mathfrak{I}^-_{\mathfrak{E}} \circ \mathfrak{I}^-_{\mathfrak{B}})(\mathfrak{z}) \geq (\mathfrak{I}^-_{\mathfrak{E}} \wedge \mathfrak{I}^-_{\mathfrak{B}})(\mathfrak{z})$, $(\mathfrak{F}^-_{\mathfrak{E}} \circ \mathfrak{F}^-_{\mathfrak{B}})(\mathfrak{z}) \leq (\mathfrak{F}^-_{\mathfrak{E}} \wedge \mathfrak{F}^-_{\mathfrak{B}})(\mathfrak{z})$. Hence, $\mathfrak{E} \sqcap \mathfrak{B} \sqsubseteq \mathfrak{E}\bar{\circ}\mathfrak{B}$.

(2) $\Rightarrow$ (1) Let $\mathfrak{K}$ and $\mathfrak{L}$ be a RID and an ID of $\mathfrak{G}$ respectively. Then by Theorem 2.1, $\chi_{\mathfrak{K}} = (\chi^+_{\mathfrak{K}}, \chi^-_{\mathfrak{K}})$ and $\chi_{\mathfrak{L}} = (\chi^+_{\mathfrak{L}}, \chi^-_{\mathfrak{L}})$ is an NSBF RID and NSBF ID of $\mathfrak{G}$ respectively. By presumption and Lemma 4.1, we have

$$
\begin{aligned}
1 = (\mathfrak{T}^+_{\chi_{\mathfrak{K}\cap\mathfrak{L}}})(\mathfrak{z}) = (\mathfrak{T}^+_{\chi_{\mathfrak{K}}} \sqcap \mathfrak{T}^+_{\chi_{\mathfrak{L}}})(\mathfrak{z}) \leq (\mathfrak{T}^+_{\chi_{\mathfrak{K}}} \circ \mathfrak{T}^+_{\chi_{\mathfrak{L}}})(\mathfrak{z}) = (\mathfrak{T}^+_{\chi_{\mathfrak{K}\mathfrak{L}}})(\mathfrak{z}), \\
0 = (\mathfrak{I}^+_{\chi_{\mathfrak{K}\cap\mathfrak{L}}})(\mathfrak{z}) = (\mathfrak{I}^+_{\chi_{\mathfrak{K}}} \sqcap \mathfrak{I}^+_{\chi_{\mathfrak{L}}})(\mathfrak{z}) \geq (\mathfrak{I}^+_{\chi_{\mathfrak{K}}} \circ \mathfrak{I}^+_{\chi_{\mathfrak{L}}}))(\mathfrak{z}) = (\mathfrak{I}^+_{\chi_{\mathfrak{K}\mathfrak{L}}})(\mathfrak{z}), \\
1 = (\mathfrak{F}^+_{\chi_{\mathfrak{K}\cap\mathfrak{L}}})(\mathfrak{z}) = (\mathfrak{F}^+_{\chi_{\mathfrak{K}}} \sqcap \mathfrak{F}^+_{\chi_{\mathfrak{L}}})(\mathfrak{z}) \leq ((\mathfrak{F}^+_{\chi_{\mathfrak{K}}} \circ \mathfrak{F}^+_{\chi_{\mathfrak{L}}}))(\mathfrak{z}) = (\mathfrak{F}^+_{\chi_{\mathfrak{K}\mathfrak{L}}})(\mathfrak{z}),
\end{aligned}
$$

and

$$
\begin{aligned}
0 = (\mathfrak{T}^-_{\chi_{\mathfrak{K}\cap\mathfrak{L}}})(\mathfrak{z}) = (\mathfrak{T}^-_{\chi_{\mathfrak{K}}} \sqcap \mathfrak{T}^-_{\chi_{\mathfrak{L}}})(\mathfrak{z}) \geq ((\mathfrak{T}^-_{\chi_{\mathfrak{K}}} \circ \mathfrak{T}^-_{\chi_{\mathfrak{L}}}))(\mathfrak{z}) = (\mathfrak{T}^-_{\chi_{\mathfrak{K}\mathfrak{L}}})(\mathfrak{z}), \\
1 = (\mathfrak{I}^-_{\chi_{\mathfrak{K}\cap\mathfrak{L}}})(\mathfrak{z}) = (\mathfrak{I}^-_{\chi_{\mathfrak{K}}} \sqcap \mathfrak{I}^-_{\chi_{\mathfrak{L}}})(\mathfrak{z}) \leq ((\mathfrak{I}^-_{\chi_{\mathfrak{K}}} \circ \mathfrak{I}^-_{\chi_{\mathfrak{L}}}))(\mathfrak{z}) = (\mathfrak{I}^-_{\chi_{\mathfrak{K}\mathfrak{L}}})(\mathfrak{z}), \\
0 = (\mathfrak{F}^-_{\chi_{\mathfrak{t}\cap\mathfrak{L}}})(\mathfrak{z}) = (\mathfrak{F}^-_{\chi_{\mathfrak{K}}} \sqcap \mathfrak{F}^-_{\chi_{\mathfrak{L}}})(\mathfrak{z}) \geq ((\mathfrak{F}^+_{\chi_{\mathfrak{K}}} \circ \mathfrak{F}^-_{\chi_{\mathfrak{L}}}))(\mathfrak{z}) = (\mathfrak{F}^-_{\chi_{\mathfrak{K}\mathfrak{L}}})(\mathfrak{z}).
\end{aligned}
$$

Thus, $\mathfrak{z} \in \mathfrak{K}\mathfrak{L}$ and so, $\mathfrak{K} \cap \mathfrak{L} \subseteq \mathfrak{K}\mathfrak{L}$. It follows that by Lemma 4.2, $\mathfrak{G}$ is WR. □

The ensuing lemma will be used to examine in Theorem 4.2.

Lemma 4.3. *[15] Let $\mathfrak{G}$ be an SG. Then the ensuing pronouncements are equivalent:*

(1) $\mathfrak{G}$ *is WR.*

(2) $\mathfrak{K} \cap \mathfrak{U} \subseteq \mathfrak{K}\mathfrak{U}$ *for every BID* $\mathfrak{K}$ *and every ID* $\mathfrak{U}$ *of* $\mathfrak{G}$.

Theorem 4.2. *Let $\mathfrak{G}$ be an SG. Then the ensuing are equivalent:*

(1) $\mathfrak{G}$ *is WR,*

(2) $\mathfrak{E} \sqcap \mathfrak{B} \sqsubseteq \mathfrak{E}\bar{\circ}\mathfrak{B}$, *for every NSBF B ID* $\mathfrak{E} = (\mathfrak{E}^-, \mathfrak{E}^+)$ *and every NSBF ID* $\mathfrak{B} = (\mathfrak{B}^-, \mathfrak{B}^+)$ *of* $\mathfrak{G}$.

(3) $\mathfrak{E} \sqcap \mathfrak{B} \sqsubseteq \mathfrak{E}\bar{\circ}\mathfrak{B}$, *for every NSBF B ID* $\mathfrak{E} = (\mathfrak{E}^-, \mathfrak{E}^+)$ *and every NSBF INID* $\mathfrak{B} = (\mathfrak{B}^-, \mathfrak{B}^+)$ *of* $\mathfrak{G}$.

(4) $\mathfrak{E} \sqcap \mathfrak{B} \sqsubseteq \mathfrak{E}\bar{\circ}\mathfrak{B}$, *for every NSBF GB ID* $\mathfrak{E}$ *and every NSBF INID* $\mathfrak{B} = (\mathfrak{B}^-, \mathfrak{B}^+)$ *of* $\mathfrak{G}$.

Proof. $(1) \Rightarrow (4)$ Let $\mathfrak{E} = (\mathfrak{E}^-, \mathfrak{E}^+)$ and $\mathfrak{B} = (\mathfrak{B}^-, \mathfrak{B}^+)$ be an NSBF GB ID and an NSBF INID of $\mathfrak{G}$ respectively. Let $\mathfrak{z} \in \mathfrak{S}$. Then there exists $\mathfrak{r}, \mathfrak{n} \in \mathfrak{G}$ such that $\mathfrak{z} = \mathfrak{z}\mathfrak{r}\mathfrak{z}\mathfrak{n}$. Thus,

$$\begin{aligned}
(\mathfrak{T}^+_{\mathfrak{E}} \bar{\circ} \mathfrak{T}^+_{\mathfrak{B}})(\mathfrak{z}) &= \bigvee_{\mathfrak{z}=\mathfrak{m}\mathfrak{q}} \{\mathfrak{T}^+_{\mathfrak{E}}(\mathfrak{m}) \wedge \mathfrak{T}^+_{\mathfrak{B}}(\mathfrak{q})\} \\
&= \bigvee_{\mathfrak{z}\mathfrak{r}\mathfrak{z}\mathfrak{n}=\mathfrak{m}\mathfrak{q}} \{\mathfrak{T}^+_{\mathfrak{E}}(\mathfrak{m}) \wedge \mathfrak{T}^+_{\mathfrak{B}}(\mathfrak{q})\} \\
&\geq \mathfrak{T}^+_{\mathfrak{E}}(\mathfrak{z}) \wedge \mathfrak{T}^+_{\mathfrak{B}}(\mathfrak{r}\mathfrak{z}\mathfrak{n}) \\
&\geq \mathfrak{T}^+_{\mathfrak{E}}(\mathfrak{z}) \wedge \wedge \mathfrak{T}^+_{\mathfrak{B}}(\mathfrak{z}) \\
&= (\mathfrak{T}^+_{\mathfrak{E}} \wedge \mathfrak{T}^+_{\mathfrak{E}})(\mathfrak{z})
\end{aligned}$$

$$\begin{aligned}
(\mathfrak{I}^+_{\mathfrak{E}} \bar{\circ} \mathfrak{I}^+_{\mathfrak{B}})(\mathfrak{z}) &= \bigwedge_{\mathfrak{z}=\mathfrak{m}\mathfrak{q}} \{\mathfrak{I}^+_{\mathfrak{E}}(\mathfrak{m}) \vee \mathfrak{I}^+_{\mathfrak{B}}(\mathfrak{q})\} \\
&= \bigwedge_{\mathfrak{z}\mathfrak{r}\mathfrak{z}\mathfrak{n}=\mathfrak{m}\mathfrak{q}} \{\mathfrak{I}^+_{\mathfrak{E}}(\mathfrak{m}) \vee \mathfrak{I}^+_{\mathfrak{B}}(\mathfrak{q})\} \\
&\leq \mathfrak{I}^+_{\mathfrak{E}}(\mathfrak{z}) \vee \mathfrak{I}^+_{\mathfrak{B}}(\mathfrak{r}\mathfrak{z}\mathfrak{n}) \\
&\leq \mathfrak{I}^+_{\mathfrak{E}}(\mathfrak{z}) \vee \mathfrak{I}^+_{\mathfrak{B}}(\mathfrak{z}) \\
&= (\mathfrak{I}^+_{\mathfrak{E}} \vee \mathfrak{I}^+_{\mathfrak{E}})(\mathfrak{z}),
\end{aligned}$$

$$\begin{aligned}
(\mathfrak{F}^+_{\mathfrak{E}} \bar{\circ} \mathfrak{F}^+_{\mathfrak{B}})(\mathfrak{z}) &= \bigvee_{\mathfrak{z}=\mathfrak{m}\mathfrak{q}} \{\mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{m}) \wedge \mathfrak{F}^+_{\mathfrak{B}}(\mathfrak{q})\} \\
&= \bigvee_{\mathfrak{z}\mathfrak{r}\mathfrak{z}\mathfrak{n}=\mathfrak{m}\mathfrak{q}} \{\mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{m}) \wedge \mathfrak{F}^+_{\mathfrak{B}}(\mathfrak{q})\} \\
&\geq \mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{z}) \wedge \mathfrak{F}^+_{\mathfrak{B}}(\mathfrak{r}\mathfrak{z}\mathfrak{n}) \\
&\geq \mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{z}) \wedge \mathfrak{F}^+_{\mathfrak{B}}(\mathfrak{z}) \\
&= (\mathfrak{F}^+_{\mathfrak{E}} \wedge \mathfrak{F}^+_{\mathfrak{E}})(\mathfrak{z}),
\end{aligned}$$

and

$$\begin{aligned}
(\mathfrak{T}^-_{\mathfrak{E}} \bar{\circ} \mathfrak{T}^-_{\mathfrak{B}})(\mathfrak{z}) &= \bigwedge_{\mathfrak{z}=\mathfrak{m}\mathfrak{q}} \{\mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{m}) \vee \mathfrak{T}^-_{\mathfrak{B}}(\mathfrak{q})\} \\
&= \bigwedge_{\mathfrak{z}\mathfrak{r}\mathfrak{z}\mathfrak{n}=\mathfrak{m}\mathfrak{q}} \{\mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{m}) \vee \mathfrak{T}^-_{\mathfrak{B}}(\mathfrak{q})\} \\
&\leq \mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{z}) \vee \mathfrak{T}^-_{\mathfrak{B}}(\mathfrak{r}\mathfrak{z}\mathfrak{n}) \\
&\leq \mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{z}) \vee \mathfrak{T}^-_{\mathfrak{B}}(\mathfrak{z}) \\
&= (\mathfrak{T}^-_{\mathfrak{E}} \vee \mathfrak{T}^-_{\mathfrak{E}})(\mathfrak{z}),
\end{aligned}$$

$$\begin{aligned}
(\mathfrak{I}^-_{\mathfrak{E}} \bar{\circ} \mathfrak{I}^-_{\mathfrak{B}})(\mathfrak{z}) &= \bigvee_{\mathfrak{z}=\mathfrak{m}\mathfrak{q}} \{\mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{m}) \wedge \mathfrak{I}^-_{\mathfrak{B}}(\mathfrak{q})\} \\
&= \bigvee_{\mathfrak{z}\mathfrak{r}\mathfrak{z}\mathfrak{n}=\mathfrak{m}\mathfrak{q}} \{\mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{m}) \wedge \mathfrak{I}^-_{\mathfrak{B}}(\mathfrak{q})\} \\
&\geq \mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{z}) \wedge \mathfrak{I}^-_{\mathfrak{B}}(\mathfrak{r}\mathfrak{z}\mathfrak{n}) \\
&\geq \mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{z}) \wedge \mathfrak{I}^-_{\mathfrak{B}}(\mathfrak{z}) \\
&= (\mathfrak{I}^-_{\mathfrak{E}} \wedge \mathfrak{I}^-_{\mathfrak{E}})(\mathfrak{z}),
\end{aligned}$$

$$\begin{aligned}
(\mathfrak{F}^-_{\mathfrak{E}} \bar{\circ} \mathfrak{F}^-_{\mathfrak{B}})(\mathfrak{z}) &= \bigwedge_{\mathfrak{z}=\mathfrak{m}\mathfrak{q}} \{\mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{m}) \vee \mathfrak{T}^-_{\mathfrak{B}}(\mathfrak{q})\} \\
&= \bigwedge_{\mathfrak{z}\mathfrak{r}\mathfrak{z}\mathfrak{n}=\mathfrak{m}\mathfrak{q}} \{\mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{m}) \vee \mathfrak{F}^-_{\mathfrak{B}}(\mathfrak{q})\} \\
&\leq \mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{z}) \vee \mathfrak{F}^-_{\mathfrak{B}}(\mathfrak{r}\mathfrak{z}\mathfrak{n}) \\
&\leq \mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{z}) \vee \mathfrak{F}^-_{\mathfrak{B}}(\mathfrak{z}) \\
&= (\mathfrak{F}^-_{\mathfrak{E}} \wedge \mathfrak{F}^-_{\mathfrak{E}})(\mathfrak{z}).
\end{aligned}$$

Hence, $(\mathfrak{T}^+_{\mathfrak{E}}\bar{\circ}\mathfrak{T}^+_{\mathfrak{B}}(\mathfrak{z}) \geq (\mathfrak{T}^+_{\mathfrak{E}} \wedge \mathfrak{T}^+_{\mathfrak{E}})(\mathfrak{z})$, $(\mathfrak{I}^-_{\mathfrak{E}}\bar{\circ}\mathfrak{I}^-_{\mathfrak{B}}(\mathfrak{z}) \leq (\mathfrak{I}^-_{\mathfrak{E}} \vee \mathfrak{I}^-_{\mathfrak{E}})(\mathfrak{z})$, $(\mathfrak{F}^+_{\mathfrak{E}}\bar{\circ}\mathfrak{F}^+_{\mathfrak{B}}(\mathfrak{z}) \geq (\mathfrak{F}^+_{\mathfrak{E}} \wedge \mathfrak{F}^+_{\mathfrak{E}})(\mathfrak{z})$ and $(\mathfrak{T}^-_{\mathfrak{E}}\bar{\circ}\mathfrak{T}^-_{\mathfrak{B}}(\mathfrak{z}) \leq (\mathfrak{T}^-_{\mathfrak{E}} \vee \mathfrak{T}^-_{\mathfrak{E}})(\mathfrak{z})$, $(\mathfrak{I}^+_{\mathfrak{E}}\bar{\circ}\mathfrak{I}^-_{\mathfrak{B}}(\mathfrak{z}) \geq (\mathfrak{I}^-_{\mathfrak{E}} \wedge \mathfrak{I}^-_{\mathfrak{E}})(\mathfrak{z})$, $(\mathfrak{F}^-_{\mathfrak{E}}\bar{\circ}\mathfrak{F}^-_{\mathfrak{B}}(\mathfrak{z}) \leq (\mathfrak{F}^+_{\mathfrak{E}} \vee \mathfrak{F}^-_{\mathfrak{E}})(\mathfrak{z})$. Thus, $\mathfrak{E} \sqcap \mathfrak{B} \sqsubseteq \mathfrak{E}\bar{\circ}\mathfrak{B}$.

(4) $\Rightarrow$ (3) $\Rightarrow$ (2) This is obvious because every NSBF B ID of $\mathfrak{G}$ is an NSBF GB ID of $\mathfrak{G}$ and every NSBF ID of $\mathfrak{G}$ is an NSBF INID of $\mathfrak{G}$.

(2) $\Rightarrow$ (1). Let $\mathfrak{K}$ be a GB ID of $\mathfrak{G}$ and $\mathfrak{U}$ be an ID of $\mathfrak{G}$. Then by Theorem 2.1 and 3.2 $\chi_{\mathfrak{K}} = (\chi^+_{\mathfrak{K}}, \chi^-_{\mathfrak{K}})$ and $\chi_{\mathfrak{U}} = (\chi^+_{\mathfrak{U}}, \chi^-_{\mathfrak{U}})$ is an NSBF GB ID and is an NSBF ID of $\mathfrak{G}$. By presumption and Lemma 4.1, we have

$$1 = (\mathfrak{T}^+_{\chi_{\mathfrak{K}\cap\mathfrak{U}}})(\mathfrak{z}) = (\mathfrak{T}^+_{\chi_{\mathfrak{K}}} \sqcap \mathfrak{T}^+_{\chi_{\mathfrak{U}}})(\mathfrak{z}) \leq ((\mathfrak{T}^+_{\chi_{\mathfrak{K}}} \circ \mathfrak{T}^+_{\chi_{\mathfrak{U}}})(\mathfrak{z}) = (\mathfrak{T}^+_{\chi_{\mathfrak{K}\mathfrak{U}}})(\mathfrak{z}),$$

$$0 = (\mathfrak{I}^+_{\chi_{\mathfrak{K}\cap\mathfrak{U}}})(\mathfrak{z}) = (\mathfrak{I}^+_{\chi_{\mathfrak{K}}} \sqcap \mathfrak{U}^+_{\chi_{\mathfrak{L}}})(\mathfrak{z}) \geq ((\mathfrak{I}^+_{\chi_{\mathfrak{K}}} \circ \mathfrak{I}^+_{\chi_{\mathfrak{U}}}))(\mathfrak{z}) = (\mathfrak{I}^+_{\chi_{\mathfrak{K}\mathfrak{U}}})(\mathfrak{z}),$$

$$1 = (\mathfrak{F}^+_{\chi_{\mathfrak{K}\cap\mathfrak{U}}})(\mathfrak{z}) = (\mathfrak{F}^+_{\chi_{\mathfrak{K}}} \sqcap \mathfrak{F}^+_{\chi_{\mathfrak{U}}})(\mathfrak{z}) \leq ((\mathfrak{F}^+_{\chi_{\mathfrak{K}}} \circ \mathfrak{F}^+_{\chi_{\mathfrak{U}}}))(\mathfrak{z}) = (\mathfrak{F}^+_{\chi_{\mathfrak{K}\mathfrak{U}}})(\mathfrak{z}),$$

and

$$-1 = (\mathfrak{T}^-_{\chi_{\mathfrak{K}\cap\mathfrak{U}}})(\mathfrak{z}) = (\mathfrak{T}^-_{\chi_{\mathfrak{K}}} \sqcap \mathfrak{T}^-_{\chi_{\mathfrak{U}}})(\mathfrak{z}) \geq ((\mathfrak{T}^-_{\chi_{\mathfrak{K}}} \circ \mathfrak{T}^-_{\chi_{\mathfrak{U}}}))(\mathfrak{z}) = (\mathfrak{T}^-_{\chi_{\mathfrak{K}\mathfrak{U}}})(\mathfrak{z}),$$

$$0 = (\mathfrak{I}^-_{\chi_{\mathfrak{K}\cap\mathfrak{U}}})(\mathfrak{z}) = (\mathfrak{I}^-_{\chi_{\mathfrak{K}}} \sqcap \mathfrak{I}^-_{\chi_{\mathfrak{U}}})(\mathfrak{z}) \leq ((\mathfrak{I}^-_{\chi_{\mathfrak{K}}} \circ \mathfrak{I}^-_{\chi_{\mathfrak{U}}}))(\mathfrak{z}) = (\mathfrak{I}^-_{\chi_{\mathfrak{K}\mathfrak{U}}})(\mathfrak{z}),$$

$$-1 = (\mathfrak{F}^-_{\chi_{\mathfrak{k}\cap\mathfrak{U}}})(\mathfrak{z}) = (\mathfrak{F}^-_{\chi_{\mathfrak{K}}} \sqcap \mathfrak{F}^-_{\chi_{\mathfrak{U}}})(\mathfrak{z}) \geq ((\mathfrak{F}^+_{\chi_{\mathfrak{K}}} \circ \mathfrak{F}^-_{\chi_{\mathfrak{U}}}))(\mathfrak{z}) = (\mathfrak{F}^-_{\chi_{\mathfrak{K}\mathfrak{U}}})(\mathfrak{z}).$$

Thus, $\mathfrak{z} \in \mathfrak{K}\mathfrak{U}$. Hence, $\mathfrak{K} \cap \mathfrak{U} = \mathfrak{K}\mathfrak{U}$. Therefore by Lemma 4.3, $\mathfrak{G}$ is WR. □

The ensuing lemma allows us to examine Theorem 4.3.

Lemma 4.4. *[15] Let $\mathfrak{G}$ be an SG. Then the ensuing pronouncements are equivalent:*

(1) *$\mathfrak{G}$ is WR.*
(2) *$\mathfrak{U} \cap \mathfrak{K} \cap \mathfrak{R} \subseteq \mathfrak{U}\mathfrak{K}\mathfrak{R}$ for every BID $\mathfrak{U}$, every ID $\mathfrak{K}$ and every RID $\mathfrak{R}$ of $\mathfrak{G}$.*

Theorem 4.3. *Let $\mathfrak{G}$ be an SG. Then the ensuing pronouncements are equivalent:*

(1) *$\mathfrak{G}$ is WR.*
(2) *$\mathfrak{E} \sqcap \mathfrak{B} \sqcap \mathfrak{N} \sqsubseteq \mathfrak{E}\bar{\circ}\mathfrak{B}\bar{\circ}\mathfrak{N}$ for every NSBF B ID $\mathfrak{E} = (\mathfrak{E}^-, \mathfrak{E}^+)$, every NSBF ID $\mathfrak{B} = (\mathfrak{B}^-, \mathfrak{B}^+)$ and every NSBF RID $\mathfrak{N} = (\mathfrak{N}^-, \mathfrak{N}^+)$ of $\mathfrak{G}$.*
(3) *$\mathfrak{E} \sqcap \mathfrak{B} \sqcap \mathfrak{N} \sqsubseteq \mathfrak{E}\bar{\circ}\mathfrak{B}\bar{\circ}\mathfrak{N}$ for every NSBF GB ID $\mathfrak{E} = (\mathfrak{E}^-, \mathfrak{E}^+)$, every NSBF INID $\mathfrak{B} = (\mathfrak{B}^-, \mathfrak{B}^+)$ and every NSBF RID $\mathfrak{N} = (\mathfrak{N}^-, \mathfrak{N}^+)$ of $\mathfrak{G}$.*

Proof. (1) $\Rightarrow$ (3) Suppose that $\mathfrak{E} = (\mathfrak{E}^-, \mathfrak{E}^+)$ is an NSBF GB ID, $\mathfrak{B} = (\mathfrak{B}^-, \mathfrak{B}^+)$ is an NSBF INID and $\mathfrak{N} = (\mathfrak{N}^-, \mathfrak{N}^+)$ is an NSBF RID of $\mathfrak{G}$. Let $\mathfrak{z} \in \mathfrak{G}$. Then there exist $\mathfrak{r}, \mathfrak{c} \in \mathfrak{G}$ such that $\mathfrak{z} = \mathfrak{z}\mathfrak{r}\mathfrak{z}\mathfrak{c}$.

Thus,

$$\begin{aligned}
(\mathfrak{T}^{+}_{\mathfrak{E}} \circ \mathfrak{T}^{+}_{\mathfrak{B}} \circ \mathfrak{T}^{+}_{\mathfrak{N}})(\mathfrak{z}) &= ((\bigvee_{\mathfrak{z}=\mathfrak{a}\mathfrak{b}} \{\mathfrak{T}^{+}_{\mathfrak{E}}(\mathfrak{a}) \wedge (\mathfrak{T}^{+}_{\mathfrak{B}} \circ \mathfrak{T}^{+}_{\mathfrak{N}})(\mathfrak{b})\}) \\
&= ((\bigvee_{\mathfrak{z}\mathfrak{r}\mathfrak{z}\mathfrak{c}=\mathfrak{a}\mathfrak{b}} \{\mathfrak{T}^{+}_{\mathfrak{E}}(\mathfrak{a}) \wedge (\mathfrak{T}^{+}_{\mathfrak{B}} \circ \mathfrak{T}^{+}_{\mathfrak{N}})(\mathfrak{b})\}) \\
&\geq \mathfrak{T}^{+}_{\mathfrak{E}}(\mathfrak{z}) \wedge (\mathfrak{T}^{+}_{\mathfrak{B}} \circ \mathfrak{T}^{+}_{\mathfrak{N}})(\mathfrak{r}\mathfrak{z}\mathfrak{c}) \\
&= \mathfrak{T}^{+}_{\mathfrak{E}}(\mathfrak{z}) \wedge \bigvee_{\mathfrak{r}\mathfrak{z}\mathfrak{c}=\mathfrak{m}\mathfrak{s}} \{\mathfrak{T}^{+}_{\mathfrak{B}}(\mathfrak{m}) \wedge \mathfrak{T}^{+}_{\mathfrak{N}}(\mathfrak{s})\} \\
&= \mathfrak{T}^{+}_{\mathfrak{E}}(\mathfrak{z}) \wedge (\bigvee_{\mathfrak{r}\mathfrak{z}\mathfrak{r}\mathfrak{z}\mathfrak{c}^{2}=\mathfrak{m}\mathfrak{s}} \mathfrak{T}^{+}_{\mathfrak{B}}(\mathfrak{m}) \wedge \mathfrak{T}^{+}_{\mathfrak{N}}(\mathfrak{s})) \\
&= \mathfrak{T}^{+}_{\mathfrak{E}}(\mathfrak{z}) \wedge (\mathfrak{T}^{+}_{\mathfrak{B}}(\mathfrak{r}\mathfrak{z}\mathfrak{r}) \wedge \mathfrak{T}^{+}_{\mathfrak{N}}(\mathfrak{z}\mathfrak{c}^{2})) \\
&\geq \mathfrak{T}^{+}_{\mathfrak{E}}(\mathfrak{z}) \wedge (\mathfrak{T}^{+}_{\mathfrak{B}}(\mathfrak{z}) \wedge \mathfrak{T}^{+}_{\mathfrak{N}}(\mathfrak{z})) \\
&= (\mathfrak{T}^{+}_{\mathfrak{E}} \wedge \mathfrak{T}^{+}_{\mathfrak{B}} \wedge \mathfrak{T}^{+}_{\mathfrak{N}})(\mathfrak{z}),
\end{aligned}$$

$$\begin{aligned}
(\mathfrak{I}^{+}_{\mathfrak{E}} \circ \mathfrak{I}^{+}_{\mathfrak{B}} \circ \mathfrak{I}^{+}_{\mathfrak{N}})(\mathfrak{z}) &= ((\bigwedge_{\mathfrak{z}=\mathfrak{a}\mathfrak{b}} \{\mathfrak{I}^{+}_{\mathfrak{E}}(\mathfrak{a}) \vee (\mathfrak{I}^{+}_{\mathfrak{B}} \circ \mathfrak{T}^{+}_{\mathfrak{N}})(\mathfrak{b})\} \\
&= ((\bigwedge_{\mathfrak{z}\mathfrak{r}\mathfrak{z}\mathfrak{c}=\mathfrak{a}\mathfrak{b}} \{\mathfrak{I}^{+}_{\mathfrak{E}}(\mathfrak{a}) \vee (\mathfrak{I}^{+}_{\mathfrak{B}} \circ \mathfrak{I}^{+}_{\mathfrak{N}})(\mathfrak{b})\}) \\
&\leq \mathfrak{I}^{+}_{\mathfrak{E}}(\mathfrak{z}) \vee (\mathfrak{I}^{+}_{\mathfrak{B}} \circ \mathfrak{I}^{+}_{\mathfrak{N}})(\mathfrak{r}\mathfrak{z}\mathfrak{c}) \\
&= \mathfrak{I}^{+}_{\mathfrak{E}}(\mathfrak{z}) \vee \bigwedge_{\mathfrak{r}\mathfrak{z}\mathfrak{c}=\mathfrak{m}\mathfrak{s}} \{\mathfrak{I}^{+}_{\mathfrak{B}}(\mathfrak{m}) \vee \mathfrak{I}^{+}_{\mathfrak{N}}(\mathfrak{s})\} \\
&= \mathfrak{I}^{+}_{\mathfrak{E}}(\mathfrak{z}) \vee (\bigwedge_{\mathfrak{r}\mathfrak{z}\mathfrak{r}\mathfrak{z}\mathfrak{c}^{2}=\mathfrak{m}\mathfrak{s}} \mathfrak{I}^{+}_{\mathfrak{B}}(\mathfrak{m}) \vee \mathfrak{I}^{+}_{\mathfrak{N}}(\mathfrak{s})) \\
&= \mathfrak{I}^{+}_{\mathfrak{E}}(\mathfrak{z}) \wedge (\mathfrak{I}^{+}_{\mathfrak{B}}(\mathfrak{r}\mathfrak{z}\mathfrak{r}) \vee \mathfrak{I}^{+}_{\mathfrak{N}}(\mathfrak{z}\mathfrak{c}^{2})) \\
&\leq \mathfrak{I}^{+}_{\mathfrak{E}}(\mathfrak{z}) \vee (\mathfrak{I}^{+}_{\mathfrak{B}}(\mathfrak{z}) \vee \mathfrak{I}^{+}_{\mathfrak{N}}(\mathfrak{z})) \\
&= (\mathfrak{I}^{+}_{\mathfrak{E}} \vee \mathfrak{I}^{+}_{\mathfrak{B}} \vee \mathfrak{I}^{+}_{\mathfrak{N}})(\mathfrak{z}),
\end{aligned}$$

$$\begin{aligned}
(\mathfrak{F}^{+}_{\mathfrak{E}} \circ \mathfrak{F}^{+}_{\mathfrak{B}} \circ \mathfrak{F}^{+}_{\mathfrak{N}})(\mathfrak{z}) &= ((\bigvee_{\mathfrak{z}=\mathfrak{a}\mathfrak{b}} \{\mathfrak{F}^{+}_{\mathfrak{E}}(\mathfrak{a}) \wedge (\mathfrak{F}^{+}_{\mathfrak{B}} \circ \mathfrak{F}^{+}_{\mathfrak{N}})(\mathfrak{b})\}) \\
&= ((\bigvee_{\mathfrak{z}\mathfrak{r}\mathfrak{z}\mathfrak{c}=\mathfrak{a}\mathfrak{b}} \{\mathfrak{F}^{+}_{\mathfrak{E}}(\mathfrak{a}) \wedge (\mathfrak{F}^{+}_{\mathfrak{B}} \circ \mathfrak{F}^{+}_{\mathfrak{N}})(\mathfrak{b})\}) \\
&\geq \mathfrak{F}^{+}_{\mathfrak{E}}(\mathfrak{z}) \wedge (\mathfrak{F}^{+}_{\mathfrak{B}} \circ \mathfrak{F}^{+}_{\mathfrak{N}})(\mathfrak{r}\mathfrak{z}\mathfrak{c}) \\
&= \mathfrak{F}^{+}_{\mathfrak{E}}(\mathfrak{z}) \wedge \bigvee_{\mathfrak{r}\mathfrak{z}\mathfrak{c}=\mathfrak{m}\mathfrak{s}} \{\mathfrak{F}^{+}_{\mathfrak{B}}(\mathfrak{m}) \wedge \mathfrak{F}^{+}_{\mathfrak{N}}(\mathfrak{s})\} \\
&= \mathfrak{F}^{+}_{\mathfrak{E}}(\mathfrak{z}) \wedge (\bigvee_{\mathfrak{r}\mathfrak{z}\mathfrak{r}\mathfrak{z}\mathfrak{c}^{2}=\mathfrak{m}\mathfrak{s}} \mathfrak{F}^{+}_{\mathfrak{B}}(\mathfrak{m}) \wedge \mathfrak{F}^{+}_{\mathfrak{N}}(\mathfrak{s})) \\
&= \mathfrak{F}^{+}_{\mathfrak{E}}(\mathfrak{z}) \wedge (\mathfrak{T}^{+}_{\mathfrak{B}}(\mathfrak{r}\mathfrak{z}\mathfrak{r}) \wedge \mathfrak{F}^{+}_{\mathfrak{N}}(\mathfrak{z}\mathfrak{c}^{2})) \\
&\geq \mathfrak{F}^{+}_{\mathfrak{E}}(\mathfrak{z}) \wedge (\mathfrak{F}^{+}_{\mathfrak{B}}(\mathfrak{z}) \wedge \mathfrak{F}^{+}_{\mathfrak{N}}(\mathfrak{z})) \\
&= (\mathfrak{F}^{+}_{\mathfrak{E}} \wedge \mathfrak{F}^{+}_{\mathfrak{B}} \wedge \mathfrak{F}^{+}_{\mathfrak{N}})(\mathfrak{z}),
\end{aligned}$$

and

$$\begin{aligned}
(\mathfrak{T}^{-}_{\mathfrak{E}} \circ \mathfrak{T}^{-}_{\mathfrak{B}} \circ \mathfrak{T}^{-}_{\mathfrak{N}})(\mathfrak{z}) &= ((\bigwedge_{\mathfrak{z}=\mathfrak{a}\mathfrak{b}} \{\mathfrak{T}^{-}_{\mathfrak{E}}(\mathfrak{a}) \vee (\mathfrak{T}^{+}_{\mathfrak{B}} \circ \mathfrak{T}^{+}_{\mathfrak{N}})(\mathfrak{b})\}) \\
&= ((\bigwedge_{\mathfrak{z}\mathfrak{r}\mathfrak{z}\mathfrak{c}=\mathfrak{a}\mathfrak{b}} \{\mathfrak{T}^{-}_{\mathfrak{E}}(\mathfrak{a}) \vee (\mathfrak{T}^{-}_{\mathfrak{B}} \circ \mathfrak{T}^{-}_{\mathfrak{N}})(\mathfrak{b})\}) \\
&\leq \mathfrak{T}^{-}_{\mathfrak{E}}(\mathfrak{z}) \vee (\mathfrak{T}^{-}_{\mathfrak{B}} \circ \mathfrak{T}^{-}_{\mathfrak{N}})(\mathfrak{r}\mathfrak{z}\mathfrak{c}) \\
&= \mathfrak{T}^{-}_{\mathfrak{E}}(\mathfrak{z}) \vee \bigwedge_{\mathfrak{r}\mathfrak{z}\mathfrak{c}=\mathfrak{m}\mathfrak{s}} \{\mathfrak{T}^{-}_{\mathfrak{B}}(\mathfrak{m}) \vee \mathfrak{T}^{-}_{\mathfrak{N}}(\mathfrak{s})\}
\end{aligned}$$

$$
\begin{aligned}
&= \mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{z}) \wedge \Big(\bigwedge_{\mathfrak{r}\mathfrak{z}\mathfrak{r}\mathfrak{z}\mathfrak{c}^2=\mathfrak{m}\mathfrak{s}} \mathfrak{T}^-_{\mathfrak{B}}(\mathfrak{m}) \wedge \mathfrak{T}^-_{\mathfrak{N}}(\mathfrak{s})\Big)\\
&= \mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{z}) \vee (\mathfrak{T}^+_{\mathfrak{B}}(\mathfrak{r}\mathfrak{z}\mathfrak{r}) \vee \mathfrak{T}^+_{\mathfrak{N}}(\mathfrak{z}\mathfrak{c}^2))\\
&\leq \mathfrak{T}^-_{\mathfrak{E}}(\mathfrak{z}) \vee (\mathfrak{T}^-_{\mathfrak{B}}(\mathfrak{z}) \vee \mathfrak{T}^-_{\mathfrak{N}}(\mathfrak{z}))\\
&= (\mathfrak{T}^-_{\mathfrak{E}} \vee \mathfrak{T}^+_{\mathfrak{B}} \vee \mathfrak{T}^+_{\mathfrak{N}})(\mathfrak{z}),
\end{aligned}
$$

$$
\begin{aligned}
(\mathfrak{I}^-_{\mathfrak{E}} \circ \mathfrak{I}^-_{\mathfrak{B}} \circ \mathfrak{I}^-_{\mathfrak{N}})(\mathfrak{z}) &= \Big(\Big(\bigvee_{\mathfrak{z}=\mathfrak{a}\mathfrak{b}} \{\mathfrak{I}^+_{\mathfrak{E}}(\mathfrak{a}) \wedge (\mathfrak{I}^-_{\mathfrak{B}} \circ \mathfrak{T}^-_{\mathfrak{N}})(\mathfrak{b})\}\Big)\\
&= \Big(\Big(\bigvee_{\mathfrak{z}\mathfrak{r}\mathfrak{z}\mathfrak{c}=\mathfrak{a}\mathfrak{b}} \{\mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{a}) \wedge (\mathfrak{I}^-_{\mathfrak{B}} \circ \mathfrak{I}^-_{\mathfrak{N}})(\mathfrak{b})\}\Big)\\
&\geq \mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{z}) \wedge (\mathfrak{I}^-_{\mathfrak{B}} \circ \mathfrak{I}^-_{\mathfrak{N}})(\mathfrak{r}\mathfrak{z}\mathfrak{c})\\
&= \mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{z}) \wedge \bigvee_{\mathfrak{r}\mathfrak{z}\mathfrak{c}=\mathfrak{m}\mathfrak{s}} \{\mathfrak{I}^-_{\mathfrak{B}}(\mathfrak{m}) \wedge \mathfrak{I}^-_{\mathfrak{N}}(\mathfrak{s})\}\\
&= \mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{z}) \wedge \Big(\bigvee_{\mathfrak{r}\mathfrak{z}\mathfrak{r}\mathfrak{z}\mathfrak{c}^2=\mathfrak{m}\mathfrak{s}} \mathfrak{I}^-_{\mathfrak{B}}(\mathfrak{m}) \vee \mathfrak{I}^-_{\mathfrak{N}}(\mathfrak{s})\Big)\\
&= \mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{z}) \wedge (\mathfrak{I}^-_{\mathfrak{B}}(\mathfrak{r}\mathfrak{z}\mathfrak{r}) \vee \mathfrak{I}^+_{\mathfrak{N}}(\mathfrak{z}\mathfrak{c}^2))\\
&\geq \mathfrak{I}^-_{\mathfrak{E}}(\mathfrak{z}) \wedge (\mathfrak{I}^-_{\mathfrak{B}}(\mathfrak{z}) \wedge \mathfrak{I}^-_{\mathfrak{N}}(\mathfrak{z}))\\
&= (\mathfrak{I}^-_{\mathfrak{E}} \wedge \mathfrak{I}^-_{\mathfrak{B}} \wedge \mathfrak{I}^-_{\mathfrak{N}})(\mathfrak{z}),\\
(\mathfrak{F}^-_{\mathfrak{E}} \circ \mathfrak{F}^-_{\mathfrak{B}} \circ \mathfrak{F}^-_{\mathfrak{N}})(\mathfrak{z}) &= \Big(\Big(\bigwedge_{\mathfrak{z}=\mathfrak{a}\mathfrak{b}} \{\mathfrak{F}^+_{\mathfrak{E}}(\mathfrak{a}) \vee (\mathfrak{F}^-_{\mathfrak{B}} \circ \mathfrak{F}^-_{\mathfrak{N}})(\mathfrak{b})\}\Big)\\
&= \Big(\Big(\bigwedge_{\mathfrak{z}\mathfrak{r}\mathfrak{z}\mathfrak{c}=\mathfrak{a}\mathfrak{b}} \{\mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{a}) \vee (\mathfrak{F}^-_{\mathfrak{B}} \circ \mathfrak{F}^+_{\mathfrak{N}})(\mathfrak{b})\}\Big)\\
&\leq \mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{z}) \vee (\mathfrak{F}^-_{\mathfrak{B}} \circ \mathfrak{F}^-_{\mathfrak{N}})(\mathfrak{r}\mathfrak{z}\mathfrak{c})\\
&= \mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{z}) \vee \bigwedge_{\mathfrak{r}\mathfrak{z}\mathfrak{c}=\mathfrak{m}\mathfrak{s}} \{\mathfrak{F}^-_{\mathfrak{B}}(\mathfrak{m}) \vee \mathfrak{F}^-_{\mathfrak{N}}(\mathfrak{s})\}\\
&= \mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{z}) \vee \Big(\bigvee_{\mathfrak{r}\mathfrak{z}\mathfrak{r}\mathfrak{z}\mathfrak{c}^2=\mathfrak{m}\mathfrak{s}} \mathfrak{F}^-_{\mathfrak{B}}(\mathfrak{m}) \vee \mathfrak{F}^-_{\mathfrak{N}}(\mathfrak{s})\Big)\\
&= \mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{z}) \vee (\mathfrak{T}^-_{\mathfrak{B}}(\mathfrak{r}\mathfrak{z}\mathfrak{r}) \vee \mathfrak{F}^-_{\mathfrak{N}}(\mathfrak{z}\mathfrak{c}^2))\\
&\leq \mathfrak{F}^-_{\mathfrak{E}}(\mathfrak{z}) \vee (\mathfrak{F}^-_{\mathfrak{B}}(\mathfrak{z}) \vee \mathfrak{F}^-_{\mathfrak{N}}(\mathfrak{z}))\\
&= (\mathfrak{F}^-_{\mathfrak{E}} \vee \mathfrak{F}^-_{\mathfrak{B}} \vee \mathfrak{F}^-_{\mathfrak{N}})(\mathfrak{z}).
\end{aligned}
$$

Hence, we get that $(\mathfrak{T}^+_{\mathfrak{E}} \circ \mathfrak{T}^+_{\mathfrak{B}} \circ \mathfrak{T}^+_{\mathfrak{N}}(\mathfrak{z}) \geq (\mathfrak{T}^+_{\mathfrak{E}} \wedge \mathfrak{T}^+_{\mathfrak{B}} \wedge \mathfrak{T}^+_{\mathfrak{N}})(\mathfrak{z})$, $(\mathfrak{I}^+_{\mathfrak{E}} \circ \mathfrak{I}^+_{\mathfrak{B}} \circ \mathfrak{I}^+_{\mathfrak{N}}(\mathfrak{z}) \leq (\mathfrak{I}^+_{\mathfrak{E}} \vee \mathfrak{I}^+_{\mathfrak{B}} \vee \mathfrak{I}^+_{\mathfrak{N}})(\mathfrak{z})$, $(\mathfrak{F}^+_{\mathfrak{E}} \circ \mathfrak{F}^+_{\mathfrak{B}} \circ \mathfrak{F}^+_{\mathfrak{N}}(\mathfrak{z}) \geq (\mathfrak{F}^+_{\mathfrak{E}} \wedge \mathfrak{F}^+_{\mathfrak{B}} \wedge \mathfrak{F}^+_{\mathfrak{N}})(\mathfrak{z})$ and $(\mathfrak{T}^-_{\mathfrak{E}} \circ \mathfrak{T}^-_{\mathfrak{B}} \circ \mathfrak{T}^-_{\mathfrak{N}}(\mathfrak{z}) \leq (\mathfrak{T}^-_{\mathfrak{E}} \vee \mathfrak{T}^-_{\mathfrak{B}} \vee \mathfrak{T}^-_{\mathfrak{N}})(\mathfrak{z})$, $(\mathfrak{I}^-_{\mathfrak{E}} \circ \mathfrak{I}^-_{\mathfrak{B}} \circ \mathfrak{I}^-_{\mathfrak{N}}(\mathfrak{z}) \geq (\mathfrak{I}^-_{\mathfrak{E}} \wedge \mathfrak{I}^-_{\mathfrak{B}} \wedge \mathfrak{I}^-_{\mathfrak{N}})(\mathfrak{z})$, $(\mathfrak{F}^-_{\mathfrak{E}} \circ \mathfrak{F}^-_{\mathfrak{B}} \circ \mathfrak{F}^-_{\mathfrak{N}}(\mathfrak{z}) \leq (\mathfrak{F}^-_{\mathfrak{E}} \vee \mathfrak{F}^-_{\mathfrak{B}} \vee \mathfrak{F}^-_{\mathfrak{N}})(\mathfrak{z})$. Therefore, $\mathfrak{E} \sqcap \mathfrak{B} \sqcap \mathfrak{N} \sqsubseteq \mathfrak{E}\bar{\circ}\mathfrak{B}\bar{\circ}\mathfrak{N}$.

It is obvious that (3) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1) Let $\mathfrak{U}$ be a BID, $\mathfrak{K}$ be an ID and $\mathfrak{N}$ be a RID of $\mathfrak{G}$. Then, by Theorem 2.1 and 3.2, $\chi_{\mathfrak{U}} = (\chi^+_{\mathfrak{U}}, \chi^-_{\mathfrak{U}})$, $\chi_{\mathfrak{K}} = (\chi^+_{\mathfrak{K}}, \chi^-_{\mathfrak{K}})$, $\chi_{\mathfrak{N}} = (\chi^+_{\mathfrak{N}}, \chi^-_{\mathfrak{N}})$ is an NSBF B ID, is an NSBF ID and is an NSBF RID of $\mathfrak{G}$. By presumption and Lemma 4.1, we have

$$
\begin{aligned}
1 = (\mathfrak{T}^+_{\chi_{\mathfrak{U}\cap\mathfrak{K}\cap\mathfrak{N}}})(\mathfrak{z}) = (\mathfrak{T}^+_{\chi_{\mathfrak{U}}} \sqcap \mathfrak{T}^+_{\chi_{\mathfrak{K}}} \sqcap \mathfrak{T}^+_{\chi_{\mathfrak{N}}})(\mathfrak{z}) \leq (\mathfrak{T}^+_{\chi_{\mathfrak{U}}} \circ \mathfrak{T}^+_{\chi_{\mathfrak{K}}} \circ \mathfrak{T}^+_{\chi_{\mathfrak{N}}})(\mathfrak{z}) = (\mathfrak{T}^+_{\chi_{\mathfrak{U}\mathfrak{K}\mathfrak{N}}})(\mathfrak{z}),\\
0 = (\mathfrak{I}^+_{\chi_{\mathfrak{U}\cap\mathfrak{K}\cap\mathfrak{N}}})(\mathfrak{z}) = (\mathfrak{I}^+_{\chi_{\mathfrak{U}}} \sqcap \mathfrak{I}^+_{\chi_{\mathfrak{K}}} \sqcap \mathfrak{I}^+_{\chi_{\mathfrak{N}}})(\mathfrak{z}) \geq (\mathfrak{I}^+_{\chi_{\mathfrak{U}}} \circ \mathfrak{I}^+_{\chi_{\mathfrak{K}}} \circ \mathfrak{I}^+_{\chi_{\mathfrak{N}}})(\mathfrak{z}) = (\mathfrak{I}^+_{\chi_{\mathfrak{U}\mathfrak{K}\mathfrak{N}}})(\mathfrak{z}),\\
1 = (\mathfrak{F}^+_{\chi_{\mathfrak{U}\cap\mathfrak{K}\cap\mathfrak{N}}})(\mathfrak{z}) = (\mathfrak{F}^+_{\chi_{\mathfrak{U}}} \sqcap \mathfrak{F}^+_{\chi_{\mathfrak{K}}} \sqcap \mathfrak{F}^+_{\chi_{\mathfrak{N}}})(\mathfrak{z}) \leq (\mathfrak{T}^+_{\chi_{\mathfrak{U}}} \circ \mathfrak{F}^+_{\chi_{\mathfrak{K}}} \circ \mathfrak{F}^+_{\chi_{\mathfrak{N}}})(\mathfrak{z}) = (\mathfrak{F}^+_{\chi_{\mathfrak{U}\mathfrak{K}\mathfrak{N}}})(\mathfrak{z}),
\end{aligned}
$$

and

$$
-1 = (\mathfrak{T}^-_{\chi_{\mathfrak{U}\cap\mathfrak{K}\cap\mathfrak{N}}})(\mathfrak{z}) = (\mathfrak{T}^-_{\chi_{\mathfrak{U}}} \sqcap \mathfrak{T}^-_{\chi_{\mathfrak{K}}} \sqcap \mathfrak{T}^-_{\chi_{\mathfrak{N}}})(\mathfrak{z}) \geq (\mathfrak{T}^-_{\chi_{\mathfrak{U}}} \circ \mathfrak{T}^-_{\chi_{\mathfrak{K}}} \circ \mathfrak{T}^-_{\chi_{\mathfrak{N}}})(\mathfrak{z}) = (\mathfrak{T}^-_{\chi_{\mathfrak{U}\mathfrak{K}\mathfrak{N}}})(\mathfrak{z}),
$$

$$0 = (\mathfrak{I}^-_{\chi_{\mathfrak{U}\cap\mathfrak{K}\cap\mathfrak{R}}})(\mathfrak{z}) = (\mathfrak{I}^-_{\chi_{\mathfrak{U}}} \sqcap \mathfrak{I}^-_{\chi_{\mathfrak{K}}} \sqcap \mathfrak{I}^+_{\chi_{\mathfrak{R}}})(\mathfrak{z}) \leq (\mathfrak{I}^-_{\chi_{\mathfrak{U}}} \circ \mathfrak{I}^-_{\chi_{\mathfrak{K}}} \circ \mathfrak{I}^-_{\chi_{\mathfrak{R}}})(\mathfrak{z}) = (\mathfrak{I}^-_{\chi_{\mathfrak{U}\mathfrak{K}\mathfrak{R}}})(\mathfrak{z}),$$

$$-1 = (\mathfrak{F}^-_{\chi_{\mathfrak{U}\cap\mathfrak{K}\cap\mathfrak{R}}})(\mathfrak{z}) = (\mathfrak{F}^-_{\chi_{\mathfrak{U}}} \sqcap \mathfrak{F}^-_{\chi_{\mathfrak{K}}} \sqcap \mathfrak{F}^+_{\chi_{\mathfrak{R}}})(\mathfrak{z}) \geq (\mathfrak{T}^-_{\chi_{\mathfrak{U}}} \circ \mathfrak{F}^-_{\chi_{\mathfrak{K}}} \circ \mathfrak{F}^+_{\chi_{\mathfrak{R}}})(\mathfrak{z}) = (\mathfrak{F}^-_{\chi_{\mathfrak{U}\mathfrak{K}\mathfrak{R}}})(\mathfrak{z}).$$

Thus we have $\mathfrak{z} \in \mathfrak{U}\mathfrak{K}\mathfrak{R}$. Hence, $\mathfrak{U} \cap \mathfrak{K} \cap \mathfrak{R} \sqsubseteq \mathfrak{U}\mathfrak{K}\mathfrak{R}$. It thus follows, by Lemma 4.4, that $\mathfrak{G}$ is WR. □

5. Conclusion

In this paper, we introduced several new types of neutrosophic bipolar-valued fuzzy ideals in semigroups, including bi-ideals, generalized bi-ideals, ideals, and interior ideals. We studied their fundamental properties and established relationships among these concepts. Furthermore, we provided various characterizations of weakly regular semigroups in terms of these newly defined fuzzy structures.

The results presented in this work extend existing studies on bipolar fuzzy sets and neutrosophic models and offer a broader framework for analyzing uncertainty in algebraic systems. Future research may explore similar structures in other algebraic settings or investigate their applications in decision-making and computational intelligence.

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