

Computational Analysis of Unique Coupled Fixed Point Theorems in GFM-Spaces with an Application

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Abstract. In this paper, we establish some coupled fixed-point (FP) results in generalized fuzzy metric spaces (GFM-spaces) with a supportive three-dimensional column graph justification. We use the rectangular property of a GFM-space to prove the uniqueness of coupled FP findings in the said space under the coupled contraction conditions. In support of our findings, we provide illustrative, unique coupled FP numerical examples to validate our work. Further, we sketch a three-dimensional column graph using a numerical example to illustrate the inequality of coupled contraction in the aforementioned space. Moreover, we apply the Urysohn-type integral equations (UTIEs) to establish the existence and uniqueness of a coupled solution, thereby to unifying our work. Furthermore, we present an illustrative numerical example formulated in the setting of UTIEs which is not only verify the validity of our theoretical work but also establish the direct connection between the developed theory and its application, highlighting the reliability and effectiveness of the proposed approach.

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1. INTRODUCTION

In functional analysis, the theory of fixed point (FP) is the most active research areas. This has wide-ranging applications in many fields such as engineering, computer science, economics, etc. The theory of FP was first introduced by Banach [1] and stated a “Banach Contraction Principle” that is “a single-valued contractive type mapping on a complete metric spaces has a unique FP”. Since, then, many authors have presented their ideas about FP theory. This idea has been extended in many directions for single and multi-valued mappings and their applications. In 1965 the term fuzzy set was first established by Zadeh [2]. Kramosil and Michalek [3] introduced the notion of a fuzzy metric space (FM-space) and presented the structural properties of the said space. Later, Grabiec [4] proved the FP theorems of Banach and Edelstein’s contraction mapping theorems for self-mappings on complete and compact FM-spaces respectively. George and Veeramani [5–7] proved different results in the context of FM-spaces. Gregori and Romaguera [8] established some properties of FM-spaces. Brass [9] used real numbers to measure the distance between fuzzy sets and discussed the nonexistence of Hausdorff-like metrics for fuzzy sets. Gregori and Sapena [10] presented a stronger form of convergence sequences and proved some FP in FM-spaces.

Lakshmikantham and Ćirić [11] introduced the idea of coupled contractions and proved some coupled FP results in partially ordered metric spaces. In support of their findings, they presented initial value problems for ordinary differential equations. Further, Gordji et al. [12] presented coupled FP theorems in partially ordered Metric space with a supportive application. Cho et al. [13] proved some modified coupled FP results in ordered generalized metric spaces with a supportive application of integral equations. Huang et al. [14] studied some coupled FP and FP results in ordered metric spaces using multi-valued increasing operators. While Li et al. [15] proved some strong concept FP theorems in fuzzy metric spaces with an application of integral equations. Sintunavart and Kumam [16] proved some coupled common FP and coupled coincidence points theorems in partially ordered metric spaces. In [17], Jiang and Gu established some common coupled FP theorems with supportive applications in multiplicative metric spaces. While Shanjit et al. [18] studied some coupled FP results in partially ordered multiplicative metric space with supportive applications. Hussain et al. [19] proved some coupled FP theorems on multiplicative metric spaces with an application.

In 2010, Sun and Yang [20] presented the idea of Q-fuzzy metric-space known as a GFM-space. They established the structural properties and proved some common FP results. Pandiselvi et al. [21] proved a unique FP theorem using monotone, continuous, and non-decreasing functions in GFM-spaces. Recently, Rehman et al. [22], established some basic properties and proved some generalized unique FP results on complete GFM-space with supportive examples. Recently, Ullah et al. [23] and Rehman et al. [24] studied three-dimensional graphs in metric FP theory using different types of mappings and spaces. They supported their work by using various types of illustrative numerical examples and integral equations applications.

In this paper, collectively, we use the above concepts and prove some unique coupled FP theorems in GFM-spaces under generalized coupled contraction conditions with an illustrative numerical example. Based on numerical example, we establish a 3D graph to support our generalized unique coupled FP results in GFM-spaces. Moreover, we present an application of the Urysohn type integral equations (UTIEs) to validate our findings. In addition, we present an illustrative UTIE numerical example which is not only verifying the validity of our theoretical work but also establishing the direct connection between the developed theory and its application, highlighting the reliability and effectiveness of the proposed approach. This concept of unique coupled FP will play an important role in the theory of FP to improve more coupled FP results on complete GFM-spaces with various types of 3D graphs and various types of integral equations applications.

2. PRELIMINARIES

In this section, we recall some basic preliminary concept related to our main results. Throughout this paper, a set of natural numbers is represented by $\mathbb{N}$ and a continuous t -norm is represented τ -norm as defined in [25].

Definition 2.1. [25] An operation $*$: $[0, 1]^2 \rightarrow [0, 1]$ would be a τ -norm if $*$ fulfils the conditions,

- i- $*$ is associative, commutative, and continuous.
- ii- $1 * \varrho_1 = \varrho_1, \forall \varrho_1 \in [0, 1]$.
- iii- $\varrho_2 * \varrho_1 \leq \varrho_4 * \varrho_3$ whenever $\varrho_1 \leq \varrho_3$ and $\varrho_2 \leq \varrho_4$, for $\varrho_4, \varrho_3, \varrho_2, \varrho_1 \in [0, 1]$.

The basic continuous τ -norms are follows;

- i- The τ -norm: $\varrho_2 * \varrho_1 := \min \{\varrho_2, \varrho_1\}, \forall \varrho_2, \varrho_1 \in [0, 1]$.
- ii- Lukasiewicz τ -norm: $\varrho_2 * \varrho_1 := \max \{\varrho_2 + \varrho_1 - 1, 0\}, \forall \varrho_2, \varrho_1 \in [0, 1]$.
- iii- The product τ -norm: $\varrho_2 * \varrho_1 := \varrho_2 \varrho_1, \forall \varrho_2, \varrho_1 \in [0, 1]$.

Definition 2.2. [20] A 3-tuple $(\mathbf{C}, F_m, *)$ is said to be a GFM if $\mathbf{C}$ is any set, $*$ is a τ -norm and F_m is a fuzzy set on $\mathbf{C} \times \mathbf{C} \times \mathbf{C} \times (0, \infty)$ and fulfilling the following conditions $\forall c_3, c_2, c_1 \in \mathbf{C}$ and $s, \tau > 0$,

- i- $F_m(c_3, c_2, c_1, \tau) > 0$, and $F_m(c_3, c_2, c_1, \tau) = 1 \Leftrightarrow c_3 = c_2 = c_1$;
- ii- $F_m(c_2, c_2, c_1, \tau) \geq F_m(c_3, c_2, c_1, \tau)$ with $c_3 \neq c_2$;
- iii- $F_m(c_3, c_2, c_1, \tau) = F_m(\varphi(c_2, c_1, c_3), \tau)$; (symmetry) in which φ is a permutation.
- iv- $F_m(c_3, a, a, \tau) * F_m(a, c_2, c_1, s) \leq F_m(c_3, c_2, c_1, \tau + s)$;
- v- $F_m(c_3, c_2, c_1, \tau) : (0, \infty) \rightarrow [0, 1]$ is continuous.

A GFM-space is known as symmetric if $F_m(c_2, c_2, c_1, \tau) = F_m(c_1, c_1, c_2, \tau), \forall c_2, c_1 \in \mathbf{C}$ and $\tau > 0$.

Definition 2.3. [20–22] Let a 3-tuple $(\mathbf{C}, F_m, *)$ be a GFM-space, with $c \in \mathbf{C}$ and a sequence $\{c_i\}$ in $\mathbf{C}$. Then,

- i- $\{c_i\}$ is said to be convergent to $c \in \mathbf{C}$ if $\lim_{i \rightarrow \infty} F_m(c_i, c_i, c, \tau) = \lim_{i \rightarrow \infty} F_m(c_i, c, c, \tau) = 1$, for $\tau > 0$.

ii- $\{c_i\}$ is said to be a Cauchy sequence, if $0 < \varepsilon < 1$ and there is $i_0 \in \mathbb{N}$ s.t $F_m(c_i, c_j, c_k, \tau) > 1 - \varepsilon$,
 $\forall i, j, k \geq i_0$ and $\tau > 0$.

iii- $(\mathbf{C}, F_m, *)$ is complete, if every Cauchy sequence is convergent in $\mathbf{C}$.

iv- $\{c_i\}$ is called a contractive in $(\mathbf{C}, F_m, *)$, if there is $0 < \beta < 1$ so that

$$\frac{1}{F_m(c_i, c_{i+1}, c_{i+2}, \tau)} - 1 \leq \beta \left(\frac{1}{F_m(c_{i-1}, c_i, c_{i+1}, \tau)} - 1 \right), \quad \text{for } \tau > 0, \text{ and } i \geq 1.$$

Lemma 2.1. [5] Let $(\mathbf{C}, F_m, *)$ is a FM-space. Let $c \in \mathbf{C}$ and $\{c_i\}$ be a sequence in $\mathbf{C}$. Then $\{c_i\}$ converges to a point c iff $\lim_{i \rightarrow \infty} F_m(c_i, c_i, c, \tau) = 1$, for $\tau > 0$.

Definition 2.4. [22] Let $(\mathbf{C}, F_m, *)$ be a GFM-space and a mappings $\Gamma : \mathbf{C} \rightarrow \mathbf{C}$ is called a GF-contraction, if $\exists 0 < a < 1$ so that

$$\frac{1}{F_m(\Gamma c_1, \Gamma c_2, \Gamma c_3, \tau)} - 1 \leq a \left(\frac{1}{F_m(c_1, c_2, c_3, \tau)} - 1 \right), \quad \forall c_1, c_2, c_3 \in \mathbf{C} \text{ and } \tau > 0. \quad (2.1)$$

Proposition 2.1. [22] Let $(\mathbf{C}, F_m, *)$ be a GFM-space. Then, for all $c_1, c_2, c_3 \in \mathbf{C}$ and $\tau > 0$,

$$\begin{aligned} i- & \frac{1}{F_m(c_1, c_1, c_2, \tau)} - 1 \leq \frac{1}{F_m(c_1, c_2, c_3, \tau)} - 1 \text{ with } c_3 \neq c_1. \\ ii- & \frac{1}{F_m(c_1, c_2, c_2, \tau)} - 1 \leq 2 \left(\frac{1}{F_m(c_2, c_1, c_1, \tau)} - 1 \right). \\ iii- & \frac{1}{F_m(c_1, c_2, c_3, \tau)} - 1 \leq \left(\frac{1}{F_m(c_1, c_2, c_2, \tau)} - 1 \right) + \left(\frac{1}{F_m(c_1, c_3, c_3, \tau)} - 1 \right). \end{aligned}$$

Definition 2.5. [22] Let $(\mathbf{C}, F_m, *)$ be a GFM-space. The GFM " F_m " is rectangular if,

$$\frac{1}{F_m(c_1, c_2, c_3, \tau)} - 1 \leq \left(\frac{1}{F_m(c_1, c_2, \tilde{a}, \tau)} - 1 \right) + \left(\frac{1}{F_m(c_2, c_3, \tilde{b}, \tau)} - 1 \right)$$

for all $c_3, c_2, c_1, \tilde{a}, \tilde{b} \in \mathbf{C}$ and $\tau > 0$.

Lemma 2.2. [22] Prove that a GFM " F_m " is rectangular.

Definition 2.6. [26] Let $\mathbf{C} \neq \emptyset$ be a non-empty set. An element $(c, d) \in \mathbf{C} \times \mathbf{C}$ is called a coupled-FP of a coupled-mapping $\Gamma : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$ if: $\Gamma(c, d) = c$ and $\Gamma(d, c) = d$ for $d, c \in \mathbf{C}$.

In this paper, we investigate the use of graph theory in GFM-spaces using coupled contraction conditions. We establish unique coupled FP results in GFM-spaces with an illustrative unique coupled FP numerical example. Additionally, we provide a 3-dimensional graph based on numerical example to illustrate the theory of coupled contraction conditions in GFM-spaces. Further to support our theory, we also demonstrate an application of UTIEs for the existence of unique coupled solutions to unify our findings.

3. MAIN RESULTS

Theorem 3.1. Let $(\mathbf{C}, F_m, *)$ be a complete GFM-space in which F_m is rectangular. A coupled-mapping $\Gamma : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$ satisfies,

$$\frac{1}{F_m(\Gamma(\ell, g), \Gamma(u, v), \Gamma(p, q), \tau)} - 1 \leq \varrho_1 \left(\frac{1}{F_m(\ell, u, p, \tau)} - 1 \right) + \varrho_2 \left(\frac{1}{\mathbf{M}} - 1 \right), \quad (3.1)$$

where

$$\frac{1}{\mathbf{M}} - 1 = \left(\frac{1}{F_m(\ell, \ell, \Gamma(\ell, g), \tau)} - 1 + \frac{1}{F_m(u, u, \Gamma(u, v), \tau)} - 1 + \frac{1}{F_m(p, p, \Gamma(p, q), \tau)} - 1 \right),$$

for all $\ell, g, u, v, p, q \in \mathbf{C}$, $\tau > 0$, and $\varrho_1, \varrho_2 \in [0, 1)$ with $(\varrho_1 + 3\varrho_2) < 1$. Then, Γ has a unique coupled FP in $\mathbf{C}$.

Proof. Let $\ell_0, g_0 \in \mathbf{C}$ be the arbitrary points, let $\{\ell_\gamma\}$ and $\{g_\gamma\}$ be the sequences in $\mathbf{C}$ such that

$$\ell_{\gamma+1} = \Gamma(\ell_\gamma, g_\gamma) \quad \text{and} \quad g_{\gamma+1} = \Gamma(g_\gamma, \ell_\gamma), \quad \text{for } \gamma \geq 0. \quad (3.2)$$

Now, by view of (3.1) and (3.2), for $\tau > 0$, we have

$$\begin{aligned} \frac{1}{F_m(\ell_{\gamma+1}, \ell_{\gamma+1}, \ell_{\gamma+2}, \tau)} - 1 &= \frac{1}{F_m(\Gamma(\ell_\gamma, g_\gamma), \Gamma(\ell_\gamma, g_\gamma), \Gamma(\ell_{\gamma+1}, g_{\gamma+1}), \tau)} - 1 \\ &\leq \varrho_1 \left(\frac{1}{F_m(\ell_\gamma, \ell_\gamma, \Gamma(\ell_\gamma, g_\gamma), \tau)} - 1 \right) + \varrho_2 \left(\frac{1}{\mathbf{M}} - 1 \right), \end{aligned}$$

where

$$\begin{aligned} \frac{1}{\mathbf{M}} - 1 &= \left(\frac{1}{F_m(\ell_\gamma, \ell_\gamma, \Gamma(\ell_\gamma, g_\gamma), \tau)} - 1 + \frac{1}{F_m(\ell_\gamma, \ell_\gamma, \Gamma(\ell_\gamma, g_\gamma), \tau)} - 1 + \frac{1}{F_m(\ell_{\gamma+1}, \ell_{\gamma+1}, \Gamma(\ell_{\gamma+1}, g_{\gamma+1}), \tau)} - 1 \right) \\ &= 2 \left(\frac{1}{F_m(\ell_\gamma, \ell_\gamma, \ell_{\gamma+1}, \tau)} - 1 \right) + \left(\frac{1}{F_m(\ell_{\gamma+1}, \ell_{\gamma+1}, \ell_{\gamma+2}, \tau)} - 1 \right). \end{aligned}$$

After routine calculation, for $\tau > 0$, we have

$$\frac{1}{F_m(\ell_{\gamma+1}, \ell_{\gamma+1}, \ell_{\gamma+2}, \tau)} - 1 = \delta \left(\frac{1}{F_m(\ell_\gamma, \ell_\gamma, \ell_{\gamma+1}, \tau)} - 1 \right), \quad \text{where } \delta = \frac{\varrho_1 + 2\varrho_2}{1 - \varrho_2} < 1. \quad (3.3)$$

Now again by view of (3.1) and (3.2), for $\tau > 0$, we have

$$\begin{aligned} \frac{1}{F_m(\ell_\gamma, \ell_\gamma, \ell_{\gamma+1}, \tau)} - 1 &= \frac{1}{F_m(\Gamma(\ell_{\gamma-1}, g_{\gamma-1}), \Gamma(\ell_{\gamma-1}, g_{\gamma-1}), \Gamma(\ell_\gamma, g_\gamma), \tau)} - 1 \\ &\leq \varrho_1 \left(\frac{1}{F_m(\ell_{\gamma-1}, \ell_{\gamma-1}, \ell_\gamma, \tau)} - 1 \right) + \varrho_2 \left(\frac{1}{\mathbf{M}} - 1 \right), \end{aligned}$$

where

$$\frac{1}{\mathbf{M}} - 1 = 2 \left(\frac{1}{F_m(\ell_{\gamma-1}, \ell_{\gamma-1}, \ell_\gamma, \tau)} - 1 \right) + \left(\frac{1}{F_m(\ell_\gamma, \ell_\gamma, \ell_{\gamma+1}, \tau)} - 1 \right).$$

After routine calculation, we have

$$\frac{1}{F_m(\ell_\gamma, \ell_\gamma, \ell_{\gamma+1}, \tau)} - 1 = \delta \left(\frac{1}{F_m(\ell_{\gamma-1}, \ell_{\gamma-1}, \ell_\gamma, \tau)} - 1 \right), \quad \text{where } \delta = \frac{\varrho_1 + 2\varrho_2}{1 - \varrho_2} < 1. \quad (3.4)$$

Now, by view of (3.3) and (3.4) and then by induction, for $\tau \gg 0$, we have that

$$\begin{aligned} \frac{1}{F_m(\ell_{\gamma+1}, \ell_{\gamma+1}, \ell_{\gamma+2}, \tau)} - 1 &\leq \delta \left(\frac{1}{F_m(\ell_\gamma, \ell_\gamma, \ell_{\gamma+1}, \tau)} - 1 \right) \\ &\leq \delta^2 \left(\frac{1}{F_m(\ell_{\gamma-1}, \ell_{\gamma-1}, \ell_\gamma, \tau)} - 1 \right) \leq \dots \leq \delta^{\gamma+1} \left(\frac{1}{F_m(\ell_0, \ell_0, \ell_1, \tau)} - 1 \right) \rightarrow 0, \quad \text{as } \gamma \rightarrow \infty. \end{aligned}$$

Hence, we obtain

$$\lim_{\gamma \rightarrow \infty} F_m(\ell_{\gamma+1}, \ell_{\gamma+1}, \ell_{\gamma+2}, \tau) = 1, \quad \text{for } \tau > 0.$$

Since F_m is rectangular and $j > \gamma$, we have

$$\begin{aligned} \frac{1}{F_m(\ell_\gamma, \ell_\gamma, \ell_j, \tau)} - 1 &\leq \left(\frac{1}{F_m(\ell_\gamma, \ell_\gamma, \ell_{\gamma+1}, \tau)} - 1 \right) + \left(\frac{1}{F_m(\ell_{\gamma+1}, \ell_{\gamma+1}, \ell_{\gamma+2}, \tau)} - 1 \right) + \dots + \left(\frac{1}{F_m(\ell_{j-1}, \ell_{j-1}, \ell_j, \tau)} - 1 \right) \\ &\leq (\delta^\gamma + \delta^{\gamma+1} + \dots + \delta^{j-1}) \left(\frac{1}{F_m(\ell_0, \ell_0, \ell_1, \tau)} - 1 \right) = \frac{\delta^\gamma}{1 - \delta} \left(\frac{1}{F_m(\ell_0, \ell_0, \ell_1, \tau)} - 1 \right) \rightarrow 0, \quad \text{as } \gamma \rightarrow \infty. \end{aligned}$$

By Proposition 2.1, for $\tau > 0$, and $\gamma, j, s \in \mathbb{N}$ with $s > j > \gamma$, we have

$$\frac{1}{F_m(\ell_\gamma, \ell_j, \ell_s, \tau)} - 1 \leq \left(\frac{1}{F_m(\ell_\gamma, \ell_\gamma, \ell_j, \tau)} - 1 \right) + \left(\frac{1}{F_m(\ell_s, \ell_s, \ell_j, \tau)} - 1 \right).$$

By taking $\gamma, j, s \rightarrow \infty$, we obtain $\lim_{\gamma, j, s \rightarrow \infty} F_m(\ell_\gamma, \ell_j, \ell_s, \tau) = 1$, for $\tau > 0$. Hence, it is proven that $\{\ell_\gamma\}$ is a Cauchy sequence in $(\mathbf{C}, F_m, *)$. Similarly, for a sequence $\{\mathfrak{g}_\gamma\}_{\gamma \geq 0}$ and by using the same arguments as for a sequences $\{\ell_\gamma\}$. Then, from (3.1) and after routine calculation, for $\tau > 0$, we get

$$\frac{1}{F_m(\mathfrak{g}_{\gamma+1}, \mathfrak{g}_{\gamma+1}, \mathfrak{g}_{\gamma+2}, \tau)} - 1 = \delta \left(\frac{1}{F_m(\mathfrak{g}_\gamma, \mathfrak{g}_\gamma, \mathfrak{g}_{\gamma+1}, \tau)} - 1 \right), \quad \text{where } \delta = \frac{\varrho_1 + 2\varrho_2}{1 - \varrho_2} < 1. \quad (3.5)$$

Similarly, for $\tau > 0$, we have

$$\frac{1}{F_m(\mathfrak{g}_\gamma, \mathfrak{g}_\gamma, \mathfrak{g}_{\gamma+1}, \tau)} - 1 = \delta \left(\frac{1}{F_m(\mathfrak{g}_{\gamma-1}, \mathfrak{g}_{\gamma-1}, \mathfrak{g}_\gamma, \tau)} - 1 \right), \quad \text{where } \delta = \frac{\varrho_1 + 2\varrho_2}{1 - \varrho_2} < 1. \quad (3.6)$$

Now, from (3.5) and (3.6), then by induction for $\tau \gg 0$, we have that

$$\begin{aligned} \frac{1}{F_m(\mathfrak{g}_{\gamma+1}, \mathfrak{g}_{\gamma+1}, \mathfrak{g}_{\gamma+2}, \tau)} - 1 &\leq \delta \left(\frac{1}{F_m(\mathfrak{g}_\gamma, \mathfrak{g}_\gamma, \mathfrak{g}_{\gamma+1}, \tau)} - 1 \right) \\ &\leq \delta^2 \left(\frac{1}{F_m(\mathfrak{g}_{\gamma-1}, \mathfrak{g}_{\gamma-1}, \mathfrak{g}_\gamma, \tau)} - 1 \right) \leq \dots \leq \delta^{\gamma+1} \left(\frac{1}{F_m(\mathfrak{g}_0, \mathfrak{g}_0, \mathfrak{g}_1, \tau)} - 1 \right) \rightarrow 0, \quad \text{as } \gamma \rightarrow \infty. \end{aligned}$$

Hence, we obtain

$$\lim_{\gamma \rightarrow \infty} F_m(\mathfrak{g}_{\gamma+1}, \mathfrak{g}_{\gamma+1}, \mathfrak{g}_{\gamma+2}, \tau) = 1, \quad \text{for } \tau > 0.$$

Since F_m is rectangular, for $j > \gamma$ and for $\tau > 0$,

$$\begin{aligned} & \frac{1}{F_m(g_\gamma, g_\gamma, g_j, \tau)} - 1 \\ & \leq \left(\frac{1}{F_m(g_\gamma, g_\gamma, g_{\gamma+1}, \tau)} - 1 \right) + \left(\frac{1}{F_m(g_{\gamma+1}, g_{\gamma+1}, g_{\gamma+2}, \tau)} - 1 \right) + \cdots + \left(\frac{1}{F_m(g_{j-1}, g_{j-1}, g_j, \tau)} - 1 \right) \\ & \leq (\delta^\gamma + \delta^{\gamma+1} + \cdots + \delta^{j-1}) \left(\frac{1}{F_m(g_0, g_0, g_1, \tau)} - 1 \right) = \frac{\delta^\gamma}{1 - \delta} \left(\frac{1}{F_m(g_0, g_0, g_1, \tau)} - 1 \right) \rightarrow 0, \quad \text{as } \gamma \rightarrow \infty. \end{aligned}$$

By Proposition 2.1, for $\tau > 0$, and $\gamma, j, s \in \mathbb{N}$ with $s > j > \gamma$, we have

$$\frac{1}{F_m(g_\gamma, g_j, g_s, \tau)} - 1 \leq \left(\frac{1}{F_m(g_\gamma, g_\gamma, g_j, \tau)} - 1 \right) + \left(\frac{1}{F_m(g_s, g_s, g_j, \tau)} - 1 \right).$$

By taking $\gamma, j, s \rightarrow \infty$, we obtain $\lim_{\gamma, j, s \rightarrow \infty} F_m(g_\gamma, g_j, g_s, \tau) = 1$, for $\tau > 0$. Hence, it is proved that both sequences $\{\ell_\gamma\}$ and $\{g_\gamma\}$ are Cauchy sequences in $(\mathbf{C}, F_m, *)$. Since, by the completeness of $(\mathbf{C}, F_m, *)$, $\exists \chi, \delta \in \mathbf{C}$ such that $\ell_\gamma \rightarrow \chi$ and $g_\gamma \rightarrow \delta$ as $\gamma \rightarrow \infty$, or equivalently $\lim_{\gamma \rightarrow \infty} \ell_\gamma = \chi$ and $\lim_{\gamma \rightarrow \infty} g_\gamma = \delta$.

Now we have to prove that $\Gamma(\chi, \delta) = \chi$. Then, from (3.1), for $\tau > 0$, and by using the rectangular property and symmetric property of F_m , we have that

$$\begin{aligned} & \frac{1}{F_m(\Gamma(\chi, \delta), \chi, \chi, \tau)} - 1 \leq \left(\frac{1}{F_m(\Gamma(\chi, \delta), \ell_{\gamma+1}, \ell_{\gamma+1}, \tau)} - 1 \right) + \left(\frac{1}{F_m(\ell_{\gamma+1}, \chi, \chi, \tau)} - 1 \right) \\ & = \left(\frac{1}{F_m(\Gamma(\chi, \delta), \Gamma(\ell_\gamma, g_\gamma), \Gamma(\ell_\gamma, g_\gamma), \tau)} - 1 \right) + \left(\frac{1}{F_m(\ell_{\gamma+1}, \chi, \chi, \tau)} - 1 \right) \quad (3.7) \\ & \leq \varrho_1 \left(\frac{1}{F_m(\chi, \ell_\gamma, \ell_\gamma, \tau)} - 1 \right) + \varrho_2 \left(\frac{1}{\mathbf{M}} - 1 \right) + \left(\frac{1}{F_m(\ell_{\gamma+1}, \chi, \chi, \tau)} - 1 \right), \end{aligned}$$

where

$$\begin{aligned} \frac{1}{\mathbf{M}} - 1 &= \left(\frac{1}{F_m(\chi, \chi, \Gamma(\chi, \delta), \tau)} - 1 + \frac{1}{F_m(\ell_\gamma, \ell_\gamma, \Gamma(\ell_\gamma, g_\gamma), \tau)} - 1 + \frac{1}{F_m(\ell_\gamma, \ell_\gamma, \Gamma(\ell_\gamma, g_\gamma), \tau)} - 1 \right) \\ &= \left(\frac{1}{F_m(\chi, \chi, \Gamma(\chi, \delta), \tau)} - 1 + \frac{1}{F_m(\ell_\gamma, \ell_\gamma, \ell_{\gamma+1}, \tau)} - 1 + \frac{1}{F_m(\ell_\gamma, \ell_\gamma, \ell_{\gamma+1}, \tau)} - 1 \right) \\ &\rightarrow \left(\frac{1}{F_m(\chi, \chi, \Gamma(\chi, \delta), \tau)} - 1 \right), \quad \text{as } \gamma \rightarrow \infty. \end{aligned}$$

Also, by taking limit $\gamma \rightarrow \infty$ of (3.7), for $\tau > 0$, we obtain

$$\frac{1}{F_m(\Gamma(\chi, \delta), \chi, \chi, \tau)} - 1 \leq \varrho_2 \left(\frac{1}{F_m(\Gamma(\chi, \delta), \chi, \chi, \tau)} - 1 \right) \Rightarrow (1 - \varrho_2) \left(\frac{1}{F_m(\Gamma(\chi, \delta), \chi, \chi, \tau)} - 1 \right) \leq 0.$$

Notice that $(1 - \varrho_2) \neq 0$, therefore, we obtain $F_m(\Gamma(\chi, \delta), \chi, \chi, \tau) = 1 \Rightarrow \Gamma(\chi, \delta) = \chi$, for $\tau > 0$. Similarly, it is easy to show that $\Gamma(\delta, \chi) = \delta$. Hence we get that $(\Gamma(\chi, \delta), \Gamma(\delta, \chi)) = (\chi, \delta)$. This shows that (χ, δ) is a coupled FP of a mapping Γ in GFM-space $(\mathbf{C}, F_m, *)$.

Uniqueness: Let $(\chi', \delta') \in \mathbf{C}$ be the other coupled FP of a mapping Γ in $\mathbf{C}$, such that $\Gamma(\chi', \delta') = \chi'$

and $\Gamma(\delta', \chi') = \delta'$. Then, first, we have to prove that $\chi = \chi'$. Now, by the view of (3.1), for $\tau > 0$, we have

$$\begin{aligned} \frac{1}{F_m(\chi, \chi, \chi', \tau)} - 1 &= \frac{1}{F_m(\Gamma(\chi, g^*), \Gamma(\chi, g^*), \Gamma(\chi', g'), \tau)} - 1 \\ &\leq \varrho_1 \left(\frac{1}{F_m(\chi, \chi, \chi', \tau)} - 1 \right) + \varrho_2 \left(\frac{1}{\mathbf{M}} - 1 \right), \end{aligned}$$

where

$$\begin{aligned} \frac{1}{\mathbf{M}} - 1 &= \left(\frac{1}{F_m(\chi, \chi, \Gamma(\chi, g^*), \tau)} - 1 + \frac{1}{F_m(\chi, \chi, \Gamma(\chi, g^*), \tau)} - 1 + \frac{1}{F_m(\chi', \chi', \Gamma(\chi', g'), \tau)} - 1 \right) \\ &= \left(\frac{1}{F_m(\chi, \chi, \chi, \tau)} - 1 + \frac{1}{F_m(\chi, \chi, \chi, \tau)} - 1 + \frac{1}{F_m(\chi', \chi', \chi', \tau)} - 1 \right) = 0. \end{aligned}$$

Hence, for $\tau > 0$, we obtained

$$\frac{1}{F_m(\chi, \chi, \chi', \tau)} - 1 \leq \varrho_1 \left(\frac{1}{F_m(\chi, \chi, \chi', \tau)} - 1 \right) \Rightarrow (1 - \varrho_1) \left(\frac{1}{F_m(\chi, \chi, \chi', \tau)} - 1 \right) \leq 0.$$

As $(1 - \varrho_1) \neq 0$, therefore $F_m(\chi, \chi, \chi', \tau) = 1$ implies that $\chi = \chi'$, for $\tau > 0$. Next, we prove that $\delta = \delta'$, then, by (3.1), for $\tau > 0$, we've

$$\frac{1}{F_m(\delta, \delta, \delta', \tau)} - 1 = \frac{1}{F_m(\Gamma(\delta, \chi), \Gamma(\delta, \chi), \Gamma(\delta', \chi'), \tau)} - 1 \leq \varrho_1 \left(\frac{1}{F_m(\delta, \delta, \delta', \tau)} - 1 \right) + \varrho_2 \left(\frac{1}{\mathbf{M}} - 1 \right),$$

where

$$\begin{aligned} \frac{1}{\mathbf{M}} - 1 &= \left(\frac{1}{F_m(\delta, \delta, \Gamma(\delta, \chi), \tau)} - 1 + \frac{1}{F_m(\delta, \delta, \Gamma(\delta, \chi), \tau)} - 1 + \frac{1}{F_m(\delta', \delta', \Gamma(\delta', \chi'), \tau)} - 1 \right) \\ &= \left(\frac{1}{F_m(\delta, \delta, \delta, \tau)} - 1 + \frac{1}{F_m(\delta, \delta, \delta, \tau)} - 1 + \frac{1}{F_m(\delta', \delta', \delta', \tau)} - 1 \right) = 0. \end{aligned}$$

Hence, for $\tau > 0$, we obtained

$$\frac{1}{F_m(\delta, \delta, \delta', \tau)} - 1 \leq \varrho_1 \left(\frac{1}{F_m(\delta, \delta, \delta', \tau)} - 1 \right) \Rightarrow (1 - \varrho_1) \left(\frac{1}{F_m(\delta, \delta, \delta', \tau)} - 1 \right) \leq 0.$$

As $(1 - \varrho_1) \neq 0$, therefore, $F_m(\delta, \delta, \delta', \tau) = 1$ implies that $\delta = \delta'$, for $\tau > 0$. Hence, we get that $(\chi, \delta) = (\chi', \delta')$. Thus, the mapping Γ has a unique coupled FP in $\mathbf{C}$.

Theorem 3.2. Let $(\mathbf{C}, F_m, *)$ be a complete GFM-space in which F_m is rectangular. A coupled-mapping $\Gamma : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$ satisfies,

$$\frac{1}{F_m(\Gamma(\ell, g), \Gamma(u, v), \Gamma(p, q), \tau)} - 1 \leq \varrho_1 \left(\frac{1}{F_m(\ell, u, p, \tau)} - 1 \right) + \varrho_2 \left(\frac{1}{\mathbf{M}} - 1 \right) + \varrho_3 \left(\frac{1}{\mathbb{L}} - 1 \right), \quad (3.8)$$

where

$$\frac{1}{\mathbf{M}} - 1 = \left(\frac{1}{F_m(\ell, \ell, \Gamma(\ell, g), \tau)} - 1 + \frac{1}{F_m(u, u, \Gamma(u, v), \tau)} - 1 + \frac{1}{F_m(p, p, \Gamma(p, q), \tau)} - 1 \right),$$

and,

$$\mathbb{L} = \min \left\{ \begin{array}{l} F_m(\ell, \ell, \Gamma(\ell, g), \tau), F_m(u, u, \Gamma(u, v), \tau), F_m(p, p, \Gamma(p, q), \tau), \\ F_m(\ell, \ell, \Gamma(u, v), \tau), F_m(u, u, \Gamma(p, q), \tau), F_m(p, p, \Gamma(\ell, g), \tau) \end{array} \right\},$$

for all $\ell, g, u, v, p, q \in \mathbf{C}$, $\tau > 0$, and $\varrho_1, \varrho_2, \varrho_3 \in [0, 1)$ with $(\varrho_1 + 3\varrho_2 + 2\varrho_3) < 1$. Then, Γ has a unique coupled FP in $\mathbf{C}$.

Proof: Let $\ell_0, g_0 \in \mathbf{C}$ be the arbitrary points, let $\{\ell_\gamma\}$ and $\{g_\gamma\}$ be the sequences in $\mathbf{C}$ such that

$$\ell_{\gamma+1} = \Gamma(\ell_\gamma, g_\gamma) \quad \text{and} \quad g_{\gamma+1} = \Gamma(g_\gamma, \ell_\gamma) \quad \text{for } i \geq 0. \quad (3.9)$$

Now by view of inequality (3.8) and (3.9), we have

$$\begin{aligned} \frac{1}{F_m(\ell_{\gamma+1}, \ell_{\gamma+1}, \ell_{\gamma+2}, \tau)} - 1 &= \frac{1}{F_m(\Gamma(\ell_\gamma, g_\gamma), \Gamma(\ell_\gamma, g_\gamma), \Gamma(\ell_{\gamma+1}, g_{\gamma+1}), \tau)} - 1 \\ &\leq \varrho_1 \left(\frac{1}{F_m(\ell_\gamma, \ell_\gamma, \ell_{\gamma+1}, \tau)} - 1 \right) + \varrho_2 \left(\frac{1}{\mathbf{M}} - 1 \right) + \varrho_3 \left(\frac{1}{\mathbf{L}} - 1 \right), \end{aligned} \quad (3.10)$$

where,

$$\frac{1}{\mathbf{M}} - 1 = 2 \left(\frac{1}{F_m(\ell_\gamma, \ell_\gamma, \ell_{\gamma+1}, \tau)} - 1 \right) + \left(\frac{1}{F_m(\ell_{\gamma+1}, \ell_{\gamma+1}, \ell_{\gamma+2}, \tau)} - 1 \right).$$

And

$$\begin{aligned} \mathbf{L} &= \min \left\{ F_m(\ell_\gamma, \ell_\gamma, \Gamma(\ell_\gamma, g_\gamma), \tau), F_m(\ell_\gamma, \ell_\gamma, \Gamma(\ell_\gamma, g_\gamma), \tau), F_m(\ell_{\gamma+1}, \ell_{\gamma+1}, \Gamma(\ell_{\gamma+1}, g_{\gamma+1}), \tau), \right. \\ &\quad \left. F_m(\ell_\gamma, \ell_\gamma, \Gamma(\ell_\gamma, g_\gamma), \tau), F_m(\ell_\gamma, \ell_\gamma, \Gamma(\ell_{\gamma+1}, g_{\gamma+1}), \tau), F_m(\ell_{\gamma+1}, \ell_{\gamma+1}, \Gamma(\ell_\gamma, g_\gamma), \tau) \right\} \\ &= \min \left\{ F_m(\ell_\gamma, \ell_\gamma, \ell_{\gamma+1}, \tau), F_m(\ell_\gamma, \ell_\gamma, \ell_{\gamma+1}, \tau), F_m(\ell_{\gamma+1}, \ell_{\gamma+1}, \ell_{\gamma+2}, \tau), \right. \\ &\quad \left. F_m(\ell_\gamma, \ell_\gamma, \ell_{\gamma+1}, \tau), F_m(\ell_\gamma, \ell_\gamma, \ell_{\gamma+2}, \tau), F_m(\ell_{\gamma+1}, \ell_{\gamma+1}, \ell_{\gamma+1}, \tau) \right\}. \end{aligned}$$

Hence, we obtain

$$\mathbf{L} = \min \{ F_m(\ell_\gamma, \ell_\gamma, \ell_{\gamma+1}, \tau), F_m(\ell_{\gamma+1}, \ell_{\gamma+1}, \ell_{\gamma+2}, \tau), F_m(\ell_\gamma, \ell_\gamma, \ell_{\gamma+2}, \tau) \}, \quad \text{for } \tau > 0. \quad (3.11)$$

Now, we have the following three-cases:

- i- If $F_m(\ell_\gamma, \ell_\gamma, \ell_{\gamma+1}, \tau)$ is the minimum term in (3.11). Then, from (3.10) and after simple routine calculation, for $\tau > 0$, we obtain

$$\begin{aligned} \frac{1}{F_m(\ell_{\gamma+1}, \ell_{\gamma+1}, \ell_{\gamma+2}, \tau)} - 1 &= \frac{1}{F_m(\Gamma(\ell_\gamma, g_\gamma), \Gamma(\ell_\gamma, g_\gamma), \Gamma(\ell_{\gamma+1}, g_{\gamma+1}), \tau)} - 1 \\ &\leq (\varrho_1 + 2\varrho_2 + \varrho_3) \left(\frac{1}{F_m(\ell_\gamma, \ell_\gamma, \ell_{\gamma+1}, \tau)} - 1 \right) + \varrho_2 \left(\frac{1}{F_m(\ell_{\gamma+1}, \ell_{\gamma+1}, \ell_{\gamma+2}, \tau)} - 1 \right). \end{aligned}$$

Hence, for $\tau > 0$, we obtain

$$\frac{1}{F_m(\ell_{\gamma+1}, \ell_{\gamma+1}, \ell_{\gamma+2}, \tau)} - 1 \leq \left(\frac{\varrho_1 + 2\varrho_2 + \varrho_3}{1 - \varrho_2} \right) \left(\frac{1}{F_m(\ell_\gamma, \ell_\gamma, \ell_{\gamma+1}, \tau)} - 1 \right). \quad (3.12)$$

- ii- If $F_m(\ell_{\gamma+1}, \ell_{\gamma+1}, \ell_{\gamma+2}, \tau)$ is the minimum term in (3.11). Then, from (3.10) and after simple routine calculation, for $\tau > 0$, we obtain

$$\begin{aligned} \frac{1}{F_m(\ell_{\gamma+1}, \ell_{\gamma+1}, \ell_{\gamma+2}, \tau)} - 1 &= \frac{1}{F_m(\Gamma(\ell_\gamma, g_\gamma), \Gamma(\ell_\gamma, g_\gamma), \Gamma(\ell_{\gamma+1}, g_{\gamma+1}), \tau)} - 1 \\ &\leq (\varrho_1 + 2\varrho_2) \left(\frac{1}{F_m(\ell_\gamma, \ell_\gamma, \ell_{\gamma+1}, \tau)} - 1 \right) + (\varrho_2 + \varrho_3) \left(\frac{1}{F_m(\ell_{\gamma+1}, \ell_{\gamma+1}, \ell_{\gamma+2}, \tau)} - 1 \right). \end{aligned}$$

Hence, for $\tau > 0$, we obtain

$$\frac{1}{F_m(\ell_{\gamma+1}, \ell_{\gamma+1}, \ell_{\gamma+2}, \tau)} - 1 \leq \left(\frac{\varrho_1 + 2\varrho_2}{1 - \varrho_2 - \varrho_3} \right) \left(\frac{1}{F_m(\ell_\gamma, \ell_\gamma, \ell_{\gamma+1}, \tau)} - 1 \right). \quad (3.13)$$

iii- If $F_m(\ell_\gamma, \ell_\gamma, \ell_{\gamma+2}, \tau)$ is the minimum term in (3.11). Then, from (3.10) and after simple routine calculation using Proposition 2.1, for $\tau > 0$, we obtain

$$\begin{aligned} \frac{1}{F_m(\ell_{\gamma+1}, \ell_{\gamma+1}, \ell_{\gamma+2}, \tau)} - 1 &= \frac{1}{F_m(\Gamma(\ell_\gamma, \mathfrak{g}_\gamma), \Gamma(\ell_\gamma, \mathfrak{g}_\gamma), \Gamma(\ell_{\gamma+1}, \mathfrak{g}_{\gamma+1}), \tau)} - 1 \\ &\leq (\varrho_1 + 2\varrho_2 + \varrho_3) \left(\frac{1}{F_m(\ell_\gamma, \ell_\gamma, \ell_{\gamma+1}, \tau)} - 1 \right) + (\varrho_2 + \varrho_3) \left(\frac{1}{F_m(\ell_{\gamma+1}, \ell_{\gamma+1}, \ell_{\gamma+2}, \tau)} - 1 \right). \end{aligned}$$

Hence, for $\tau > 0$, we obtain

$$\frac{1}{F_m(\ell_{\gamma+1}, \ell_{\gamma+1}, \ell_{\gamma+2}, \tau)} - 1 \leq \left(\frac{\varrho_1 + 2\varrho_2 + \varrho_3}{1 - \varrho_2 - \varrho_3} \right) \left(\frac{1}{F_m(\ell_\gamma, \ell_\gamma, \ell_{\gamma+1}, \tau)} - 1 \right). \quad (3.14)$$

Let, we define $\delta := \max \left\{ \left(\frac{\varrho_1 + 2\varrho_2 + \varrho_3}{1 - \varrho_2 - \varrho_3} \right), \left(\frac{\varrho_1 + 2\varrho_2}{1 - \varrho_2 - \varrho_3} \right), \left(\frac{\varrho_1 + 2\varrho_2 + \varrho_3}{1 - \varrho_2 - \varrho_3} \right) \right\} < 1$. Then, from (3.12), (3.13), and (3.14), we obtain

$$\frac{1}{F_m(\ell_{\gamma+1}, \ell_{\gamma+1}, \ell_{\gamma+2}, \tau)} - 1 \leq \delta \left(\frac{1}{F_m(\ell_\gamma, \ell_\gamma, \ell_{\gamma+1}, \tau)} - 1 \right), \quad \text{for } \tau > 0. \quad (3.15)$$

Correspondingly, once again by referring (3.9) and (3.10) and using the rectangular property of F_m , then, after a straightforward calculation, for $\tau > 0$, we get

$$\frac{1}{F_m(\ell_\gamma, \ell_\gamma, \ell_{\gamma+1}, \tau)} - 1 \leq \delta \left(\frac{1}{F_m(\ell_{\gamma-1}, \ell_{\gamma-1}, \ell_\gamma, \tau)} - 1 \right), \quad \text{for } \tau > 0. \quad (3.16)$$

From (3.15) and (3.16) and then by induction, we have

$$\begin{aligned} \frac{1}{F_m(\ell_{\gamma+1}, \ell_{\gamma+1}, \ell_{\gamma+2}, \tau)} - 1 &\leq \delta \left(\frac{1}{F_m(\ell_\gamma, \ell_\gamma, \ell_{\gamma+1}, \tau)} - 1 \right) \\ &\leq \delta^2 \left(\frac{1}{F_m(\ell_{\gamma-1}, \ell_{\gamma-1}, \ell_\gamma, \tau)} - 1 \right) \leq \dots \leq \delta^{\gamma+1} \left(\frac{1}{F_m(\ell_1, \ell_1, \ell_2, \tau)} - 1 \right) \rightarrow 0, \quad \text{as } \gamma \rightarrow \infty. \end{aligned}$$

Hence, we obtain

$$\lim_{\gamma \rightarrow \infty} F_m(\ell_\gamma, \ell_\gamma, \ell_{\gamma+1}, \tau) = 1, \quad \text{for } \tau > 0.$$

Since F_m is rectangular and $\gamma > j$, we have

$$\begin{aligned} \frac{1}{F_m(\ell_\gamma, \ell_\gamma, \ell_j, \tau)} - 1 &\leq \left(\frac{1}{F_m(\ell_\gamma, \ell_\gamma, \ell_{\gamma+1}, \tau)} - 1 \right) + \left(\frac{1}{F_m(\ell_{\gamma+1}, \ell_{\gamma+1}, \ell_{\gamma+2}, \tau)} - 1 \right) + \dots + \left(\frac{1}{F_m(\ell_{j-1}, \ell_{j-1}, \ell_j, \tau)} - 1 \right) \\ &\leq (\delta^\gamma + \delta^{\gamma+1} + \dots + \delta^{j-1}) \left(\frac{1}{F_m(\ell_0, \ell_0, \ell_1, \tau)} - 1 \right) = \frac{\delta^\gamma}{1 - \delta} \left(\frac{1}{F_m(\ell_0, \ell_0, \ell_1, \tau)} - 1 \right) \rightarrow 0, \quad \text{as } \gamma \rightarrow \infty. \end{aligned}$$

By Proposition 2.1, for $\tau > 0$, and $\gamma, j, s \in \mathbb{N}$ with $s > j > \gamma$, we have

$$\frac{1}{F_m(\ell_\gamma, \ell_j, \ell_s, \tau)} - 1 \leq \left(\frac{1}{F_m(\ell_\gamma, \ell_\gamma, \ell_j, \tau)} - 1 \right) + \left(\frac{1}{F_m(\ell_s, \ell_s, \ell_j, \tau)} - 1 \right).$$

By taking $\gamma, j, s \rightarrow \infty$, we obtain $\lim_{\gamma, j, s \rightarrow \infty} F_m(\ell_\gamma, \ell_j, \ell_s, \tau) = 1$, for $\tau > 0$. Hence, it is proven that $\{\ell_\gamma\}$ is a Cauchy sequence in $(\mathbf{C}, F_m, *)$. Similarly, for a sequence $\{g_\gamma\}$ and by using the same arguments as for a sequences $\{\ell_\gamma\}$, we get

$$\frac{1}{F_m(g_{\gamma+1}, g_{\gamma+1}, g_{\gamma+2}, \tau)} - 1 \leq \delta \left(\frac{1}{F_m(g_\gamma, g_\gamma, g_{\gamma+1}, \tau)} - 1 \right), \quad \text{for } \tau > 0. \quad (3.17)$$

where $\delta = \max \left\{ \left(\frac{\varrho_1 + 2\varrho_2 + \varrho_3}{1 - \varrho_2} \right), \left(\frac{\varrho_1 + 2\varrho_2}{1 - \varrho_2 - \varrho_3} \right), \left(\frac{\varrho_1 + 2\varrho_2 + \varrho_3}{1 - \varrho_2 - \varrho_3} \right) \right\} < 1$ as same as in (3.15) and (3.16). Similarly,

$$\frac{1}{F_m(g_\gamma, g_\gamma, g_{\gamma+1}, \tau)} - 1 \leq \delta \left(\frac{1}{F_m(g_{\gamma-1}, g_{\gamma-1}, g_\gamma, \tau)} - 1 \right), \quad \text{for } \tau > 0. \quad (3.18)$$

Now, from (3.17) and (3.18), and then by induction, for $\tau \gg 0$, we have that

$$\begin{aligned} \frac{1}{F_m(g_{\gamma+1}, g_{\gamma+1}, g_{\gamma+2}, \tau)} - 1 &\leq \delta \left(\frac{1}{F_m(g_\gamma, g_\gamma, g_{\gamma+1}, \tau)} - 1 \right) \\ &\leq \delta^2 \left(\frac{1}{F_m(g_{\gamma-1}, g_{\gamma-1}, g_\gamma, \tau)} - 1 \right) \leq \dots \leq \delta^{i+1} \left(\frac{1}{F_m(g_0, g_0, g_1, \tau)} - 1 \right) \rightarrow 0, \quad \text{as } i \rightarrow \infty. \end{aligned}$$

Hence, we obtain

$$\lim_{\gamma \rightarrow \infty} F_m(g_{\gamma+1}, g_{\gamma+1}, g_{\gamma+2}, \tau) = 1, \quad \text{for } \tau > 0.$$

Since F_m is rectangular, for $j > \gamma$ and for $\tau > 0$,

$$\begin{aligned} \frac{1}{F_m(g_\gamma, g_\gamma, g_j, \tau)} - 1 &\leq \left(\frac{1}{F_m(g_\gamma, g_\gamma, g_{\gamma+1}, \tau)} - 1 \right) + \left(\frac{1}{F_m(g_{\gamma+1}, g_{\gamma+1}, g_{\gamma+2}, \tau)} - 1 \right) + \dots + \left(\frac{1}{F_m(g_{j-1}, g_{j-1}, g_j, \tau)} - 1 \right) \\ &\leq (\delta^\gamma + \delta^{\gamma+1} + \dots + \delta^{j-1}) \left(\frac{1}{F_m(g_0, g_0, g_1, \tau)} - 1 \right) = \frac{\delta^\gamma}{1 - \delta} \left(\frac{1}{F_m(g_0, g_0, g_1, \tau)} - 1 \right) \rightarrow 0, \quad \text{as } \gamma \rightarrow \infty. \end{aligned}$$

By Proposition 2.1, for $\tau > 0$, and $\gamma, j, s \in \mathbb{N}$ with $s > j > \gamma$, we have

$$\frac{1}{F_m(g_\gamma, g_j, g_s, \tau)} - 1 \leq \left(\frac{1}{F_m(g_\gamma, g_\gamma, g_j, \tau)} - 1 \right) + \left(\frac{1}{F_m(g_s, g_s, g_j, \tau)} - 1 \right).$$

By taking $\gamma, j, s \rightarrow \infty$, we obtain $\lim_{\gamma, j, s \rightarrow \infty} F_m(g_\gamma, g_k, g_s, \tau) = 1$, for $\tau > 0$. Hence, it is proved that the both sequences $\{\ell_\gamma\}$ and $\{g_\gamma\}$ are Cauchy sequences in $(\mathbf{C}, F_m, *)$. Since, by the completeness of $(\mathbf{C}, F_m, *)$, $\exists \chi, \delta \in \mathbf{C}$ such that $\ell_\gamma \rightarrow \chi$ and $g_\gamma \rightarrow \delta$ as $\gamma \rightarrow \infty$, or equivalently $\lim_{\gamma \rightarrow \infty} \ell_\gamma = \chi$ and $\lim_{\gamma \rightarrow \infty} g_\gamma = \delta$.

Now we have to prove that $\Gamma(\chi, \delta) = \chi$. Then, from (3.9) and (3.10), for $\tau > 0$, and by using the rectangular property and symmetric property of F_m , we have that

$$\begin{aligned} \frac{1}{F_m(\Gamma(\chi, \delta), \chi, \chi, \tau)} - 1 &\leq \left(\frac{1}{F_m(\Gamma(\chi, \delta), \ell_{\gamma+1}, \ell_{\gamma+1}, \tau)} - 1 \right) + \left(\frac{1}{F_m(\ell_{\gamma+1}, \chi, \chi, \tau)} - 1 \right) \\ &= \left(\frac{1}{F_m(\Gamma(\chi, \delta), \Gamma(\ell_\gamma, g_\gamma), \Gamma(\ell_\gamma, g_\gamma), \tau)} - 1 \right) + \left(\frac{1}{F_m(\ell_{\gamma+1}, \chi, \chi, \tau)} - 1 \right) \\ &\leq \varrho_1 \left(\frac{1}{F_m(\chi, \ell_\gamma, \ell_\gamma, \tau)} - 1 \right) + \varrho_2 \left(\frac{1}{\mathbf{M}} - 1 \right) + \varrho_3 \left(\frac{1}{\mathbf{L}} - 1 \right) + \left(\frac{1}{F_m(\ell_{\gamma+1}, \chi, \chi, \tau)} - 1 \right), \end{aligned} \quad (3.19)$$

where

$$\begin{aligned} \frac{1}{\mathbf{M}} - 1 &= \left(\frac{1}{F_m(\chi, \chi, \Gamma(\chi, \delta), \tau)} - 1 + \frac{1}{F_m(\ell_\gamma, \ell_\gamma, \Gamma(\ell_\gamma, g_\gamma), \tau)} - 1 + \frac{1}{F_m(\ell_\gamma, \ell_\gamma, \Gamma(\ell_\gamma, g_\gamma), \tau)} - 1 \right) \\ &= \left(\frac{1}{F_m(\chi, \chi, \Gamma(\chi, \delta), \tau)} - 1 + \frac{1}{F_m(\ell_\gamma, \ell_\gamma, \ell_{\gamma+1}, \tau)} - 1 + \frac{1}{F_m(\ell_\gamma, \ell_\gamma, \ell_{\gamma+1}, \tau)} - 1 \right) \\ &\rightarrow \left(\frac{1}{F_m(\chi, \chi, \Gamma(\chi, \delta), \tau)} - 1 \right), \quad \text{as } \gamma \rightarrow \infty. \end{aligned} \quad (3.20)$$

And

$$\begin{aligned} \mathbb{L} &= \min \left\{ \begin{array}{l} F_m(\chi, \chi, \Gamma(\chi, \delta), \tau), F_m(\ell_\gamma, \ell_\gamma, \Gamma(\ell_\gamma, g_\gamma), \tau), F_m(\ell_\gamma, \ell_\gamma, \Gamma(\ell_\gamma, g_\gamma), \tau), \\ F_m(\chi, \chi, \Gamma(\ell_\gamma, g_\gamma), \tau), F_m(\ell_\gamma, \ell_\gamma, \Gamma(\ell_\gamma, g_\gamma), \tau), F_m(\ell_\gamma, \ell_\gamma, \Gamma(\chi, \delta), \tau) \end{array} \right\} \\ &= \min \left\{ \begin{array}{l} F_m(\chi, \chi, \Gamma(\chi, \delta), \tau), F_m(\ell_\gamma, \ell_\gamma, \ell_{\gamma+1}, \tau), F_m(\ell_\gamma, \ell_\gamma, \ell_{\gamma+1}, \tau), \\ F_m(\chi, \chi, \ell_{\gamma+1}, \tau), F_m(\ell_\gamma, \ell_\gamma, \ell_{\gamma+1}, \tau), F_m(\ell_\gamma, \ell_\gamma, \Gamma(\chi, \delta), \tau) \end{array} \right\} \\ &\rightarrow F_m(\chi, \chi, \Gamma(\chi, \delta), \tau), \quad \text{as } \gamma \rightarrow \infty. \end{aligned} \quad (3.21)$$

Now, from (3.19), (3.20), and (3.21), and taking limit $\gamma \rightarrow \infty$, we get

$$\begin{aligned} \frac{1}{F_m(\Gamma(\chi, \delta), \chi, \chi, \tau)} - 1 &\leq (\varrho_2 + \varrho_3) \left(\frac{1}{F_m(\chi, \chi, \Gamma(\chi, \delta), \tau)} - 1 \right) \\ &\Rightarrow (1 - (\varrho_2 + \varrho_3)) \left(\frac{1}{F_m(\Gamma(\chi, \delta), \chi, \chi, \tau)} - 1 \right) \leq 0, \quad \text{for } \tau > 0. \end{aligned}$$

This is a contradiction. Notice that, $(1 - (\varrho_2 + \varrho_3)) \neq 0$, therefore, we obtain $F_m(\Gamma(\chi, \delta), \chi, \chi, \tau) = 1$ this implies that $\Gamma(\chi, \delta) = \chi$ for $\tau > 0$. Similarly, we can show that $\Gamma(\delta, \chi) = \delta$. Hence we get that $(\Gamma(\chi, \delta), \Gamma(\delta, \chi)) = (\chi, \delta)$, this means that (χ, δ) is a coupled FP of a mapping Γ in GFM-space $(\mathbf{C}, F_m, *)$.

Uniqueness: Let $(\chi', \delta') \in \mathbf{C}$ be the other coupled FP of a mapping Γ in $\mathbf{C}$ such that $\Gamma(\chi', \delta') = \chi'$ and $\Gamma(\delta', \chi') = \delta'$. Then, first, we have to prove that $\chi = \chi'$. Now, by the view of (3.8), for $\tau > 0$, we have

$$\begin{aligned} \frac{1}{F_m(\chi, \chi, \chi', \tau)} - 1 &= \frac{1}{F_m(\Gamma(\chi, \delta), \Gamma(\chi, \delta), \Gamma(\chi', \delta'), \tau)} - 1 \\ &\leq \varrho_1 \left(\frac{1}{F_m(\chi, \chi, \chi', \tau)} - 1 \right) + \varrho_2 \left(\frac{1}{\mathbf{M}} - 1 \right) + \varrho_3 \left(\frac{1}{\mathbb{L}} - 1 \right), \end{aligned} \quad (3.22)$$

where

$$\begin{aligned} \frac{1}{\mathbf{M}} - 1 &= \left(\frac{1}{F_m(\chi, \chi, \Gamma(\chi, \delta), \tau)} - 1 + \frac{1}{F_m(\chi, \chi, \Gamma(\chi, \delta), \tau)} - 1 + \frac{1}{F_m(\chi', \chi', \Gamma(\chi', \delta'), \tau)} - 1 \right) \\ &= \left(\frac{1}{F_m(\chi, \chi, \chi, \tau)} - 1 + \frac{1}{F_m(\chi, \chi, \chi, \tau)} - 1 + \frac{1}{F_m(\chi', \chi', \chi', \tau)} - 1 \right) = 0. \end{aligned}$$

And

$$\begin{aligned}
 \mathbb{L} &= \min \left\{ F_m(\chi, \chi, \Gamma(\chi, \delta), \tau), F_m(\chi, \chi, \Gamma(\chi, \delta'), \tau), F_m(\chi', \chi', \Gamma(\chi', \delta'), \tau), \right. \\
 &\quad \left. F_m(\chi, \chi, \Gamma(\chi, \delta), \tau), F_m(\chi, \chi, \Gamma(\chi', \delta'), \tau), F_m(\chi', \chi', \Gamma(\chi, \delta), \tau) \right\} \\
 &= \min \left\{ F_m(\chi, \chi, \chi, \tau), F_m(\chi, \chi, \chi, \tau), F_m(\chi', \chi', \chi', \tau), \right. \\
 &\quad \left. F_m(\chi, \chi, \chi, \tau), F_m(\chi, \chi, \chi', \tau), F_m(\chi', \chi', \chi, \tau) \right\} \\
 &= \min \{ F_m(\chi, \chi, \chi', \tau), F_m(\chi', \chi', \chi, \tau) \}.
 \end{aligned} \tag{3.23}$$

Now, we have the following two-cases:

i- If $F_m(\chi, \chi, \chi', \tau)$ is the minimum term in (3.23). Then, from (3.22) and (3.23), we have that

$$\begin{aligned}
 \frac{1}{F_m(\chi, \chi, \chi', \tau)} - 1 &\leq (\varrho_1 + \varrho_3) \left(\frac{1}{F_m(\chi, \chi, \chi', \tau)} - 1 \right) \\
 \Rightarrow (1 - (\varrho_1 + \varrho_3)) &\left(\frac{1}{F_m(\chi, \chi, \chi', \tau)} - 1 \right) \leq 0, \quad \text{for } \tau > 0.
 \end{aligned}$$

This is a contradiction. Notice that $(1 - (\varrho_1 + \varrho_3)) \neq 0$, therefore $F_m(\chi, \chi, \chi', \tau) = 1 \Rightarrow \chi = \chi'$, for $\tau > 0$.

ii- If $F_m(\chi', \chi', \chi, \tau)$ is the minimum term in (3.23). Then, from (3.22) and (3.23), and by using Proposition 2.1, we have

$$\begin{aligned}
 \frac{1}{F_m(\chi, \chi, \chi', \tau)} - 1 &\leq \varrho_1 \left(\frac{1}{F_m(\chi, \chi, \chi', \tau)} - 1 \right) + \varrho_3 \left(\frac{1}{F_m(\chi', \chi', \chi, \tau)} - 1 \right) \\
 &\leq (\varrho_1 + 2\varrho_3) \left(\frac{1}{F_m(\chi, \chi, \chi', \tau)} - 1 \right), \quad \text{for } \tau > 0.
 \end{aligned}$$

Hence, we obtain

$$(1 - \varrho_1 - 2\varrho_3) \left(\frac{1}{F_m(\chi, \chi, \chi', \tau)} - 1 \right) \leq 0, \quad \text{for } \tau > 0.$$

This is a contradiction. As $(1 - (\varrho_1 + 2\varrho_3)) \neq 0$, therefore, $F_m(\chi, \chi, \chi', \tau) = 1 \Rightarrow \chi = \chi'$, for $\tau > 0$.

Similarly, we can prove that $\delta = \delta'$. Hence, we get that $(\chi, \delta) = (\chi', \delta')$. Thus, the mapping Γ has a unique coupled FP in $\mathbf{C}$.

Corollary 3.1. Let $(\mathbf{C}, F_m, *)$ be a complete GFM-space in which F_m is rectangular. A coupled-mapping $\Gamma : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$ satisfies,

$$\frac{1}{F_m(\Gamma(\ell, g), \Gamma(u, v), \Gamma(p, q), \tau)} - 1 \leq \varrho_1 \left(\frac{1}{F_m(\ell, u, p, \tau)} - 1 \right) + \varrho_3 \left(\frac{1}{\mathbb{L}} - 1 \right), \tag{3.24}$$

where

$$\mathbb{L} = \min \left\{ F_m(\ell, \ell, \Gamma(\ell, g), \tau), F_m(u, u, \Gamma(u, v), \tau), F_m(p, p, \Gamma(p, q), \tau), \right. \\
 \left. F_m(\ell, \ell, \Gamma(u, v), \tau), F_m(u, u, \Gamma(p, q), \tau), F_m(p, p, \Gamma(\ell, g), \tau) \right\},$$

for all $\ell, g, u, v, p, q \in \mathbf{C}$, $\tau > 0$ and $\varrho_1, \varrho_3 \in [0, 1)$ with $(\varrho_1 + 2\varrho_3) < 1$. Then, Γ has a unique coupled FP in $\mathbf{C}$.

Example 3.1. Let $\mathbf{C} = [0, 1]$, $*$ is a continuous τ -norm and $F_m : \mathbf{C} \times \mathbf{C} \times \mathbf{C} \times (0, \infty) \rightarrow [0, 1]$ is defined as

$$F_m(\ell, g, \delta, \tau) = \frac{\tau}{\tau + |\ell - g| + |g - \delta| + |\delta - \ell|}, \quad \forall \ell, g, \delta \in \mathbf{C} \text{ and } \tau > 0.$$

Then, it is easy to prove that F_m is rectangular and a 3-tuple $(\mathbf{C}, F_m, *)$ is a complete GFM-space. Now, a mapping $\Gamma : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$ is defined by $\Gamma(\ell, g) = \frac{3\ell^3}{5} + \frac{5}{64} \forall \ell, g \in [0, 1]$. Then, $\forall \ell, g, u, v, p, q \in \mathbf{C}$ and $\tau > 0$, we have

$$\begin{aligned} \frac{1}{F_m(\Gamma(\ell, g), \Gamma(u, v), \Gamma(p, q), \tau)} - 1 &= \frac{1}{\tau} (|\Gamma(\ell, g) - \Gamma(u, v)| + |\Gamma(u, v) - \Gamma(p, q)| + |\Gamma(p, q) - \Gamma(\ell, g)|) \\ &= \frac{1}{\tau} \left(\left| \frac{3\ell^3}{5} - \frac{3u^3}{5} \right| + \left| \frac{3u^3}{5} - \frac{3p^3}{5} \right| + \left| \frac{3p^3}{5} - \frac{3\ell^3}{5} \right| \right) \\ &= \frac{3}{5\tau} (|(\ell - u)(\ell^2 + \ell u + u^2) + (u - p)(u^2 + up + p^2) + (p - \ell)(p^2 + p\ell + \ell^2)|) \\ &\leq \frac{3}{5\tau} (|3(\ell - u) + 3(u - p) + 3(p - \ell)|) \\ &= \frac{9}{5} \left(\frac{|\ell - u| + |u - p| + |p - \ell|}{\tau} \right) = \frac{9}{5} \left(\frac{1}{F_m(\ell, u, p, \tau)} - 1 \right), \text{ for } \tau > 0. \end{aligned}$$

Hence, it is showed that a mapping Γ is not a coupled-contractive in GFM-space and has a unique coupled FP, i.e., $\Gamma(\ell, g) = \Gamma\left(\frac{5}{4}, \frac{5}{4}\right) = \frac{3(5/4)^3}{5} + \frac{5}{64} = \frac{5}{4} \notin \mathbf{C} = [0, 1]$.

Next, we calculate the approximate coupled FPs of a mapping Γ . Since, a mapping Γ is defined as $\Gamma(\ell) = \frac{3\ell^3}{5} + \frac{5}{64}$, $\ell \in [0, 1]$. To determine the approximate coupled FPs of a mapping Γ , we solve $\Gamma(\ell, g) = \ell$. Thus,

$$\ell = \frac{3\ell^3}{5} + \frac{5}{64} \Rightarrow 192\ell^3 - 320\ell + 25 = 0 \Rightarrow \ell^3 - \frac{5}{3}\ell + \frac{25}{192} = 0.$$

This is a cubic equation is in the standard depressed form: $\ell^3 + p\ell + q = 0$, where $p = -\frac{5}{3}$ and $q = \frac{25}{192}$. Using the trigonometric method: As, $\Delta = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3$, by substituting the value of p and q , we obtain $\Delta < 0$. So the cubic equation has three real roots. Now, we use the trigonometric method to solve the cubic equation

$$\ell_n = 2\sqrt{-\frac{p}{3}} \cos\left(\frac{1}{3} \arccos\left(\frac{3q}{2p} \sqrt{-\frac{3}{p}}\right) - \frac{2n\pi}{3}\right), \quad n = 0, 1, 2.$$

Since, $p = -\frac{5}{3}$ and $q = \frac{25}{192}$. Then, after simplification, we obtain

$$\ell_n = \frac{2\sqrt{5}}{3} \cos\left(\frac{1}{3} \arccos\left(-\frac{45}{128\sqrt{5}}\right) - \frac{2n\pi}{3}\right), \quad n = 0, 1, 2.$$

Hence, after calculation, we obtained the three approximate coupled FPs, that are, $\{1.25174, 0.07833, -1.33007\}$. Since, the mapping Γ is defined on $[0, 1]$. Thus, the mapping Γ has a approximately unique coupled FP in $[0, 1]$ is 0.07833.

Next, we present an illustrative numerical example which satisfies all the coupled-contractive conditions of our results in GFM-spaces.

Example 3.2. Let $\mathbf{C} = [0, 1]$, $*$ is a continuous τ -norm and $F_m : \mathbf{C} \times \mathbf{C} \times \mathbf{C} \times (0, \infty) \rightarrow [0, 1]$ is defined as

$$F_m(\ell, g, \delta, \tau) = \frac{\tau}{\tau + |\ell - g| + |g - \delta| + |\delta - \ell|}, \quad \forall \ell, g, \delta \in \mathbf{C} \quad \text{and } \tau > 0.$$

Then, it is easy to prove that F_m is rectangular and a 3-tuple $(\mathbf{C}, F_m, *)$ is a complete GFM-space. Now, a mapping $\Gamma : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$ is defined by $\Gamma(\ell, g) = \frac{\ell}{7} \forall \ell, g \in [0, 1]$. Then, $\forall \ell, g, u, v, p, q \in \mathbf{C}$ and $\tau > 0$, we have

$$\begin{aligned} \frac{1}{F_m(\Gamma(\ell, g), \Gamma(u, v), \Gamma(p, q), \tau)} - 1 &= \frac{1}{\tau} (|\Gamma(\ell, g) - \Gamma(u, v)| + |\Gamma(u, v) - \Gamma(p, q)| + |\Gamma(p, q) - \Gamma(\ell, g)|) \\ &= \frac{1}{\tau} \left(\left| \frac{\ell^2}{7} - \frac{u^2}{7} \right| + \left| \frac{u^2}{7} - \frac{p^2}{7} \right| + \left| \frac{p^2}{7} - \frac{\ell^2}{7} \right| \right) \\ &= \frac{1}{7\tau} (|(\ell - u)(\ell + u) + (u - p)(u + p) + (p - \ell)(p + \ell)|) \\ &\leq \frac{2}{7} \left(\frac{|\ell - u| + |u - p| + |p - \ell|}{\tau} \right) = \frac{2}{7} \left(\frac{1}{F_m(\ell, u, p, \tau)} - 1 \right), \text{ for } \tau > 0. \end{aligned}$$

Hence, it is showed that a mapping Γ is coupled-contractive in GFM-space. Now, we justify the coupled-contraction condition (3.8), for $\tau > 0$, we have

$$\begin{aligned} \frac{1}{F_m(\Gamma(\ell, g), \Gamma(u, v), \Gamma(p, q), \tau)} - 1 &= \frac{2}{7} \left(\frac{1}{F_m(\ell, u, p, \tau)} - 1 \right) \\ &\leq \frac{2}{7} \left(\frac{1}{F_m(\ell, u, p, \tau)} - 1 \right) + \frac{1}{14} \left(\frac{2(|\ell^2 - 7\ell| + |u^2 - 7u| + |p^2 - 7p|)}{7\tau} \right) \\ &\quad + \frac{1}{21} \left(\frac{1}{\min \left\{ \frac{\frac{7\tau}{7\tau+2|\ell^2-7\ell|}, \frac{7\tau}{7\tau+2|u^2-7u|}, \frac{7\tau}{7\tau+2|p^2-7p|}, \frac{7\tau}{7\tau+2|u^2-7\ell|}, \frac{7\tau}{7\tau+2|p^2-7u|}, \frac{7\tau}{7\tau+2|\ell^2-7p|} \right\}} - 1 \right) \\ &\leq \frac{2}{7} \left(\frac{1}{F_m(\ell, u, p, \tau)} - 1 \right) \\ &\quad + \frac{1}{14} \left(\frac{1}{F_m(\ell, \ell, \Gamma(\ell, g), \tau)} - 1 + \frac{1}{F_m(u, u, \Gamma(u, v), \tau)} - 1 + \frac{1}{F_m(p, p, \Gamma(p, q), \tau)} - 1 \right) \\ &\quad + \frac{1}{21} \left(\frac{1}{\mathbb{L}(\Gamma(\ell, g), \Gamma(u, v), \Gamma(p, q), \tau)} - 1 \right), \end{aligned}$$

where

$$\begin{aligned} \mathbb{L} &= \min \left\{ F_m(\ell, \ell, \Gamma(\ell, g), \tau), F_m(u, u, \Gamma(u, v), \tau), F_m(p, p, \Gamma(p, q), \tau), \right. \\ &\quad \left. F_m(\ell, \ell, \Gamma(u, v), \tau), F_m(u, u, \Gamma(p, q), \tau), F_m(p, p, \Gamma(\ell, g), \tau) \right\} \\ &= \min \left\{ \frac{7\tau}{7\tau+2|\ell^2-7\ell|}, \frac{7\tau}{7\tau+2|u^2-7u|}, \frac{7\tau}{7\tau+2|p^2-7p|}, \frac{7\tau}{7\tau+2|u^2-7\ell|}, \frac{7\tau}{7\tau+2|p^2-7u|}, \frac{7\tau}{7\tau+2|\ell^2-7p|} \right\} > 0, \end{aligned}$$

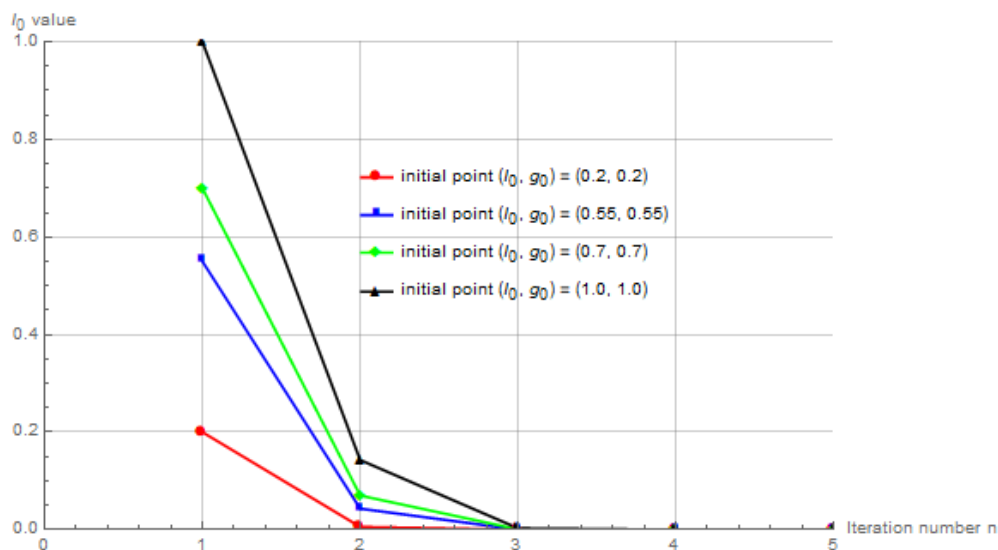


FIGURE 1. Couple FP graph of a mapping $\Gamma(\ell, g) = \frac{\ell}{7}$, for $\ell, g \in [0, 1]$.

TABLE 1. Numerical calculations of a coupled-contractive condition (3.8)

Cases	Case-1	Case-2	Case-3	Case-4	Case-5	Case-6	Case-7
(ℓ, g)	$(0,0)$	$(0.22,0.22)$	$(0.44,0.55)$	$(0.62,0.69)$	$(0.73,0.73)$	$(0.82,0.89)$	$(1,1)$
(u, v)	$(0,0)$	$(0.22,0.44)$	$(0.55,0.55)$	$(0.7,0.7)$	$(0.73,0.79)$	$(0.83,0.9)$	$(1,1)$
(p, q)	$(0,0)$	$(0.44,0.44)$	$(0.55,0.62)$	$(0.7,0.73)$	$(0.82,0.82)$	$(0.91,1.00)$	$(1,1)$
$\Gamma(\ell, g)$	0	0.0069	0.0276	0.0549	0.0761	0.096	0.1428
$\Gamma(u, v)$	0	0.0069	0.0432	0.07	0.0761	0.098	0.1428
$\Gamma(p, q)$	0	0.0276	0.0432	0.07	0.096	0.1183	0.1428
L.H.S.(3.8)	0	0.2517	0.1257	0.0914	0.1028	0.1029	0
R.H.S.(3.8)	0	0.5731	0.6323	0.7354	0.6959	0.9	0.7447

for all $\ell, g, u, v, p, q \in \mathbf{C}$ and $\tau > 0$. Hence, from the above, we conclude that all the conditions of Theorem 3.2 are satisfied with $\varrho_1 = 2/7$, $\varrho_2 = 1/14$, and $\varrho_3 = 1/21$, and $(\varrho_1 + 3\varrho_2 + 2\varrho_3) = \frac{25}{42} < 1$. The mapping Γ has a unique coupled FP in $\mathbf{C}$, i.e., $\Gamma(0,0) = 0 \in [0, 1]$.

In Figure 1, we use different initial values of the variables $\ell, g \in [0, 1]$ in a coupled mapping $\Gamma(\ell, g) = \frac{\ell}{7}$ and shown that the coupled mapping Γ has iteratively convergence to a unique coupled FP, i.e., $0 \in [0, 1]$.

In Table 1, we calculate the values of a mapping Γ and terms of the inequality (3.8) of Theorem 3.2 using different values of the variables $\ell, g, u, v, p, q \in [0, 1]$ with $\varrho_1 = \frac{2}{7}$, $\varrho_2 = \frac{1}{14}$, $\varrho_3 = \frac{1}{21}$, and $\tau = 0.5$. The last two rows in Table 1 is validating the inequality (3.8) of Theorem 3.2.

Based on Table 1, we sketch a three-dimensional column graph, that is, Figure 2, which is justifying the inequality (3.8) of Theorem 3.2. In Figure 2, the terms $\Gamma(\ell, g)$, $\Gamma(u, v)$, and $\Gamma(p, q)$ are represented by $T(c, d)$, $T(u, v)$, and $T(p, q)$ respectively. In Figure 2, we show a graphical sketch of a mapping Γ and the

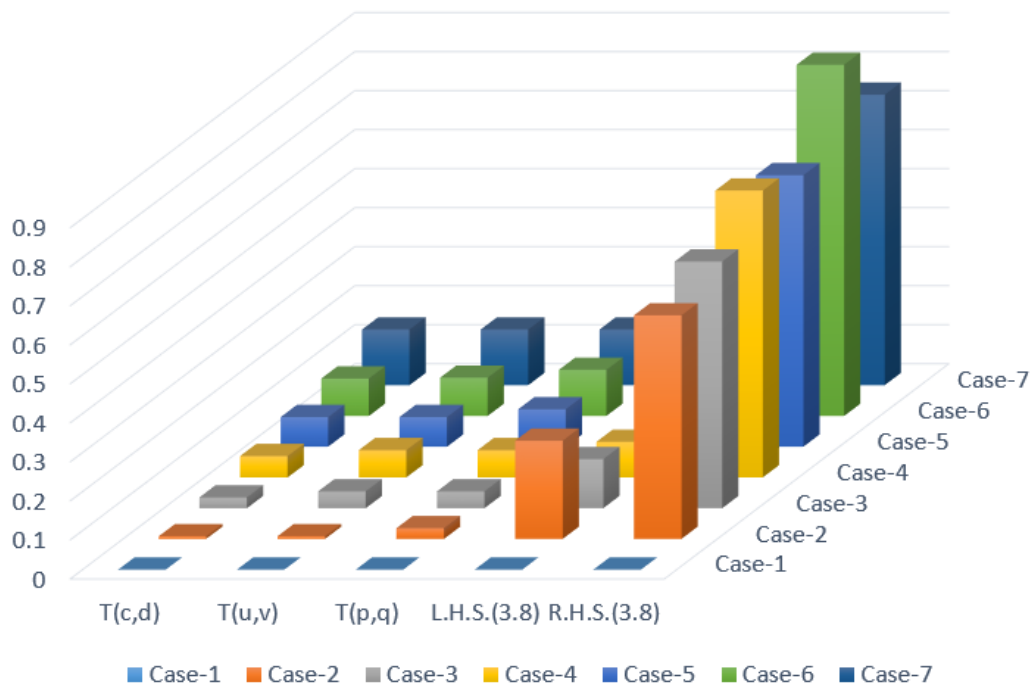


FIGURE 2. 3D Graph presentation of a coupled-contractive condition (3.8)

terms of the inequality (3.8) using different values of the variables $\ell, g, u, v, p, q \in [0, 1]$. Figure 2 illustrates the graphical justification of inequality (3.8) using the parametric constants $\varrho_1 = 2/7$, $\varrho_2 = 1/14$, and $\varrho_3 = 1/21$ with $\tau = 0.5$. In Figure 2, the left and right hand sides of the inequality (3.8) are graphically shown and justified for all cases.

4. APPLICATION

This is an application of Urysohn type integral equations (UTIEs) to support our main work. By using the concept of Shamas et al. [27], we prove that the solution of the UTIEs have a unique coupled FP of the integral equations to our main outcomes. To make our findings more understandable, we now define the terms that follow using the supremum norm, Let $\mathbf{C} = C([0, 1], \mathfrak{R})$ represent the space of all continuous and real-valued functions defined on a closed interval $[0, 1]$. The supremum norm is defined by:

$$\|\ell\|_{\infty} = \sup_{y \in [0, 1]} |\ell(y)|, \quad \text{where } \ell \in \mathbf{C} = (C[0, 1], \mathfrak{R}).$$

Now, we define a GFM $F_m : \mathbf{C} \times \mathbf{C} \times \mathbf{C} \times (0, \infty) \rightarrow [0, 1]$ by

$$F_m(\ell, u, p, \tau) = \frac{\tau}{\tau + D(\ell, u, p)}, \quad (4.1)$$

where

$$\begin{aligned} D(\ell, u, p) &= \max \left\{ \sup_{y \in [0,1]} |\ell(y) - u(y)|, \sup_{y \in [0,1]} |u(y) - p(y)|, \sup_{y \in [0,1]} |p(y) - \ell(y)| \right\} \\ &= \max \{ \|\ell - u\|_\infty, \|u - p\|_\infty, \|p - \ell\|_\infty \}, \quad \forall \ell, u, p \in \mathbf{C} = C([0,1], \mathfrak{R}). \end{aligned}$$

Then, it is easy to show that $(\mathbf{C}, F_m, *)$ is a complete GFM-space. Now we define the system of UTIEs as

$$\ell(y) = f(y) + \int_0^1 \mathcal{K}(y, s, \ell(s)) ds, \quad g(y) = f(y) + \int_0^1 \mathcal{K}(y, s, g(s)) ds, \quad (4.2)$$

for $y \in [0, 1] \subset \mathfrak{R}$ and $\mathcal{K}$ is a continuous mapping $\mathcal{K} : [0, 1] \times [0, 1] \times \mathfrak{R} \times \mathfrak{R} \rightarrow \mathfrak{R}$.

Theorem 4.1. Suppose that $\exists \lambda \in [0, 1]$ such that

$$\max \{ \|\Gamma(\ell, g) - \Gamma(u, v)\|_\infty, \|\Gamma(u, v) - \Gamma(p, q)\|_\infty, \|\Gamma(p, q) - \Gamma(\ell, g)\|_\infty \} \leq \lambda A, \quad (4.3)$$

where

$$A = \max\{A_1, A_2, A_3\}, \quad (4.4)$$

with

$$\begin{aligned} A_1 &= \max \{ \|\ell - u\|_\infty, \|u - p\|_\infty, \|p - \ell\|_\infty \}, \\ A_2 &= (\|\ell - f - \mathbb{W}_{(\ell, g)}\|_\infty + \|u - f - \mathbb{W}_{(u, v)}\|_\infty + \|p - f - \mathbb{W}_{(p, q)}\|_\infty), \\ A_3 &= \tau \left[\frac{1}{\min \left\{ \frac{\tau}{\tau + \|\ell - f - \mathbb{W}_{(\ell, g)}\|_\infty}, \frac{\tau}{\tau + \|u - f - \mathbb{W}_{(u, v)}\|_\infty}, \frac{\tau}{\tau + \|p - f - \mathbb{W}_{(p, q)}\|_\infty}, \right.} \right. \\ &\quad \left. \left. \frac{\tau}{\tau + \|\ell - f - \mathbb{W}_{(u, v)}\|_\infty}, \frac{\tau}{\tau + \|u - f - \mathbb{W}_{(p, q)}\|_\infty}, \frac{\tau}{\tau + \|p - f - \mathbb{W}_{(\ell, g)}\|_\infty} \right\}} - 1 \right], \end{aligned}$$

for all $\ell, g, u, v, p, q, f, \mathbb{W}_{(\ell, g)}, \mathbb{W}_{(u, v)}, \mathbb{W}_{(p, q)} \in \mathbf{C} = C([0, 1], \mathfrak{R})$ and $\tau > 0$. Then, the UTIEs (4.2) have a unique coupled solution in $\mathbf{C}$.

Proof: Consider the coupled mapping $\Gamma : C([0, 1], \mathfrak{R}) \times C([0, 1], \mathfrak{R}) \rightarrow C([0, 1], \mathfrak{R})$ defined by:

$$\begin{aligned} \Gamma(\ell, g)(y) &= f(y) + \mathbb{W}_{(\ell, g)}(y), \\ \Gamma(u, v)(y) &= f(y) + \mathbb{W}_{(u, v)}(y), \\ \Gamma(p, q)(y) &= f(y) + \mathbb{W}_{(p, q)}(y), \end{aligned} \quad (4.5)$$

where

$$\begin{aligned} \mathbb{W}_{(\ell, g)}(y) &= \int_0^1 \mathcal{K}(y, s, \ell(s)) ds, \\ \mathbb{W}_{(u, v)}(y) &= \int_0^1 \mathcal{K}(y, s, u(s)) ds, \end{aligned}$$

$$\mathbb{W}_{(p,q)}(y) = \int_0^1 \mathcal{K}(y, s, p(s)) ds,$$

for all $\ell, g, u, v, p, q, f, \mathbb{W}_{(\ell,g)}, \mathbb{W}_{(u,v)}, \mathbb{W}_{(p,q)} \in \mathbf{C} = C([0, 1], \mathfrak{R})$. Now, we apply Theorem 3.2, for $\tau > 0$, we have

$$\frac{1}{F_m(\Gamma(\ell, g), \Gamma(u, v), \Gamma(p, q), \tau)} - 1 = \frac{1}{\tau} \max \left\{ \begin{array}{l} \|\Gamma(\ell, g) - \Gamma(u, v)\|_\infty, \\ \|\Gamma(u, v) - \Gamma(p, q)\|_\infty, \\ \|\Gamma(p, q) - \Gamma(\ell, g)\|_\infty \end{array} \right\} \leq \frac{1}{\tau} \lambda \mathbb{A}, \quad (4.6)$$

$\forall \ell, g, u, v, p, q \in \mathbf{C}$. Then, we may have the following three main cases:

i- If A_1 is the maximum term in (4.4), then, $\mathbb{A} = A_1$. Now, from (4.6), we have

$$\frac{1}{F_m(\Gamma(\ell, g), \Gamma(u, v), \Gamma(p, q), \tau)} - 1 \leq \frac{1}{\tau} \lambda \mathbb{A} = \frac{1}{\tau} \lambda A_1 = \lambda \left(\frac{1}{F_m(\ell, u, p, \tau)} - 1 \right),$$

$\forall \ell, g, u, v, p, q \in \mathbf{C}$ and $\tau > 0$. Hence, a coupled mapping Γ satisfies all the hypotheses of Theorem 3.1 and Theorem 3.2 with $\lambda = \varrho_1$ and $\varrho_2 = \varrho_3 = 0$. Thus, the UTIEs (4.2) have a unique coupled solution in $\mathbf{C}$ indeed be the unique coupled FP of the coupled mapping Γ in $\mathbf{C}$.

ii- If A_2 is the maximum term in (4.4), then, $\mathbb{A} = B_2$. Now, from (4.6), we have

$$\begin{aligned} \frac{1}{F_m(\Gamma(\ell, g), \Gamma(u, v), \Gamma(p, q), \tau)} - 1 &\leq \frac{1}{\tau} \lambda \mathbb{A} = \frac{1}{\tau} \lambda A_2 \\ &= \lambda \left(\frac{\|\ell - f + \mathbb{W}_{(\ell,g)}\|_\infty + \|u - f + \mathbb{W}_{(u,v)}\|_\infty + \|p - f + \mathbb{W}_{(p,q)}\|_\infty}{\tau} \right) = \lambda \left(\frac{1}{\mathbf{M}} - 1 \right), \end{aligned}$$

where

$$\begin{aligned} \frac{1}{\mathbf{M}} - 1 &= \left(\frac{1}{F_m(\ell, \ell, \Gamma(\ell, g), \tau)} - 1 + \frac{1}{F_m(u, u, \Gamma(u, v), \tau)} - 1 + \frac{1}{F_m(p, p, \Gamma(p, q), \tau)} - 1 \right) \\ &= \left(\frac{1}{\frac{\tau}{\tau + \max\{0, \|\ell - \Gamma(\ell, g)\|_\infty\}}} - 1 + \frac{1}{\frac{\tau}{\tau + \max\{0, \|u - \Gamma(u, v)\|_\infty\}}} - 1 + \frac{1}{\frac{\tau}{\tau + \max\{0, \|p - \Gamma(p, q)\|_\infty\}}} - 1 \right) \\ &= \left(\frac{1}{\frac{\tau}{\tau + \|\ell - \Gamma(\ell, g)\|_\infty}} - 1 + \frac{1}{\frac{\tau}{\tau + \|u - \Gamma(u, v)\|_\infty}} - 1 + \frac{1}{\frac{\tau}{\tau + \|p - \Gamma(p, q)\|_\infty}} - 1 \right) \\ &= \frac{1}{\tau} ((\tau + \|\ell - \Gamma(\ell, g)\|_\infty - \tau) + (\tau + \|u - \Gamma(u, v)\|_\infty - \tau) + (\tau + \|p - \Gamma(p, q)\|_\infty - \tau)) \\ &= \frac{1}{\tau} (\|\ell - \Gamma(\ell, g)\|_\infty + \|u - \Gamma(u, v)\|_\infty + \|p - \Gamma(p, q)\|_\infty) \\ &= \frac{1}{\tau} (\|\ell - f - \mathbb{W}_{(\ell,g)}\|_\infty + \|u - f - \mathbb{W}_{(u,v)}\|_\infty + \|p - f - \mathbb{W}_{(p,q)}\|_\infty), \end{aligned}$$

for all $\ell, g, u, v, p, q, f, \mathbb{W}_{(\ell,g)}, \mathbb{W}_{(u,v)}, \mathbb{W}_{(p,q)} \in \mathbf{C}$ and $\tau > 0$. Hence, a coupled mapping Γ satisfies all the hypotheses of Theorem 3.1 and Theorem 3.2 with $\lambda = \varrho_2$ and $\varrho_1 = \varrho_3 = 0$. Thus, the UTIEs (4.2) have a unique coupled solution in $\mathbf{C}$ indeed be the unique coupled FP of the coupled mapping Γ .

iii- If A_3 is the maximum term in (4.4), then, $\mathbb{A} = A_3$. Now, from (4.6), we have

$$\frac{1}{F_m(\Gamma(\ell, g), \Gamma(u, v), \Gamma(p, q), \tau)} - 1 \leq \lambda \frac{\mathbb{A}}{\tau} = \lambda \frac{A_3}{\tau}$$

$$= \lambda \left(\frac{1}{\min \left\{ \frac{\tau}{\tau + \|\ell - f - W_{(\ell, g)}\|_\infty}, \frac{\tau}{\tau + \|u - f - W_{(u, v)}\|_\infty}, \frac{\tau}{\tau + \|p - f - W_{(p, q)}\|_\infty}, \frac{\tau}{\tau + \|\ell - f - W_{(u, v)}\|_\infty}, \frac{\tau}{\tau + \|u - f - W_{(p, q)}\|_\infty}, \frac{\tau}{\tau + \|p - f - W_{(\ell, g)}\|_\infty} \right\}} - 1 \right) = \lambda \left(\frac{1}{\mathbb{L}} - 1 \right),$$

where

$$\begin{aligned} \mathbb{L} &= \min \left\{ \begin{array}{l} F_m(\ell, \ell, \Gamma(\ell, g), \tau), F_m(u, u, \Gamma(u, v), \tau), F_m(p, p, \Gamma(p, q), \tau), \\ F_m(\ell, \ell, \Gamma(u, v), \tau), F_m(u, u, \Gamma(p, q), \tau), F_m(p, p, \Gamma(\ell, g), \tau) \end{array} \right\} \\ &= \min \left\{ \begin{array}{l} \frac{\tau}{\tau + D(\ell, \ell, \Gamma(\ell, g))}, \frac{\tau}{\tau + D(u, u, \Gamma(u, v))}, \frac{\tau}{\tau + D(p, p, \Gamma(p, q))}, \\ \frac{\tau}{\tau + D(\ell, \ell, \Gamma(u, v))}, \frac{\tau}{\tau + D(u, u, \Gamma(p, q))}, \frac{\tau}{\tau + D(p, p, \Gamma(\ell, g))} \end{array} \right\} \\ &= \min \left\{ \begin{array}{l} \frac{\tau}{\tau + \max\{\|\ell - \ell\|_\infty, \|\ell - \Gamma(\ell, g)\|_\infty, \|\Gamma(\ell, g) - \ell\|_\infty\}}, \frac{\tau}{\tau + \max\{\|u - u\|_\infty, \|u - \Gamma(u, v)\|_\infty, \|\Gamma(u, v) - u\|_\infty\}}, \\ \frac{\tau}{\tau + \max\{\|p - p\|_\infty, \|p - \Gamma(p, q)\|_\infty, \|\Gamma(p, q) - p\|_\infty\}}, \frac{\tau}{\tau + \max\{\|\ell - \ell\|_\infty, \|\ell - \Gamma(u, v)\|_\infty, \|\Gamma(u, v) - \ell\|_\infty\}}, \\ \frac{\tau}{\tau + \max\{\|u - u\|_\infty, \|u - \Gamma(p, q)\|_\infty, \|\Gamma(p, q) - u\|_\infty\}}, \frac{\tau}{\tau + \max\{\|p - p\|_\infty, \|p - \Gamma(\ell, g)\|_\infty, \|\Gamma(\ell, g) - p\|_\infty\}} \end{array} \right\} \\ &= \min \left\{ \begin{array}{l} \frac{\tau}{\tau + \max\{0, \|\ell - \Gamma(\ell, g)\|_\infty\}}, \frac{\tau}{\tau + \max\{0, \|u - \Gamma(u, v)\|_\infty\}}, \frac{\tau}{\tau + \max\{0, \|p - \Gamma(p, q)\|_\infty\}}, \\ \frac{\tau}{\tau + \max\{0, \|\ell - \Gamma(u, v)\|_\infty\}}, \frac{\tau}{\tau + \max\{0, \|u - \Gamma(p, q)\|_\infty\}}, \frac{\tau}{\tau + \max\{0, \|p - \Gamma(\ell, g)\|_\infty\}} \end{array} \right\} \\ &= \min \left\{ \begin{array}{l} \frac{\tau}{\tau + \|\ell - \Gamma(\ell, g)\|_\infty}, \frac{\tau}{\tau + \|u - \Gamma(u, v)\|_\infty}, \frac{\tau}{\tau + \|p - \Gamma(p, q)\|_\infty}, \\ \frac{\tau}{\tau + \|\ell - \Gamma(u, v)\|_\infty}, \frac{\tau}{\tau + \|u - \Gamma(p, q)\|_\infty}, \frac{\tau}{\tau + \|p - \Gamma(\ell, g)\|_\infty} \end{array} \right\} \\ &= \min \left\{ \begin{array}{l} \frac{\tau}{\tau + \|\ell - f - W_{(\ell, g)}\|_\infty}, \frac{\tau}{\tau + \|u - f - W_{(u, v)}\|_\infty}, \frac{\tau}{\tau + \|p - f - W_{(p, q)}\|_\infty}, \\ \frac{\tau}{\tau + \|\ell - f - W_{(u, v)}\|_\infty}, \frac{\tau}{\tau + \|u - f - W_{(p, q)}\|_\infty}, \frac{\tau}{\tau + \|p - f - W_{(\ell, g)}\|_\infty} \end{array} \right\}, \end{aligned}$$

for all $\ell, g, u, v, p, q, f, W_{(\ell, g)}, W_{(u, v)}, W_{(p, q)} \in \mathbf{C}$ and $\tau > 0$. Hence, a coupled mapping Γ satisfies all the hypotheses of Theorem 3.2 with $\lambda = \varrho_3$ and $\varrho_1 = \varrho_2 = 0$. Thus, the UTIEs (4.2) have a unique coupled solution in $\mathbf{C}$ indeed be the unique coupled FP of the coupled mapping Γ .

Next, we present a numerical example which satisfies all the condition of Theorem 3.1, Theorem 3.2 and Theorem 4.1.

Example 4.1. Let $\mathbf{C} = C([0, 1], \mathfrak{R})$ be the Banach space of real-valued continuous functions on $[0, 1]$ and the supremum norm is defined by

$$\|\ell\|_\infty = \sup_{y \in [0, 1]} |\ell(y)|, \quad \text{where } \ell \in \mathbf{C} = (C[0, 1], \mathfrak{R}).$$

Since, a GFM $F_m : \mathbf{C} \times \mathbf{C} \times \mathbf{C} \times (0, \infty) \rightarrow [0, 1]$ is defined by

$$F_m(\ell, u, p, \tau) = \frac{\tau}{\tau + D(\ell, u, p)}, \quad (4.7)$$

where

$$D(\ell, u, p) = \max \{ \|\ell - u\|_\infty, \|u - p\|_\infty, \|p - \ell\|_\infty \}, \quad \forall \ell, u, p \in \mathbf{C} = C([0, 1], \mathfrak{R}) \text{ and } \tau > 0.$$

Consider the UTIEs are

$$\ell(y) = \int_0^1 \frac{e^{-s}}{5} \frac{\ell(s)}{1 + |\ell(s)|} ds + \frac{y}{2} \quad \text{and} \quad g(y) = \int_0^1 \frac{e^{-s}}{5} \frac{g(s)}{1 + |g(s)|} ds + \frac{y}{2}, \quad (4.8)$$

for all $y \in [0, 1]$ and $\ell(y), g(y) \in \mathbf{C} = C([0, 1], \mathfrak{R})$. Now, the coupled operator $\Gamma : C([0, 1], \mathfrak{R}) \times C([0, 1], \mathfrak{R}) \rightarrow C([0, 1], \mathfrak{R})$ is defined by

$$(\Gamma(\ell, g))(y) = \int_0^1 \frac{e^{-s}}{5} \frac{\ell(s)}{1 + |\ell(s)|} ds + \frac{y}{2}.$$

Let the function ψ on $\mathfrak{R}$ be defined as $\psi(\rho) = \frac{\rho}{1-|\rho|}$ for $\rho \in \mathfrak{R}$, and we known that

$$|\psi(a) - \psi(b)| \leq |a - b| \quad \forall a, b \in \mathfrak{R}.$$

Now, we justify the inequality (4.3) of Theorem 4.1, we have

$$\begin{aligned} |T(\ell, g)(y) - T(u, v)(y)| &\leq \left| \int_0^1 \frac{e^{-s}}{5} \frac{\ell(s)}{1 + |\ell(s)|} ds - \int_0^1 \frac{e^{-s}}{5} \frac{u(s)}{1 + |u(s)|} ds \right| \\ &\leq \int_0^1 \frac{e^{-s}}{5} \left| \frac{\ell(s)}{1 + |\ell(s)|} - \frac{u(s)}{1 + |u(s)|} \right| ds \\ &\leq \int_0^1 \frac{e^{-s}}{5} |\ell(s) - u(s)| ds \\ &\leq \frac{1}{5} \|x - u\|_\infty \int_0^1 e^{-s} ds = \frac{1 - e^{-1}}{5} \|x - u\|_\infty. \end{aligned}$$

Now, by using the supremum norm over $y \in [0, 1]$, we obtain

$$\|\Gamma(\ell, g) - \Gamma(u, v)\|_\infty \leq \frac{1 - e^{-1}}{5} \|\ell - u\|_\infty.$$

Similarly,

$$\|\Gamma(u, v) - \Gamma(p, q)\|_\infty \leq \frac{1 - e^{-1}}{5} \|u - p\|_\infty \quad \text{and} \quad \|\Gamma(p, q) - \Gamma(\ell, g)\|_\infty \leq \frac{1 - e^{-1}}{5} \|p - \ell\|_\infty.$$

Then,

$$\begin{aligned} D(\Gamma(\ell, g), \Gamma(u, v), \Gamma(p, q)) &= \max \{ \|\Gamma(\ell, g) - \Gamma(u, v)\|_\infty, \|\Gamma(u, v) - \Gamma(p, q)\|_\infty, \|\Gamma(p, q) - \Gamma(\ell, g)\|_\infty \} \\ &\leq \max \left\{ \frac{1 - e^{-1}}{5} \|\ell - u\|_\infty, \frac{1 - e^{-1}}{5} \|u - p\|_\infty, \frac{1 - e^{-1}}{5} \|p - \ell\|_\infty \right\} \\ &\leq \frac{1 - e^{-1}}{5} \max \{ \|\ell - u\|_\infty, \|u - p\|_\infty, \|p - \ell\|_\infty \} \\ &= \frac{1 - e^{-1}}{5} D(\ell, u, p). \end{aligned}$$

Hence,

$$D(\Gamma(\ell, g), \Gamma(u, v), \Gamma(p, q)) \leq \frac{1 - e^{-1}}{5} D(\ell, u, p).$$

Thus, the condition (4.3) of Theorem 4.1 is satisfied with $\lambda = \frac{1 - e^{-1}}{5} \approx 0.1264 < 1$. Now, we justify the conditions of Theorem 3.1 and Theorem 3.2, then from (4.7), for $\tau > 0$, we have

$$\begin{aligned} \frac{1}{F_m(\Gamma(\ell, g), \Gamma(u, v), \Gamma(p, q), \tau)} - 1 &= \frac{1}{\tau} D(\Gamma(\ell, g), \Gamma(u, v), \Gamma(p, q)) \\ &\leq \frac{1}{\tau} \frac{1 - e^{-1}}{5} D(\ell, u, p) = \frac{1 - e^{-1}}{5} \left(\frac{1}{F_m(\ell, u, p, \tau)} - 1 \right), \end{aligned}$$

for all $\ell, g, u, v, p, q \in \mathbf{C} = (C[0, 1], \mathfrak{R})$. Hence, all the conditions of Theorem 3.1 and Theorem 3.2 are satisfied with $\varrho_1 = \lambda = \frac{1 - e^{-1}}{5} \approx 0.1264 < 1$ and $\varrho_2 = \varrho_3 = 0$, then, $(\varrho_1 + 3\varrho_2 + 2\varrho_3) = \frac{1 - e^{-1}}{5} \approx 0.1264 < 1$. Next we calculate the approximate FP which will be the unique solution of the UTIEs and indeed be the unique coupled FP of a mapping Γ in $\mathbf{C} = (C[0, 1], \mathfrak{R})$. Let ℓ^* be the unique solution of UTIE and let start with $\ell_0(y) = 0$, we obtain $\ell_1(y) = \frac{y}{2}$ and the next iteration is

$$\ell_2(y) = \int_0^1 \frac{e^{-s}}{5} \frac{\frac{s}{2}}{1 + |\frac{s}{2}|} ds + \frac{y}{2} \approx 0.0216 + \frac{y}{2}.$$

Continuing the iterative convergence process, we can get a unique solution of the UTIE i.e., $\ell^*(y) = g^*(y)$ and indeed be the unique coupled FP of a coupled mapping Γ such that $(\Gamma(\ell^*, g^*), \Gamma(g^*, \ell^*)) = (\ell^*, g^*) \in \mathbf{C} \times \mathbf{C}$. Thus, all the hypothesis of Theorem 3.1, Theorem 3.2, and Theorem 4.1 are satisfied. Therefore, the system has a unique coupled FP and hence a unique solution.

5. CONCLUSION

We established some unique coupled FP-results in GFM-spaces and demonstrate unique coupled FP-theorems under the modified contraction conditions. In support of our findings, we established illustrative unique coupled FP numerical examples to unify our work. Based on unique coupled FP numerical example, we presented a three-dimensional column graph to justify the inequality (3.8) (coupled contraction) of Theorem 3.2. Moreover, we presented an application of the UTIEs to unify the theory of coupled contractions in aforementioned space. Furthermore, we presented a numerical example in the framework of a UTIE to demonstrate the applicability of our obtained results. The numerical Example 4.1 plays a crucial role in bridging the gap between the abstract FP theory and its practical application. This example conforms that the proposed work effectively applied to the study of the UTIEs and illustrated that how the theoretical conditions ensure the existence and uniqueness of solution. Therefore, the illustrated numerical example not only validates theoretical work but also establish the direct connection between the developed theory and its application, highlighting the reliability and effectiveness of the proposed approach. Based on this new approach, one can prove more modified and general contractive type coupled FP results with supportive various types of three-dimensional graphs and various types of integral and differential equations applications.

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