

The Inverse Unit Two-Parameter Mirra Distribution: Properties, Estimation, and Applications

Ibrahim Hassan Alkhairy¹, Jabir Bengalath^{2,*}, Sule Omeiza Bashiru³, Hassan Alsuhabi¹,
Mahmoud H. Abu-Moussa⁴, Ramy Aldallal⁵, Eslam Hussam^{6,7}, Ahmed M. Gemeay⁸

¹*Department of Mathematics, Al-Qunfudah University College, Umm Al-Qura University, Mecca, Saudi Arabia*

²*Academics, Baby Memorial Hospital Ltd. (BMH), Kozhikode (Calicut) 673004, Kerala, India*

³*Department of Mathematics and Statistics, Confluence University of Science and Technology, Osara, Kogi State, Nigeria*

⁴*Faculty of Education and Arts, Sohar University, Sohar, Oman*

⁵*Department of Management, College of Business Administration in Hawtat Bani Tamim, Prince Sattam bin Abdulaziz University, Saudi Arabia*

⁶*Department of Accounting, College of Business Administration in Hawtat Bani Tamim, Prince Sattam bin Abdulaziz University, Saudi Arabia*

⁷*Department of Mathematics, Faculty of Science, Capital University, Cairo, Egypt*

⁸*Department of Mathematics, Faculty of Science, Tanta University, Tanta 31527, Egypt*

*Corresponding author: jabirbengu@gmail.com

Abstract. This paper proposes a new probability distribution called the inverse unit two-parameter Mirra distribution (IUTMD), obtained by applying the inverse transformation scheme to the unit two-parameter Mirra distribution. The proposed model is mathematically simple, preserves the two-parameter structure of the baseline distribution, and provides greater flexibility for modeling positive continuous data. It is defined on the positive real line. Several structural properties of the proposed distribution are derived in closed form, including the survival function, hazard rate function, ordinary moments, incomplete moments, conditional moments, and extropy. The hazard rate function is capable of exhibiting increasing, decreasing, and bathtub shapes, making the proposed model suitable for lifetime and reliability applications. The unknown parameters are estimated using fifteen estimation methods, and a comprehensive Monte Carlo simulation study is conducted to investigate the finite-sample performance of these estimators under different parameter settings and sample sizes. The applicability of the proposed model is illustrated using two real datasets

Received: Jul. 26, 2026.

2020 *Mathematics Subject Classification.* 60E05; 62E20; 62F10.

Key words and phrases. inverse transformation scheme; unit two-parameter Mirra distribution; hazard rate function; maximum likelihood estimation; Monte Carlo simulation; goodness-of-fit.

involving precipitation measurements and the lifetimes of electronic components. The performance of the proposed distribution is compared with several existing competing distributions using standard goodness-of-fit measures and information criteria. The results show that the proposed distribution provides a better fit to the datasets considered and offers a flexible and useful model for positive continuous data.

1. INTRODUCTION

Probability distributions are fundamental tools in statistical modeling. They help researchers understand the behaviour of random variables and model uncertainty in many real-world phenomena. They have been applied in engineering, medicine, economics, environmental science, reliability, finance, agriculture, and many other fields. In reliability engineering, probability distributions are used to study the lifetime of machines and electronic devices. They are also applied to rainfall, river flows, air pollution, and other environmental data. In finance, they are used to model asset returns and financial risk. The quality of statistical inference often depends on how well a probability distribution fits the observed data. Therefore, the search for more flexible probability distributions continues to receive considerable attention in statistical research.

Many classical probability distributions have been proposed over the years. However, they are often unable to describe the different shapes observed in real data. Real data may have different levels of skewness, tail behaviour, and hazard rate shapes that existing models cannot describe well. As a result, using an unsuitable model may lead to poor parameter estimates, unreliable predictions, and incorrect statistical conclusions. For this reason, many new probability distributions have been developed to provide greater flexibility for modelling different types of data. Some recent examples include the cosine Topp-Leone XShanker distribution [31], the Modi-Lomax distribution [33], the new odd type-inverse Rayleigh distribution [32], the alpha-power DUS Lindley distribution [35], transmuted power unit inverse Lindley distribution [36], and the half-logistic Garhy distribution [34].

Several methods have been proposed for constructing new probability distributions. These include generator families such as the cosine inverse Lomax generator [20], the new logarithmic pie power generator [21], and the new gamma odd Lindley generator [22], as well as parameter induction methods, transformation methods, and inversion methods. Many of these methods improve model flexibility by introducing one or more additional parameters. However, additional parameters may increase the difficulty of parameter estimation, computational cost, and the risk of parameter identifiability problems. Therefore, there is still interest in construction methods that improve model flexibility without increasing the number of parameters.

The inverse transformation scheme is one of the approaches used to construct new probability distributions. This approach has been used to develop several inverse distributions, such as the inverse Xgamma [30], inverse Hamza [27], inverse exponential power [23], inverse type II Topp-Leone half-logistic [24], inverse Dagum [26], inverted Gompertz-Fréchet [28], inverse power Cauchy [29], and inverse power type II Topp-Leone half-logistic [25] distributions. It transforms a baseline distribution into a new distribution while preserving the original number of parameters.

Consequently, it provides an alternative way of obtaining more flexible models without increasing model complexity.

Recently, [1] proposed the unit two-parameter Mirra distribution (UTMD), a flexible two-parameter distribution defined on the unit interval $(0,1)$. The distribution has useful statistical properties and is suitable for modelling bounded data. However, many datasets arising in reliability, engineering, medicine, and environmental studies are measured on the positive real line rather than on the unit interval. This limits the direct application of the unit two-parameter Mirra distribution. In addition, there is a growing need for simple lifetime models that can accommodate complex hazard rate shapes without increasing the number of model parameters.

To address these limitations, this paper proposes the inverse unit two-parameter Mirra distribution (IUTMD) by applying the inverse transformation scheme to the unit two-parameter Mirra distribution. The proposed distribution is defined on the positive real line ($x > 0$) and retains the same two parameters as the baseline model. Although the model has a simple structure, it provides greater flexibility for modelling positive data. An important feature of the proposed distribution is its ability to model bathtub-shaped, increasing, and decreasing hazard rates. The bathtub-shaped hazard pattern is common in reliability studies, where failures occur at the early stage, followed by a period of stable operation, and finally an increasing failure rate due to wear-out. Increasing and decreasing hazard rates are also observed in many practical applications involving mechanical systems, electronic components, industrial equipment, and manufactured products.

Several mathematical properties of the proposed distribution are derived. The unknown parameters of the proposed distribution are estimated using fifteen estimation methods, including the maximum likelihood estimator and several minimum distance and goodness-of-fit-based estimators. A comprehensive Monte Carlo simulation study is carried out to examine the finite-sample performance of these estimators under different sample sizes and parameter settings. The results provide useful information on the accuracy and efficiency of the different estimation methods.

Finally, the usefulness of the proposed distribution is illustrated using two real datasets involving precipitation measurements and the lifetimes of electronic components. Its performance is compared with several existing competing distributions using standard goodness-of-fit measures and information criteria. The results show that the proposed IUTMD provides a better fit to the datasets considered, demonstrating its flexibility and usefulness for modelling positive continuous data.

The organization of the remaining sections of this study is illustrated in Figure 1.

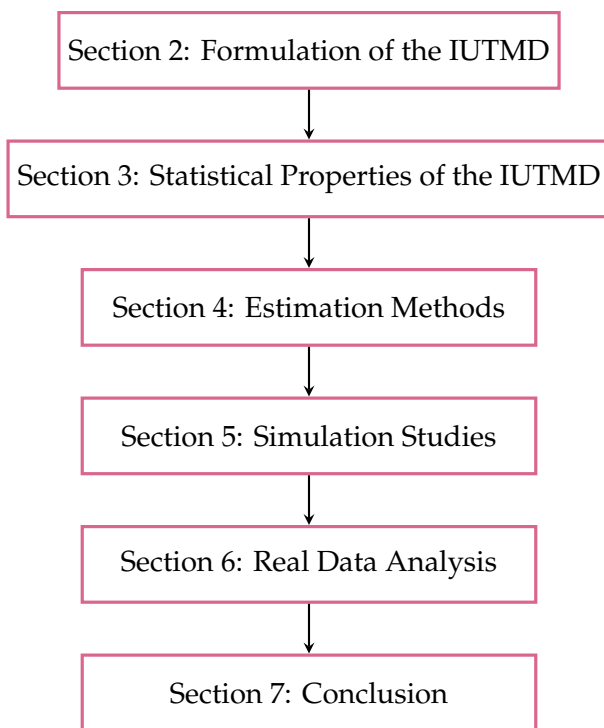


FIGURE 1. Flowchart of the Remaining Sections of the Study.

2. MODEL FORMULATION

Al-Omari et al. [1] presented the unit two-parameter Mirra distribution (UTMD) and defined its cumulative distribution function (CDF) as follows

$$W(x) = e^{-\frac{\lambda(1-x)}{x}} \left(1 + \frac{\lambda\phi}{\lambda^2 + \phi} \left(\frac{\lambda(1-x)^2}{2x^2} + \frac{1-x}{x} \right) \right); 0 < x < 1, \lambda > 0, \phi > 0. \quad (2.1)$$

We applied the inverse transformation to the CDF (2.1) to obtain a new distribution called inverse unit two-parameter Mirra distribution (IUTMD) with CDF given by

$$F(x) = \frac{e^{-\lambda x} (-2\lambda^2 - \phi(\lambda x(\lambda x + 2) + 2))}{2(\lambda^2 + \phi)} + 1; x > 0, \lambda > 0, \phi > 0, \quad (2.2)$$

and its probability density function (PDF) is defined as follows

$$f(x) = \frac{\lambda^3 e^{\lambda(-x)} (x^2 \phi + 2)}{2(\lambda^2 + \phi)}. \quad (2.3)$$

The survival function of the IUTMD is given by

$$s(x) = -\frac{e^{-\lambda x} (-2\lambda^2 - \phi(\lambda x(\lambda x + 2) + 2))}{2(\lambda^2 + \phi)}$$

The hazard function of the IUTMD is given by

$$h(x) = \frac{\frac{\lambda^3 e^{\lambda(-x)} (x^2 \phi + 2)}{2(\lambda^2 + \phi)}}{1 - \left(\frac{e^{-\lambda x} (-2\lambda^2 - \phi(\lambda x(\lambda x + 2) + 2))}{2(\lambda^2 + \phi)} + 1 \right)} = \frac{\lambda^3 (\phi x^2 + 2)}{2\lambda^2 + \phi (\lambda x(\lambda x + 2) + 2)}$$

Figure 2 shows the CDF plot of the IUTMD. The CDF increases smoothly from 0 to 1 as x increases, indicating the expected behavior of a valid CDF.

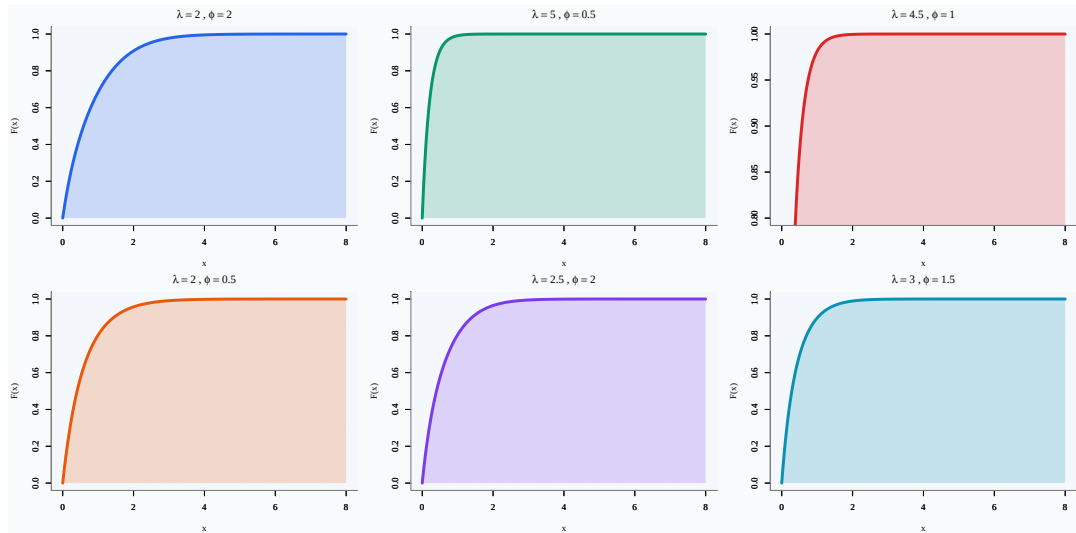


FIGURE 2. Plots of the CDF of the IUTMD.

Figure 3 shows that the IUTMD is positively skewed.

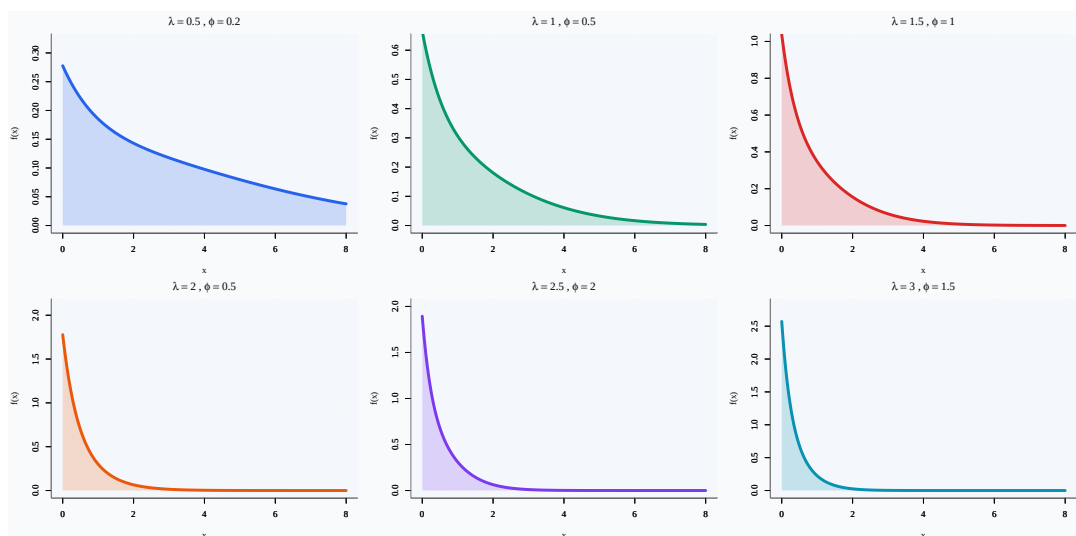


FIGURE 3. Plots of the PDF of the IUTMD.

Figure 4 shows that the hazard function of the IUTMD can take increasing, decreasing, and bathtub shapes for different parameter values. This flexibility makes the distribution suitable for modeling data with different failure rate patterns.

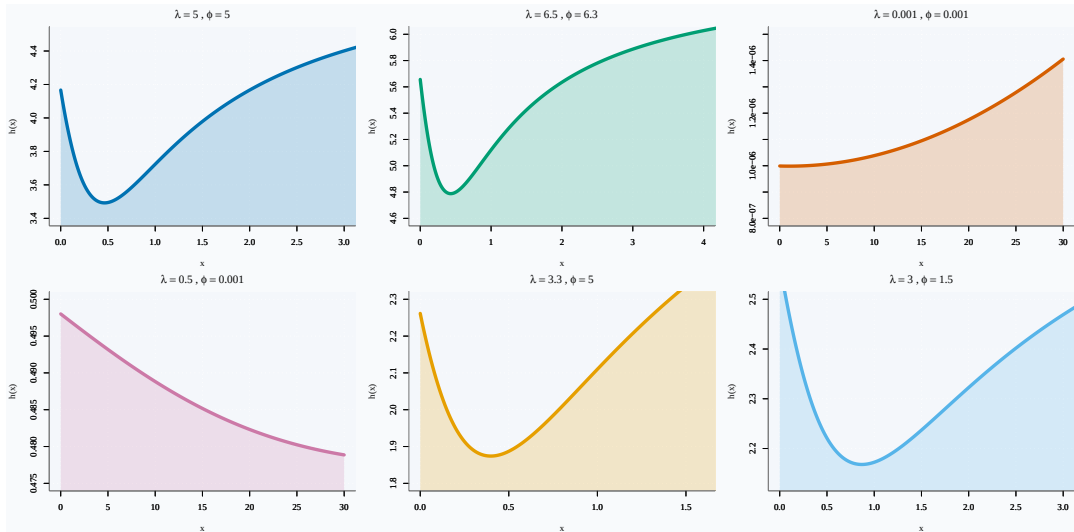


FIGURE 4. Plots of the hazard function of the IUTMD.

3. STATISTICAL PROPERTIES

This section presents some important statistical properties of the IUTMD.

3.1. Moments. The moments of a distribution provide useful information about its main characteristics, such as the mean, variance, skewness, and kurtosis. The r th raw moment of the IUTMD is obtained by integrating x^r with respect to the PDF and is given by

$$\mu'_r = \int_0^\infty x^r f(x) dx = \int_0^\infty x^r \frac{\lambda^3 e^{\lambda(-x)} (x^2 \phi + 2)}{2(\lambda^2 + \phi)} dx = \frac{\lambda^{-r} (2\lambda^2 \Gamma(r+1) + \phi \Gamma(r+3))}{2(\lambda^2 + \phi)},$$

if $r > -1$.

The first four moments are obtained as follows

$$\begin{aligned} \mu'_1 &= \int_0^\infty x f(x) dx = \int_0^\infty x \frac{\lambda^3 e^{\lambda(-x)} (\phi x^2 + 2)}{2(\lambda^2 + \phi)} dx = \frac{\lambda^2 + 3\phi}{\lambda^3 + \lambda\phi}, \\ \mu'_2 &= \int_0^\infty x^2 f(x) dx = \int_0^\infty x^2 \frac{\lambda^3 e^{\lambda(-x)} (x^2 \phi + 2)}{2(\lambda^2 + \phi)} dx = \frac{12}{\lambda^2} - \frac{10}{\lambda^2 + \phi}, \\ \mu'_3 &= \int_0^\infty x^3 f(x) dx = \int_0^\infty x^3 \frac{\lambda^3 e^{\lambda(-x)} (x^2 \phi + 2)}{2(\lambda^2 + \phi)} dx = \frac{6(\lambda^2 + 10\phi)}{\lambda^3(\lambda^2 + \phi)}, \\ \mu'_4 &= \int_0^\infty x^4 f(x) dx = \int_0^\infty x^4 \frac{\lambda^3 e^{\lambda(-x)} (x^2 \phi + 2)}{2(\lambda^2 + \phi)} dx = \frac{24(\lambda^2 + 15\phi)}{\lambda^4(\lambda^2 + \phi)}. \end{aligned}$$

The variance of the IUTMD is given by

$$Var(x) = \mu'_2 - (\mu'_1)^2 = \left(\frac{12}{\lambda^2} - \frac{10}{\lambda^2 + \phi} \right) - \left(\frac{\lambda^2 + 3\phi}{\lambda^3 + \lambda\phi} \right)^2.$$

The skewness (S_k) and kurtosis (K_u) of the proposed model can be obtained by using the following expressions

$$S_k = \frac{\mu'_3 - 3\mu'_1\mu'_2 + 2(\mu'_1)^3}{(\mu'_2 - (\mu'_1)^2)^{3/2}},$$

$$K_u = \frac{\mu'_4 - 4\mu'_1\mu'_3 + 6(\mu'_1)^2\mu'_2 - 3(\mu'_1)^4}{(\mu'_2 - (\mu'_1)^2)^2}.$$

The numerical results in Table 1 show the effect of the parameters λ and ϕ on the first four raw moments and the summary measures of the proposed distribution. When $\phi = 0.5$ is fixed, increasing λ decreases the first four raw moments, the mean, and the variance. This indicates that larger values of λ make the distribution more concentrated around smaller values. In contrast, the skewness generally increases, reaches its highest listed value at $\lambda = 3$, and then decreases slightly. The kurtosis initially decreases slightly, subsequently increases, and then decreases slightly for the largest values of λ .

When $\lambda = 2.5$ is fixed, increasing ϕ increases the first four raw moments, the mean, and the variance. This shows that larger values of ϕ increase both the average value and the spread of the distribution. The skewness and kurtosis initially increase slightly and then decrease as ϕ increases. This indicates that, after the initial increase, larger values of ϕ generally make the distribution less skewed and less peaked.

Overall, the results show that both parameters have clear effects on the shape and spread of the proposed distribution, making it flexible for modelling data with different characteristics.

Table 1: Numerical values of the first four raw moments and summary measures of the IUTMD.

λ	ϕ	μ'_1	μ'_2	μ'_3	μ'_4	Variance	Skewness	Kurtosis
0.2000	0.5000	14.2593	281.4815	7000.0000	209444.4444	78.1550	1.0962	4.8387
0.5000	0.5000	4.6667	34.6667	336.0000	3968.0000	12.8889	1.1654	4.8335
0.8000	0.5000	2.3465	9.9781	57.9770	418.3799	4.4720	1.4356	5.6451
1.1000	0.5000	1.4407	4.0694	16.3708	83.4954	1.9937	1.6920	6.7694
1.5000	0.5000	0.9091	1.6970	4.6869	16.8081	0.8705	1.9224	8.0899
2.0000	0.5000	0.6111	0.7778	1.5000	3.8333	0.4043	2.0635	9.1210
2.5000	0.5000	0.4593	0.4385	0.6400	1.2516	0.2276	2.1141	9.6009
3.0000	0.5000	0.3684	0.2807	0.3275	0.5146	0.1450	2.1243	9.7710

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Table 1 – continued from previous page

λ	ϕ	μ'_1	μ'_2	μ'_3	μ'_4	Variance	Skewness	Kurtosis
4.0000	0.5000	0.2652	0.1439	0.1193	0.1335	0.0736	2.1072	9.7497
5.0000	0.5000	0.2078	0.0878	0.0565	0.0489	0.0446	2.0836	9.6146
2.5000	0.1000	0.4126	0.3452	0.4384	0.7499	0.1750	2.0718	9.5370
2.5000	0.3000	0.4366	0.3933	0.5423	1.0084	0.2026	2.1227	9.7926
2.5000	0.4000	0.4481	0.4162	0.5919	1.1318	0.2154	2.1230	9.7264
2.5000	0.7000	0.4806	0.4812	0.7321	1.4807	0.2502	2.0806	9.2682
2.5000	1.0000	0.5103	0.5407	0.8607	1.8008	0.2802	2.0136	8.7259
2.5000	1.5000	0.5548	0.6297	1.0529	2.2792	0.3218	1.8973	7.9287
2.5000	2.0000	0.5939	0.7079	1.2218	2.6996	0.3551	1.7935	7.3101
2.5000	2.5000	0.6286	0.7771	1.3714	3.0720	0.3820	1.7052	6.8365
2.5000	3.0000	0.6595	0.8389	1.5049	3.4041	0.4040	1.6305	6.4698
2.5000	4.0000	0.7122	0.9444	1.7327	3.9711	0.4372	1.5132	5.9511

3.2. Inverse Moments. Inverse moments are useful in reliability analysis, actuarial science, and risk modeling. They are particularly helpful for studying measures that depend on the reciprocal of the random variable. The r th inverse moment of the IUTMD is obtained by integrating x^{-r} with respect to the PDF and is given by

$$\mu_{-r} = \int_0^{\infty} x^{-r} f(x) dx = \int_0^{\infty} x^{-r} \frac{\lambda^3 e^{\lambda(-x)} (x^2 \phi + 2)}{2(\lambda^2 + \phi)} dx = \frac{\lambda^r \Gamma(1-r) (2\lambda^2 + (r-2)(r-1)\phi)}{2(\lambda^2 + \phi)},$$

if $r < 1$.

3.3. Incomplete Moment. Incomplete moments describe the behaviour of a distribution over a specified range of values. They are useful in reliability studies and in measuring income inequality. The incomplete moment of the IUTMD is given by

$$\begin{aligned} I.M &= \int_0^t x^s f(x) dx = \int_0^t x^s \frac{\lambda^3 e^{\lambda(-x)} (x^2 \phi + 2)}{2(\lambda^2 + \phi)} dx \\ &= \frac{\lambda^{-s} (2\lambda^2 (\Gamma(s+1) - \Gamma(s+1, t\lambda)) + \phi (\Gamma(s+3) - \Gamma(s+3, t\lambda)))}{2(\lambda^2 + \phi)}, \end{aligned}$$

if $s > -1$.

3.4. Conditional Moments. Conditional moments describe the behaviour of a random variable above a specified threshold. They are widely used in reliability analysis, survival analysis, and risk assessment. The conditional moment of the IUTMD is given by

$$C.M = \frac{1}{S(t)} \int_t^{\infty} x^s f(x) dx = \frac{1}{S(t)} \int_t^{\infty} x^s \frac{\lambda^3 e^{\lambda(-x)} (x^2 \phi + 2)}{2(\lambda^2 + \phi)} dx$$

$$= \frac{1}{S(t)} \frac{\lambda^{-s} (2\lambda^2 \Gamma(s+1, t\lambda) + \phi \Gamma(s+3, t\lambda))}{2(\lambda^2 + \phi)}.$$

3.5. Mean Residual Life Function. The mean residual life function gives the expected remaining lifetime of a unit that has survived up to a specified time t . It is widely used in reliability and survival analysis to study the remaining life of a system. The mean residual life function of the IUTMD is given by

$$\begin{aligned} MRL &= \frac{1}{S(t)} \int_t^\infty x f(x) dx - t = \frac{1}{S(t)} \int_t^\infty x \frac{\lambda^3 e^{\lambda(-x)} (x^2 \phi + 2)}{2(\lambda^2 + \phi)} dx - t \\ &= \frac{1}{S(t)} \frac{e^{\lambda(-t)} (2\lambda^2 (\lambda t + 1) + \phi (\lambda t (\lambda t (\lambda t + 3) + 6) + 6))}{2\lambda (\lambda^2 + \phi)} - t. \end{aligned}$$

3.6. Mean Inactive Time. The mean inactive time gives the expected time elapsed since failure, given that the failure occurred before a specified time t . It is a useful measure in reliability and survival analysis for studying past lifetime. The mean inactive time of the IUTMD is given by

$$\begin{aligned} MIT &= t - \frac{1}{F(t)} \int_0^t x f(x) dx = t - \frac{1}{F(t)} \int_0^t x \frac{\lambda^3 e^{\lambda(-x)} (x^2 \phi + 2)}{2(\lambda^2 + \phi)} dx \\ &= t - \frac{1}{F(t)} \frac{e^{\lambda(-t)} (2(\lambda^2 + 3\phi) e^{\lambda t} + \lambda(-t\phi(\lambda t(\lambda t + 3) + 6) - 2\lambda(\lambda t + 1)) - 6\phi)}{2\lambda(\lambda^2 + \phi)}. \end{aligned}$$

3.7. Entropy. Entropy is a measure of uncertainty that complements entropy by quantifying the concentration of a probability distribution. It provides additional information about the uncertainty and variability of a random variable. The entropy of the IUTMD is obtained as

$$E = -\frac{1}{2} \left[\int_0^\infty (f(x))^2 dx \right] = -\frac{1}{2} \left[\int_0^\infty \left(\frac{\lambda^3 e^{\lambda(-x)} (x^2 \phi + 2)}{2(\lambda^2 + \phi)} \right)^2 dx \right] = -\frac{1}{2} \left[\frac{\lambda(8\lambda^4 + 4\lambda^2 \phi + 3\phi^2)}{16(\lambda^2 + \phi)^2} \right].$$

3.8. Weighted Entropy. Weighted entropy is a weighted version of entropy that gives more importance to larger observations. The weighted entropy of the IUTMD is given by

$$WE = -\frac{1}{2} \left[\int_0^\infty x (f(x))^2 dx \right] = -\frac{1}{2} \left[\int_0^\infty x \left(\frac{\lambda^3 e^{\lambda(-x)} (x^2 \phi + 2)}{2(\lambda^2 + \phi)} \right)^2 dx \right] = -\frac{1}{2} \left[\frac{8\lambda^4 + 12\lambda^2 \phi + 15\phi^2}{32(\lambda^2 + \phi)^2} \right].$$

The numerical values in Table 2 show the behaviour of the entropy and weighted entropy of the proposed distribution for different parameter values. When $\phi = 0.5$ is fixed, the entropy decreases as λ increases. For example, it changes from -0.0181 at $\lambda = 0.2$ to -1.2137 at $\lambda = 5.0$. In contrast, the weighted entropy initially increases from -0.2145 to about -0.1193 and then decreases slightly for larger values of λ . When $\lambda = 2.5$ is fixed, the entropy increases steadily as ϕ increases, changing from -0.6104 at $\phi = 0.1$ to -0.3424 at $\phi = 4.0$. The weighted entropy changes only slightly with increasing ϕ . It increases slightly up to about $\phi = 1.5$ and then decreases gradually for larger values of ϕ . These results show that the parameter λ has a stronger effect on the entropy, while the weighted entropy is less affected by changes in both parameters.

TABLE 2. Extropy and weighted extropy of the IUTMD for selected parameter values.

λ	ϕ	E	WE
0.2000	0.5000	-0.0181	-0.2145
0.5000	0.5000	-0.0486	-0.1597
0.8000	0.5000	-0.1021	-0.1307
1.1000	0.5000	-0.1750	-0.1214
1.5000	0.5000	-0.2836	-0.1193
2.0000	0.5000	-0.4221	-0.1202
2.5000	0.5000	-0.5586	-0.1213
3.0000	0.5000	-0.6926	-0.1222
4.0000	0.5000	-0.9553	-0.1233
5.0000	0.5000	-1.2137	-0.1238
2.5000	0.1000	-0.6104	-0.1241
2.5000	0.3000	-0.5832	-0.1225
2.5000	0.4000	-0.5706	-0.1219
2.5000	0.7000	-0.5361	-0.1204
2.5000	1.0000	-0.5061	-0.1196
2.5000	1.5000	-0.4640	-0.1193
2.5000	2.0000	-0.4299	-0.1199
2.5000	2.5000	-0.4018	-0.1212
2.5000	3.0000	-0.3785	-0.1228
2.5000	4.0000	-0.3424	-0.1268

4. ESTIMATION METHODS

In this section, we estimate the parameters of the IUTMD using various estimation techniques in the literature, which are explained in detail in the following subsections.

4.1. Maximum likelihood estimation (MLE). Let $X_1, X_2, \dots, X_n$ be a random sample of size n drawn from the IUTMD model with PDF given in Eq. (2.3). The likelihood function for the observed sample $\mathbf{x} = (x_1, \dots, x_n)$ is

$$\begin{aligned}
 L(\lambda, \phi) &= \prod_{i=1}^n \frac{\lambda^3 e^{-\lambda x_i} (\phi x_i^2 + 2)}{2(\lambda^2 + \phi)}, \\
 &= \frac{\lambda^{3n} \exp(-\lambda \sum_{i=1}^n x_i) \prod_{i=1}^n (\phi x_i^2 + 2)}{[2(\lambda^2 + \phi)]^n}.
 \end{aligned}$$

The corresponding log-likelihood function is

$$\ell(\lambda, \phi) = 3n \ln \lambda - \lambda \sum_{i=1}^n x_i + \sum_{i=1}^n \ln(\phi x_i^2 + 2) - n \ln 2 - n \ln(\lambda^2 + \phi).$$

The maximum likelihood estimators $\hat{\lambda}$ and $\hat{\phi}$ are obtained by solving the system of equations obtained by setting the partial derivatives of the log-likelihood function $\ell(\lambda, \phi)$ equal to zero.

Thus

$$\frac{\partial \ell}{\partial \lambda} = \frac{3n}{\lambda} - \sum_{i=1}^n x_i - \frac{2n\lambda}{\lambda^2 + \phi}.$$

Setting $\partial \ell / \partial \lambda = 0$ gives the first score equation

$$\frac{3n}{\lambda} - \sum_{i=1}^n x_i = \frac{2n\lambda}{\lambda^2 + \phi}. \quad (4.1)$$

$$\frac{\partial \ell}{\partial \phi} = \sum_{i=1}^n \frac{x_i^2}{\phi x_i^2 + 2} - \frac{n}{\lambda^2 + \phi}.$$

Setting $\partial \ell / \partial \phi = 0$ yields the second score equation:

$$\sum_{i=1}^n \frac{x_i^2}{\phi x_i^2 + 2} = \frac{n}{\lambda^2 + \phi}. \quad (4.2)$$

The system of nonlinear Eqs. (4.1)–(4.2) does not admit closed-form solutions. Therefore, the MLEs $\hat{\lambda}$ and $\hat{\phi}$ must be obtained numerically, such as by the Newton–Raphson or Fisher scoring methods in R.

4.2. Anderson-Darling estimation (ADE). The ADE technique was proposed by [2] to minimize a weighted goodness-of-fit statistic that places a greater emphasis on the tail of the distribution. The parameter estimates for the IUTMD can be obtained by minimizing the ADE statistic, and they are given by the formula

$$\mathcal{A}_n(x_i; \lambda, \phi) = -n - \frac{1}{n} \sum_{i=1}^n (2i-1) [\log F(x_{i:n}) + \log S(x_{n+1-i:n})],$$

where $x_{i:n}$ denotes the i th order statistic, and $F(x)$ and $S(x)$ are the CDF and survival function of the IUTMD, respectively.

4.3. Cramér–von Mises estimation (CVME). The CVME technique was proposed by [3]. This method estimates the parameters by finding the values that minimize the CVME criterion, which is particularly sensitive to discrepancies across the entire range of the empirical CDF. The parameters $\hat{\lambda}$ and $\hat{\phi}$ of the IUTMD are obtained by minimizing the following expression

$$C_n(x_i; \lambda, \phi) = \frac{1}{12n} + \sum_{i=1}^n \left[F(x_{i:n}) - \frac{2i-1}{2n} \right]^2.$$

4.4. Maximum product of spacings estimation (MPSE). The MPSE method, introduced by [4], serves as an alternative to traditional MLE for estimating the parameters of probability distributions. Instead of maximizing the likelihood, the maximum product of spacings method works by maximizing the product of the spacings. For the IUTMD, parameter estimates are obtained by minimizing the negative log-product of the spacings

$$S_n(x_i; \lambda, \phi) = \frac{1}{n+1} \sum_{i=1}^{n+1} \log I_i(x_i),$$

where $I_i(x_i) = F(x_{i:n}) - F(x_{i-1:n})$, $F(x_{0:n}) = 0$ and $F(x_{n+1:n}) = 1$.

4.5. Ordinary least squares estimation (OLSE). The OLSE method, proposed by [5], is commonly used to estimate the parameters of probability distributions. It minimizes the sum of squared differences between the empirical CDF and the model's theoretical CDF. For the IUTMD, the parameters are obtained by minimizing the least squares statistic

$$Q_n(x_i; \lambda, \phi) = \sum_{i=1}^n \left[F(x_{i:n}) - \frac{i}{n+1} \right]^2.$$

4.6. Right-tail Anderson Darling estimation (RTADE). The RTADE method by [2] emphasizes accurate modelling of the right tail (upper tail) of the distribution. It is particularly suitable for the IUTMD when right tail behaviour is critical. The estimated parameters $\hat{\lambda}$ and $\hat{\phi}$ for the IUTMD are obtained by minimizing the RTADE criterion

$$\mathcal{A}_n^R(x_i; \lambda, \phi) = \frac{n}{2} - 2 \sum_{i=1}^n F(x_{i:n}) - \frac{1}{n} \sum_{i=1}^n (2i-1) \log S(x_{i:n}).$$

4.7. Weighted least squares estimation (WLSE). The WLSE method, introduced by [5], serves as a robust alternative to ordinary least squares. It incorporates weights into the squared difference between the empirical and theoretical CDFs, assigning greater weights to observations near the tails of the distribution.

For the IUTMD, parameter estimates are obtained by minimizing the WLSE criterion

$$\mathcal{L}_n^{\text{WLS}}(x_i; \lambda, \phi) = \sum_{i=1}^n \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_{i:n}) - \frac{i}{n+1} \right]^2.$$

4.8. Left tail Anderson Darling estimation (LTADE). The LTADE method focuses on optimizing how well the theoretical model matches the observed data, specifically in the lower (left) tail of the distribution, as outlined in [6]. For the IUTMD, the parameter estimates $\hat{\lambda}$ and $\hat{\phi}$ are obtained by minimizing the LTADE criterion, which is expressed as

$$\mathcal{A}_n^L(x_i; \lambda, \phi) = -\frac{3}{2}n + 2 \sum_{i=1}^n F(x_{i:n}) - \frac{1}{n} \sum_{i=1}^n (2i-1) \log F(x_{i:n}).$$

4.9. Minimum spacing absolute distance estimation (MSADE). The MSADE approach introduced by [7] is based on minimizing the total absolute deviations between the observed spacings and their theoretical expected value of $\frac{1}{n+1}$. For the IUTMD, the parameters are estimated by minimizing the corresponding MSADE criterion, which is given by

$$\mathcal{M}_n^{\text{SAD}}(x_i; \lambda, \phi) = \sum_{i=1}^{n+1} \left| I_i - \frac{1}{n+1} \right|.$$

4.10. Minimum spacing absolute-log distance estimation (MSALDE). The MSALDE method, proposed by [7], employs a logarithmic transformation of the spacing to estimate parameters, intending to achieve balanced treatment across all spacings. For the IUTMD, parameters are estimated by minimizing the MSALDE criterion, which is expressed as follows

$$\mathcal{M}_n^{\text{SALD}}(x_i; \lambda, \phi) = \sum_{i=1}^{n+1} \left| \log I_i - \log \frac{1}{n+1} \right|.$$

4.11. Anderson Darling left tail second order estimation (ADSOE). This refined approach, as introduced in [8], examines the second-order characteristics of the distribution within the left tail, thereby offering a more detailed evaluation of the model's suitability near the lower extremes. The parameters of the IUTMD are determined through minimization of the ADSOE criterion, whose expression is provided as follows

$$\mathcal{A}_{n,2}^L(x_i; \lambda, \phi) = 2 \sum_{i=1}^n \log F(x_i) + \frac{1}{n} \sum_{i=1}^n \frac{(2i-1)}{F(x_i)}.$$

4.12. Kolmogorov estimation (KE). The KE method [8] is favoured for its simplicity and effectiveness in measuring the largest discrepancies between the empirical and theoretical CDFs. The parameters of the IUTMD can be estimated via the Kolmogorov estimators by minimizing the following expression

$$\mathcal{K}_n(x_i; \lambda, \phi) = \text{MAX}_{1 \leq i \leq n} \left[\frac{i}{n} - F(x_i), F(x_i) - \frac{i-1}{n} \right].$$

4.13. Minimum spacing square distance estimation (MSSDE). The MSSDE method, proposed by [7], is based on the sum of squared differences between the observed spacings and their corresponding expected values. For the IUTMD, parameter estimation is achieved by minimizing the MSSDE statistic, given by the following expression:

$$\mathcal{M}_n^{\text{SSD}}(x_i; \lambda, \phi) = \sum_{i=1}^{n+1} \left(I_i - \frac{1}{n+1} \right)^2.$$

4.14. Minimum spacing square-log distance estimation (MSSLDE). The MSSLDE approach [7] is based on employing the squared differences derived from log-transformed spacing. The parameters of the IUTMD are obtained through minimization of the MSSLDE criterion

$$\mathcal{M}_n^{\text{SSLD}}(x_i; \lambda, \phi) = \sum_{i=1}^{n+1} \left(\log I_i - \log \frac{1}{n+1} \right)^2.$$

4.15. Minimum spacing LINEX distance estimation (MSLNDE). The MSLNDE method [7] utilizes an asymmetric LINEX-type loss function to derive the parameter estimates. The parameters of the IUTMD are obtained by minimizing the corresponding MSLNDE criterion

$$\mathcal{M}_n^{\text{SLX}}(x_i; \lambda, \phi) = \sum_{i=1}^{n+1} \left[e^{I_i - \frac{1}{n+1}} - \left(I_i - \frac{1}{n+1} \right) - 1 \right].$$

5. NUMERICAL SIMULATION

The performance of the estimators of the proposed model was evaluated through a comprehensive Monte Carlo simulation study. All computations were performed using the R software environment. The following measures were used to assess the accuracy of the estimators:

- Average bias (BIAS), $|Bias(\widehat{\omega})| = \frac{1}{L} \sum_{i=1}^L |\widehat{\omega} - \omega|$.
- Mean squared errors (MSE), $MSE = \frac{1}{L} \sum_{i=1}^L (\widehat{\omega} - \omega)^2$.
- Mean relative errors (MRE), $MRE = \frac{1}{L} \sum_{i=1}^L |\widehat{\omega} - \omega| / \omega$.
- Average absolute difference (D_{abs}), $D_{abs} = \frac{1}{nH} \sum_{i=1}^H \sum_{j=1}^n |F(x_{ij}; \omega) - F(x_{ij}; \widehat{\omega})|$.
- Maximum absolute difference (D_{max}), $D_{max} = \frac{1}{H} \sum_{i=1}^H \max_{j=1, \dots, n} |F(x_{ij}; \omega) - F(x_{ij}; \widehat{\omega})|$.
- Average scaled absolute error, $ASAE = \frac{1}{n} \sum_{i=1}^n \frac{|x_i - \hat{x}_i|}{x_n - x_1}$, $\omega = (\lambda, \phi)$.

The numerical results of the simulation study are presented in Tables 3 to 7. These tables consider five different parameter combinations, with each combination evaluated at various sample sizes using 1,000 replications. Figures 5 to 10 are used to illustrate the results presented in Table 3. The Monte Carlo study indicates that all estimators become more stable as the sample size increases, while the values of all measures decrease accordingly. This pattern indicates that the estimators generally become more accurate as the sample size increases. The simulation results presented in Table 8 indicate that the MLE has the best overall performance among all the estimation methods, followed by the WLSE. The ranks of the estimation methods are presented in Table 8.

TABLE 3. Numerical values of simulation measures for $\lambda = 0.35$, $\phi = 0.65$.

n	Est.	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
20	BIAS($\hat{\lambda}$)	0.04288 ⁽¹⁾	0.04755 ⁽³⁾	0.04755 ⁽⁴⁾	0.0553 ⁽⁹⁾	0.05548 ⁽¹⁰⁾	0.04843 ⁽⁵⁾	0.0514 ⁽⁷⁾	0.04992 ⁽⁶⁾	0.06119 ⁽¹³⁾	0.06072 ⁽¹²⁾	0.05855 ⁽¹¹⁾	0.0451 ⁽²⁾	0.06913 ⁽¹⁴⁾	0.05328 ⁽⁸⁾	0.0709 ⁽¹⁵⁾
20	BIAS($\hat{\phi}$)	0.30556 ⁽¹⁾	0.32146 ⁽³⁾	0.32764 ⁽⁵⁾	0.34906 ⁽⁸⁾	0.36489 ⁽¹²⁾	0.3329 ⁽⁶⁾	0.34425 ⁽⁷⁾	0.32655 ⁽⁴⁾	0.35796 ⁽¹⁰⁾	0.3625 ⁽¹¹⁾	0.38737 ⁽¹³⁾	0.31496 ⁽²⁾	0.39016 ⁽¹⁴⁾	0.351 ⁽⁹⁾	0.39398 ⁽¹⁵⁾
20	MSE($\hat{\lambda}$)	0.00289 ⁽¹⁾	0.00351 ⁽³⁾	0.00363 ⁽⁴⁾	0.00507 ⁽¹⁰⁾	0.0047 ⁽⁹⁾	0.00369 ⁽⁵⁾	0.00414 ⁽⁷⁾	0.00395 ⁽⁶⁾	0.00575 ⁽¹⁰⁾	0.00591 ⁽¹³⁾	0.00522 ⁽¹¹⁾	0.0032 ⁽²⁾	0.00733 ⁽¹⁴⁾	0.00469 ⁽⁸⁾	0.00782 ⁽¹⁵⁾
20	MSE($\hat{\phi}$)	0.12315 ⁽¹⁾	0.13463 ⁽³⁾	0.14001 ⁽⁵⁾	0.1555 ⁽⁸⁾	0.1651 ⁽¹¹⁾	0.14252 ⁽⁶⁾	0.15071 ⁽⁷⁾	0.13963 ⁽⁴⁾	0.1629 ⁽¹⁰⁾	0.16689 ⁽¹²⁾	0.18417 ⁽¹³⁾	0.13113 ⁽²⁾	0.18617 ⁽¹⁴⁾	0.15573 ⁽⁹⁾	0.18799 ⁽¹⁵⁾
20	MRE($\hat{\lambda}$)	0.12251 ⁽¹⁾	0.13582 ⁽³⁾	0.13587 ⁽⁴⁾	0.15799 ⁽⁹⁾	0.15851 ⁽¹⁰⁾	0.13837 ⁽⁵⁾	0.14686 ⁽⁷⁾	0.14262 ⁽⁶⁾	0.17484 ⁽¹³⁾	0.1735 ⁽¹²⁾	0.16728 ⁽¹¹⁾	0.12887 ⁽²⁾	0.19752 ⁽¹⁴⁾	0.15222 ⁽⁸⁾	0.20258 ⁽¹⁵⁾
20	MRE($\hat{\phi}$)	0.47009 ⁽¹⁾	0.49455 ⁽³⁾	0.50406 ⁽⁵⁾	0.53702 ⁽⁸⁾	0.56138 ⁽¹²⁾	0.51215 ⁽⁶⁾	0.52962 ⁽⁷⁾	0.50238 ⁽⁴⁾	0.55071 ⁽¹⁰⁾	0.55769 ⁽¹¹⁾	0.59595 ⁽¹³⁾	0.48455 ⁽²⁾	0.60025 ⁽¹⁴⁾	0.54 ⁽⁹⁾	0.60612 ⁽¹⁵⁾
20	D_{abs}	0.05112 ⁽¹⁾	0.05545 ⁽³⁾	0.05431 ⁽⁴⁾	0.05615 ⁽⁷⁾	0.05688 ⁽⁹⁾	0.05398 ⁽⁵⁾	0.05577 ⁽⁶⁾	0.05674 ⁽⁸⁾	0.06795 ⁽¹⁴⁾	0.06533 ⁽¹²⁾	0.07202 ⁽¹⁵⁾	0.0531 ⁽²⁾	0.06569 ⁽¹³⁾	0.06016 ⁽¹⁰⁾	0.06493 ⁽¹¹⁾
20	D_{max}	0.08105 ⁽¹⁾	0.08713 ⁽⁵⁾	0.08635 ⁽⁴⁾	0.0885 ⁽⁷⁾	0.09102 ⁽⁹⁾	0.08623 ⁽³⁾	0.08826 ⁽⁶⁾	0.08965 ⁽⁸⁾	0.10703 ⁽¹⁴⁾	0.10361 ⁽¹¹⁾	0.11685 ⁽¹⁵⁾	0.08475 ⁽²⁾	0.10525 ⁽¹³⁾	0.09501 ⁽¹⁰⁾	0.10445 ⁽¹²⁾
20	ASAE	0.0443 ⁽⁷⁾	0.04193 ⁽⁵⁾	0.04275 ⁽⁶⁾	0.04099 ⁽⁴⁾	0.04035 ⁽³⁾	0.04026 ⁽²⁾	0.03975 ⁽¹⁾	0.04556 ⁽⁹⁾	0.05599 ⁽¹²⁾	0.05181 ⁽¹¹⁾	0.06319 ⁽¹⁵⁾	0.04539 ⁽⁸⁾	0.05908 ⁽¹³⁾	0.04929 ⁽¹⁰⁾	0.06006 ⁽¹⁴⁾
20	$\Sigma Ranks$	15 ⁽¹⁾	33 ⁽³⁾	41 ^(4.5)	70 ⁽⁸⁾	85 ⁽¹⁰⁾	41 ^(4.5)	55 ^(6.5)	55 ^(6.5)	108 ⁽¹²⁾	105 ⁽¹¹⁾	117 ⁽¹³⁾	24 ⁽²⁾	123 ⁽¹⁴⁾	81 ⁽⁹⁾	127 ⁽¹⁵⁾
65	BIAS($\hat{\lambda}$)	0.02419 ⁽¹⁾	0.02636 ⁽⁴⁾	0.02718 ⁽⁸⁾	0.02624 ⁽³⁾	0.02766 ⁽⁹⁾	0.0249 ⁽²⁾	0.02645 ⁽⁵⁾	0.02707 ⁽⁷⁾	0.03384 ⁽¹²⁾	0.03172 ⁽¹¹⁾	0.03603 ⁽¹³⁾	0.0267 ⁽⁶⁾	0.03711 ⁽¹⁵⁾	0.02869 ⁽¹⁰⁾	0.03688 ⁽¹⁴⁾
65	BIAS($\hat{\phi}$)	0.24249 ⁽²⁾	0.24179 ⁽¹⁾	0.26463 ⁽⁸⁾	0.2539 ⁽⁵⁾	0.26043 ⁽⁷⁾	0.26707 ⁽⁹⁾	0.25177 ⁽⁴⁾	0.24686 ⁽³⁾	0.26932 ⁽¹¹⁾	0.27833 ⁽¹²⁾	0.30208 ⁽¹⁵⁾	0.25877 ⁽⁶⁾	0.30132 ⁽¹⁴⁾	0.26744 ⁽¹⁰⁾	0.29618 ⁽¹³⁾
65	MSE($\hat{\lambda}$)	0.00093 ⁽¹⁾	0.00107 ⁽³⁾	0.00115 ⁽⁸⁾	0.00109 ⁽⁵⁾	0.00118 ⁽⁹⁾	0.00098 ⁽²⁾	0.00108 ⁽⁴⁾	0.00113 ⁽⁷⁾	0.00183 ⁽¹²⁾	0.00152 ⁽¹¹⁾	0.00202 ⁽¹³⁾	0.00112 ⁽⁶⁾	0.00213 ⁽¹⁵⁾	0.0013 ⁽¹⁰⁾	0.00212 ⁽¹⁴⁾
65	MSE($\hat{\phi}$)	0.08345 ⁽¹⁾	0.08487 ⁽²⁾	0.09477 ⁽⁸⁾	0.08662 ⁽⁴⁾	0.09236 ⁽⁷⁾	0.0963 ⁽⁹⁾	0.08663 ⁽⁵⁾	0.08548 ⁽³⁾	0.1019 ⁽¹¹⁾	0.10502 ⁽¹²⁾	0.12142 ⁽¹⁵⁾	0.09208 ⁽⁶⁾	0.11834 ⁽¹⁴⁾	0.09819 ⁽¹⁰⁾	0.11476 ⁽¹³⁾
65	MRE($\hat{\lambda}$)	0.0691 ⁽¹⁾	0.07767 ⁽³⁾	0.07496 ⁽⁸⁾	0.07902 ⁽⁹⁾	0.07713 ⁽²⁾	0.07965 ⁽⁵⁾	0.07713 ⁽²⁾	0.07965 ⁽⁵⁾	0.09669 ⁽¹²⁾	0.09063 ⁽¹¹⁾	0.10294 ⁽¹³⁾	0.07628 ⁽⁶⁾	0.10602 ⁽¹⁵⁾	0.08197 ⁽¹⁰⁾	0.10536 ⁽¹⁴⁾
65	MRE($\hat{\phi}$)	0.37306 ⁽²⁾	0.37199 ⁽¹⁾	0.40712 ⁽⁸⁾	0.39061 ⁽⁵⁾	0.40066 ⁽⁷⁾	0.41087 ⁽⁹⁾	0.38733 ⁽⁴⁾	0.37979 ⁽³⁾	0.41434 ⁽¹¹⁾	0.42819 ⁽¹²⁾	0.46474 ⁽¹⁵⁾	0.39811 ⁽⁶⁾	0.46357 ⁽¹⁴⁾	0.41145 ⁽¹⁰⁾	0.45567 ⁽¹³⁾
65	D_{abs}	0.0298 ⁽¹⁾	0.03144 ⁽⁶⁾	0.03112 ⁽⁵⁾	0.03046 ⁽²⁾	0.03145 ⁽⁷⁾	0.03108 ⁽⁴⁾	0.03085 ⁽³⁾	0.03237 ⁽⁹⁾	0.03799 ⁽¹²⁾	0.03604 ⁽¹¹⁾	0.03906 ⁽¹³⁾	0.03206 ⁽⁸⁾	0.03919 ⁽¹⁴⁾	0.03318 ⁽¹⁰⁾	0.03983 ⁽¹⁵⁾
65	D_{max}	0.04702 ⁽¹⁾	0.04972 ⁽⁴⁾	0.05017 ⁽⁶⁾	0.04878 ⁽²⁾	0.05023 ⁽⁷⁾	0.04982 ⁽⁵⁾	0.04932 ⁽³⁾	0.05072 ⁽⁸⁾	0.0607 ⁽¹²⁾	0.0584 ⁽¹¹⁾	0.06415 ⁽¹⁴⁾	0.05124 ⁽⁹⁾	0.06368 ⁽¹³⁾	0.05354 ⁽¹⁰⁾	0.06417 ⁽¹⁵⁾
65	ASAE	0.02129 ⁽⁷⁾	0.01998 ⁽³⁾	0.02012 ⁽⁴⁾	0.02051 ⁽⁶⁾	0.02017 ⁽⁵⁾	0.01964 ⁽¹⁾	0.01975 ⁽²⁾	0.02243 ⁽⁹⁾	0.02803 ⁽¹²⁾	0.02673 ⁽¹¹⁾	0.03007 ⁽¹⁴⁾	0.02177 ⁽⁸⁾	0.03034 ⁽¹⁵⁾	0.02471 ⁽¹⁰⁾	0.02983 ⁽¹³⁾
65	$\Sigma Ranks$	17 ⁽¹⁾	28 ⁽²⁾	35 ^(3.5)	67 ⁽⁹⁾	43 ⁽⁵⁾	35 ^(3.5)	45 ⁽⁶⁾	105 ⁽¹²⁾	105 ⁽¹²⁾	102 ⁽¹¹⁾	125 ⁽¹⁴⁾	61 ⁽⁷⁾	129 ⁽¹⁵⁾	90 ⁽¹⁰⁾	124 ⁽¹³⁾
130	BIAS($\hat{\lambda}$)	0.01662 ⁽¹⁾	0.01831 ⁽⁵⁾	0.01935 ⁽⁹⁾	0.01805 ⁽³⁾	0.01922 ⁽⁸⁾	0.01857 ⁽⁶⁾	0.01768 ⁽²⁾	0.01876 ⁽⁷⁾	0.02275 ⁽¹²⁾	0.02166 ⁽¹¹⁾	0.02818 ⁽¹⁵⁾	0.01819 ⁽⁴⁾	0.02524 ⁽¹³⁾	0.02083 ⁽¹⁰⁾	0.02533 ⁽¹⁴⁾
130	BIAS($\hat{\phi}$)	0.20359 ⁽¹⁾	0.21261 ⁽⁴⁾	0.21256 ⁽³⁾	0.21326 ⁽⁵⁾	0.22656 ⁽⁸⁾	0.22813 ⁽⁹⁾	0.2178 ⁽⁶⁾	0.21124 ⁽²⁾	0.23725 ⁽¹²⁾	0.23276 ⁽¹¹⁾	0.26559 ⁽¹⁵⁾	0.23007 ⁽¹⁰⁾	0.24631 ⁽¹³⁾	0.22369 ⁽⁷⁾	0.25257 ⁽¹⁴⁾
130	MSE($\hat{\lambda}$)	0.00044 ⁽¹⁾	0.00052 ⁽⁴⁾	0.00056 ^(7.5)	0.00051 ⁽³⁾	0.00058 ⁽⁹⁾	0.00054 ⁽⁶⁾	5e-04 ⁽²⁾	0.00056 ^(7.5)	0.00081 ⁽¹²⁾	0.00073 ⁽¹¹⁾	0.00124 ⁽¹⁵⁾	0.00053 ⁽⁵⁾	0.001 ^(13.5)	0.00069 ⁽¹⁰⁾	0.001 ^(13.5)
130	MSE($\hat{\phi}$)	0.06177 ⁽⁵⁾	0.06612 ⁽⁸⁾	0.06516 ⁽⁴⁾	0.06508 ⁽³⁾	0.07255 ⁽⁷⁾	0.07289 ⁽⁸⁾	0.06942 ⁽⁶⁾	0.06483 ⁽²⁾	0.07999 ⁽¹²⁾	0.07682 ⁽¹⁰⁾	0.09798 ⁽¹⁵⁾	0.0771 ⁽¹¹⁾	0.08262 ⁽¹³⁾	0.07295 ⁽⁹⁾	0.08698 ⁽¹⁴⁾
130	MRE($\hat{\lambda}$)	0.04748 ⁽¹⁾	0.0523 ⁽⁵⁾	0.05529 ⁽⁹⁾	0.05158 ⁽³⁾	0.05492 ⁽⁸⁾	0.05304 ⁽⁶⁾	0.05051 ⁽²⁾	0.05051 ⁽²⁾	0.06501 ⁽¹²⁾	0.0619 ⁽¹¹⁾	0.0805 ⁽¹⁵⁾	0.05196 ⁽⁴⁾	0.0721 ⁽¹³⁾	0.0595 ⁽¹⁰⁾	0.07237 ⁽¹⁴⁾
130	MRE($\hat{\phi}$)	0.31322 ⁽¹⁾	0.32709 ⁽⁴⁾	0.32701 ⁽³⁾	0.3281 ⁽⁵⁾	0.34855 ⁽⁸⁾	0.35098 ⁽⁹⁾	0.33507 ⁽⁶⁾	0.32498 ⁽²⁾	0.365 ⁽¹²⁾	0.35809 ⁽¹¹⁾	0.40861 ⁽¹⁵⁾	0.35395 ⁽¹⁰⁾	0.37894 ⁽¹³⁾	0.34414 ⁽⁷⁾	0.38857 ⁽¹⁴⁾
130	D_{abs}	0.02029 ⁽¹⁾	0.02214 ⁽⁴⁾	0.02288 ^(8.5)	0.02216 ⁽³⁾	0.02266 ⁽⁶⁾	0.0227 ⁽⁷⁾	0.02125 ⁽²⁾	0.02226 ⁽⁵⁾	0.02775 ⁽¹²⁾	0.02539 ⁽¹¹⁾	0.02818 ⁽¹⁴⁾	0.02288 ^(8.5)	0.02858 ⁽¹⁵⁾	0.02433 ⁽¹⁰⁾	0.02797 ⁽¹³⁾
130	D_{max}	0.03245 ⁽¹⁾	0.03542 ⁽⁴⁾	0.0364 ⁽⁶⁾	0.03465 ⁽³⁾	0.03667 ⁽⁸⁾	0.03684 ⁽⁹⁾	0.03422 ⁽²⁾	0.03554 ⁽⁵⁾	0.04429 ⁽¹²⁾	0.04066 ⁽¹¹⁾	0.04655 ⁽¹⁵⁾	0.03662 ⁽⁷⁾	0.0462 ⁽¹⁴⁾	0.039 ⁽¹⁰⁾	0.04552 ⁽¹³⁾
130	ASAE	0.01391 ⁽⁷⁾	0.01322 ⁽³⁾	0.01341 ⁽⁴⁾	0.01375 ⁽⁶⁾	0.01344 ⁽⁵⁾	0.01271 ⁽¹⁾	0.01298 ⁽²⁾	0.01421 ⁽⁹⁾	0.01813 ⁽¹²⁾	0.01724 ⁽¹¹⁾	0.0198 ⁽¹⁵⁾	0.01414 ⁽⁸⁾	0.01945 ⁽¹⁴⁾	0.0161 ⁽¹⁰⁾	0.01943 ⁽¹³⁾
130	$\Sigma Ranks$	15 ⁽¹⁾	37 ⁽³⁾	53 ⁽⁶⁾	33 ⁽²⁾	66 ⁽⁸⁾	60 ⁽⁷⁾	43 ⁽⁴⁾	45.5 ⁽⁵⁾	107 ⁽¹²⁾	97 ⁽¹¹⁾	133 ⁽¹⁵⁾	66.5 ⁽⁹⁾	120.5 ⁽¹³⁾	82 ⁽¹⁰⁾	121.5 ⁽¹⁴⁾
200	BIAS($\hat{\lambda}$)	0.0141 ⁽¹⁾	0.01475 ⁽⁴⁾	0.01505 ⁽⁷⁾	0.0145 ⁽³⁾	0.01599 ⁽⁹⁾	0.0149 ⁽⁶⁾	0.01426 ⁽²⁾	0.01554 ⁽⁸⁾	0.01818 ⁽¹²⁾	0.01727 ⁽¹¹⁾	0.02357 ⁽¹⁵⁾	0.01477 ⁽⁵⁾	0.02079 ⁽¹³⁾	0.01609 ⁽¹⁰⁾	0.02114 ⁽¹⁴⁾
200	BIAS($\hat{\phi}$)	0.18773 ⁽²⁾	0.19339 ⁽⁴⁾	0.20045 ⁽⁸⁾	0.173 ⁽¹⁾	0.2071 ⁽¹¹⁾	0.20408 ⁽⁹⁾	0.19315 ⁽³⁾	0.19886 ⁽⁷⁾	0.21278 ⁽¹²⁾	0.20593 ⁽¹⁰⁾	0.24407 ⁽¹⁵⁾	0.19347 ⁽⁵⁾	0.21793 ⁽¹³⁾	0.19865 ⁽⁶⁾	0.22026 ⁽¹⁴⁾
200	MSE($\hat{\lambda}$)	0.00031 ⁽¹⁾	0.00035 ⁽³⁾	0.00036 ⁽⁷⁾	0.00033 ^(2.5)	0.00039 ⁽⁹⁾	0.00035 ⁽⁵⁾	0.00033 ^(2.5)	0.00037 ⁽⁸⁾	0.00052 ⁽¹²⁾	0.00046 ⁽¹¹⁾	0.00092 ⁽¹⁵⁾	0.00035 ⁽⁵⁾	0.00068 ⁽¹³⁾	4e-04 ⁽¹⁰⁾	0.00069 ⁽¹⁴⁾
200	MSE($\hat{\phi}$)	0.05524 ⁽²⁾	0.05718 ⁽⁴⁾	0.06129 ⁽⁹⁾	0.04709 ⁽¹⁾	0.06265 ⁽¹¹⁾	0.061 ⁽⁷⁾	0.05675 ⁽¹¹⁾	0.05675 ⁽¹¹⁾	0.0666 ⁽¹²⁾	0.06612 ⁽⁸⁾	0.0856 ⁽¹⁵⁾	0.05803 ⁽⁵⁾	0.06732 ⁽¹³⁾	0.06014 ⁽⁶⁾	0.06842 ⁽¹⁴⁾
200	MRE($\hat{\lambda}$)	0.04028 ⁽¹⁾	0.04214 ⁽⁴⁾	0.04301 ⁽⁷⁾	0.04144 ⁽³⁾	0.04568 ⁽⁹⁾	0.04258 ⁽⁶⁾	0.04073 ⁽²⁾	0.04439 ⁽⁸⁾	0.05193 ⁽¹²⁾	0.04934 ⁽¹¹⁾	0.06735 ⁽¹⁵⁾	0.04219 ⁽⁵⁾	0.05941 ⁽¹³⁾	0.04597 ⁽¹⁰⁾	0.06039 ⁽¹⁴⁾
200	MRE($\hat{\phi}$)	0.28881 ⁽²⁾	0.29753 ⁽⁴⁾	0.30839 ⁽⁸⁾	0.26616 ⁽¹⁾	0.31861 ⁽¹¹⁾	0.31396 ⁽⁹⁾	0.29716 ⁽³⁾	0.30594 ⁽⁷⁾	0.32736 ⁽¹²⁾	0.31681 ⁽¹⁰⁾	0.3755 ⁽¹⁵⁾	0.29764 ⁽⁵⁾	0.33527 ⁽¹³⁾	0.30561 ⁽⁶⁾	0.33886 ⁽¹⁴⁾
200	D_{abs}	0.01749 ⁽²⁾	0.01763 ⁽⁵⁾	0.01935 ⁽⁹⁾	0.01725 ⁽¹⁾	0.01827 ⁽⁶⁾	0.01758 ⁽⁴⁾	0.01754 ⁽³⁾	0.01803 ⁽⁶⁾	0.02191 ⁽¹²⁾	0.02078 ⁽¹¹⁾	0.02197 ⁽¹³⁾	0.01867 ⁽⁸⁾	0.0234 ⁽¹⁴⁾	0.01964 ⁽¹⁰⁾	0.02363 ⁽¹⁵⁾
200	D_{max}	0.02785 ⁽²⁾	0.02847 ⁽⁴⁾	0.03065 ⁽⁹⁾	0.02769 ⁽¹⁾	0.02986 ⁽⁸⁾	0.0288 ⁽⁵⁾	0.02812 ⁽³⁾	0.02905 ⁽⁶⁾	0.03513 ⁽¹²⁾	0.03346 ⁽¹¹⁾	0.03675 ⁽¹³⁾	0.0297 ⁽⁷⁾	0.03779 ⁽¹⁴⁾	0.0312 ⁽¹⁰⁾	0.03813 ⁽¹⁵⁾
200	ASAE	0.01061 ⁽⁶⁾	0.01008 ⁽²⁾	0.01037 ⁽⁴⁾	0.01071 ⁽⁷⁾	0.01039 ⁽⁵⁾	0.00999 ⁽¹⁾	0.01012 ⁽³⁾	0.01098 ⁽⁸⁾	0.01414 ⁽¹²⁾	0.01307 ⁽¹¹⁾	0.01539 ⁽¹³⁾	0.01102 ⁽⁹⁾	0.01544 ⁽¹⁴⁾	0.01229 ⁽¹⁰⁾	0.01564 ⁽¹⁵⁾
200	$\Sigma Ranks$	19 ⁽¹⁾	36 ⁽⁴⁾	68 ^(7.5)	20.5 ⁽²⁾	80 ⁽⁹⁾	52 ⁽⁵⁾	24.5 ⁽³⁾	68 ^(7.5)	107 ⁽¹²⁾	93 ⁽¹¹⁾	128 ^(14.5)	54 ⁽⁶⁾	119 ⁽¹³⁾	83 ⁽¹⁰⁾	128 ^(14.5)
270	BIAS($\hat{\lambda}$)	0.01197 ⁽¹⁾	0.01248 ⁽²⁾	0.01365 ⁽⁷⁾	0.01294 ⁽⁵⁾	0.01297 ⁽⁶⁾	0.01293 ⁽⁴⁾	0.01281 ⁽³⁾	0.0139 ⁽¹⁰⁾	0.01575 ⁽¹²⁾	0.01469 ⁽¹¹⁾	0.02039 ⁽¹⁵⁾	0.01387 ⁽⁹⁾	0.01755 ⁽¹⁴⁾	0.01379 ⁽⁸⁾	0.01659 ⁽¹³⁾

TABLE 4. Numerical values of simulation measures for $\lambda = 1.25$, $\phi = 0.95$.

n	Est.	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
20	BIAS($\hat{\lambda}$)	0.21187 ⁽²⁾	0.22984 ⁽⁴⁾	0.22287 ⁽³⁾	0.28179 ⁽¹¹⁾	0.23884 ⁽⁶⁾	0.25238 ⁽⁸⁾	0.24069 ⁽⁷⁾	0.22954 ⁽⁵⁾	0.28244 ⁽¹²⁾	0.31387 ⁽¹³⁾	0.26904 ⁽⁹⁾	0.19592 ⁽¹⁾	0.32605 ⁽¹⁴⁾	0.27688 ⁽¹⁰⁾	0.3342 ⁽¹⁵⁾
20	BIAS($\hat{\phi}$)	0.50324 ⁽⁴⁾	0.56441 ⁽⁷⁾	0.51113 ⁽⁵⁾	0.60834 ⁽¹³⁾	0.5499 ⁽⁶⁾	0.59065 ⁽¹⁰⁾	0.57031 ⁽⁸⁾	0.45099 ⁽²⁾	0.57404 ⁽⁹⁾	0.63368 ⁽¹⁵⁾	0.45776 ⁽³⁾	0.44707 ⁽¹⁾	0.60146 ⁽¹²⁾	0.5986 ⁽¹¹⁾	0.62014 ⁽¹⁴⁾
20	MSE($\hat{\lambda}$)	0.06917 ⁽⁴⁾	0.07807 ⁽⁴⁾	0.07559 ⁽³⁾	0.11507 ⁽¹¹⁾	0.08362 ⁽⁵⁾	0.09341 ⁽⁸⁾	0.08638 ⁽⁶⁾	0.08766 ⁽⁷⁾	0.11486 ⁽¹⁰⁾	0.13876 ⁽¹³⁾	0.11544 ⁽¹²⁾	0.06355 ⁽¹⁾	0.14583 ⁽¹⁴⁾	0.11166 ⁽⁹⁾	0.15172 ⁽¹⁵⁾
20	MSE($\hat{\phi}$)	0.32971 ⁽⁵⁾	0.38583 ⁽⁷⁾	0.3281 ⁽⁴⁾	0.44731 ⁽¹³⁾	0.36343 ⁽⁶⁾	0.42246 ⁽¹⁰⁾	0.3883 ⁽⁸⁾	0.27219 ⁽²⁾	0.40778 ⁽⁹⁾	0.47946 ⁽¹⁵⁾	0.2873 ⁽³⁾	0.26501 ⁽¹⁾	0.4352 ⁽¹¹⁾	0.44104 ⁽¹²⁾	0.45785 ⁽¹⁴⁾
20	MRE($\hat{\lambda}$)	0.1695 ⁽²⁾	0.18357 ⁽⁴⁾	0.1783 ⁽³⁾	0.22543 ⁽¹¹⁾	0.19107 ⁽⁶⁾	0.20191 ⁽⁸⁾	0.19256 ⁽⁷⁾	0.18363 ⁽⁵⁾	0.22595 ⁽¹²⁾	0.25109 ⁽¹³⁾	0.21523 ⁽⁹⁾	0.15674 ⁽¹⁾	0.26084 ⁽¹⁴⁾	0.22151 ⁽¹⁰⁾	0.26736 ⁽¹⁵⁾
20	MRE($\hat{\phi}$)	0.52973 ⁽⁴⁾	0.59411 ⁽⁷⁾	0.53803 ⁽⁵⁾	0.64036 ⁽¹³⁾	0.57884 ⁽⁶⁾	0.62174 ⁽¹⁰⁾	0.60032 ⁽⁸⁾	0.47472 ⁽²⁾	0.60425 ⁽⁹⁾	0.66703 ⁽¹⁵⁾	0.48185 ⁽³⁾	0.4706 ⁽¹⁾	0.63311 ⁽¹²⁾	0.63011 ⁽¹¹⁾	0.65278 ⁽¹⁴⁾
20	D_{abs}	0.04318 ⁽²⁾	0.04595 ⁽⁴⁾	0.05184 ⁽⁹⁾	0.04295 ⁽¹⁾	0.05136 ⁽⁸⁾	0.04578 ⁽³⁾	0.04768 ⁽⁶⁾	0.05351 ⁽¹¹⁾	0.05576 ⁽¹²⁾	0.04772 ⁽⁷⁾	0.05906 ⁽¹³⁾	0.05326 ⁽¹⁰⁾	0.06279 ⁽¹⁵⁾	0.04601 ⁽⁵⁾	0.06037 ⁽¹⁴⁾
20	D_{max}	0.07067 ⁽²⁾	0.07526 ⁽⁶⁾	0.0847 ⁽⁹⁾	0.06874 ⁽¹⁾	0.08195 ⁽⁸⁾	0.07472 ⁽⁴⁾	0.07714 ⁽⁷⁾	0.08785 ⁽¹²⁾	0.08758 ⁽¹¹⁾	0.07477 ⁽⁵⁾	0.09453 ⁽¹⁴⁾	0.08663 ⁽¹⁰⁾	0.09684 ⁽¹⁵⁾	0.07358 ⁽³⁾	0.09451 ⁽¹³⁾
20	ASAE	0.04648 ⁽⁵⁾	0.04659 ⁽⁶⁾	0.04809 ⁽⁸⁾	0.04094 ⁽¹⁾	0.04565 ⁽⁴⁾	0.04326 ⁽²⁾	0.04378 ⁽³⁾	0.05595 ⁽¹²⁾	0.05279 ⁽¹⁰⁾	0.04766 ⁽⁷⁾	0.07482 ⁽¹⁵⁾	0.05458 ⁽¹¹⁾	0.06426 ⁽¹³⁾	0.04937 ⁽⁹⁾	0.06459 ⁽¹⁴⁾
20	Σ Ranks	28 ⁽¹⁾	49 ^(3,5)	49 ^(3,5)	75 ⁽⁹⁾	55 ⁽⁵⁾	63 ⁽⁸⁾	60 ⁽⁷⁾	58 ⁽⁶⁾	94 ⁽¹²⁾	103 ⁽¹³⁾	81 ⁽¹¹⁾	37 ⁽²⁾	120 ⁽¹⁴⁾	80 ⁽¹⁰⁾	128 ⁽¹⁵⁾
65	BIAS($\hat{\lambda}$)	0.15857 ⁽³⁾	0.16431 ⁽⁴⁾	0.17084 ⁽⁷⁾	0.22241 ⁽¹²⁾	0.16637 ⁽⁶⁾	0.17249 ⁽⁸⁾	0.16433 ⁽⁵⁾	0.15512 ⁽²⁾	0.20612 ⁽¹¹⁾	0.22806 ⁽¹³⁾	0.20423 ⁽¹⁰⁾	0.12898 ⁽¹⁾	0.24619 ⁽¹⁴⁾	0.19606 ⁽⁹⁾	0.24762 ⁽¹⁵⁾
65	BIAS($\hat{\phi}$)	0.46194 ⁽⁴⁾	0.48152 ⁽⁶⁾	0.49527 ⁽⁸⁾	0.54867 ⁽¹²⁾	0.49189 ⁽⁷⁾	0.49703 ⁽⁹⁾	0.47375 ⁽⁵⁾	0.4485 ⁽³⁾	0.50912 ⁽¹⁰⁾	0.56142 ⁽¹³⁾	0.43816 ⁽²⁾	0.41015 ⁽¹⁾	0.56184 ⁽¹⁵⁾	0.51683 ⁽¹¹⁾	0.56154 ⁽¹⁴⁾
65	MSE($\hat{\lambda}$)	0.04165 ⁽³⁾	0.04484 ⁽⁶⁾	0.04737 ⁽⁸⁾	0.07866 ⁽¹²⁾	0.04387 ⁽⁵⁾	0.04736 ⁽⁷⁾	0.0428 ⁽⁴⁾	0.03954 ⁽²⁾	0.06556 ⁽¹⁰⁾	0.08091 ⁽¹³⁾	0.06914 ⁽¹¹⁾	0.0264 ⁽¹⁾	0.08915 ⁽¹⁴⁾	0.0639 ⁽⁹⁾	0.08975 ⁽¹⁵⁾
65	MSE($\hat{\phi}$)	0.28749 ⁽⁴⁾	0.30309 ⁽⁶⁾	0.31745 ⁽⁸⁾	0.38949 ⁽¹²⁾	0.30531 ⁽⁷⁾	0.32426 ⁽⁹⁾	0.29394 ⁽⁵⁾	0.26812 ⁽²⁾	0.34647 ⁽¹⁰⁾	0.39618 ⁽¹³⁾	0.27371 ⁽³⁾	0.2366 ⁽¹⁾	0.39396 ⁽¹⁴⁾	0.35177 ⁽¹¹⁾	0.39331 ⁽¹³⁾
65	MRE($\hat{\lambda}$)	0.12686 ⁽³⁾	0.13145 ⁽⁴⁾	0.13667 ⁽⁷⁾	0.17793 ⁽¹²⁾	0.1331 ⁽⁶⁾	0.13799 ⁽⁸⁾	0.13147 ⁽⁵⁾	0.1241 ⁽²⁾	0.16489 ⁽¹¹⁾	0.18245 ⁽¹³⁾	0.16339 ⁽¹⁰⁾	0.10318 ⁽¹⁾	0.19695 ⁽¹⁴⁾	0.15685 ⁽⁹⁾	0.1981 ⁽¹⁵⁾
65	MRE($\hat{\phi}$)	0.48625 ⁽⁴⁾	0.50686 ⁽⁶⁾	0.52134 ⁽⁸⁾	0.57755 ⁽¹²⁾	0.51778 ⁽⁷⁾	0.52319 ⁽⁹⁾	0.49868 ⁽⁵⁾	0.4721 ⁽³⁾	0.53592 ⁽¹⁰⁾	0.59097 ⁽¹³⁾	0.46122 ⁽²⁾	0.43174 ⁽¹⁾	0.59141 ⁽¹⁵⁾	0.54403 ⁽¹¹⁾	0.5911 ⁽¹⁴⁾
65	D_{abs}	0.02598 ⁽¹⁾	0.02798 ⁽⁴⁾	0.02858 ⁽⁶⁾	0.02632 ⁽²⁾	0.02882 ⁽⁷⁾	0.02672 ⁽³⁾	0.02852 ⁽⁵⁾	0.03012 ⁽¹⁰⁾	0.03311 ⁽¹²⁾	0.02953 ⁽⁹⁾	0.03874 ⁽¹⁵⁾	0.03188 ⁽¹¹⁾	0.03454 ⁽¹⁵⁾	0.02883 ⁽⁸⁾	0.03525 ⁽¹⁴⁾
65	D_{max}	0.04342 ⁽¹⁾	0.04663 ⁽⁴⁾	0.04775 ⁽⁶⁾	0.04405 ⁽²⁾	0.04798 ⁽⁸⁾	0.04476 ⁽³⁾	0.04747 ⁽⁵⁾	0.05018 ⁽¹⁰⁾	0.0538 ⁽¹²⁾	0.04872 ⁽⁹⁾	0.06258 ⁽¹⁵⁾	0.05219 ⁽¹⁾	0.05625 ⁽¹³⁾	0.04778 ⁽⁷⁾	0.05704 ⁽¹⁴⁾
65	ASAE	0.01994 ⁽⁵⁾	0.01989 ⁽⁶⁾	0.02104 ⁽⁷⁾	0.01863 ⁽¹⁾	0.02096 ⁽⁶⁾	0.01902 ⁽²⁾	0.0198 ⁽³⁾	0.02404 ⁽¹⁰⁾	0.02603 ⁽¹²⁾	0.02419 ⁽¹⁰⁾	0.03809 ⁽¹⁵⁾	0.02453 ⁽¹¹⁾	0.0294 ⁽¹³⁾	0.02289 ⁽⁸⁾	0.02945 ⁽¹⁴⁾
65	Σ Ranks	28 ⁽¹⁾	45 ⁽³⁾	44 ⁽³⁾	77 ⁽⁹⁾	58 ⁽⁶⁾	58 ⁽⁶⁾	42 ⁽³⁾	43 ⁽⁴⁾	98 ⁽¹²⁾	108 ⁽¹³⁾	83 ^(10,5)	39 ⁽²⁾	125 ⁽¹⁴⁾	83 ^(10,5)	128 ⁽¹⁵⁾
130	BIAS($\hat{\lambda}$)	0.12005 ⁽²⁾	0.13426 ⁽⁴⁾	0.1427 ⁽⁶⁾	0.16539 ⁽¹¹⁾	0.14787 ⁽⁸⁾	0.13552 ⁽⁵⁾	0.13116 ⁽³⁾	0.14493 ⁽⁷⁾	0.16336 ⁽¹⁰⁾	0.17869 ⁽¹²⁾	0.18155 ⁽¹³⁾	0.11788 ⁽¹⁾	0.20005 ⁽¹⁵⁾	0.15607 ⁽⁹⁾	0.19806 ⁽¹⁴⁾
130	BIAS($\hat{\phi}$)	0.39729 ⁽¹⁾	0.4409 ⁽⁷⁾	0.45421 ⁽⁹⁾	0.45462 ⁽¹⁰⁾	0.42365 ⁽⁵⁾	0.42234 ⁽⁴⁾	0.44035 ⁽⁶⁾	0.45697 ⁽¹¹⁾	0.48066 ⁽¹³⁾	0.41028 ⁽³⁾	0.39752 ⁽²⁾	0.51993 ⁽¹⁵⁾	0.44231 ⁽⁸⁾	0.49946 ⁽¹⁴⁾	
130	MSE($\hat{\lambda}$)	0.02556 ⁽²⁾	0.03151 ⁽⁵⁾	0.03252 ⁽⁶⁾	0.04787 ⁽¹¹⁾	0.03573 ⁽⁷⁾	0.03118 ⁽⁴⁾	0.02809 ⁽³⁾	0.03641 ⁽⁸⁾	0.04545 ⁽¹⁰⁾	0.05453 ⁽¹³⁾	0.05335 ⁽¹²⁾	0.02235 ⁽¹⁾	0.06363 ⁽¹⁵⁾	0.04334 ⁽⁹⁾	0.06213 ⁽¹⁴⁾
130	MSE($\hat{\phi}$)	0.22462 ⁽²⁾	0.26379 ⁽⁷⁾	0.27078 ⁽⁸⁾	0.29602 ⁽¹²⁾	0.27272 ⁽⁹⁾	0.25086 ⁽⁵⁾	0.24458 ⁽³⁾	0.26207 ⁽⁶⁾	0.28983 ⁽¹¹⁾	0.31695 ⁽¹³⁾	0.24747 ⁽⁴⁾	0.2236 ⁽¹⁾	0.34906 ⁽¹⁵⁾	0.27563 ⁽¹⁰⁾	0.32939 ⁽¹⁴⁾
130	MRE($\hat{\lambda}$)	0.09604 ⁽²⁾	0.10741 ⁽⁴⁾	0.11416 ⁽⁶⁾	0.13231 ⁽¹¹⁾	0.11829 ⁽⁸⁾	0.10841 ⁽⁵⁾	0.10493 ⁽³⁾	0.11594 ⁽⁷⁾	0.13069 ⁽¹⁰⁾	0.14295 ⁽¹²⁾	0.14524 ⁽¹³⁾	0.09343 ⁽¹⁾	0.16004 ⁽¹⁵⁾	0.12485 ⁽⁹⁾	0.15845 ⁽¹⁴⁾
130	MRE($\hat{\phi}$)	0.4182 ⁽¹⁾	0.46411 ⁽⁷⁾	0.47811 ⁽⁹⁾	0.48626 ⁽¹²⁾	0.47855 ⁽¹⁰⁾	0.44594 ⁽⁵⁾	0.44557 ⁽⁴⁾	0.46353 ⁽⁶⁾	0.48102 ⁽¹¹⁾	0.50595 ⁽¹³⁾	0.43187 ⁽³⁾	0.41844 ⁽²⁾	0.54729 ⁽¹⁵⁾	0.46559 ⁽⁸⁾	0.52575 ⁽¹⁴⁾
130	D_{abs}	0.01984 ⁽²⁾	0.02065 ⁽⁵⁾	0.02107 ⁽⁷⁾	0.01966 ⁽¹⁾	0.01991 ⁽³⁾	0.02002 ⁽⁴⁾	0.02081 ⁽⁶⁾	0.02151 ⁽⁸⁾	0.02388 ⁽¹²⁾	0.02156 ⁽⁹⁾	0.03053 ⁽¹⁵⁾	0.02263 ⁽¹⁰⁾	0.02472 ⁽¹³⁾	0.02263 ⁽¹¹⁾	0.02564 ⁽¹⁴⁾
130	D_{max}	0.03341 ⁽²⁾	0.03509 ⁽⁵⁾	0.03596 ⁽⁷⁾	0.0334 ⁽¹⁾	0.03408 ⁽⁴⁾	0.03379 ⁽³⁾	0.0352 ⁽⁶⁾	0.03661 ⁽⁹⁾	0.03992 ⁽¹²⁾	0.03656 ⁽⁸⁾	0.04936 ⁽¹⁵⁾	0.03764 ⁽¹⁰⁾	0.04153 ⁽¹³⁾	0.03794 ⁽¹¹⁾	0.04273 ⁽¹⁴⁾
130	ASAE	0.01266 ⁽⁵⁾	0.01261 ⁽⁴⁾	0.01212 ⁽⁶⁾	0.01212 ⁽¹⁾	0.01213 ⁽⁷⁾	0.01219 ⁽²⁾	0.01237 ⁽³⁾	0.01474 ⁽⁹⁾	0.01675 ⁽¹²⁾	0.01555 ⁽¹¹⁾	0.02506 ⁽¹⁵⁾	0.0155 ⁽¹⁰⁾	0.01891 ⁽¹³⁾	0.01459 ⁽⁸⁾	0.01896 ⁽¹⁴⁾
130	Σ Ranks	19 ⁽¹⁾	48 ⁽⁵⁾	64 ⁽⁶⁾	72 ⁽⁹⁾	66 ^(7,5)	38 ^(3,5)	35 ⁽²⁾	66 ^(7,5)	99 ⁽¹²⁾	104 ⁽¹³⁾	93 ⁽¹¹⁾	38 ^(3,5)	129 ⁽¹⁵⁾	83 ⁽¹⁰⁾	126 ⁽¹⁴⁾
200	BIAS($\hat{\lambda}$)	0.10164 ⁽¹⁾	0.1086 ⁽²⁾	0.12633 ⁽⁷⁾	0.12719 ⁽⁸⁾	0.12973 ⁽⁹⁾	0.11414 ⁽⁴⁾	0.11868 ⁽⁵⁾	0.12006 ⁽⁶⁾	0.14192 ⁽¹¹⁾	0.14536 ⁽¹²⁾	0.1727 ⁽¹⁴⁾	0.11034 ⁽³⁾	0.1772 ⁽¹⁵⁾	0.13556 ⁽¹⁰⁾	0.16658 ⁽¹³⁾
200	BIAS($\hat{\phi}$)	0.36156 ⁽¹⁾	0.39056 ⁽⁴⁾	0.41605 ⁽¹¹⁾	0.39505 ⁽⁵⁾	0.41511 ⁽¹⁰⁾	0.38547 ⁽³⁾	0.39982 ⁽⁶⁾	0.40683 ⁽⁹⁾	0.42405 ⁽¹³⁾	0.42189 ⁽¹²⁾	0.40641 ⁽⁸⁾	0.38229 ⁽²⁾	0.47097 ⁽¹⁵⁾	0.40251 ⁽⁷⁾	0.465 ⁽¹⁴⁾
200	MSE($\hat{\lambda}$)	0.0183 ⁽¹⁾	0.01998 ⁽³⁾	0.02622 ⁽⁷⁾	0.03136 ⁽⁹⁾	0.02741 ⁽⁸⁾	0.02348 ⁽⁵⁾	0.02335 ⁽⁴⁾	0.02619 ⁽⁶⁾	0.0348 ⁽¹⁰⁾	0.03876 ⁽¹²⁾	0.04958 ⁽¹⁴⁾	0.01912 ⁽²⁾	0.05208 ⁽¹⁵⁾	0.03493 ⁽¹¹⁾	0.04596 ⁽¹³⁾
200	MSE($\hat{\phi}$)	0.19141 ⁽¹⁾	0.2141 ⁽³⁾	0.23882 ⁽⁹⁾	0.22763 ⁽⁶⁾	0.23715 ⁽⁸⁾	0.21449 ⁽⁴⁾	0.22312 ⁽⁵⁾	0.23178 ⁽⁷⁾	0.25196 ⁽¹²⁾	0.25854 ⁽¹³⁾	0.24625 ⁽¹¹⁾	0.21049 ⁽²⁾	0.30099 ⁽¹⁵⁾	0.24035 ⁽¹⁰⁾	0.29076 ⁽¹⁴⁾
200	MRE($\hat{\lambda}$)	0.08131 ⁽¹⁾	0.08688 ⁽²⁾	0.10106 ⁽⁷⁾	0.10175 ⁽⁸⁾	0.10378 ⁽⁹⁾	0.09131 ⁽⁴⁾	0.09494 ⁽⁵⁾	0.09605 ⁽⁶⁾	0.11353 ⁽¹¹⁾	0.11629 ⁽¹²⁾	0.13816 ⁽¹⁴⁾	0.08827 ⁽³⁾	0.14176 ⁽¹⁵⁾	0.10845 ⁽¹⁰⁾	0.13327 ⁽¹³⁾
200	MRE($\hat{\phi}$)	0.38058 ⁽¹⁾	0.41112 ⁽⁴⁾	0.43795 ⁽¹¹⁾	0.41584 ⁽⁵⁾	0.43696 ⁽¹⁰⁾	0.40575 ⁽³⁾	0.42086 ⁽⁶⁾	0.42825 ⁽⁹⁾	0.44637 ⁽¹³⁾	0.44409 ⁽¹²⁾	0.4278 ⁽⁸⁾	0.40241 ⁽²⁾	0.49575 ⁽¹⁵⁾	0.4237 ⁽⁷⁾	0.48947 ⁽¹⁴⁾
200	D_{abs}	0.01571 ⁽¹⁾	0.01748 ⁽²⁾	0.01616 ⁽³⁾	0.01633 ⁽⁴⁾	0.01719 ⁽⁶⁾	0.01584 ⁽²⁾	0.01666 ⁽⁵⁾	0.01787 ⁽⁹⁾	0.01991 ⁽¹²⁾	0.01875 ⁽¹¹⁾	0.02657 ⁽¹⁵⁾	0.01773 ⁽⁸⁾	0.02091 ⁽¹⁴⁾	0.01805 ⁽¹⁰⁾	0.02068 ⁽¹³⁾
200	D_{max}	0.02683 ⁽¹⁾	0.02992 ⁽⁷⁾	0.02804 ⁽³⁾	0.02976 ⁽⁶⁾	0.02884 ⁽⁵⁾	0.0272 ⁽⁴⁾	0.02864 ⁽⁸⁾	0.03078 ⁽¹⁰⁾	0.03362 ⁽¹²⁾	0.03182 ⁽¹¹⁾	0.04329 ⁽¹⁵⁾	0.03009 ⁽⁸⁾	0.03553 ⁽¹⁴⁾	0.03062 ⁽⁹⁾	0.0352 ⁽¹³⁾
200	ASAE	0.00959 ⁽⁴⁾	0.0096 ⁽⁵⁾	0.01005 ⁽⁶⁾	0.00911 ⁽¹⁾	0.01013 ⁽⁷⁾	0.00931 ⁽²⁾	0.0094 ⁽³⁾	0.01112 ⁽⁸⁾	0.01305 ⁽¹²⁾	0.01211 ⁽¹¹⁾	0.02005 ⁽¹⁵⁾	0.01151 ⁽¹⁰⁾	0.01463 ⁽¹³⁾	0.01132 ⁽⁹⁾	0.0147 ⁽¹⁴⁾
200	Σ Ranks	12 ⁽¹⁾	37 ⁽³⁾	65 ⁽⁷⁾	49 ⁽⁶⁾	73 ⁽⁹⁾	29 ⁽²⁾	44 ⁽⁵⁾	70 ⁽⁸⁾	106 ^(11,5)	106 ^(11,5)	114 ⁽¹³⁾	40 ⁽⁴⁾	131 ⁽¹⁵⁾	83 ⁽¹⁰⁾	121 ⁽¹⁴⁾
270	BIAS($\hat{\lambda}$)	0.0951 ⁽¹⁾	0.1034 ⁽⁵⁾	0.10921 ⁽⁶⁾	0.12003 ⁽¹¹⁾	0.11481 ⁽⁷⁾	0.10232 ⁽⁴⁾	0.10172 ⁽³⁾	0.11713 ⁽⁸⁾	0.11735 ⁽⁹⁾	0.13233 ⁽¹²⁾	0.16471 ⁽¹⁵⁾	0.09818 ⁽²⁾	0.15104 ⁽¹⁴⁾	0.11937 ⁽¹⁰⁾	0.15046 ⁽¹³⁾
270	BIAS($\hat{\phi}$)															

TABLE 5. Numerical values of simulation measures for $\lambda = 1.75$, $\phi = 2.5$.

n	Est.	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
20	BIAS($\hat{\lambda}$)	0.31315 ^[4]	0.3469 ^[6]	0.29512 ^[2]	0.39249 ^[11]	0.35462 ^[7]	0.35846 ^[8]	0.31813 ^[5]	0.30077 ^[3]	0.39295 ^[12]	0.41227 ^[13]	0.39178 ^[10]	0.25338 ^[1]	0.4554 ^[15]	0.38074 ^[9]	0.43805 ^[14]
20	BIAS($\hat{\phi}$)	1.36354 ^[5]	1.48525 ^[8]	1.32957 ^[4]	1.54056 ^[11]	1.57425 ^[13]	1.5064 ^[9]	1.38903 ^[6]	1.32344 ^[3]	1.4573 ^[7]	1.58096 ^[14]	1.05675 ^[1]	1.2175 ^[2]	1.5619 ^[12]	1.61937 ^[15]	1.53523 ^[10]
20	MSE($\hat{\lambda}$)	0.14937 ^[5]	0.17833 ^[6]	0.12174 ^[2]	0.2337 ^[1]	0.17987 ^[7]	0.18553 ^[8]	0.14602 ^[4]	0.13381 ^[3]	0.22237 ^[10]	0.25529 ^[13]	0.24939 ^[12]	0.1008 ^[1]	0.28008 ^[15]	0.20282 ^[9]	0.26717 ^[14]
20	MSE($\hat{\phi}$)	2.40491 ^[5]	2.72433 ^[8]	2.24826 ^[4]	3.00057 ^[13]	2.93026 ^[10]	2.77599 ^[9]	2.43327 ^[6]	2.14722 ^[3]	2.70495 ^[7]	3.09972 ^[14]	1.61076 ^[1]	1.93587 ^[2]	2.95162 ^[11]	3.12616 ^[15]	2.98308 ^[12]
20	MRE($\hat{\lambda}$)	0.17894 ^[4]	0.19823 ^[6]	0.16864 ^[2]	0.22428 ^[11]	0.20264 ^[7]	0.20484 ^[8]	0.18179 ^[5]	0.17187 ^[3]	0.22454 ^[12]	0.23558 ^[13]	0.22388 ^[10]	0.14479 ^[1]	0.26023 ^[15]	0.21756 ^[9]	0.25032 ^[14]
20	MRE($\hat{\phi}$)	0.54542 ^[5]	0.5941 ^[8]	0.53183 ^[4]	0.61622 ^[11]	0.6297 ^[13]	0.60256 ^[9]	0.55561 ^[6]	0.52938 ^[3]	0.58292 ^[7]	0.63239 ^[14]	0.4227 ^[1]	0.487 ^[2]	0.62476 ^[12]	0.64775 ^[15]	0.61409 ^[10]
20	D_{abs}	0.04122 ^[1]	0.05222 ^[7]	0.05568 ^[10]	0.04859 ^[6]	0.04776 ^[4]	0.04665 ^[3]	0.0482 ^[5]	0.05353 ^[9]	0.05741 ^[11]	0.05235 ^[8]	0.06534 ^[15]	0.05757 ^[12]	0.05821 ^[13]	0.04504 ^[2]	0.06203 ^[14]
20	D_{max}	0.06997 ^[1]	0.08561 ^[7]	0.0901 ^[10]	0.07866 ^[5]	0.07825 ^[4]	0.07765 ^[3]	0.07943 ^[6]	0.08796 ^[9]	0.09187 ^[11]	0.08678 ^[8]	0.10652 ^[15]	0.09579 ^[13]	0.09462 ^[12]	0.07585 ^[2]	0.0998 ^[14]
20	ASAE	0.0456 ^[6]	0.04355 ^[5]	0.04723 ^[8]	0.04017 ^[1]	0.0424 ^[4]	0.04063 ^[2]	0.04102 ^[3]	0.05163 ^[11]	0.05063 ^[10]	0.04566 ^[7]	0.08019 ^[15]	0.05216 ^[12]	0.0628 ^[13]	0.04787 ^[9]	0.06314 ^[14]
20	Σ Ranks	36 ^[1]	61 ^[7]	46 ^[3]	80 ^[9,5]	69 ^[8]	59 ^[6]	46 ^[3]	47 ^[5]	87 ^[12]	104 ^[13]	80 ^[9,5]	46 ^[3]	118 ^[15]	85 ^[11]	116 ^[14]
65	BIAS($\hat{\lambda}$)	0.16018 ^[1]	0.19719 ^[4]	0.19606 ^[3]	0.30498 ^[12]	0.24694 ^[7]	0.24667 ^[6]	0.2547 ^[9]	0.22068 ^[5]	0.32523 ^[13]	0.3038 ^[11]	0.25396 ^[8]	0.1792 ^[2]	0.34776 ^[14]	0.28124 ^[10]	0.36052 ^[15]
65	BIAS($\hat{\phi}$)	1.05569 ^[3]	1.18341 ^[6]	1.07544 ^[4]	1.36879 ^[13]	1.23903 ^[8]	1.30585 ^[9]	1.17529 ^[5]	1.18676 ^[7]	1.36351 ^[12]	1.31551 ^[10]	0.92508 ^[1]	0.98843 ^[2]	1.50571 ^[15]	1.36123 ^[11]	1.43742 ^[14]
65	MSE($\hat{\lambda}$)	0.04261 ^[1]	0.07141 ^[4]	0.06707 ^[3]	0.15187 ^[11]	0.0939 ^[6]	0.09472 ^[7]	0.11331 ^[9]	0.07445 ^[5]	0.15384 ^[12]	0.15781 ^[13]	0.11107 ^[8]	0.05495 ^[2]	0.17588 ^[14]	0.13757 ^[10]	0.18849 ^[15]
65	MSE($\hat{\phi}$)	1.50834 ^[1]	1.87885 ^[6]	1.66093 ^[4]	2.4663 ^[13]	2.04503 ^[8]	2.20571 ^[9]	1.95266 ^[7]	1.85101 ^[5]	2.40271 ^[11]	2.33369 ^[10]	1.28258 ^[1]	1.48438 ^[2]	2.78203 ^[15]	2.43955 ^[12]	2.6681 ^[14]
65	MRE($\hat{\lambda}$)	0.09153 ^[1]	0.11268 ^[4]	0.11203 ^[3]	0.17427 ^[12]	0.14111 ^[7]	0.14096 ^[6]	0.1261 ^[5]	0.18585 ^[13]	0.1736 ^[11]	0.14512 ^[8]	0.1024 ^[2]	0.19872 ^[10]	0.16071 ^[15]	0.20601 ^[9]	0.26061 ^[14]
65	MRE($\hat{\phi}$)	0.42228 ^[3]	0.47336 ^[6]	0.43018 ^[4]	0.54751 ^[13]	0.49561 ^[8]	0.52234 ^[9]	0.47011 ^[5]	0.4747 ^[7]	0.5454 ^[12]	0.52621 ^[10]	0.37003 ^[1]	0.39537 ^[2]	0.60228 ^[15]	0.54449 ^[11]	0.57497 ^[14]
65	D_{abs}	0.02775 ^[6]	0.02703 ^[5]	0.02693 ^[3]	0.02637 ^[1]	0.02694 ^[4]	0.02668 ^[2]	0.0279 ^[7]	0.02993 ^[9]	0.03929 ^[14]	0.03397 ^[11]	0.04153 ^[15]	0.03023 ^[10]	0.03428 ^[15]	0.02951 ^[8]	0.03701 ^[14]
65	D_{max}	0.046 ^[4]	0.04572 ^[3]	0.04538 ^[1]	0.04565 ^[2]	0.04691 ^[6]	0.04619 ^[5]	0.04763 ^[7]	0.05099 ^[10]	0.06497 ^[14]	0.05659 ^[11]	0.06697 ^[15]	0.04982 ^[9]	0.05864 ^[12]	0.04974 ^[8]	0.06275 ^[13]
65	ASAE	0.01929 ^[3]	0.02009 ^[5]	0.02076 ^[6]	0.01913 ^[2]	0.02079 ^[7]	0.0195 ^[4]	0.01826 ^[1]	0.0228 ^[9]	0.02493 ^[12]	0.02378 ^[11]	0.03591 ^[15]	0.02359 ^[10]	0.02757 ^[13]	0.02233 ^[8]	0.02904 ^[14]
65	Σ Ranks	25 ^[1]	43 ^[4]	31 ^[2]	79 ^[10]	61 ^[7]	57 ^[5]	113 ^[13]	62 ^[8]	113 ^[12]	98 ^[12]	72 ^[9]	41 ^[3]	124 ^[14]	88 ^[11]	127 ^[15]
130	BIAS($\hat{\lambda}$)	0.14704 ^[2]	0.18646 ^[9]	0.1604 ^[6]	0.14884 ^[4]	0.1882 ^[10]	0.13538 ^[3]	0.14738 ^[5]	0.16732 ^[7]	0.21004 ^[12]	0.22849 ^[13]	0.19167 ^[11]	0.15542 ^[5]	0.26544 ^[15]	0.18208 ^[8]	0.23378 ^[14]
130	BIAS($\hat{\phi}$)	0.92344 ^[3]	1.06292 ^[9]	1.02773 ^[7]	1.03359 ^[8]	1.16312 ^[12]	0.97129 ^[4]	1.01359 ^[5]	1.11738 ^[11]	1.02676 ^[6]	1.23328 ^[14]	0.85435 ^[1]	0.88607 ^[2]	1.24408 ^[15]	1.10109 ^[10]	1.19591 ^[13]
130	MSE($\hat{\lambda}$)	0.0348 ^[2]	0.05545 ^[9]	0.04541 ^[7]	0.04004 ^[5]	0.05245 ^[8]	0.03137 ^[1]	0.03852 ^[3]	0.04116 ^[6]	0.06964 ^[11]	0.09428 ^[13]	0.07381 ^[12]	0.03898 ^[4]	0.11612 ^[15]	0.06522 ^[10]	0.09702 ^[14]
130	MSE($\hat{\phi}$)	1.22834 ^[2]	1.59788 ^[9]	1.497 ^[6]	1.53692 ^[8]	1.77654 ^[12]	1.32927 ^[4]	1.41859 ^[5]	1.66569 ^[10]	1.53512 ^[7]	2.03664 ^[14]	1.16056 ^[1]	1.26774 ^[3]	2.07557 ^[15]	1.68388 ^[11]	1.92077 ^[13]
130	MRE($\hat{\lambda}$)	0.08402 ^[2]	0.10655 ^[9]	0.09166 ^[6]	0.08505 ^[4]	0.10754 ^[10]	0.07736 ^[1]	0.08422 ^[3]	0.09561 ^[7]	0.12003 ^[12]	0.13056 ^[13]	0.10953 ^[11]	0.08881 ^[5]	0.15168 ^[15]	0.10405 ^[8]	0.13359 ^[14]
130	MRE($\hat{\phi}$)	0.36938 ^[3]	0.42517 ^[9]	0.41109 ^[7]	0.41344 ^[8]	0.46525 ^[12]	0.38852 ^[4]	0.40544 ^[5]	0.44695 ^[11]	0.41071 ^[6]	0.49331 ^[14]	0.34174 ^[1]	0.35443 ^[2]	0.49763 ^[15]	0.44044 ^[10]	0.47836 ^[13]
130	D_{abs}	0.01926 ^[2]	0.02015 ^[9]	0.02015 ^[4]	0.0202 ^[5]	0.0216 ^[8]	0.02141 ^[7]	0.01885 ^[3]	0.02426 ^[12]	0.02504 ^[13]	0.02306 ^[14]	0.02308 ^[15]	0.0214 ^[6]	0.02377 ^[11]	0.02595 ^[14]	0.02359 ^[10]
130	D_{max}	0.03288 ^[1]	0.03445 ^[3]	0.03457 ^[4]	0.03489 ^[5]	0.03758 ^[8]	0.03665 ^[7]	0.03289 ^[2]	0.04171 ^[12]	0.04236 ^[13]	0.03962 ^[9]	0.05354 ^[15]	0.03625 ^[6]	0.04148 ^[11]	0.04372 ^[14]	0.04074 ^[10]
130	ASAE	0.01229 ^[1]	0.01245 ^[4]	0.01276 ^[6]	0.01228 ^[1]	0.0131 ^[7]	0.01254 ^[5]	0.01514 ^[10]	0.01681 ^[12]	0.01617 ^[11]	0.02587 ^[15]	0.01489 ^[8]	0.01856 ^[13]	0.01507 ^[9]	0.01507 ^[9]	0.0191 ^[14]
130	Σ Ranks	19 ^[1]	64 ^[7]	53 ^[6]	48 ^[5]	87 ^[10]	34 ^[3]	30 ^[2]	86 ^[9]	92 ^[11]	110 ^[13]	82 ^[8]	41 ^[4]	125 ^[15]	94 ^[12]	115 ^[14]
200	BIAS($\hat{\lambda}$)	0.10281 ^[1]	0.12101 ^[2]	0.15067 ^[8]	0.14154 ^[5]	0.15587 ^[10]	0.12503 ^[3]	0.14216 ^[6]	0.15581 ^[9]	0.14085 ^[4]	0.19459 ^[15]	0.18737 ^[14]	0.14474 ^[7]	0.18483 ^[12]	0.1794 ^[11]	0.18686 ^[13]
200	BIAS($\hat{\phi}$)	0.7937 ^[1]	0.83026 ^[3]	0.86565 ^[4]	0.88031 ^[6]	0.99631 ^[13]	0.89683 ^[8]	0.96806 ^[10]	0.97522 ^[11]	0.88602 ^[7]	1.0941 ^[15]	0.7999 ^[2]	0.87892 ^[5]	0.99513 ^[12]	0.94447 ^[9]	1.0425 ^[14]
200	MSE($\hat{\lambda}$)	0.01833 ^[1]	0.02716 ^[3]	0.03585 ^[9]	0.03342 ^[5]	0.03824 ^[10]	0.02418 ^[2]	0.03248 ^[4]	0.03464 ^[7]	0.03483 ^[8]	0.0717 ^[15]	0.06659 ^[13]	0.034 ^[6]	0.06853 ^[14]	0.05632 ^[11]	0.06515 ^[12]
200	MSE($\hat{\phi}$)	0.95336 ^[1]	1.09076 ^[3]	1.15128 ^[5]	1.17894 ^[6]	1.44113 ^[12]	1.21309 ^[8]	1.34951 ^[10]	1.30842 ^[9]	1.3666 ^[4]	1.70418 ^[15]	1.01827 ^[2]	1.20777 ^[7]	1.48675 ^[13]	1.36011 ^[11]	1.53907 ^[14]
200	MRE($\hat{\lambda}$)	0.05875 ^[1]	0.06915 ^[2]	0.0861 ^[8]	0.08088 ^[5]	0.08907 ^[10]	0.07145 ^[3]	0.08124 ^[6]	0.08904 ^[9]	0.08048 ^[4]	0.11119 ^[15]	0.10707 ^[14]	0.08271 ^[7]	0.10562 ^[12]	0.10251 ^[11]	0.10678 ^[13]
200	MRE($\hat{\phi}$)	0.31748 ^[1]	0.33211 ^[3]	0.34626 ^[4]	0.35212 ^[6]	0.39852 ^[13]	0.35945 ^[8]	0.38434 ^[10]	0.39009 ^[11]	0.35441 ^[7]	0.43764 ^[15]	0.31996 ^[2]	0.35157 ^[5]	0.39805 ^[12]	0.37779 ^[9]	0.417 ^[14]
200	D_{abs}	0.01682 ^[5]	0.01664 ^[4]	0.01699 ^[6]	0.01741 ^[7]	0.0158 ^[2]	0.01841 ^[10]	0.01583 ^[3]	0.01563 ^[1]	0.02179 ^[14]	0.02164 ^[13]	0.02866 ^[15]	0.01832 ^[9]	0.01847 ^[11]	0.01806 ^[8]	0.02141 ^[12]
200	D_{max}	0.0284 ^[5]	0.0283 ^[3]	0.02911 ^[6]	0.02962 ^[7]	0.02778 ^[1]	0.03105 ^[8]	0.02839 ^[4]	0.02807 ^[2]	0.03632 ^[12]	0.03714 ^[14]	0.04593 ^[15]	0.03134 ^[10]	0.03229 ^[11]	0.03121 ^[9]	0.03652 ^[13]
200	ASAE	0.00949 ^[3]	0.00968 ^[5]	0.01027 ^[6]	0.00935 ^[1,5]	0.01039 ^[7]	0.00935 ^[1,5]	0.00963 ^[4]	0.01052 ^[8]	0.01333 ^[12]	0.01259 ^[11]	0.01873 ^[15]	0.01115 ^[9]	0.01416 ^[13]	0.01171 ^[10]	0.01524 ^[14]
200	Σ Ranks	19 ^[1]	28 ^[2]	56 ^[5]	48.5 ^[3]	78 ^[10]	51.5 ^[4]	57 ^[6]	67 ^[8]	72 ^[9]	128 ^[15]	92 ^[12]	65 ^[7]	110 ^[13]	89 ^[11]	119 ^[14]
270	BIAS($\hat{\lambda}$)	0.09953 ^[1]	0.11716 ^[5]	0.11931 ^[7]	0.11434 ^[3]	0.13893 ^[10]	0.11486 ^[4]	0.11352 ^[2]	0.14179 ^[12]	0.12644 ^[8]	0.13599 ^[9]	0.16503 ^[15]	0.11818 ^[6]	0.14162 ^[11]	0.14528 ^[13]	0.15897 ^[14]
270	BIAS($\hat{\phi}$)	0.68569 ^[1]	0.77144 ^[3]	0.86												

TABLE 6. Numerical values of simulation measures for $\lambda = 0.8, \phi = 1.5$.

n	Est.	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSE	MSLDE	MSLNDE
20	BIAS($\hat{\lambda}$)	0.11943 ⁽³⁾	0.12089 ⁽⁴⁾	0.11696 ⁽²⁾	0.14753 ⁽¹⁰⁾	0.14956 ⁽¹²⁾	0.12561 ⁽⁵⁾	0.14186 ⁽⁹⁾	0.12915 ⁽⁶⁾	0.14838 ⁽¹¹⁾	0.15928 ⁽¹³⁾	0.13833 ⁽⁷⁾	0.11058 ⁽¹⁾	0.19032 ⁽¹⁴⁾	0.14105 ⁽⁸⁾	0.19314 ⁽¹⁵⁾
20	BIAS($\hat{\phi}$)	0.68874 ⁽¹⁾	0.71993 ⁽³⁾	0.77616 ⁽⁵⁾	0.79906 ⁽⁹⁾	0.85517 ⁽¹³⁾	0.74941 ⁽⁴⁾	0.78223 ⁽⁸⁾	0.78142 ⁽⁷⁾	0.77848 ⁽⁶⁾	0.82553 ⁽¹¹⁾	0.81629 ⁽¹⁰⁾	0.69039 ⁽²⁾	0.94288 ⁽¹⁵⁾	0.88095 ⁽¹²⁾	0.84086 ⁽¹²⁾
20	MSE($\hat{\lambda}$)	0.02363 ⁽⁴⁾	0.02326 ⁽³⁾	0.02298 ⁽²⁾	0.03364 ⁽⁹⁾	0.03242 ⁽⁸⁾	0.02702 ⁽⁶⁾	0.03065 ⁽⁷⁾	0.02583 ⁽⁵⁾	0.03484 ⁽¹¹⁾	0.04483 ⁽¹³⁾	0.03478 ⁽¹⁰⁾	0.01888 ⁽¹⁾	0.05186 ⁽¹⁴⁾	0.03527 ⁽¹²⁾	0.05727 ⁽¹⁵⁾
20	MSE($\hat{\phi}$)	0.65473 ⁽²⁾	0.69095 ⁽³⁾	0.76045 ⁽⁵⁾	0.8231 ⁽¹⁰⁾	0.91202 ⁽¹¹⁾	0.74241 ⁽⁴⁾	0.82078 ⁽⁹⁾	0.81476 ⁽⁷⁾	0.81327 ⁽⁶⁾	0.93296 ⁽¹³⁾	0.81601 ⁽⁸⁾	0.65256 ⁽¹⁾	1.06751 ⁽¹⁵⁾	0.97402 ⁽¹⁴⁾	0.92079 ⁽¹²⁾
20	MRE($\hat{\lambda}$)	0.14928 ⁽³⁾	0.15111 ⁽⁴⁾	0.1462 ⁽²⁾	0.18441 ⁽¹⁰⁾	0.18694 ⁽¹²⁾	0.15701 ⁽⁵⁾	0.17733 ⁽⁸⁾	0.16143 ⁽⁶⁾	0.18547 ⁽¹¹⁾	0.1991 ⁽¹³⁾	0.17291 ⁽⁷⁾	0.13823 ⁽¹⁾	0.2379 ⁽¹⁴⁾	0.17631 ⁽⁸⁾	0.24143 ⁽¹⁵⁾
20	MRE($\hat{\phi}$)	0.45916 ⁽¹⁾	0.47996 ⁽³⁾	0.51744 ⁽⁵⁾	0.5327 ⁽⁹⁾	0.57011 ⁽¹³⁾	0.49961 ⁽⁴⁾	0.52148 ⁽⁸⁾	0.52095 ⁽⁷⁾	0.51898 ⁽⁶⁾	0.55035 ⁽¹¹⁾	0.54419 ⁽¹⁰⁾	0.46026 ⁽²⁾	0.62859 ⁽¹⁵⁾	0.5873 ⁽¹⁴⁾	0.56057 ⁽¹²⁾
20	D_{abs}	0.05291 ⁽⁵⁾	0.05137 ⁽³⁾	0.05072 ⁽¹⁾	0.05576 ⁽⁹⁾	0.0555 ⁽⁷⁾	0.05206 ⁽⁴⁾	0.05514 ⁽⁶⁾	0.05564 ⁽⁸⁾	0.06403 ⁽¹⁴⁾	0.05669 ⁽¹⁰⁾	0.07956 ⁽¹⁵⁾	0.05099 ⁽²⁾	0.05769 ⁽¹²⁾	0.06114 ⁽¹³⁾	0.05703 ⁽¹¹⁾
20	D_{max}	0.08594 ⁽⁵⁾	0.08372 ⁽³⁾	0.08312 ⁽²⁾	0.09097 ⁽⁹⁾	0.09028 ⁽⁷⁾	0.08483 ⁽⁴⁾	0.09031 ⁽⁸⁾	0.09214 ⁽¹⁰⁾	0.10445 ⁽¹⁴⁾	0.08959 ⁽⁶⁾	0.13002 ⁽¹⁵⁾	0.08286 ⁽¹⁾	0.09858 ⁽¹³⁾	0.09735 ⁽¹²⁾	0.09556 ⁽¹¹⁾
20	ASAE	0.04213 ⁽⁷⁾	0.0412 ⁽⁵⁾	0.04139 ⁽⁶⁾	0.04068 ⁽⁴⁾	0.04015 ⁽³⁾	0.03913 ⁽²⁾	0.03833 ⁽¹⁾	0.04679 ⁽⁸⁾	0.05268 ⁽¹²⁾	0.04854 ⁽¹¹⁾	0.07239 ⁽¹⁵⁾	0.04846 ⁽¹⁰⁾	0.05956 ⁽¹³⁾	0.04765 ⁽⁹⁾	0.05991 ⁽¹⁴⁾
20	$\Sigma Ranks$	31 ^(3,5)	31 ^(3,5)	30 ⁽²⁾	79 ⁽⁸⁾	86 ⁽⁹⁾	38 ⁽⁵⁾	65 ⁽⁷⁾	64 ⁽⁶⁾	91 ⁽¹⁰⁾	101 ⁽¹²⁾	97 ⁽¹¹⁾	21 ⁽¹⁾	125 ⁽¹⁵⁾	104 ⁽¹³⁾	117 ⁽¹⁴⁾
65	BIAS($\hat{\lambda}$)	0.0702 ⁽⁴⁾	0.07343 ⁽⁶⁾	0.06953 ⁽³⁾	0.08827 ⁽¹⁰⁾	0.07286 ⁽⁵⁾	0.06879 ⁽²⁾	0.06688 ⁽¹⁾	0.08641 ⁽⁹⁾	0.0936 ⁽¹²⁾	0.08921 ⁽¹¹⁾	0.09576 ⁽¹³⁾	0.07853 ⁽⁷⁾	0.10563 ⁽¹⁴⁾	0.08013 ⁽⁸⁾	0.12451 ⁽¹⁵⁾
65	BIAS($\hat{\phi}$)	0.64724 ⁽⁹⁾	0.55852 ⁽²⁾	0.63943 ⁽⁸⁾	0.61143 ⁽⁶⁾	0.56482 ⁽⁴⁾	0.56339 ⁽³⁾	0.54499 ⁽¹⁾	0.64843 ⁽¹⁰⁾	0.65177 ⁽¹¹⁾	0.72582 ⁽¹³⁾	0.76318 ⁽¹⁴⁾	0.56984 ⁽⁵⁾	0.7127 ⁽¹²⁾	0.62264 ⁽⁷⁾	0.77789 ⁽¹⁵⁾
65	MSE($\hat{\lambda}$)	0.00728 ⁽²⁾	0.00856 ⁽⁴⁾	0.00878 ⁽³⁾	0.01189 ⁽¹⁰⁾	0.00901 ⁽⁷⁾	0.00766 ⁽³⁾	0.00703 ⁽¹⁾	0.01084 ⁽⁹⁾	0.01406 ⁽¹²⁾	0.01451 ⁽¹³⁾	0.01399 ⁽¹¹⁾	0.00884 ⁽⁶⁾	0.01982 ⁽¹⁴⁾	0.00956 ⁽⁸⁾	0.0263 ⁽¹⁵⁾
65	MSE($\hat{\phi}$)	0.56978 ⁽¹⁰⁾	0.45008 ⁽³⁾	0.55808 ⁽⁷⁾	0.51569 ⁽⁶⁾	0.45313 ⁽⁴⁾	0.4431 ⁽²⁾	0.42601 ⁽¹⁾	0.56162 ⁽⁹⁾	0.60446 ⁽¹¹⁾	0.68519 ⁽¹³⁾	0.73029 ⁽¹⁴⁾	0.46907 ⁽⁵⁾	0.65367 ⁽¹²⁾	0.55948 ⁽⁸⁾	0.79187 ⁽¹⁵⁾
65	MRE($\hat{\lambda}$)	0.08775 ⁽⁴⁾	0.09178 ⁽⁶⁾	0.08692 ⁽³⁾	0.11034 ⁽¹⁰⁾	0.09107 ⁽⁵⁾	0.08599 ⁽²⁾	0.0836 ⁽¹⁾	0.11082 ⁽⁹⁾	0.117 ⁽¹²⁾	0.11151 ⁽¹¹⁾	0.11969 ⁽¹³⁾	0.09817 ⁽⁷⁾	0.13203 ⁽¹⁴⁾	0.10016 ⁽⁸⁾	0.15563 ⁽¹⁵⁾
65	MRE($\hat{\phi}$)	0.43149 ⁽⁹⁾	0.37235 ⁽²⁾	0.42629 ⁽⁸⁾	0.40762 ⁽⁶⁾	0.37655 ⁽⁴⁾	0.3756 ⁽³⁾	0.36333 ⁽¹⁾	0.43228 ⁽¹⁰⁾	0.43451 ⁽¹¹⁾	0.48388 ⁽¹³⁾	0.50878 ⁽¹⁴⁾	0.37989 ⁽⁵⁾	0.47513 ⁽¹²⁾	0.4151 ⁽⁷⁾	0.51859 ⁽¹⁵⁾
65	D_{abs}	0.03012 ⁽⁴⁾	0.02863 ⁽²⁾	0.03516 ⁽⁹⁾	0.03298 ⁽⁶⁾	0.02849 ⁽¹⁾	0.0315 ⁽⁵⁾	0.02867 ⁽³⁾	0.03593 ⁽¹¹⁾	0.03855 ⁽¹⁵⁾	0.03708 ⁽¹³⁾	0.03647 ⁽¹²⁾	0.03384 ⁽⁷⁾	0.03816 ⁽¹⁴⁾	0.03441 ⁽⁸⁾	0.03554 ⁽¹⁰⁾
65	D_{max}	0.05087 ⁽⁴⁾	0.04767 ⁽²⁾	0.05677 ⁽⁸⁾	0.05463 ⁽⁶⁾	0.04759 ⁽¹⁾	0.0514 ⁽⁵⁾	0.04787 ⁽³⁾	0.05972 ⁽¹⁰⁾	0.06313 ⁽¹⁴⁾	0.06289 ⁽¹³⁾	0.06201 ⁽¹²⁾	0.05542 ⁽⁷⁾	0.06331 ⁽¹⁵⁾	0.05682 ⁽⁹⁾	0.0612 ⁽¹¹⁾
65	ASAE	0.02101 ⁽⁷⁾	0.0191 ⁽¹⁾	0.01944 ⁽²⁾	0.01974 ⁽³⁾	0.01998 ^(4,5)	0.02088 ⁽⁸⁾	0.01998 ^(4,5)	0.02199 ⁽⁹⁾	0.02601 ⁽¹¹⁾	0.02608 ⁽¹²⁾	0.02769 ⁽¹³⁾	0.02164 ⁽⁶⁾	0.02913 ⁽¹⁴⁾	0.02358 ⁽¹⁰⁾	0.02915 ⁽¹⁵⁾
65	$\Sigma Ranks$	53 ^(5,5)	28 ⁽²⁾	53 ^(5,5)	63 ⁽⁸⁾	35.5 ⁽⁴⁾	31 ⁽³⁾	16.5 ⁽¹⁾	86 ⁽¹⁰⁾	109 ⁽¹¹⁾	112 ⁽¹²⁾	116 ⁽¹³⁾	57 ⁽⁷⁾	121 ⁽¹⁴⁾	73 ⁽⁹⁾	126 ⁽¹⁵⁾
130	BIAS($\hat{\lambda}$)	0.05002 ⁽⁵⁾	0.04376 ⁽¹⁾	0.05355 ⁽⁸⁾	0.0513 ⁽⁷⁾	0.04993 ⁽⁴⁾	0.04425 ⁽²⁾	0.0493 ⁽³⁾	0.05417 ⁽⁹⁾	0.06011 ⁽¹²⁾	0.05571 ⁽¹¹⁾	0.07283 ⁽¹⁵⁾	0.05525 ⁽¹⁰⁾	0.07264 ⁽¹⁴⁾	0.05106 ⁽⁶⁾	0.07232 ⁽¹³⁾
130	BIAS($\hat{\phi}$)	0.50081 ⁽⁸⁾	0.4855 ⁽⁶⁾	0.5423 ⁽¹²⁾	0.44806 ⁽¹⁾	0.47649 ⁽⁵⁾	0.46643 ⁽³⁾	0.45956 ⁽²⁾	0.52691 ⁽¹⁰⁾	0.49564 ⁽⁷⁾	0.53087 ⁽¹¹⁾	0.58336 ⁽¹⁴⁾	0.51238 ⁽⁹⁾	0.55344 ⁽¹³⁾	0.47585 ⁽⁴⁾	0.61724 ⁽¹⁵⁾
130	MSE($\hat{\lambda}$)	0.00373 ⁽⁴⁾	0.00319 ⁽²⁾	0.00448 ⁽⁸⁾	0.00401 ⁽⁶⁾	0.0038 ⁽⁵⁾	0.00316 ⁽¹⁾	0.00364 ⁽³⁾	0.00454 ⁽⁹⁾	0.0055 ⁽¹²⁾	0.00528 ⁽¹¹⁾	0.00789 ⁽¹³⁾	0.00473 ⁽¹⁰⁾	0.00889 ⁽¹⁴⁾	0.00439 ⁽⁷⁾	0.00945 ⁽¹⁵⁾
130	MSE($\hat{\phi}$)	0.35572 ⁽⁷⁾	0.3595 ⁽⁸⁾	0.41786 ⁽¹²⁾	0.31043 ⁽²⁾	0.34088 ⁽⁴⁾	0.33392 ⁽³⁾	0.29502 ⁽¹⁾	0.40647 ⁽¹⁰⁾	0.34984 ⁽⁶⁾	0.41031 ⁽¹¹⁾	0.4736 ⁽¹⁴⁾	0.39923 ⁽⁹⁾	0.4494 ⁽¹³⁾	0.34385 ⁽⁵⁾	0.51338 ⁽¹⁵⁾
130	MRE($\hat{\lambda}$)	0.06252 ⁽⁵⁾	0.0547 ⁽¹⁾	0.06694 ⁽⁸⁾	0.06413 ⁽⁷⁾	0.06241 ⁽⁴⁾	0.05532 ⁽²⁾	0.06162 ⁽³⁾	0.06771 ⁽⁹⁾	0.07513 ⁽¹²⁾	0.06963 ⁽¹¹⁾	0.09103 ⁽¹⁵⁾	0.06906 ⁽¹⁰⁾	0.0908 ⁽¹⁴⁾	0.06382 ⁽⁶⁾	0.0904 ⁽¹³⁾
130	MRE($\hat{\phi}$)	0.33387 ⁽⁸⁾	0.32367 ⁽⁶⁾	0.36154 ⁽¹²⁾	0.29871 ⁽¹⁾	0.31766 ⁽⁵⁾	0.31095 ⁽³⁾	0.30638 ⁽²⁾	0.35128 ⁽¹⁰⁾	0.33043 ⁽⁷⁾	0.35391 ⁽¹¹⁾	0.38891 ⁽¹⁴⁾	0.34159 ⁽⁹⁾	0.36896 ⁽¹³⁾	0.31723 ⁽⁴⁾	0.41149 ⁽¹⁵⁾
130	D_{abs}	0.02319 ⁽⁷⁾	0.01977 ⁽¹⁾	0.0235 ⁽⁸⁾	0.02142 ⁽²⁾	0.02301 ⁽⁴⁾	0.02177 ⁽¹⁰⁾	0.02154 ⁽³⁾	0.0248 ⁽¹¹⁾	0.0209 ⁽¹⁵⁾	0.02473 ⁽⁹⁾	0.02272 ⁽¹²⁾	0.02206 ⁽⁴⁾	0.02203 ⁽¹³⁾	0.02307 ⁽⁶⁾	0.02798 ⁽¹⁵⁾
130	D_{max}	0.03856 ⁽⁷⁾	0.03373 ⁽¹⁾	0.03956 ⁽⁸⁾	0.03517 ⁽²⁾	0.03827 ⁽⁴⁾	0.04005 ⁽⁹⁾	0.03592 ⁽³⁾	0.0408 ⁽¹⁰⁾	0.04809 ⁽¹⁵⁾	0.04141 ⁽¹¹⁾	0.04559 ⁽¹²⁾	0.03692 ⁽⁴⁾	0.04729 ⁽¹³⁾	0.03824 ⁽⁵⁾	0.04749 ⁽¹⁵⁾
130	ASAE	0.01318 ⁽⁶⁾	0.01285 ⁽⁴⁾	0.01278 ⁽³⁾	0.01317 ⁽⁵⁾	0.01369 ⁽⁷⁾	0.01246 ⁽²⁾	0.01237 ⁽¹⁾	0.01481 ⁽⁹⁾	0.0191 ⁽¹³⁾	0.0171 ⁽¹¹⁾	0.0183 ⁽¹²⁾	0.01393 ⁽⁸⁾	0.02002 ⁽¹⁴⁾	0.01667 ⁽¹⁰⁾	0.02067 ⁽¹⁵⁾
130	$\Sigma Ranks$	57 ⁽⁷⁾	30 ⁽²⁾	79 ⁽⁹⁾	33 ⁽³⁾	45 ⁽⁵⁾	35 ⁽⁴⁾	21 ⁽¹⁾	87 ⁽¹⁰⁾	99 ⁽¹²⁾	97 ⁽¹¹⁾	121 ⁽¹³⁾	73 ⁽⁸⁾	122 ⁽¹⁴⁾	53 ⁽⁶⁾	128 ⁽¹⁵⁾
200	BIAS($\hat{\lambda}$)	0.04462 ⁽¹¹⁾	0.0362 ⁽³⁾	0.04032 ⁽⁸⁾	0.03474 ⁽²⁾	0.03962 ⁽⁷⁾	0.03926 ⁽⁵⁾	0.03465 ⁽¹⁾	0.03749 ⁽⁴⁾	0.04988 ⁽¹²⁾	0.04182 ⁽⁹⁾	0.05924 ⁽¹⁴⁾	0.03938 ⁽⁶⁾	0.06039 ⁽¹⁵⁾	0.04375 ⁽¹⁰⁾	0.05418 ⁽¹³⁾
200	BIAS($\hat{\phi}$)	0.4664 ⁽¹⁰⁾	0.42391 ⁽⁶⁾	0.46007 ⁽⁹⁾	0.39212 ⁽¹²⁾	0.4589 ⁽⁸⁾	0.43849 ⁽⁷⁾	0.41626 ⁽⁴⁾	0.39364 ⁽³⁾	0.47887 ⁽¹¹⁾	0.49932 ⁽¹²⁾	0.53167 ⁽¹⁵⁾	0.38352 ⁽¹⁾	0.51171 ⁽¹³⁾	0.42148 ⁽⁵⁾	0.52837 ⁽¹⁴⁾
200	MSE($\hat{\lambda}$)	0.00317 ⁽¹¹⁾	0.00222 ⁽³⁾	0.00265 ⁽⁸⁾	0.00187 ⁽¹⁾	0.0023 ⁽⁴⁾	0.00239 ⁽⁵⁾	0.00204 ⁽²⁾	0.00242 ⁽⁶⁾	0.00419 ⁽¹²⁾	0.00261 ⁽⁷⁾	0.00585 ⁽¹⁵⁾	0.0028 ⁽⁹⁾	0.00561 ⁽¹⁴⁾	0.00309 ⁽¹⁰⁾	0.00498 ⁽¹³⁾
200	MSE($\hat{\phi}$)	0.31957 ⁽⁸⁾	0.27594 ⁽⁶⁾	0.32062 ⁽⁹⁾	0.21149 ⁽¹¹⁾	0.33202 ⁽¹⁰⁾	0.27669 ⁽⁷⁾	0.25979 ⁽⁴⁾	0.24725 ⁽²⁾	0.33651 ⁽¹¹⁾	0.36221 ⁽¹²⁾	0.42489 ⁽¹⁵⁾	0.25955 ⁽³⁾	0.37889 ⁽¹³⁾	0.26328 ⁽⁵⁾	0.38487 ⁽¹⁴⁾
200	MRE($\hat{\lambda}$)	0.05578 ⁽¹¹⁾	0.04524 ⁽³⁾	0.0504 ⁽⁸⁾	0.04343 ⁽²⁾	0.04953 ⁽⁷⁾	0.04908 ⁽⁵⁾	0.04332 ⁽¹⁾	0.04686 ⁽⁴⁾	0.06235 ⁽¹²⁾	0.05227 ⁽⁹⁾	0.07405 ⁽¹⁴⁾	0.04922 ⁽⁶⁾	0.07549 ⁽¹⁵⁾	0.05469 ⁽¹⁰⁾	0.06773 ⁽¹³⁾
200	MRE($\hat{\phi}$)	0.31094 ⁽¹⁰⁾	0.28261 ⁽⁶⁾	0.30672 ⁽⁹⁾	0.26142 ⁽¹¹⁾	0.30593 ⁽⁸⁾	0.29233 ⁽⁷⁾	0.27751 ⁽⁴⁾	0.26242 ⁽³⁾	0.31924 ⁽¹¹⁾	0.33288 ⁽¹²⁾	0.35445 ⁽¹⁵⁾	0.25568 ⁽¹⁾	0.34114 ⁽¹³⁾	0.28098 ⁽⁵⁾	0.35225 ⁽¹⁴⁾
200	D_{abs}	0.01754 ⁽⁴⁾	0.01603 ⁽¹⁾	0.01903 ⁽⁸⁾	0.01628 ⁽²⁾	0.01872 ⁽⁷⁾	0.0207 ⁽¹⁰⁾	0.01788 ⁽⁵⁾	0.01715 ⁽³⁾	0.02569 ⁽¹⁵⁾	0.02061 ⁽⁹⁾	0.02213 ⁽¹²⁾	0.01809 ⁽⁶⁾	0.02563 ⁽¹⁴⁾	0.02191 ⁽¹¹⁾	0.02533 ⁽¹³⁾
200	D_{max}	0.03003 ⁽⁵⁾	0.02727 ⁽¹⁾	0.03216 ⁽⁸⁾	0.02748 ⁽²⁾	0.03137 ⁽⁷⁾	0.03367 ⁽⁹⁾	0.03004 ⁽⁶⁾	0.02879 ⁽³⁾	0.04164 ⁽¹²⁾	0.03508 ⁽¹⁰⁾	0.03771 ⁽¹²⁾	0.02991 ⁽⁴⁾	0.04283 ⁽¹⁵⁾	0.03631 ⁽¹¹⁾	0.04209 ⁽¹⁴⁾
200	ASAE	0.01002 ⁽⁵⁾	0.00948 ⁽²⁾	0.00972 ⁽³⁾	0.01033 ⁽⁶⁾	0.01038 ⁽⁷⁾	0.00943 ⁽¹⁾	0.00992 ⁽⁴⁾	0.01008 ⁽⁸⁾	0.01414 ⁽¹²⁾	0.01242 ⁽¹¹⁾	0.01589 ⁽¹⁴⁾	0.01134 ⁽⁹⁾	0.01629 ⁽¹⁵⁾	0.01232 ⁽¹⁰⁾	0.01533 ⁽¹³⁾
200	$\Sigma Ranks$	75 ⁽⁹⁾	31 ^(2,5)	70 ⁽⁸⁾	20 ⁽¹⁾	65 ⁽⁷⁾	56 ⁽⁶⁾	31 ^(2,5)	36 ⁽⁴⁾	109 ⁽¹²⁾	91 ⁽¹¹⁾	126 ⁽¹⁴⁾	45 ⁽⁵⁾	127 ⁽¹⁵⁾	77 ⁽¹⁰⁾	121 ⁽¹³⁾
270	BIAS($\hat{\lambda}$)	0.03856 ⁽⁸⁾	0.03025 ⁽¹⁾	0.03943 ⁽¹⁰⁾	0.03254 ⁽³⁾	0.03567 ⁽⁶⁾	0.03282 ⁽⁴⁾	0.03182 ⁽²⁾	0.03598 ⁽⁷⁾	0.04654 ⁽¹²⁾	0.03885 ⁽⁹⁾	0.05381 ⁽¹⁵⁾	0.03398 ⁽⁵⁾	0.05113 ⁽¹³⁾	0.03967 ⁽¹¹⁾	0.05117 ⁽¹⁴⁾
270	BIAS(															

TABLE 7. Numerical values of simulation measures for $\lambda = 0.5, \phi = 3.5$.

n	Est.	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
20	BIAS($\hat{\lambda}$)	0.06832 ⁽²⁾	0.05248 ⁽¹⁾	0.06832 ⁽⁵⁾	0.07231 ⁽⁸⁾	0.07369 ⁽⁹⁾	0.07089 ⁽⁷⁾	0.0687 ⁽⁶⁾	0.06478 ⁽³⁾	0.0759 ⁽¹²⁾	0.08197 ⁽¹³⁾	0.07494 ⁽¹⁰⁾	0.06549 ⁽⁴⁾	0.09977 ⁽¹⁵⁾	0.07517 ⁽¹¹⁾	0.09238 ⁽¹⁴⁾
20	BIAS($\hat{\phi}$)	1.91471 ⁽²⁾	1.88306 ⁽¹⁾	2.07531 ⁽⁵⁾	2.27165 ⁽¹²⁾	2.25819 ⁽¹¹⁾	2.27745 ⁽¹³⁾	2.24413 ⁽⁹⁾	1.96021 ⁽³⁾	2.0971 ⁽⁶⁾	2.21814 ⁽⁸⁾	2.01032 ⁽⁴⁾	2.25183 ⁽¹⁰⁾	2.41439 ⁽¹⁴⁾	2.13484 ⁽⁷⁾	2.4763 ⁽¹⁵⁾
20	MSE($\hat{\lambda}$)	0.00618 ⁽³⁾	0.0043 ⁽¹⁾	0.00713 ⁽⁵⁾	0.00996 ⁽¹³⁾	0.00794 ⁽⁸⁾	0.00721 ⁽⁷⁾	0.00717 ⁽⁶⁾	0.00579 ⁽²⁾	0.00852 ^(9,5)	0.00967 ⁽¹¹⁾	0.00852 ^(9,5)	0.00657 ⁽⁴⁾	0.01514 ⁽¹⁵⁾	0.00983 ⁽¹²⁾	0.01344 ⁽¹⁴⁾
20	MSE($\hat{\phi}$)	4.44351 ⁽²⁾	4.40022 ⁽¹⁾	5.05243 ⁽⁴⁾	5.91626 ⁽¹¹⁾	5.92371 ⁽¹²⁾	6.00884 ⁽¹³⁾	5.83571 ⁽¹⁰⁾	4.68578 ⁽³⁾	5.41254 ⁽⁶⁾	5.69336 ⁽⁸⁾	5.06367 ⁽⁵⁾	5.72142 ⁽⁹⁾	6.53999 ⁽¹⁴⁾	5.50651 ⁽⁷⁾	6.82831 ⁽¹⁵⁾
20	MRE($\hat{\lambda}$)	0.12681 ⁽²⁾	0.10496 ⁽¹⁾	0.13663 ⁽⁵⁾	0.14461 ⁽⁸⁾	0.14737 ⁽⁹⁾	0.14178 ⁽⁷⁾	0.13739 ⁽⁶⁾	0.12957 ⁽²⁾	0.15181 ⁽¹²⁾	0.16394 ⁽¹³⁾	0.14987 ⁽¹⁰⁾	0.13098 ⁽⁴⁾	0.19954 ⁽¹⁵⁾	0.15034 ⁽¹¹⁾	0.18475 ⁽¹⁴⁾
20	MRE($\hat{\phi}$)	0.54706 ⁽²⁾	0.53802 ⁽¹⁾	0.59295 ⁽⁵⁾	0.64904 ⁽¹²⁾	0.6452 ⁽¹¹⁾	0.6507 ⁽¹³⁾	0.64118 ⁽⁹⁾	0.56006 ⁽³⁾	0.59917 ⁽⁶⁾	0.63375 ⁽⁸⁾	0.57438 ⁽⁴⁾	0.64338 ⁽¹⁰⁾	0.68983 ⁽¹⁴⁾	0.60995 ⁽⁷⁾	0.70751 ⁽¹⁵⁾
20	D_{abs}	0.05454 ⁽⁴⁾	0.05003 ⁽¹⁾	0.06151 ⁽⁹⁾	0.05325 ⁽²⁾	0.05889 ⁽⁶⁾	0.05985 ⁽⁷⁾	0.06009 ⁽⁸⁾	0.05375 ⁽³⁾	0.06822 ⁽¹²⁾	0.06205 ⁽¹⁰⁾	0.0731 ⁽¹⁵⁾	0.05755 ⁽⁵⁾	0.06893 ⁽¹³⁾	0.06301 ⁽¹¹⁾	0.06924 ⁽¹⁴⁾
20	D_{max}	0.08406 ⁽⁴⁾	0.07664 ⁽¹⁾	0.09455 ⁽⁸⁾	0.08391 ⁽³⁾	0.09339 ⁽⁷⁾	0.09496 ⁽⁹⁾	0.09296 ⁽⁶⁾	0.08258 ⁽²⁾	0.10563 ⁽¹²⁾	0.09715 ⁽¹⁰⁾	0.11327 ⁽¹⁵⁾	0.08953 ⁽⁵⁾	0.10901 ⁽¹⁴⁾	0.09867 ⁽¹¹⁾	0.10655 ⁽¹³⁾
20	ASAE	0.04442 ⁽⁷⁾	0.0416 ⁽²⁾	0.04246 ⁽⁵⁾	0.04301 ⁽⁶⁾	0.04242 ⁽⁴⁾	0.0414 ⁽¹⁾	0.04206 ⁽³⁾	0.0449 ⁽⁸⁾	0.05704 ⁽¹²⁾	0.05209 ⁽¹¹⁾	0.06498 ⁽¹⁵⁾	0.04501 ⁽⁹⁾	0.06139 ⁽¹⁴⁾	0.05132 ⁽¹⁰⁾	0.06007 ⁽¹³⁾
20	Σ Ranks	28 ⁽²⁾	10 ⁽¹⁾	51 ⁽⁴⁾	75 ⁽⁷⁾	77 ^(8,5)	77 ^(8,5)	63 ⁽⁶⁾	30 ⁽³⁾	87.5 ^(11,5)	92 ⁽¹³⁾	87.5 ^(11,5)	60 ⁽⁵⁾	128 ⁽¹⁵⁾	87 ⁽¹⁰⁾	127 ⁽¹⁴⁾
65	BIAS($\hat{\lambda}$)	0.03018 ⁽¹⁾	0.03538 ⁽⁴⁾	0.03794 ⁽⁶⁾	0.03749 ⁽⁵⁾	0.03938 ⁽⁹⁾	0.03842 ⁽⁷⁾	0.03371 ⁽³⁾	0.03883 ⁽⁸⁾	0.04127 ⁽¹¹⁾	0.04231 ⁽¹²⁾	0.0479 ⁽¹⁴⁾	0.03281 ⁽²⁾	0.0468 ⁽¹³⁾	0.03949 ⁽¹⁰⁾	0.05177 ⁽¹⁵⁾
65	BIAS($\hat{\phi}$)	1.52496 ⁽³⁾	1.37973 ⁽¹⁾	1.64104 ⁽⁶⁾	1.66647 ⁽⁸⁾	1.80865 ⁽¹³⁾	1.68186 ⁽¹⁰⁾	1.65155 ⁽⁷⁾	1.59888 ⁽⁵⁾	1.66805 ⁽⁹⁾	1.78975 ⁽¹²⁾	1.72599 ⁽¹¹⁾	1.47976 ⁽²⁾	1.88593 ⁽¹⁴⁾	1.58487 ⁽⁴⁾	1.91174 ⁽¹⁵⁾
65	MSE($\hat{\lambda}$)	0.00154 ⁽¹⁾	0.00188 ⁽⁴⁾	0.00217 ⁽⁶⁾	0.00211 ⁽⁵⁾	0.00227 ⁽¹⁰⁾	0.00219 ⁽⁷⁾	0.00172 ⁽³⁾	0.00222 ⁽⁹⁾	0.00256 ⁽¹¹⁾	0.00308 ⁽¹²⁾	0.00353 ⁽¹⁴⁾	0.00157 ⁽²⁾	0.00339 ⁽¹³⁾	0.00221 ⁽⁸⁾	0.00444 ⁽¹⁵⁾
65	MSE($\hat{\phi}$)	3.10747 ⁽³⁾	2.62501 ⁽¹⁾	3.58535 ⁽¹⁰⁾	3.3038 ⁽⁶⁾	3.89159 ⁽¹²⁾	3.49039 ⁽⁷⁾	3.53523 ⁽⁴⁾	3.29395 ⁽⁴⁾	3.58454 ⁽⁹⁾	4.01668 ⁽¹³⁾	3.88642 ⁽¹¹⁾	3.03943 ⁽²⁾	4.26229 ⁽¹⁴⁾	3.29886 ⁽⁵⁾	4.40475 ⁽¹⁵⁾
65	MRE($\hat{\lambda}$)	0.06036 ⁽¹⁾	0.07077 ⁽⁴⁾	0.07588 ⁽⁶⁾	0.07499 ⁽⁵⁾	0.07877 ⁽⁹⁾	0.07672 ⁽⁸⁾	0.07766 ⁽⁸⁾	0.08255 ⁽¹¹⁾	0.08463 ⁽¹²⁾	0.0958 ⁽¹⁴⁾	0.06562 ⁽²⁾	0.09359 ⁽¹³⁾	0.07898 ⁽¹⁰⁾	0.10355 ⁽¹⁵⁾	
65	MRE($\hat{\phi}$)	0.4357 ⁽³⁾	0.39421 ⁽¹⁾	0.46887 ⁽⁶⁾	0.47614 ⁽⁸⁾	0.51676 ⁽¹³⁾	0.48053 ⁽¹⁰⁾	0.47187 ⁽⁷⁾	0.45682 ⁽⁵⁾	0.47659 ⁽⁹⁾	0.51136 ⁽¹²⁾	0.49314 ⁽¹¹⁾	0.42279 ⁽²⁾	0.53884 ⁽¹⁴⁾	0.45282 ⁽⁴⁾	0.54621 ⁽¹⁵⁾
65	D_{abs}	0.0256 ⁽¹⁾	0.03185 ⁽⁶⁾	0.03137 ⁽⁵⁾	0.03097 ⁽⁴⁾	0.03593 ⁽¹⁰⁾	0.03454 ⁽⁸⁾	0.02833 ⁽²⁾	0.03418 ⁽⁷⁾	0.03768 ⁽¹¹⁾	0.03822 ⁽¹²⁾	0.04129 ⁽¹⁴⁾	0.02988 ⁽³⁾	0.03906 ⁽¹³⁾	0.03511 ⁽⁹⁾	0.0433 ⁽¹⁵⁾
65	D_{max}	0.04041 ⁽¹⁾	0.04924 ⁽⁴⁾	0.04924 ⁽⁵⁾	0.04944 ⁽⁶⁾	0.05607 ⁽¹⁰⁾	0.05328 ⁽⁸⁾	0.04437 ⁽²⁾	0.05203 ⁽⁷⁾	0.05983 ⁽¹¹⁾	0.05984 ⁽¹²⁾	0.0658 ⁽¹⁴⁾	0.04632 ⁽³⁾	0.06292 ⁽¹³⁾	0.05513 ⁽⁹⁾	0.06745 ⁽¹⁵⁾
65	ASAE	0.02248 ⁽⁸⁾	0.02039 ⁽²⁾	0.02105 ⁽⁶⁾	0.02073 ⁽⁵⁾	0.02072 ⁽⁴⁾	0.02002 ⁽¹⁾	0.02061 ⁽³⁾	0.02204 ⁽⁷⁾	0.02784 ⁽¹²⁾	0.02505 ⁽¹¹⁾	0.02989 ⁽¹⁴⁾	0.02293 ⁽⁹⁾	0.02887 ⁽¹³⁾	0.02395 ⁽¹⁰⁾	0.03125 ⁽¹⁵⁾
65	Σ Ranks	22 ⁽¹⁾	27 ^(2,5)	56 ⁽⁶⁾	52 ⁽⁵⁾	90 ⁽¹⁰⁾	65 ⁽⁸⁾	38 ⁽⁴⁾	60 ⁽⁷⁾	94 ⁽¹¹⁾	108 ⁽¹²⁾	117 ⁽¹³⁾	27 ^(2,5)	128 ⁽¹⁵⁾	69 ⁽⁹⁾	135 ⁽¹⁵⁾
130	BIAS($\hat{\lambda}$)	0.02257 ⁽³⁾	0.02269 ⁽⁴⁾	0.02562 ⁽⁹⁾	0.02375 ⁽⁷⁾	0.0214 ⁽¹⁾	0.02459 ⁽⁸⁾	0.02149 ⁽²⁾	0.02314 ⁽⁶⁾	0.03189 ⁽¹²⁾	0.02815 ⁽¹¹⁾	0.03595 ⁽¹⁵⁾	0.02716 ⁽¹⁰⁾	0.03354 ⁽¹³⁾	0.02284 ⁽⁵⁾	0.03425 ⁽¹⁴⁾
130	BIAS($\hat{\phi}$)	1.38532 ⁽⁷⁾	1.3211 ⁽¹⁾	1.43189 ⁽⁹⁾	1.37798 ⁽⁵⁾	1.44799 ⁽¹⁰⁾	1.51455 ⁽¹³⁾	1.34329 ⁽³⁾	1.33165 ⁽²⁾	1.53683 ⁽¹⁴⁾	1.42143 ⁽⁸⁾	1.47555 ⁽¹¹⁾	1.38163 ⁽⁶⁾	1.50518 ⁽¹²⁾	1.35996 ⁽⁴⁾	1.65711 ⁽¹⁵⁾
130	MSE($\hat{\lambda}$)	0.00077 ^(2,5)	0.00077 ^(2,5)	0.00105 ⁽⁹⁾	0.00094 ⁽⁷⁾	0.00077 ^(2,5)	0.00099 ⁽⁸⁾	0.00077 ^(2,5)	0.00089 ⁽⁶⁾	0.00152 ⁽¹²⁾	0.00128 ⁽¹¹⁾	0.00223 ⁽¹⁵⁾	0.00108 ⁽¹⁰⁾	0.00163 ⁽¹³⁾	0.00087 ⁽⁵⁾	0.00182 ⁽¹⁴⁾
130	MSE($\hat{\phi}$)	2.52207 ⁽¹⁾	2.38137 ⁽⁴⁾	2.78323 ⁽⁹⁾	2.62942 ⁽⁸⁾	2.78828 ⁽¹⁰⁾	2.96868 ⁽¹²⁾	2.38418 ⁽²⁾	2.51607 ⁽³⁾	3.05366 ⁽¹⁴⁾	2.60746 ⁽⁷⁾	2.98931 ⁽¹³⁾	2.58218 ⁽⁶⁾	2.83343 ⁽¹¹⁾	2.54589 ⁽⁵⁾	3.31891 ⁽¹⁵⁾
130	MRE($\hat{\lambda}$)	0.05104 ⁽³⁾	0.04538 ⁽⁴⁾	0.05125 ⁽⁹⁾	0.0475 ⁽⁷⁾	0.0428 ⁽¹⁾	0.04917 ⁽⁸⁾	0.04297 ⁽²⁾	0.04629 ⁽⁶⁾	0.06378 ⁽¹²⁾	0.0563 ⁽¹¹⁾	0.0719 ⁽¹⁵⁾	0.05432 ⁽¹⁰⁾	0.06707 ⁽¹³⁾	0.04567 ⁽⁵⁾	0.0685 ⁽¹⁴⁾
130	MRE($\hat{\phi}$)	0.39581 ⁽⁷⁾	0.37746 ⁽¹⁾	0.40911 ⁽⁹⁾	0.39371 ⁽⁵⁾	0.41371 ⁽¹⁰⁾	0.43273 ⁽¹³⁾	0.3838 ⁽³⁾	0.38047 ⁽²⁾	0.43909 ⁽¹⁴⁾	0.40612 ⁽⁸⁾	0.42159 ⁽¹¹⁾	0.39475 ⁽⁶⁾	0.43005 ⁽¹²⁾	0.38856 ⁽⁴⁾	0.47346 ⁽¹⁵⁾
130	D_{abs}	0.02063 ⁽³⁾	0.01975 ⁽¹⁾	0.02384 ⁽¹⁰⁾	0.02127 ⁽⁶⁾	0.02214 ⁽⁸⁾	0.02089 ⁽⁵⁾	0.01986 ⁽²⁾	0.02153 ⁽⁷⁾	0.02669 ⁽¹³⁾	0.0251 ⁽¹¹⁾	0.0264 ⁽¹²⁾	0.02337 ⁽⁹⁾	0.03002 ⁽¹⁴⁾	0.0207 ⁽⁴⁾	0.0319 ⁽¹⁵⁾
130	D_{max}	0.03191 ⁽³⁾	0.031 ⁽¹⁾	0.03687 ⁽¹⁰⁾	0.03317 ⁽⁶⁾	0.03442 ⁽⁷⁾	0.03444 ⁽⁸⁾	0.03112 ⁽²⁾	0.03232 ⁽⁴⁾	0.04282 ⁽¹³⁾	0.03873 ⁽¹¹⁾	0.04205 ⁽¹²⁾	0.03637 ⁽⁹⁾	0.04653 ⁽¹⁴⁾	0.0328 ⁽⁵⁾	0.04921 ⁽¹⁵⁾
130	ASAE	0.01419 ⁽⁷⁾	0.01329 ⁽³⁾	0.01418 ⁽⁶⁾	0.0142 ⁽⁸⁾	0.01336 ⁽⁴⁾	0.01328 ⁽²⁾	0.01273 ⁽¹⁾	0.01481 ⁽⁹⁾	0.01792 ⁽¹²⁾	0.0164 ⁽¹⁰⁾	0.01907 ⁽¹³⁾	0.01386 ⁽⁵⁾	0.02023 ⁽¹⁵⁾	0.01699 ⁽¹¹⁾	0.01912 ⁽¹⁴⁾
130	Σ Ranks	39.5 ⁽³⁾	18.5 ⁽¹⁾	80 ⁽¹⁰⁾	59 ⁽⁷⁾	53.5 ⁽⁶⁾	77 ⁽⁹⁾	19.5 ⁽²⁾	45 ⁽⁴⁾	116 ⁽¹²⁾	88 ⁽¹¹⁾	117 ^(13,5)	71 ⁽⁸⁾	117 ^(13,5)	48 ⁽⁵⁾	131 ⁽¹⁵⁾
200	BIAS($\hat{\lambda}$)	0.01719 ⁽¹⁾	0.01849 ⁽³⁾	0.01801 ⁽²⁾	0.01951 ⁽⁶⁾	0.02124 ⁽⁹⁾	0.02 ⁽⁷⁾	0.01859 ⁽⁴⁾	0.01932 ⁽⁵⁾	0.02307 ⁽¹²⁾	0.02167 ⁽¹⁰⁾	0.03084 ⁽¹⁵⁾	0.02106 ⁽⁸⁾	0.02589 ⁽¹⁴⁾	0.02184 ⁽¹¹⁾	0.02587 ⁽¹³⁾
200	BIAS($\hat{\phi}$)	1.13876 ⁽²⁾	1.16623 ⁽³⁾	1.31195 ⁽⁹⁾	1.12912 ⁽¹⁾	1.23827 ⁽⁶⁾	1.27689 ⁽⁷⁾	1.29409 ⁽⁸⁾	1.20912 ⁽⁴⁾	1.22674 ⁽⁵⁾	1.34671 ⁽¹²⁾	1.34722 ⁽¹⁴⁾	1.34184 ⁽¹¹⁾	1.47526 ⁽¹⁵⁾	1.35975 ⁽¹³⁾	1.3333 ⁽¹⁰⁾
200	MSE($\hat{\lambda}$)	0.00047 ⁽¹⁾	0.00053 ⁽²⁾	0.00052 ⁽²⁾	0.00058 ⁽⁸⁾	0.00067 ⁽⁵⁾	6e-04 ⁽⁶⁾	0.00056 ⁽⁴⁾	0.00063 ⁽⁷⁾	0.00083 ⁽¹²⁾	7e-04 ⁽⁹⁾	0.00185 ⁽¹⁵⁾	0.00072 ^(10,5)	0.00109 ⁽¹⁴⁾	0.00072 ^(10,5)	0.00102 ⁽¹³⁾
200	MSE($\hat{\phi}$)	1.82033 ⁽¹⁾	2.00701 ⁽⁴⁾	2.27689 ⁽⁹⁾	1.93152 ⁽²⁾	2.2606 ⁽⁵⁾	2.26649 ⁽⁸⁾	2.28834 ⁽¹⁰⁾	1.99464 ⁽³⁾	2.10284 ⁽⁶⁾	2.37903 ⁽¹²⁾	2.70429 ⁽¹⁴⁾	2.30799 ⁽¹¹⁾	2.72184 ⁽¹⁵⁾	2.50084 ⁽¹³⁾	2.20777 ⁽⁷⁾
200	MRE($\hat{\lambda}$)	0.03437 ⁽¹⁾	0.03698 ⁽³⁾	0.03603 ⁽²⁾	0.03902 ⁽⁶⁾	0.04247 ⁽⁹⁾	0.03999 ⁽⁷⁾	0.03718 ⁽⁴⁾	0.03863 ⁽⁵⁾	0.04613 ⁽¹²⁾	0.04334 ⁽¹⁰⁾	0.06168 ⁽¹⁵⁾	0.04212 ⁽⁸⁾	0.05179 ⁽¹⁴⁾	0.04368 ⁽¹¹⁾	0.05173 ⁽¹³⁾
200	MRE($\hat{\phi}$)	0.32536 ⁽²⁾	0.33321 ⁽³⁾	0.37484 ⁽⁹⁾	0.32261 ⁽¹⁾	0.35379 ⁽⁶⁾	0.36483 ⁽⁷⁾	0.36974 ⁽⁸⁾	0.34546 ⁽⁴⁾	0.3505 ⁽⁵⁾	0.38477 ⁽¹²⁾	0.41063 ⁽¹⁴⁾	0.38338 ⁽¹¹⁾	0.4215 ⁽¹⁵⁾	0.3885 ⁽¹³⁾	0.38094 ⁽¹⁰⁾
200	D_{abs}	0.01593 ⁽³⁾	0.01679 ⁽³⁾	0.01614 ⁽²⁾	0.01805 ⁽⁵⁾	0.01848 ⁽⁸⁾	0.01909 ⁽⁹⁾	0.01701 ⁽⁴⁾	0.01817 ⁽⁶⁾	0.02089 ⁽¹²⁾	0.01962 ⁽¹⁰⁾	0.02504 ⁽¹⁵⁾	0.02006 ⁽¹¹⁾	0.0222 ⁽¹³⁾	0.01842 ⁽⁷⁾	0.02422 ⁽¹⁴⁾
200	D_{max}	0.02478 ⁽¹⁾	0.0261 ⁽⁴⁾	0.0255 ⁽²⁾	0.02728 ⁽⁵⁾	0.02822 ⁽⁷⁾	0.03004 ⁽⁹⁾	0.02606 ⁽³⁾	0.0275 ⁽⁶⁾	0.03202 ⁽¹²⁾	0.03052 ⁽¹⁰⁾	0.03894 ⁽¹⁵⁾	0.03078 ⁽¹¹⁾	0.03462 ⁽¹³⁾	0.02838 ⁽⁸⁾	0.03714 ⁽¹⁴⁾
200	ASAE	0.01069 ⁽⁶⁾	0.01017 ⁽²⁾	0.01066 ⁽⁵⁾	0.01084 ⁽⁷⁾	0.009971 ⁽¹⁾	0.01049 ⁽⁴⁾	0.01037 ⁽³⁾	0.01128 ⁽⁹⁾	0.01451 ⁽¹²⁾	0.01332 ⁽¹¹⁾	0.01606 ⁽¹⁵⁾	0.01124 ⁽⁸⁾	0.01571 ⁽¹³⁾	0.01315 ⁽¹⁰⁾	0.01579 ⁽¹⁴⁾
200	Σ Ranks	16 ⁽¹⁾	28 ⁽²⁾	42 ⁽⁴⁾	38 ⁽³⁾	58 ⁽⁷⁾	72 ⁽⁸⁾	48 ^(5,5)	48 ^(5,5)	86 ⁽⁹⁾	102 ⁽¹²⁾	130 ⁽¹⁵⁾	87.5 ⁽¹⁰⁾	124 ⁽¹⁴⁾	94.5 ⁽¹¹⁾	106 ⁽¹³⁾
270	BIAS($\hat{\lambda}$)	0.0153 ⁽¹⁾	0.01745 ⁽⁸⁾	0.0173 ⁽⁷⁾	0.01688 ⁽⁵⁾	0.01949 ⁽¹²⁾	0.01566 ⁽²⁾	0.0163 ⁽³⁾	0.01698 ⁽⁶⁾	0.01866 ⁽¹¹⁾	0.01764 ⁽⁹⁾	0.02446 ⁽¹⁵⁾	0.01799 ⁽¹⁰⁾	0.02026 ⁽¹³⁾	0.01645 ⁽⁴⁾	0.021

TABLE 8. Partial and overall ranks for all estimation methods of our proposed model.

Parameter	n	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE	ADSOE	KE	MSSDE	MSSLDE	MSLNDE
$\lambda = 0.35, \phi = 0.65$	20	1.0	3.0	4.5	8.0	10.0	4.5	6.5	6.5	12.0	11.0	13.0	2.0	14.0	9.0	15.0
	65	1.0	2.0	8.0	3.5	9.0	5.0	3.5	6.0	12.0	11.0	14.0	7.0	15.0	10.0	13.0
	130	1.0	3.0	6.0	2.0	8.0	7.0	4.0	5.0	12.0	11.0	15.0	9.0	13.0	10.0	14.0
	200	1.0	4.0	7.5	2.0	9.0	5.0	3.0	7.5	12.0	11.0	14.5	6.0	13.0	10.0	14.5
	270	2.0	1.0	8.0	3.0	5.0	6.0	4.0	9.0	12.0	11.0	15.0	10.0	14.0	7.0	13.0
	350	2.0	4.0	8.0	3.0	5.0	7.0	1.0	6.0	12.0	10.0	15.0	11.0	14.0	9.0	13.0
$\lambda = 1.25, \phi = 0.95$	20	1.0	3.5	3.5	9.0	5.0	8.0	7.0	6.0	12.0	13.0	11.0	2.0	14.0	10.0	15.0
	65	1.0	5.0	8.0	9.0	7.0	6.0	3.0	4.0	12.0	13.0	10.5	2.0	14.0	10.5	15.0
	130	1.0	5.0	6.0	9.0	7.5	3.5	2.0	7.5	12.0	13.0	11.0	3.5	15.0	10.0	14.0
	200	1.0	3.0	7.0	6.0	9.0	2.0	5.0	8.0	11.5	11.5	13.0	4.0	15.0	10.0	14.0
	270	1.0	5.0	6.0	7.5	7.5	2.5	2.5	9.0	11.0	12.0	15.0	4.0	13.5	10.0	13.5
	350	1.0	5.0	8.0	4.0	9.0	3.0	2.0	10.0	12.0	11.0	15.0	6.0	13.0	7.0	14.0
$\lambda = 1.75, \phi = 2.5$	20	1.0	7.0	3.0	9.5	8.0	6.0	3.0	5.0	12.0	13.0	9.5	3.0	15.0	11.0	14.0
	65	1.0	4.0	2.0	10.0	7.0	5.0	6.0	8.0	13.0	12.0	9.0	3.0	14.0	11.0	15.0
	130	1.0	7.0	6.0	5.0	10.0	3.0	2.0	9.0	11.0	13.0	8.0	4.0	15.0	12.0	14.0
	200	1.0	2.0	5.0	3.0	10.0	4.0	6.0	8.0	9.0	15.0	12.0	7.0	13.0	11.0	14.0
	270	1.0	5.0	7.0	2.0	8.0	4.0	3.0	10.0	11.0	9.0	13.5	6.0	12.0	13.5	15.0
	350	2.0	3.0	7.0	1.0	8.0	5.0	4.0	14.0	10.0	11.0	13.0	9.0	12.0	6.0	15.0
$\lambda = 0.8, \phi = 1.5$	20	3.5	3.5	2.0	8.0	9.0	5.0	7.0	6.0	10.0	12.0	11.0	1.0	15.0	13.0	14.0
	65	5.5	2.0	5.5	8.0	4.0	3.0	1.0	10.0	11.0	12.0	13.0	7.0	14.0	9.0	15.0
	130	7.0	2.0	9.0	3.0	5.0	4.0	1.0	10.0	12.0	11.0	13.0	8.0	14.0	6.0	15.0
	200	9.0	2.5	8.0	1.0	7.0	6.0	2.5	4.0	12.0	11.0	14.0	5.0	15.0	10.0	13.0
	270	8.0	2.5	9.0	2.5	6.0	4.0	1.0	5.0	12.0	11.0	14.0	7.0	13.0	10.0	15.0
	350	7.0	3.0	9.0	5.0	6.0	2.0	1.0	4.0	12.0	10.0	15.0	8.0	13.0	11.0	14.0
$\lambda = 0.5, \phi = 3.5$	20	2.0	1.0	4.0	7.0	8.5	8.5	6.0	3.0	11.5	13.0	11.5	5.0	15.0	10.0	14.0
	65	1.0	2.5	6.0	5.0	10.0	8.0	4.0	7.0	11.0	12.0	13.0	2.5	14.0	9.0	15.0
	130	3.0	1.0	10.0	7.0	6.0	9.0	2.0	4.0	12.0	11.0	13.5	8.0	13.5	5.0	15.0
	200	1.0	2.0	4.0	3.0	7.0	8.0	5.5	5.5	9.0	12.0	15.0	10.0	14.0	11.0	13.0
	270	1.0	5.0	9.0	4.0	11.0	6.0	2.0	7.0	12.0	8.0	13.0	10.0	14.0	3.0	15.0
	350	1.0	7.5	7.5	3.0	10.0	6.0	2.0	4.0	12.0	9.0	13.0	11.0	14.0	5.0	15.0
$\sum$ Ranks		70.0	106.0	193.5	153.0	231.5	156.0	102.5	208.0	345.0	343.5	386.0	181.0	417.0	279.0	428.0
Overall Rank		1	3	7	4	9	5	2	8	12	11	13	6	14	10	15

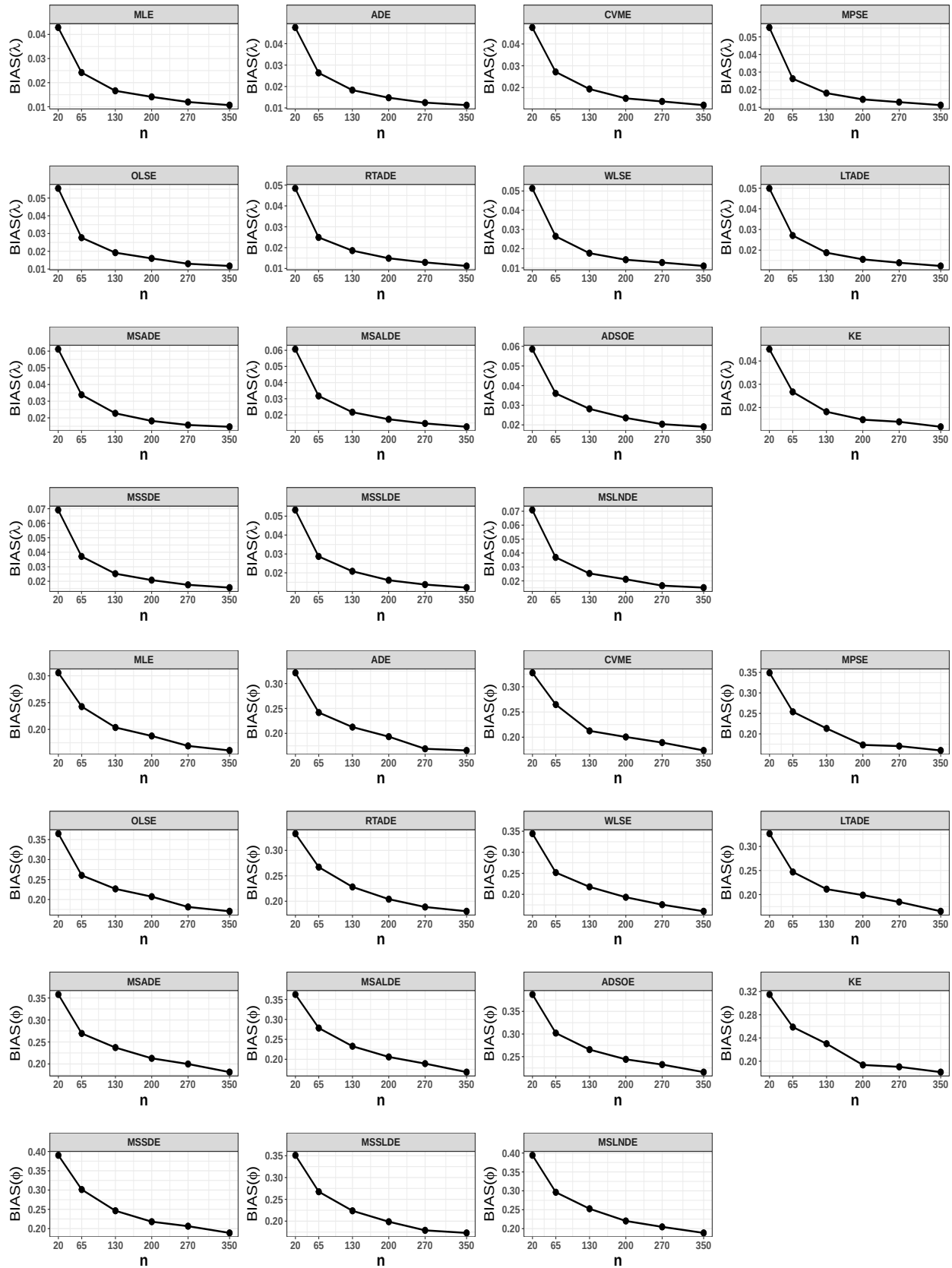


FIGURE 5. A graphical representation for BIAS values presented in Table 3.

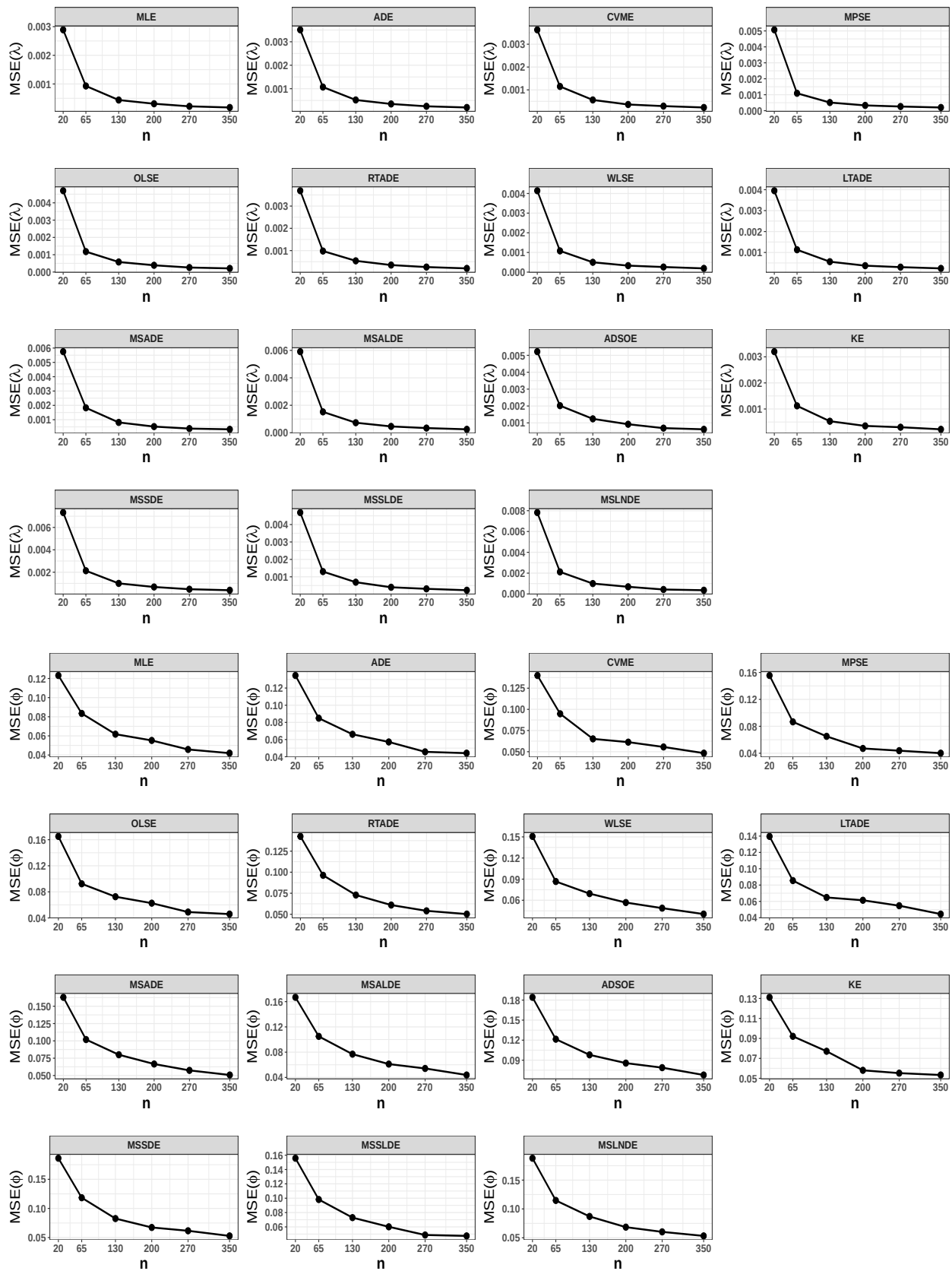


FIGURE 6. A graphical representation for MSE values presented in Table 3.

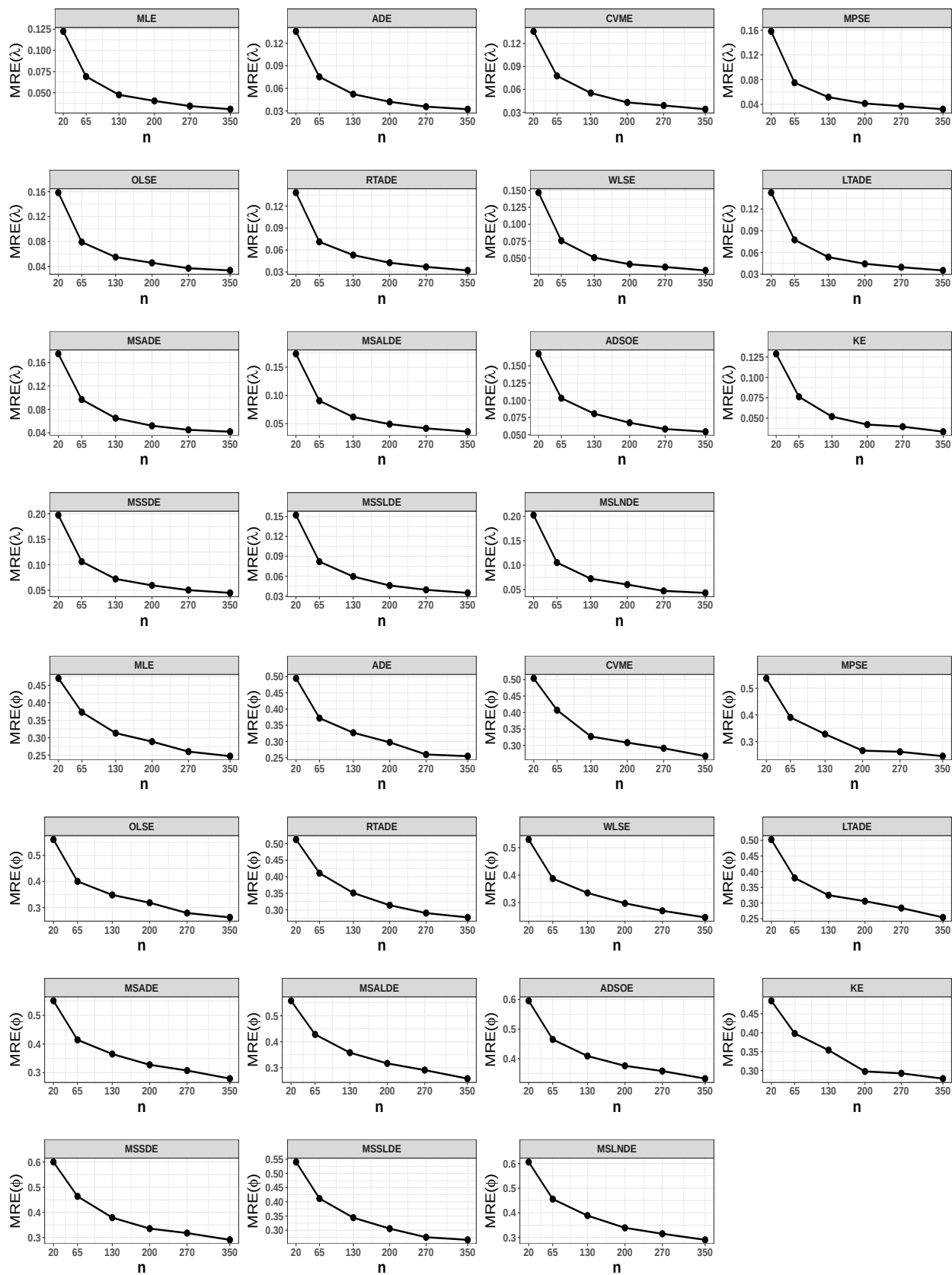


FIGURE 7. A graphical representation for MRE values presented in Table 3.

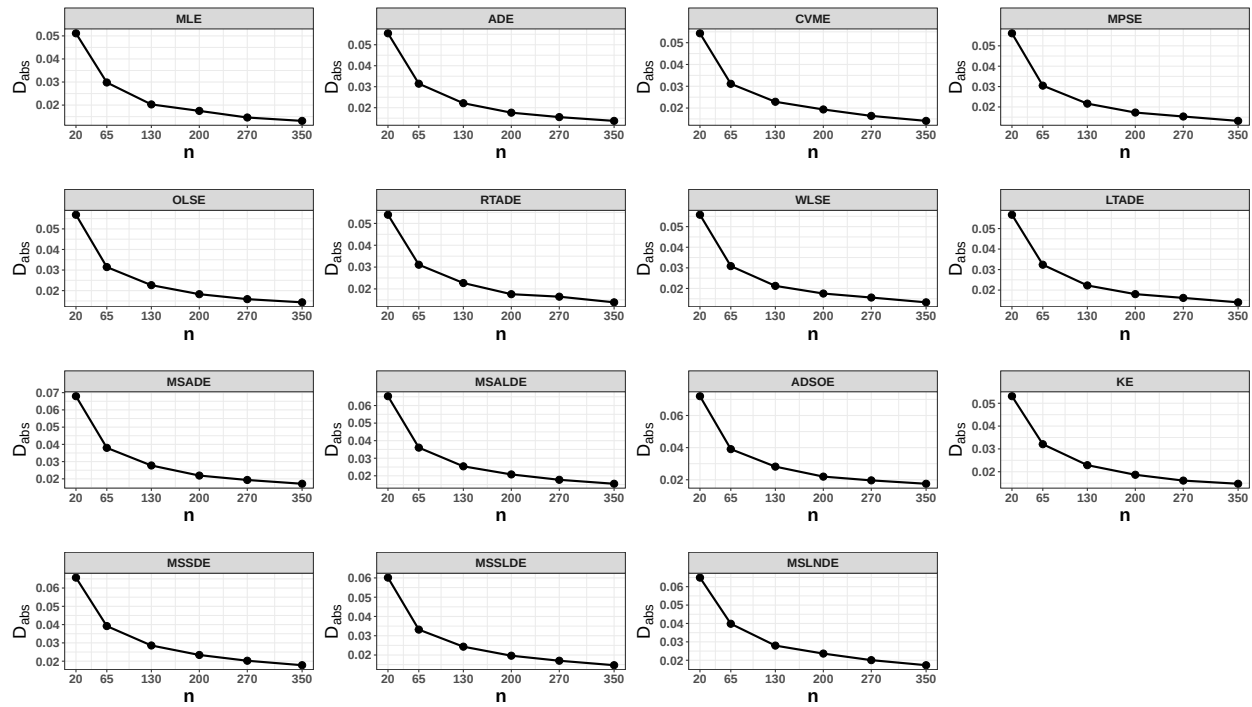


FIGURE 8. A graphical representation for D_{abs} values presented in Table 3.

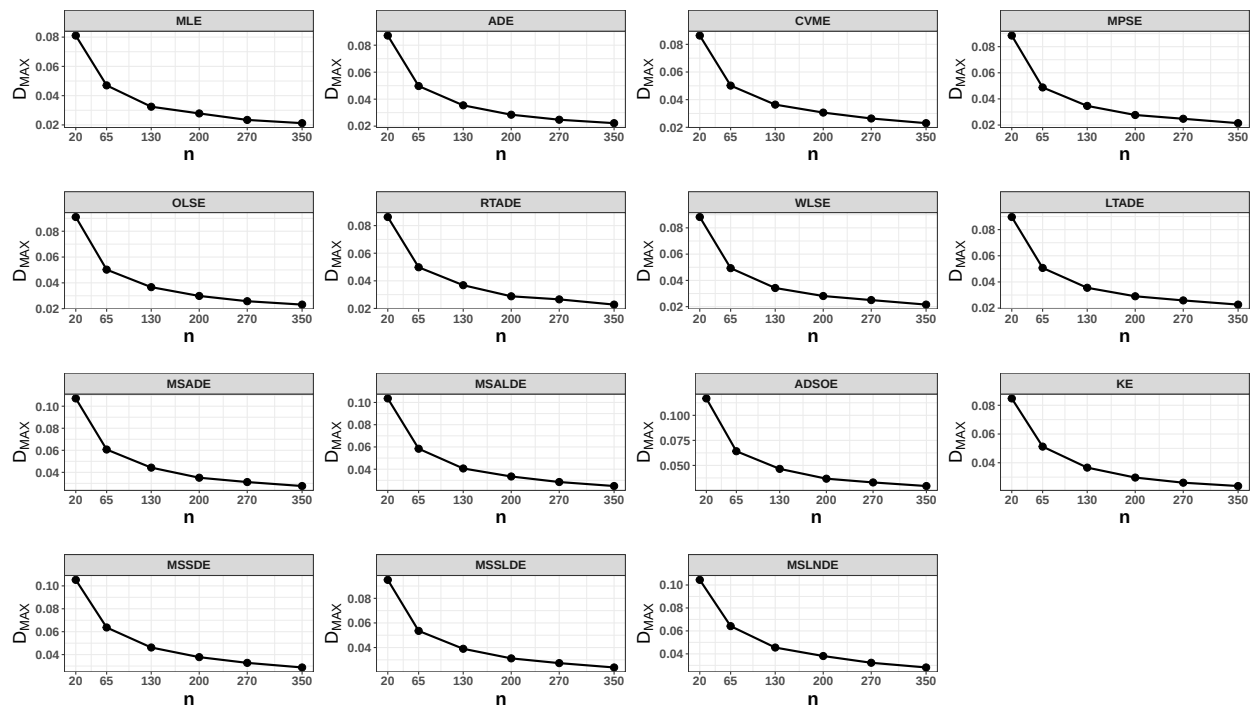


FIGURE 9. A graphical representation for D_{max} values presented in Table 3.

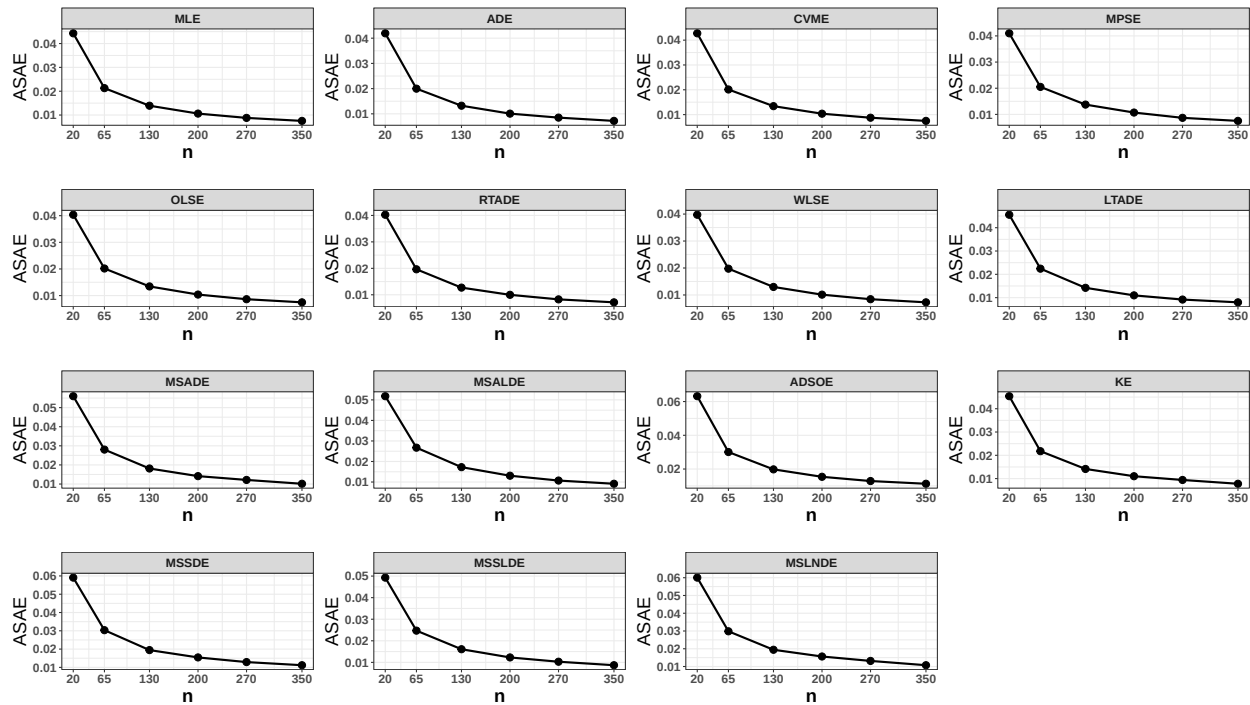


FIGURE 10. A graphical representation for ASAE values presented in Table 3.

6. REAL DATA ANALYSIS

In this section, we demonstrate the significance and flexibility of the proposed IUTMD model in fitting real-world data across diverse domains. To illustrate its practical usefulness, we consider two real datasets.

The first dataset contains measurements of precipitation (in inches) recorded during March at the Minneapolis-St. Paul weather station. This dataset was also used by [9], and the recorded observations are as follows:

0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52,
1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05.

The second dataset consists of a complete set of observations that was previously studied and discussed in [10]. It reports the lifetimes of 20 electronic components. The recorded lifetimes are:

0.03, 0.12, 0.22, 0.35, 0.73, 0.79, 1.25, 1.41, 1.52, 1.79,
1.80, 1.94, 2.38, 2.40, 2.87, 2.99, 3.14, 3.17, 4.72, 5.09.

To better understand the characteristics of these datasets, several graphical representations are presented in Figures 11 and 12, respectively. These visualizations include box plots, violin plots, and total time on test (TTT) plots.

For model comparison, we consider the competing models, namely, the Lindley distribution (LD) [18], quasi-Lindley distribution (QLD) [11], Weibull distribution (WD) [12], Gamma distribution (GD) [13], Log-logistic distribution (LLD) [16], Rayleigh distribution (RD) [14], Burr XII distribution (BXIID) [15], Inverse Lindley distribution (ILD) [17], and Dagum distribution (DD) [19].

To make a robust comparison and selection of the best-fitting model among the competing distributions, we have utilised four information criteria, namely the Akaike information criterion (AIC), Bayesian information criterion (BIC), Consistent Akaike information criterion (CAIC), and Hannan-Quinn information criterion (HQIC). Using several model selection criteria is important because each criterion applies a different level of penalty for model complexity. The AIC mainly aims to reduce the Kullback-Leibler divergence, while the BIC and HQIC apply stronger penalties as the sample size n increases. Among them, the BIC is more conservative and is especially useful for avoiding overfitting. Therefore, using all these criteria together helps confirm the reliability and superiority of the proposed model from different statistical viewpoints. This also ensures that the selected model achieves a good balance between model fit and the number of parameters.

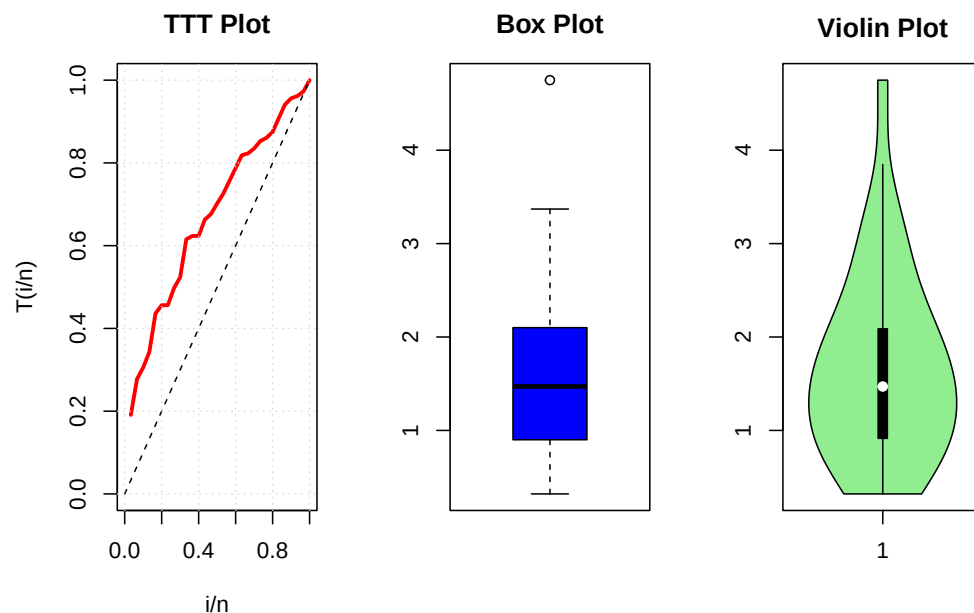


FIGURE 11. Nonparametric plots of the first dataset.

The first dataset is positively skewed, with a potential upper outlier revealed by the box plot. Thus, this observation is further supported by the violin plot, which shows a longer upper tail and a higher density of observations in the lower-to-middle range of the data. Also, it has an increasing hazard rate (IHR) behaviour that can be observed from the TTT plot. The second dataset is moderately positively skewed. Its TTT plot suggests an IHR pattern.

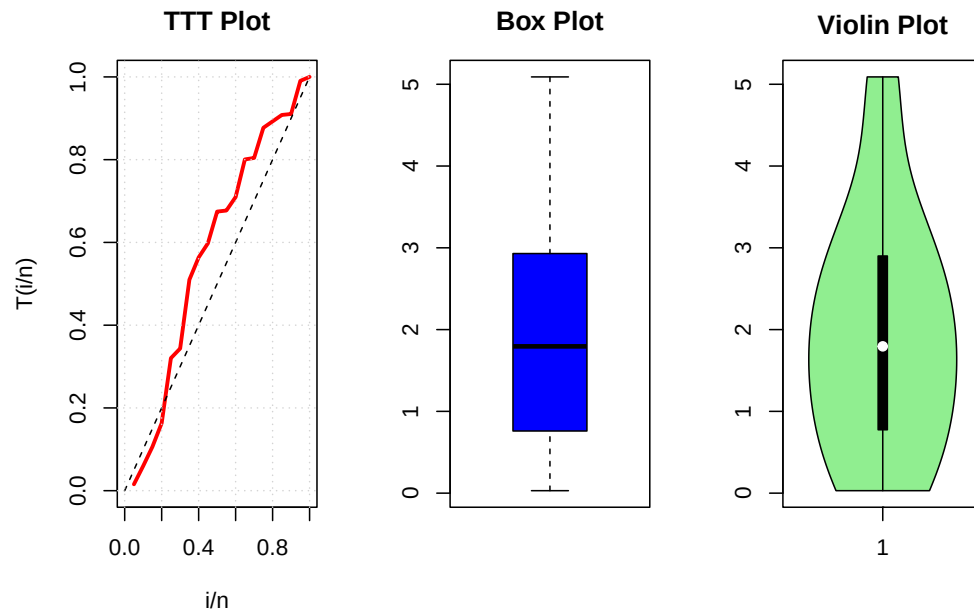


FIGURE 12. Nonparametric plots of the second dataset.

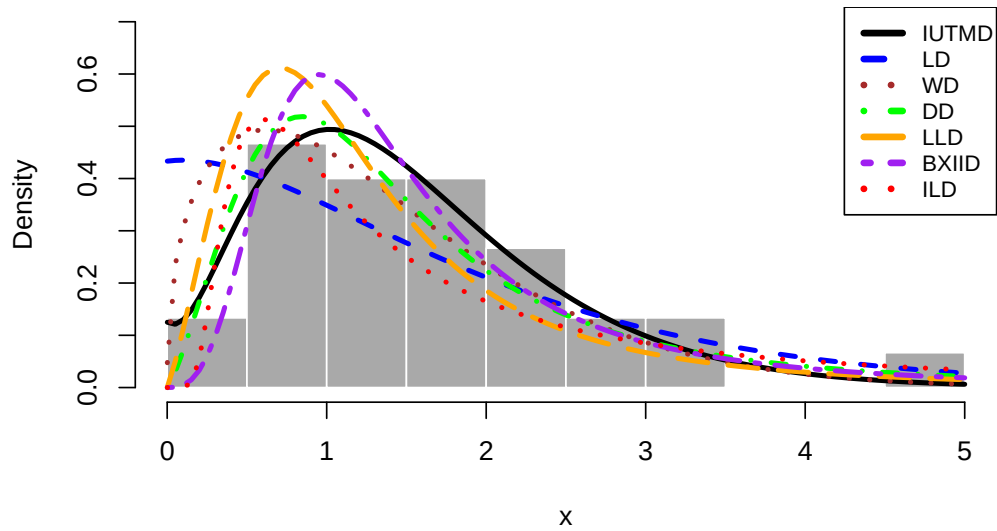


FIGURE 13. Fitted PDFs of the IUTMD and competing models for the first dataset

TABLE 10. Parameter estimates, model selection criteria, and goodness-of-fit statistics for the IUTMD and competing models for the second dataset

Distribution	Parameter Estimates	$\log L$	AIC	BIC	CAIC	HQIC	KS (p-value)	CVM (p-value)	AD (p-value)
IUTMD	$\lambda = 1.2004$ $\phi = 2.8202$	-31.7681	67.5363	69.5278	71.5278	67.9251	0.0893 (0.9927)	0.0341(0.9648)	0.2463(0.9722)
QLD	$\lambda = 0.8285$ $\alpha = 0.6566$	-32.5362	69.0724	71.0639	73.0639	69.4612	0.1230 (0.8866)	0.0614(0.8108)	0.3830(0.8639)
WD	$\alpha = 1.1961$ $\beta = 2.0389$	-32.7869	69.5739	71.5654	73.5654	69.9627	0.1270 (0.8637)	0.0687(0.765)	0.4526(0.7934)
GD	$\lambda = 1.1628$ $\sigma = 1.6643$	-33.0700	70.1401	72.1316	74.1316	70.5288	0.1531 (0.6809)	0.0919(0.6305)	0.5117(0.7327)
LLD	$\alpha = 1.4621$ $\beta = 1.4975$	-35.9225	75.8445	77.8359	79.8364	76.2337	0.1415 (0.7667)	0.1040(0.5699)	0.8187 (0.4661)
RD	$\theta = 1.6889$	-37.5192	77.0384	78.0342	79.0342	77.2328	0.1963 (0.3742)	0.1450(0.4077)	2.2422(0.0685)
BXIID	$\beta = 1.1605$ $k = 1.3195$	-36.7067	77.4135	79.4050	81.405	77.8022	0.2240 (0.2304)	0.2194(0.2336)	1.1568(0.284)
DD	$\alpha = 3659.67$ $\beta = 0.1355$ $k = 5.9638$	-31.6494	69.2988	72.2860	75.2860	69.8819	0.0936 (0.9876)	0.0478(0.9548)	0.2940(0.9621)

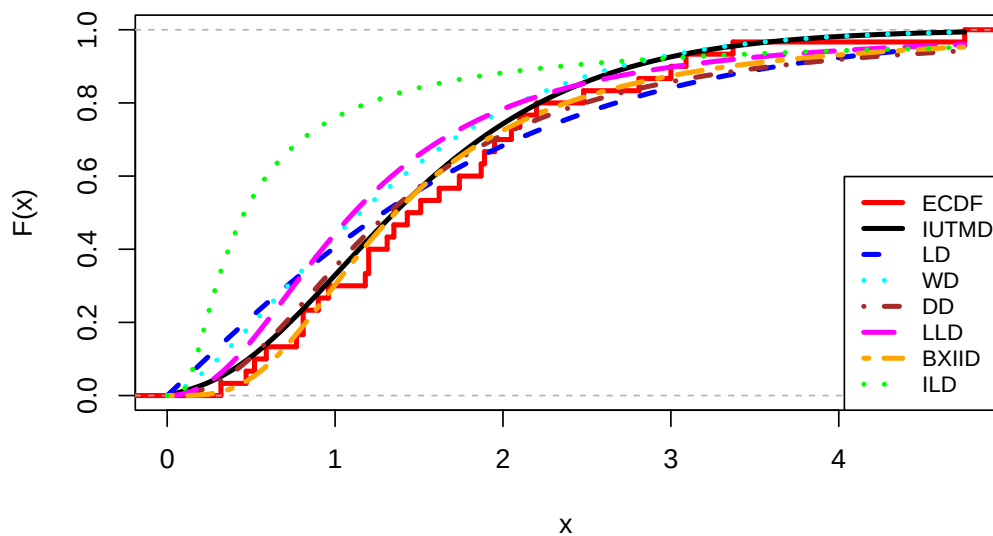


FIGURE 14. Fitted CDFs of the IUTMD and competing models for the first dataset

From Table 9, the IUTMD provides the best fit among all competing models, as indicated by its lowest information criterion values, such as AIC (80.200), BIC (83.0024), CAIC (85.0024), and HQIC (81.0965). Furthermore, the IUTMD attains comparatively satisfactory goodness-of-fit measures, and the associated p-values for the KS, CVM, and AD tests are sufficiently large, indicating no significant departure between the observed data and the fitted IUTMD. The WD and LLD emerge as close competitors to the IUTMD model, as they also yield lower goodness-of-fit measures and acceptable p-values. Therefore, their penalized criteria values are consistently larger than those of the IUTMD, indicating comparatively lower performance. In particular, the WD provides AIC = 81.2865, and the LLD model gives AIC = 81.8615; thus, both exceed the corresponding values of the IUTMD. The LD performs less well than the IUTMD, while the DD provides an acceptable fit but has higher information-criterion values than the IUTMD. Although these models provide moderate goodness-of-fit measures, their larger AIC, BIC and CAIC values indicate less efficiency in describing the data. Furthermore, the BXIID provides an acceptable fit, although its information-criterion values are higher and its goodness-of-fit statistics are less favorable than those of the IUTMD. Among all fitted models, the ILD shows the worst performance. It has the highest AIC, BIC (93.8435), and CAIC (94.8434) values, together with the largest goodness-of-fit statistics and the smallest corresponding p-values among the competing models, indicating the poorest fit to the dataset. Overall, the proposed IUTMD consistently outperforms other competing models based on the information criteria and goodness-of-fit measures.

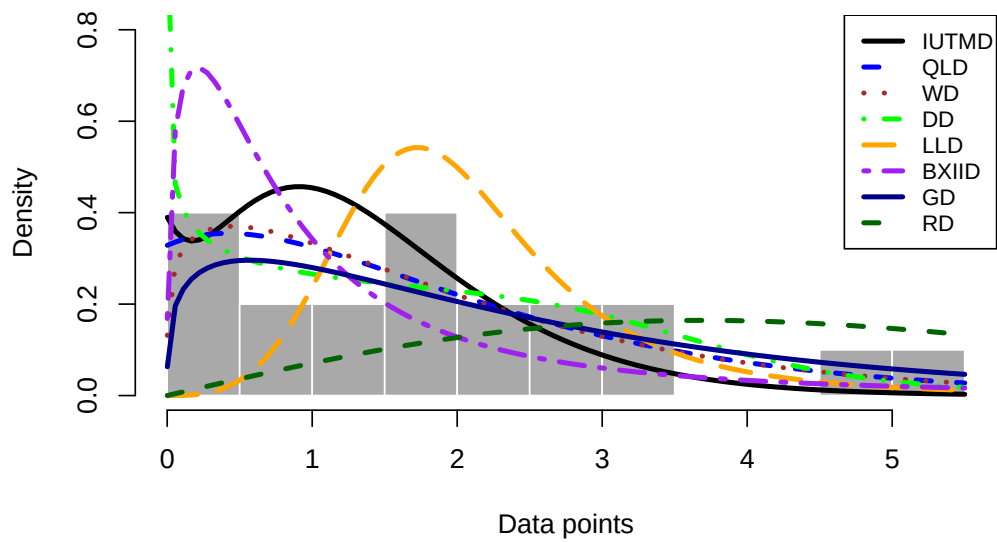


FIGURE 15. Fitted PDFs of the IUTMD and competing models for the second dataset

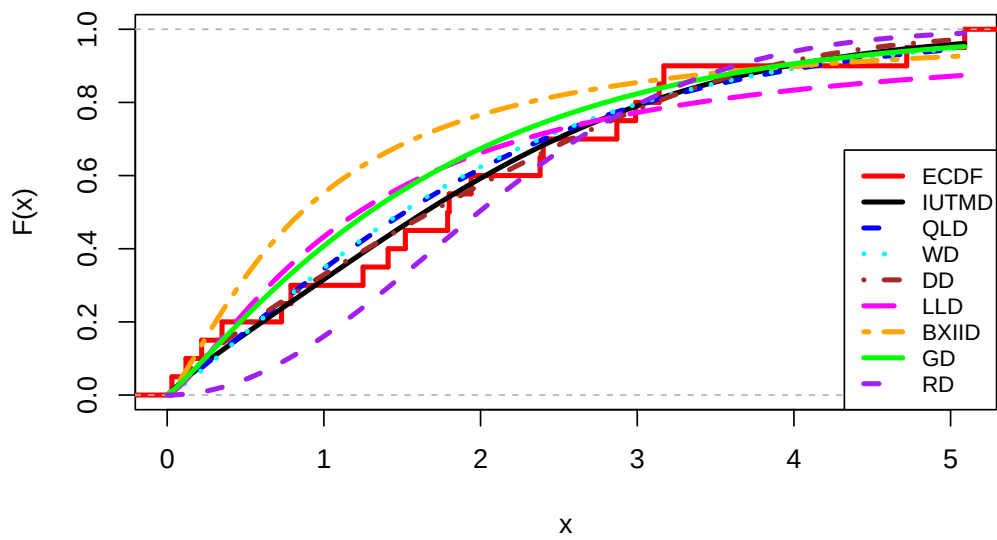


FIGURE 16. Fitted CDFs of the IUTMD and competing models for the second dataset

From Table 10, the proposed IUTMD provides the best fit among all the competing models. This is evident from its lowest information criterion values, such as $AIC = 67.5363$, $BIC = 69.5278$, $CAIC = 71.5278$ and $HQIC = 67.9251$, together with the largest goodness-of-fit p-values for the KS, CVM and AD tests, indicating no significant departure between the observed data and the fitted distribution. The QLD, DD, WD, and GD also provide acceptable fits, although their information-criterion values are higher than those of the IUTMD. The LLD model shows moderate fitting performance, yielding higher AIC, BIC, and HQIC values than the IUTMD model. Although the goodness-of-fit p-values remain acceptable, the larger penalized criteria indicate that it is comparatively less efficient in describing the dataset. The RD and BXIID models demonstrate weaker performance relative to the proposed model. Their higher information-criterion values and comparatively less favorable goodness-of-fit statistics indicate a weaker fit to the data. Overall, these results indicate that the proposed IUTMD provides the best fit among the models considered for the second dataset.

7. CONCLUSION

This paper proposed a new two-parameter distribution, namely the Inverse Unit Two-Parameter Mirra Distribution (IUTMD), which was derived by applying the inverse transformation scheme to the Unit Two-Parameter Mirra Distribution. The proposed model extends the support of the baseline distribution to the positive real line while preserving its simple functional form and two-parameter structure. The hazard rate function is flexible and can exhibit increasing, decreasing, and bathtub shapes.

Several mathematical properties of the proposed distribution were derived in closed form. These results provide a solid theoretical foundation for the proposed model and facilitate its application in statistical analysis.

The unknown parameters of the proposed distribution were estimated using fifteen different estimation methods. A comprehensive Monte Carlo simulation study was carried out to evaluate the finite-sample performance of the estimators under different parameter values and sample sizes. The simulation results showed that the estimation methods performed well and became more accurate as the sample size increased.

The proposed distribution was applied to two real datasets to demonstrate its practical usefulness. The proposed model was compared with several existing competing models using standard goodness-of-fit measures and information criteria and was found to provide the best overall fit for the datasets analysed. These results demonstrate that the Inverse Unit Two-Parameter Mirra Distribution is a flexible and useful model for positive continuous data.

Future research may focus on extending the proposed distribution to regression models, censored data settings, or multivariate distributions. The proposed distribution may also be applied in other areas where positive continuous data commonly arise, such as reliability engineering, environmental studies, finance, and survival analysis.

Authors' Contributions: All authors have worked equally to write and review the manuscript.

Acknowledgments: The authors extend their appreciation to Umm Al-Qura University, Saudi Arabia for funding this research work through grant number: 26UQU4300011GSSR03.

Funding: This research work was funded by Umm Al-Qura University, Saudi Arabia under grant number: 26UQU4300011GSSR03.

Data Availability Statement: The data that supports the findings of this study are available within the article.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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