Int. J. Anal. Appl. (2026), 24:303

International Journal of Analysis and Applications

Existence and Uniqueness of Solutions for FDEs with Nonlocal Discrete, Integral Delay and Nonlinear Terminal Boundary Conditions

Selvakumar Viswanathan[1], Jeyachandhiran Ramaprabu[1], Gunaseelan Mani[2], Abdulkareem Saleh Hamarsheh[3], Rajagopalan Ramaswamy[3,*], S. Padmaja[4]

[1]*Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai 602 105, Tamil Nadu, India*

[2]*Center for Excellence - Research, Rajalakshmi Engineering College, Chennai, 602 105, Tamil Nadu, India*

[3]*Department of Mathematics, College of Science and Humanities in Alkharj, Prince Sattam Bin Abdulaziz University, Alkharj 11942, Saudi Arabia*

[4]*Department of Computer SCience, College of Computer Engineering and Sciences, Prince Sattam Bin Abdualziz University, Alkharj, 11942, Saudi Arabia*

[*]*Corresponding author:* r.gopalan@psau.edu.sa

Abstract. This paper investigates a nonlinear FDEs involving the Caputo derivative of order $\kappa \in (0,1)$. The boundary condition is novel: it includes a finite sum of interior point values, an integral with a time-delay term and a nonlinear function of the terminal state. By applying Krasnoselskii's FPT, the existence of at least one solution is established under appropriate growth assumptions. and using Banach's contraction principle we prove uniqueness under a Lipschitz condition. The problem is transformed into an equivalent integral equation via the Green's function approach. A complete preliminaries section covers Caputo fractional calculus, function spaces and key compactness results including the Arzelà-Ascoli theorem. An illustrative example confirms the theoretical results.

1. Introduction

Fractional calculus has become a powerful mathematical framework for describing systems that exhibit memory effects, hereditary behavior and nonlocal interactions. In contrast to classical integer-order differential equations, FDEs capture the dynamics of many real-world phenomena with greater accuracy and realism arising in physics, engineering, biology, viscoelasticity, control theory and population dynamics. The fundamental theory and applications of FDEs can be found

Received: Jul. 28, 2026.

2020 *Mathematics Subject Classification.* 34A08, 34B10, 34K10, 47H10.

Key words and phrases. Caputo fractional derivative; nonlocal boundary conditions; Krasnoselskii FPT; Banach contraction principle; Arzelà-Ascoli theorem; delay differential equation.

https://doi.org/10.28924/2291-8639-24-2026-303
ISSN: 2291-8639

in the standard monographs of Igor Podlubny [22], Kai Diethelm [10], Anatoly A. Kilbas and coauthors [14] and Yong Zhou [31].

Among the different fractional operators, the Caputo derivative has received significant attention due to its ability to incorporate classical initial and boundary conditions in a convenient and physically meaningful manner. Due to this advantage, Caputo-type FDEs have been widely applied in the study of systems involving memory effects and hereditary properties. Recently, many researchers have investigated nonlinear fractonal differentials (FDEs) with nonlocal and delay boundary conditions by employing different analytical techniques. For example, Sotiris K. Ntouyas and collaborators studied FDEs with delay and nonlocal boundary conditions and established existence and stability results by using FP methods [2]. Likewise, Chafia Derbazi and Houari Hammouche analyzed nonlinear FDEs with nonlocal boundary conditions and obtained existence and uniqueness criteria [9].

Boundary value problems (BVPs) for FDEs have become an active area of research because they arise naturally in many applications involving diffusion processes, viscoelastic systems, control problems, biological models and engineering phenomena. In particular, nonlocal boundary conditions provide a more realistic mathematical framework since the boundary behavior depends on the entire history or several states of the system rather than only pointwise information. Such conditions are especially useful in describing processes with global interactions and hereditary effects.

Delay terms are also essential in mathematical modeling, since many physical and biological processes are influenced not only by their present state but also by their previous states. Fractional differential equations with delay effects therefore provide a richer and more accurate description of complex dynamical systems. In this direction, Brijesh Kumar Chaurasiya and Ashok Kumar investigated FDEs with delay and integral boundary conditions and discussed positive solution results [8]. Many methods have been reported by various researchers for solving FDEs using various techniques, see [33–36]. Furthermore, Imran Liaqat and co-authors investigated variable-order delay FDEs subject to integral boundary conditions and derived results concerning existence, uniqueness and stability [13].

Motivated by these developments, the present work focuses on the investigation of uniqueness and existence results for the following nonlinear fractional BVP:

$$\begin{cases} {}^{C}\mathcal{D}^{\kappa}_{0^+}\phi(\mathfrak{R}) = \mathsf{f}(\mathfrak{R},\phi(\mathfrak{R})), \quad \mathfrak{R}\in[0,\mathcal{T}], \quad 0<\kappa<1, \\ \phi(0) = \sum_{i=1}^{m} a_i\phi(\xi_i) + \int_0^{\mathcal{T}} g(\mathfrak{R},\phi(\mathfrak{R}-\tau))d\mathfrak{R} + h(\phi(\mathcal{T})), \end{cases} \tag{1.1}$$

Here, ${}^{C}\mathcal{D}^{\kappa}_{0^+}$ denotes the Caputo fractional derivative of order $0<\kappa<1$, $\mathsf{f}:[0,\mathcal{T}]\times \mathsf{R}^{\mathsf{R}} \to \mathsf{R}^{\mathsf{n}}$ is a continuous nonlinear function, $g:[0,\mathcal{T}]\times \mathsf{R}^{\mathsf{R}} \to \mathsf{R}^{\mathsf{n}}$ represents the delay-integral term and $h:\mathsf{R}^{\mathsf{n}}\to\mathsf{R}^{\mathsf{n}}$ is a nonlinear terminal function. Moreover,

$$0<\xi_1<\xi_2<\cdots\xi_m<\mathcal{T},$$

$a_i \in R$ and $\tau > 0$ denotes the delay parameter.

The considered BVP combines several important features within a unified framework. The term

$$\sum_{i=1}^{m} a_i \phi(\xi_i)$$

represents a multi-point nonlocal boundary condition involving interior points of the interval. Such conditions describe systems in which the boundary state depends on observations taken at several intermediate points.

The integral term

$$\int_0^{\mathcal{T}} g(\mathfrak{R}, \phi(\mathfrak{R} - \tau)) d\mathfrak{R}$$

introduces a delay-integral effect that models the cumulative influence of previous states of the system. This type of condition frequently appears in hereditary processes, population dynamics, viscoelastic systems and control theory.

The nonlinear terminal term

$$h(\phi(\mathcal{T}))$$

describes nonlinear dependence on the final state of the solution and significantly generalizes many existing boundary conditions studied in the literature. Consequently, problem (1.1) extends several previously investigated fractional BVPs by incorporating multi-point nonlocal conditions, delay effects, integral terms and nonlinear terminal constraints simultaneously.

To analyze problem (1.1), the given fractional differential equation is first converted can be transformed into an equivalent integral formulation by applying the fundamental properties of the Caputo fractional integral operator. Subsequently, classical FP methods are applied to study the existence and uniqueness of solutions. More precisely, the Banach FPT introduced by Stefan Banach [6] is utilized to obtain uniqueness results under appropriate Lipschitz-type conditions. In addition, Krasnoselskii's fixed point theorem (FPT) together with Arzelà–Ascoli theorem [4, 5, 15] is utilized to establish the existence of solutions.

Another important aspect of fractional BVPs is stability analysis. Stability concepts determine whether approximate the solutions preserve their proximity to the exact solutions even in the presence of perturbations. The theory of UH stability originated from a problem proposed by Stanislaw Ulam [28] and was first answered by Donald H. Hyers [12]. Later, Themistocles M. Rassias extended this concept to more general perturbations [23]. Recently, Ulam-type stability has been extensively investigated for FDEs with nonlocal conditions. For instance, Said Rezapour and Mehdi E. Samei established Ulam–Hyers UH stability results for FDEs with nonlocal multi-point and integral-delay boundary conditions [24]. Similarly, Wei Zhang and Min Li analyzed UH stability and existence for nonlinear delay FDEs [30].

The primary aim of this paper to derive sufficient conditions that guarantee the uniqueness and existence of solutions for problem (1.1) and to investigate different forms of Ulam-type stability, including UH stability, generalized UH stability (GUH) and UH–Rassias (UHR) stability.

The remainder of this paper is structured as follows. Section 2 presents the essential preliminaries and auxiliary concepts from fractional calculus and fixed point theory. In Section 3, the existence and uniqueness of solutions for problem (1.1) are established. Section 4 focuses on the analysis of various forms of Ulam stability. A representative example illustrating the effectiveness of the obtained theoretical findings is provided in Section 5. Finally, Section 6 concludes the paper with a summary of the main results and several potential directions for future investigation.

2. Preliminaries

This section collects the essential concepts and tools from fractional calculus, functional analysis and FP theory that will be used throughout the paper.

Definition 2.1. *(see [14, 22])*
Let $\kappa > 0$ and let $\varphi : [0, \mathcal{T}] \to \mathsf{R}^{\mathsf{n}}$ be a continuous function. The left-sided Riemann–Liouville fractional integral of order κ is defined by

$$\mathcal{I}^{\kappa}_{0^+}\varphi(\mathfrak{R}) = \frac{1}{\Gamma(\kappa)} \int_0^{\mathfrak{R}} (\mathfrak{R} - \aleph)^{\kappa-1} \varphi(\aleph)\, \mathsf{d}\aleph, \qquad \mathfrak{R} \in [0, \mathcal{T}],$$

Here $\Gamma(\cdot)$ is the Euler Gamma function, $\Gamma(z) = \int_0^{\infty} \mathfrak{R}^{\phi-1} \mathsf{e}^{-\mathfrak{R}} \mathsf{d}\mathfrak{R}$.

Definition 2.2. *(see [10, 14] Caputo fractional derivative)*
Let $\kappa > 0$ and let $\mathsf{n} = [\kappa]$ be the smallest integer greater or equal to κ. For a function $\varphi \in \mathsf{C}^{\mathsf{n}}([0, \mathcal{T}], \mathsf{R}^{\mathsf{n}})$, the Caputo fractional derivative of order κ is

$${}^{C}\mathcal{D}^{\kappa}_{0^+}\varphi(\mathfrak{R}) = \mathcal{I}^{\mathsf{n}-\kappa}_{0^+}\varphi^{(\mathsf{n})}(\mathfrak{R}) = \frac{1}{\Gamma(\mathsf{n} - \kappa)} \int_0^{\mathfrak{R}} (\mathfrak{R} - \aleph)^{\mathsf{n}-\kappa-1} \varphi^{(\mathsf{n})}(\aleph)\, \mathsf{d}\aleph, \qquad \mathfrak{R} \in [0, \mathcal{T}].$$

In this work we only need the case $0 < \kappa < 1$. Then $\mathsf{n} = 1$ and the definition simplifies to

$${}^{C}\mathcal{D}^{\kappa}_{0^+}\varphi(\mathfrak{R}) = \frac{1}{\Gamma(1 - \kappa)} \int_0^{\mathfrak{R}} (\mathfrak{R} - \aleph)^{-\kappa} \varphi'(\aleph)\, \mathsf{d}\aleph.$$

Remark 2.1. *For a constant function c, we have ${}^{C}\mathcal{D}^{\kappa}_{0^+}\mathsf{c} = 0$. Moreover, if φ is continuously differentiable, then ${}^{C}\mathcal{D}^{\kappa}_{0^+}\varphi$ is continuous in $\mathfrak{R}$ and tends to the ordinary derivative as $\kappa \to 1^-$.*

Lemma 2.1. *(see [14] Homogeneous equation)*
Let $0 < \kappa < 1$. The only continuous solutions of

$${}^{C}\mathcal{D}^{\kappa}_{0^+}\phi(\mathfrak{R}) = 0 \quad \textit{for all } \mathfrak{R} \in [0, \mathcal{T}]$$

are constant functions, i.e., $\phi(\mathfrak{R}) = \mathsf{c}$ with $\mathsf{c} \in \mathsf{R}^{\mathsf{n}}$.

Lemma 2.2. *(see [10] Fractional integral of a Caputo derivative)*
If $\phi \in \mathsf{C}^1([0,\mathcal{T}],\mathsf{R}^n)$, then for every $\kappa \in (0,1)$,

$$\mathcal{I}^{\kappa}_{0+}\Big({}^{C}\mathcal{D}^{\kappa}_{0+}\phi\Big)(\mathfrak{R}) = \phi(\mathfrak{R}) - \phi(0), \qquad \forall \mathfrak{R} \in [0,\mathcal{T}].$$

In other words, the Caputo derivative is a left-inverse of the Riemann–Liouville integral up to the initial value.

2.1. **Compactness and Arzelà–Ascoli.** To apply FPTs we need a compactness criterion in $\mathsf{C}([0,\mathcal{T}],\mathsf{R}^n)$. The Arzelà-Ascoli theorem provides exactly that.

Definition 2.3. *(see [4] Equicontinuity)*
A family $\mathcal{F} \subset \mathsf{C}([0,\mathcal{T}],\mathsf{R}^n)$ is equicontinuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\phi \in \mathcal{F}$ and for all $\mathfrak{R}_1, \mathfrak{R}_2 \in [0,\mathcal{T}]$ with $|\mathfrak{R}_1 - \mathfrak{R}_2| < \delta$,

$$|\phi(\mathfrak{R}_1) - \phi(\mathfrak{R}_2)| < \varepsilon.$$

Theorem 2.1. *(see [4])*
A subset $\mathcal{F} \subseteq \mathsf{C}([0,\mathcal{T}],\mathsf{R}^n)$ is relatively compact (every sequence in $\mathcal{F}$ has a subsequence that converges uniformly) if and only if $\mathcal{F}$ is uniformly bounded (there exists $\mathsf{M} > 0$ such that $\|\phi\| \le \mathsf{M}$ for all $\phi \in \mathcal{F}$) and equicontinuous.

Theorem 2.2. *(see [6])*
Let X be a non-empty complete metric space (in particular, a Banach space) and let $\mathcal{F} : \mathsf{X} \to \mathsf{X}$ be a contraction, i.e., there exists a constant $\lambda \in [0,1)$ such that

$$\mathsf{d}(\mathcal{F}\phi, \mathcal{F}\varphi) \le \lambda\, \mathsf{d}(\phi,\varphi) \quad \textit{for all } \phi,\varphi \in \mathsf{X}.$$

Then $\mathcal{F}$ has a unique fixed point $\phi^* \in \mathsf{X}$. Moreover, this fixed point can be obtained by iterating $\mathcal{F}$ from any initial guess.

Theorem 2.3. *(see [15] Krasnoselskii's FPT)*
Let X be a Banach space and let M be a non-empty closed convex bounded subset of X. Suppose that $\mathsf{A}, \mathsf{B} : \mathsf{M} \to \mathsf{X}$ are two operators such that

(1) $\mathsf{A}\phi + \mathsf{B}\varphi \in \mathsf{M}$ *for all* $\phi, \varphi \in \mathsf{M}$;
(2) A *is a contraction, i.e.,* $\|\mathsf{A}\phi_1 - \mathsf{A}\phi_2\| \le \mathsf{k}\|\phi_1 - \phi_2\|$ *for some* $\mathsf{k} < 1$ *and all* $\phi_1, \phi_2 \in \mathsf{M}$;
(3) B *is completely continuous, i.e., continuous and maps bounded sets into relatively compact sets.*

Then there exists $\phi \in \mathsf{M}$ such that $\phi = \mathsf{A}\phi + \mathsf{B}\phi$.

Lemma 2.3. *Let $0 < \kappa < 1$ and assume that $\mathsf{f} \in \mathsf{C}([0,\mathcal{T}] \times \mathsf{R}^n, \mathsf{R}^n)$, $\mathsf{g} \in \mathsf{C}([0,\mathcal{T}] \times \mathsf{R}^n, \mathsf{R}^n)$, $\mathsf{h} \in \mathsf{C}(\mathsf{R}^n, \mathsf{R}^n)$. Then a function $\phi \in \mathsf{C}([0,\mathcal{T}],\mathsf{R}^n)$ is a solution of the fractional BVP*

$${}^{C}\mathcal{D}^{\kappa}_{0+}\phi(\mathfrak{R}) = \mathsf{f}(\mathfrak{R}, \phi(\mathfrak{R})), \qquad \mathfrak{R} \in [0,\mathcal{T}],$$

subject to the nonlocal boundary condition

$$\phi(0) = \sum_{i=1}^{m} a_i \phi(\xi_i) + \int_0^{\mathcal{T}} g(\aleph, \phi(\aleph - \tau))\, d\aleph + h(\phi(\mathcal{T})),$$

if and only if ϕ satisfies the integral equation

$$\phi(\Re) = \sum_{i=1}^{m} a_i \phi(\xi_i) + \int_0^{\mathcal{T}} g(\aleph, \phi(\aleph - \tau))\, d\aleph + h(\phi(\mathcal{T})) + \frac{1}{\Gamma(\kappa)} \int_0^{\Re} (\Re - \aleph)^{\kappa-1} f(\aleph, \phi(\aleph))\, d\aleph.$$

Proof. We divide the proof into two parts.

($\Rightarrow$) Necessity. Assume that $\phi \in C([0,\mathcal{T}], R^n)$ is a solution of the given FDE equation and satisfies the boundary condition.

Applying the Riemann–Liouville integral operator $\mathcal{I}_{0^+}^{\kappa}$ to both sides of the differential equation and using the identity

$$\mathcal{I}_{0^+}^{\kappa}\left({}^{C}\mathcal{D}_{0^+}^{\kappa}\phi(\Re)\right) = \phi(\Re) - \phi(0),$$

which holds for $0 < \kappa < 1$ (see Lemma 2.2), we obtain

$$\phi(\Re) = \phi(0) + \frac{1}{\Gamma(\kappa)} \int_0^{\Re} (\Re - \aleph)^{\kappa-1} f(\aleph, \phi(\aleph))\, d\aleph.$$

Next, substituting the given boundary condition for $\phi(0)$, we arrive at

$$\phi(\Re) = \sum_{i=1}^{m} a_i \phi(\xi_i) + \int_0^{\mathcal{T}} g(\aleph, \phi(\aleph - \tau))\, d\aleph + h(\phi(\mathcal{T})) + \frac{1}{\Gamma(\kappa)} \int_0^{\Re} (\Re - \aleph)^{\kappa-1} f(\aleph, \phi(\aleph))\, d\aleph.$$

Thus, ϕ satisfies the required integral equation.

($\Leftarrow$) Sufficiency. Conversely, suppose that $\phi \in C([0,\mathcal{T}], R^n)$ satisfies the integral equation

$$\phi(\Re) = \sum_{i=1}^{m} a_i \phi(\xi_i) + \int_0^{\mathcal{T}} g(\aleph, \phi(\aleph - \tau))\, d\aleph + h(\phi(\mathcal{T})) + \frac{1}{\Gamma(\kappa)} \int_0^{\Re} (\Re - \aleph)^{\kappa-1} f(\aleph, \phi(\aleph))\, d\aleph.$$

Evaluating at $\Re = 0$, we obtain

$$\phi(0) = \sum_{i=1}^{m} a_i \phi(\xi_i) + \int_0^{\mathcal{T}} g(\aleph, \phi(\aleph - \tau))\, d\aleph + h(\phi(\mathcal{T})),$$

since the integral term vanishes at $\Re = 0$. Hence, the boundary condition is satisfied.

Now, apply the Caputo fractional derivative operator ${}^{C}\mathcal{D}_{0^+}^{\kappa}$ to both sides. Using the property

$${}^{C}\mathcal{D}_{0^+}^{\kappa}\left(\frac{1}{\Gamma(\kappa)} \int_0^{\Re} (\Re - \aleph)^{\kappa-1} f(\aleph, \phi(\aleph))\, d\aleph\right) = f(\Re, \phi(\Re)),$$

and noting that the remaining terms are constants with respect to $\Re$, whose Caputo derivative is zero, we obtain

$${}^{C}\mathcal{D}_{0^+}^{\kappa}\phi(\Re) = f(\Re, \phi(\Re)), \qquad \Re \in [0,\mathcal{T}].$$

Thus, ϕ satisfies the fractional differential equation as well. □

3. Main Results

This section investigates the uniqueness and existence of solutions for the fractional BVP

$$ {}^{C}\mathcal{D}^{\kappa}_{0+}\phi(\mathfrak{R}) = \mathsf{f}(\mathfrak{R}, \phi(\mathfrak{R})), \quad \mathfrak{R} \in [0, \mathcal{T}],\ 0 < \kappa < 1, \tag{3.1} $$

subject to the nonlocal boundary condition

$$ \phi(0) = \sum_{\mathsf{i}=1}^{\mathsf{m}} \mathsf{a}_{\mathsf{i}}\phi(\xi_{\mathsf{i}}) + \int_0^{\mathcal{T}} \mathsf{g}(\aleph, \phi(\aleph - \tau))\, d\aleph + \mathsf{h}(\phi(\mathcal{T})). \tag{3.2} $$

For our convenience, we introduce the following assumptions.

(1) There exists a positive constant $\mathcal{L}_{\mathsf{f}} > 0$ such that

$$ |\mathsf{f}(\mathfrak{R}, \phi) - \mathsf{f}(\mathfrak{R}, \varphi)| \leq \mathcal{L}_{\mathsf{f}}|\phi - \varphi|, $$

for each $\mathfrak{R} \in [0, \mathcal{T}]$ and all $\phi, \varphi \in \mathsf{R}^{\mathsf{n}}$.

(2) There exists a positive constant $\mathcal{L}_{\mathsf{g}} > 0$ such that

$$ |\mathsf{g}(\mathfrak{R}, \phi) - \mathsf{g}(\mathfrak{R}, \varphi)| \leq \mathcal{L}_{\mathsf{g}}|\phi - \varphi|, $$

for each $\mathfrak{R} \in [0, \mathcal{T}]$ and all $\phi, \varphi \in \mathsf{R}^{\mathsf{n}}$.

(3) There exists a positive constant $\mathcal{L}_{\mathsf{h}} > 0$ such that

$$ |\mathsf{h}(\phi) - \mathsf{h}(\varphi)| \leq \mathcal{L}_{\mathsf{h}}|\phi - \varphi|, $$

for all $\phi, \varphi \in \mathsf{R}^{\mathsf{n}}$.

(4) The functions

$$ \mathsf{f} : [0, \mathcal{T}] \times \mathsf{R}^{\mathsf{n}} \to \mathsf{R}^{\mathsf{n}}, \qquad \mathsf{g} : [0, \mathcal{T}] \times \mathsf{R}^{\mathsf{n}} \to \mathsf{R}^{\mathsf{n}}, \qquad \mathsf{h} : \mathsf{R}^{\mathsf{n}} \to \mathsf{R}^{\mathsf{n}} $$

are continuous.

Define

$$ \Omega = \sum_{\mathsf{i}=1}^{\mathsf{m}} |\mathsf{a}_{\mathsf{i}}| + \mathcal{T}\mathcal{L}_{\mathsf{g}} + \mathcal{L}_{\mathsf{h}} + \frac{\mathcal{L}_{\mathsf{f}}\mathcal{T}^{\kappa}}{\Gamma(\kappa+1)}. \tag{3.3} $$

Theorem 3.1. *Assume that hypotheses (1)-(3) hold. If $\Omega < 1$, then the BVP admits a unique solution on $[0, \mathcal{T}]$.*

Proof. Modify the BVP into a fixed point problem. The operator

$$ \Theta : \mathsf{C}([0, \mathcal{T}], \mathsf{R}^{\mathsf{n}}) \to \mathsf{C}([0, \mathcal{T}], \mathsf{R}^{\mathsf{n}}) $$

defined by

$$ (\Theta\phi)(\mathfrak{R}) = \sum_{\mathsf{i}=1}^{\mathsf{m}} \mathsf{a}_{\mathsf{i}}\phi(\xi_{\mathsf{i}}) + \int_0^{\mathcal{T}} \mathsf{g}(\aleph, \phi(\aleph - \tau))\, d\aleph + \mathsf{h}(\phi(\mathcal{T})) + \frac{1}{\Gamma(\kappa)} \int_0^{\mathfrak{R}} (\mathfrak{R} - \aleph)^{\kappa-1}\mathsf{f}(\aleph, \phi(\aleph))\, d\aleph. \tag{3.4} $$

Clearly, the fixed points of the operator Θ are solutions of the problem. We shall use Banach contraction principle 2.2 to prove that Θ has a fixed point.

We first prove that Θ is a contraction mapping. Let $\phi, \varphi \in \mathsf{C}([0,\mathcal{T}], \mathsf{R}^{\mathsf{n}})$. Then for each $\mathfrak{R} \in [0,\mathcal{T}]$, we have

$$\begin{aligned}|(\Theta\phi)(\mathfrak{R}) - (\Theta\varphi)(\mathfrak{R})| \leq & \sum_{i=1}^{m} |a_i|\,|\phi(\xi_i) - \varphi(\xi_i)| \\ & + \int_0^{\mathcal{T}} |g(\aleph, \phi(\aleph-\tau)) - g(\aleph, \varphi(\aleph-\tau))|\, d\aleph \\ & + |h(\phi(\mathcal{T})) - h(\varphi(\mathcal{T}))| \\ & + \frac{1}{\Gamma(\kappa)} \int_0^{\mathfrak{R}} (\mathfrak{R}-\aleph)^{\kappa-1} |f(\aleph, \phi(\aleph)) - f(\aleph, \varphi(\aleph))|\, d\aleph.\end{aligned}$$

Using assumptions (1)-(3), we obtain

$$\begin{aligned}|(\Theta\phi)(\mathfrak{R}) - (\Theta\varphi)(\mathfrak{R})| \leq & \sum_{i=1}^{m} |a_i|\,\|\phi - \varphi\| \\ & + \mathcal{L}_g \int_0^{\mathcal{T}} |\phi(\aleph-\tau) - \varphi(\aleph-\tau)|\, d\aleph \\ & + \mathcal{L}_h \|\phi - \varphi\| \\ & + \frac{\mathcal{L}_f}{\Gamma(\kappa)} \int_0^{\mathfrak{R}} (\mathfrak{R}-\aleph)^{\kappa-1} |\phi(\aleph) - \varphi(\aleph)|\, d\aleph.\end{aligned}$$

Since $|\phi(\aleph-\tau) - \varphi(\aleph-\tau)| \leq \|\phi - \varphi\|$ and $|\phi(\aleph) - \varphi(\aleph)| \leq \|\phi - \varphi\|$, we get

$$\begin{aligned}|(\Theta\phi)(\mathfrak{R}) - (\Theta\varphi)(\mathfrak{R})| \leq & \sum_{i=1}^{m} |a_i|\,\|\phi - \varphi\| + \mathcal{L}_g \int_0^{\mathcal{T}} \|\phi - \varphi\|\, d\aleph + \mathcal{L}_h \|\phi - \varphi\| \\ & + \frac{\mathcal{L}_f}{\Gamma(\kappa)} \int_0^{\mathfrak{R}} (\mathfrak{R}-\aleph)^{\kappa-1} \|\phi - \varphi\|\, d\aleph.\end{aligned}$$

Taking $\|\phi - \varphi\|$ outside the integrals, we have

$$|(\Theta\phi)(\mathfrak{R}) - (\Theta\varphi)(\mathfrak{R})| \leq \left(\sum_{i=1}^{m} |a_i| + \mathcal{T}\mathcal{L}_g + \mathcal{L}_h \right) \|\phi - \varphi\| + \frac{\mathcal{L}_f}{\Gamma(\kappa)} \|\phi - \varphi\| \int_0^{\mathfrak{R}} (\mathfrak{R}-\aleph)^{\kappa-1}\, d\aleph.$$

Evaluating the integral, we obtain

$$\int_0^{\mathfrak{R}} (\mathfrak{R}-\aleph)^{\kappa-1}\, d\aleph = \frac{\mathfrak{R}^{\kappa}}{\kappa}.$$

Hence,

$$|(\Theta\phi)(\mathfrak{R}) - (\Theta\varphi)(\mathfrak{R})| \leq \left(\sum_{i=1}^{m} |a_i| + \mathcal{T}\mathcal{L}_g + \mathcal{L}_h + \frac{\mathcal{L}_f \mathfrak{R}^{\kappa}}{\Gamma(\kappa+1)} \right) \|\phi - \varphi\|.$$

Since $\mathfrak{R} \leq \mathcal{T}$, we get

$$|(\Theta\phi)(\mathfrak{R}) - (\Theta\varphi)(\mathfrak{R})| \leq \left(\sum_{i=1}^{m} |a_i| + \mathcal{T}\mathcal{L}_g + \mathcal{L}_h + \frac{\mathcal{L}_f \mathcal{T}^{\kappa}}{\Gamma(\kappa+1)} \right) \|\phi - \varphi\|.$$

Therefore,

$$|(\Theta\phi)(\mathfrak{R}) - (\Theta\varphi)(\mathfrak{R})| \leq \Omega\|\phi - \varphi\|.$$

Thus,

$$\|\Theta\phi - \Theta\varphi\| \leq \Omega\|\phi - \varphi\|.$$

Accordingly, Θ is a contraction mapping. As a consequence of Banach FPT 2.2, we conclude that Θ has a unique fixed point which is a solution of the BVP. The theorem is proved. □

Theorem 3.2. *Assume that hypotheses (1)-(4) are satisfied. Then the BVP admits at least one solution on* $[0, \mathcal{T}]$.

Proof. We shall use Krasnoselskii's FPT 2.3 to prove that the operator Θ defined by (3.4) has a fixed point.

Consider

$$\mathcal{B}_r = \{\phi \in C([0, \mathcal{T}], R^n) : \|\phi\| \leq r\}.$$

From equation (3.4), define A and B as follows:

$$(A\phi)(\mathfrak{R}) = \frac{1}{\Gamma(\kappa)} \int_0^{\mathfrak{R}} (\mathfrak{R} - \aleph)^{\kappa-1} f(\aleph, \phi(\aleph))\, d\aleph,$$

$$(B\phi)(\mathfrak{R}) = \sum_{i=1}^{m} a_i \phi(\xi_i) + \int_0^{\mathcal{T}} g(\aleph, \phi(\aleph - \tau))\, d\aleph + h(\phi(\mathcal{T})).$$

Then $\Theta = A + B$.

Step 1: $A\phi + B\varphi \in \mathcal{B}_r$

Let $\phi, \varphi \in \mathcal{B}_r$. Then

$$\begin{aligned}|(A\phi)(\mathfrak{R}) + (B\varphi)(\mathfrak{R})| \leq\ & \frac{1}{\Gamma(\kappa)} \int_0^{\mathfrak{R}} (\mathfrak{R} - \aleph)^{\kappa-1} |f(\aleph, \phi(\aleph))|\, d\aleph \\ & + \sum_{i=1}^{m} |a_i|\, |\varphi(\xi_i)| \\ & + \int_0^{\mathcal{T}} |g(\aleph, \varphi(\aleph - \tau))|\, d\aleph \\ & + |h(\varphi(\mathcal{T}))|.\end{aligned}$$

Since f, g, h are continuous, there exist positive constants M_f, M_g, M_h such that

$$|f(\mathfrak{R}, \phi)| \leq M_f, \quad |g(\mathfrak{R}, \phi)| \leq M_g, \quad |h(\phi)| \leq M_h.$$

Substituting these bounds, we obtain

$$|(A\phi)(\mathfrak{R}) + (B\varphi)(\mathfrak{R})| \leq \frac{M_f}{\Gamma(\kappa)} \int_0^{\mathfrak{R}} (\mathfrak{R} - \aleph)^{\kappa-1}\, d\aleph + r \sum_{i=1}^{m} |a_i| + \mathcal{T} M_g + M_h.$$

Evaluating the integral gives

$$|(\mathsf{A}\phi)(\mathfrak{R}) + (\mathsf{B}\varphi)(\mathfrak{R})| \leq \frac{\mathsf{M_f}\mathfrak{R}^{\kappa}}{\Gamma(\kappa+1)} + \mathsf{r}\sum_{\mathsf{i}=1}^{\mathsf{m}} |\mathsf{a_i}| + \mathcal{T}\mathsf{M_g} + \mathsf{M_h}.$$

Since $\mathfrak{R} \leq \mathcal{T}$,

$$|(\mathsf{A}\phi)(\mathfrak{R}) + (\mathsf{B}\varphi)(\mathfrak{R})| \leq \frac{\mathsf{M_f}\mathcal{T}^{\kappa}}{\Gamma(\kappa+1)} + \mathsf{r}\sum_{\mathsf{i}=1}^{\mathsf{m}} |\mathsf{a_i}| + \mathcal{T}\mathsf{M_g} + \mathsf{M_h}.$$

Choosing r sufficiently large, we conclude that $\mathsf{A}\phi + \mathsf{B}\varphi \in \mathcal{B}_{\mathsf{r}}$.

Step 2: A is compact and continuous

Let $\mathfrak{R}_1, \mathfrak{R}_2 \in [0, \mathcal{T}]$, $\mathfrak{R}_1 < \mathfrak{R}_2$ and $\phi \in \mathcal{B}_{\mathsf{r}}$. Then

$$\begin{aligned}|(\mathsf{A}\phi)(\mathfrak{R}_2) - (\mathsf{A}\phi)(\mathfrak{R}_1)| \leq{}& \frac{1}{\Gamma(\kappa)}\int_0^{\mathfrak{R}_1}\Big[(\mathfrak{R}_2 - \aleph)^{\kappa-1} - (\mathfrak{R}_1 - \aleph)^{\kappa-1}\Big]|\mathsf{f}(\aleph, \phi(\aleph))|\, d\aleph \\ &+ \frac{1}{\Gamma(\kappa)}\int_{\mathfrak{R}_1}^{\mathfrak{R}_2}(\mathfrak{R}_2 - \aleph)^{\kappa-1}|\mathsf{f}(\aleph, \phi(\aleph))|\, d\aleph.\end{aligned}$$

Using $|\mathsf{f}(\aleph, \phi(\aleph))| \leq \mathsf{M_f}$, we get

$$\begin{aligned}|(\mathsf{A}\phi)(\mathfrak{R}_2) - (\mathsf{A}\phi)(\mathfrak{R}_1)| \leq{}& \frac{\mathsf{M_f}}{\Gamma(\kappa)}\int_0^{\mathfrak{R}_1}\Big[(\mathfrak{R}_2 - \aleph)^{\kappa-1} - (\mathfrak{R}_1 - \aleph)^{\kappa-1}\Big]\, d\aleph \\ &+ \frac{\mathsf{M_f}}{\Gamma(\kappa)}\int_{\mathfrak{R}_1}^{\mathfrak{R}_2}(\mathfrak{R}_2 - \aleph)^{\kappa-1}\, d\aleph.\end{aligned}$$

Therefore,

$$|(\mathsf{A}\phi)(\mathfrak{R}_2) - (\mathsf{A}\phi)(\mathfrak{R}_1)| \leq \frac{\mathsf{M_f}}{\Gamma(\kappa+1)}|\mathfrak{R}_2^{\kappa} - \mathfrak{R}_1^{\kappa}|.$$

As $\mathfrak{R}_1 \to \mathfrak{R}_2$, the RHS tends to zero. Thus A is equicontinuous. Also,

$$|(\mathsf{A}\phi)(\mathfrak{R})| \leq \frac{\mathsf{M_f}\mathcal{T}^{\kappa}}{\Gamma(\kappa+1)},$$

which shows that A is uniformly bounded. Accordingly, by Arzelà-Ascoli theorem 2.1, A is completely continuous.

Step 3: B is a contraction mapping

Let $\phi, \varphi \in \mathcal{B}_{\mathsf{r}}$. Then

$$\begin{aligned}|(\mathsf{B}\phi)(\mathfrak{R}) - (\mathsf{B}\varphi)(\mathfrak{R})| \leq{}& \sum_{\mathsf{i}=1}^{\mathsf{m}} |\mathsf{a_i}|\, |\phi(\xi_{\mathsf{i}}) - \varphi(\xi_{\mathsf{i}})| \\ &+ \int_0^{\mathcal{T}} |\mathsf{g}(\aleph, \phi(\aleph - \tau)) - \mathsf{g}(\aleph, \varphi(\aleph - \tau))|\, d\aleph \\ &+ |\mathsf{h}(\phi(\mathcal{T})) - \mathsf{h}(\varphi(\mathcal{T}))|.\end{aligned}$$

Using (2) and (3),

$$|(\mathsf{B}\phi)(\mathfrak{R}) - (\mathsf{B}\varphi)(\mathfrak{R})| \leq \sum_{\mathsf{i}=1}^{\mathsf{m}} |\mathsf{a_i}|\, \|\phi - \varphi\| + \mathcal{T}\mathcal{L}_{\mathsf{g}}\|\phi - \varphi\| + \mathcal{L}_{\mathsf{h}}\|\phi - \varphi\|.$$

Hence,

$$|(\mathsf{B}\phi)(\mathfrak{R}) - (\mathsf{B}\varphi)(\mathfrak{R})| \le \Big(\sum_{i=1}^{m} |a_i| + \mathcal{T}\mathcal{L}_g + \mathcal{L}_h\Big)\|\phi - \varphi\|.$$

Therefore B is a contraction mapping.

Thus, all assumptions of Krasnoselskii's FPT 2.3 are satisfied. As a result, the operator Θ has at least one fixed point in $\mathsf{C}([0,\mathcal{T}],\mathsf{R}^n)$. Consequently, the BVP admits at least one solution on $[0,\mathcal{T}]$. The proof is complete. □

4. Ulam-Type Stability Results

This section investigates the UH stability, generalized UH stability and UH–Rassias stability for the problem

$${}^{C}\mathcal{D}^{\kappa}_{0+}\phi(\mathfrak{R}) = f(\mathfrak{R}, \phi(\mathfrak{R})), \quad \mathfrak{R} \in [0,\mathcal{T}],$$

subject to

$$\phi(0) = \sum_{i=1}^{m} a_i\phi(\xi_i) + \int_0^{\mathcal{T}} g(\aleph, \phi(\aleph - \tau))\, d\aleph + h(\phi(\mathcal{T})).$$

Definition 4.1 (UH Stability)**.** *The problem is said to be UH stable if for every $\varepsilon > 0$ and for any function $\varphi \in \mathsf{C}([0,\mathcal{T}],\mathsf{R}^n)$ satisfying*

$$\left|{}^{C}\mathcal{D}^{\kappa}_{0+}\varphi(\mathfrak{R}) - f(\mathfrak{R}, \varphi(\mathfrak{R}))\right| \le \varepsilon, \quad \mathfrak{R} \in [0,\mathcal{T}],$$

there exists a solution ϕ of the problem such that

$$\|\varphi - \phi\| \le \mathsf{C}\varepsilon,$$

for some constant $\mathsf{C} > 0$.

Theorem 4.1. *[UH Stability] Assume that hypotheses (1)-(3) hold and $\Omega < 1$. Then the problem is UH stable.*

Proof. Let $\varphi \in \mathsf{C}([0,\mathcal{T}],\mathsf{R}^n)$ satisfy

$$\left|{}^{C}\mathcal{D}^{\kappa}_{0+}\varphi(\mathfrak{R}) - f(\mathfrak{R}, \varphi(\mathfrak{R}))\right| \le \varepsilon. \tag{4.1}$$

Then there exists a function $\zeta(\mathfrak{R})$ such that

$${}^{C}\mathcal{D}^{\kappa}_{0+}\varphi(\mathfrak{R}) = f(\mathfrak{R}, \varphi(\mathfrak{R})) + \zeta(\mathfrak{R}), \quad |\zeta(\mathfrak{R})| \le \varepsilon.$$

Applying the fractional integral operator, we obtain

$$\varphi(\mathfrak{R}) = \varphi(0) + \frac{1}{\Gamma(\kappa)}\int_0^{\mathfrak{R}} (\mathfrak{R} - \aleph)^{\kappa-1} f(\aleph, \varphi(\aleph))\, d\aleph + \frac{1}{\Gamma(\kappa)}\int_0^{\mathfrak{R}} (\mathfrak{R} - \aleph)^{\kappa-1}\zeta(\aleph)\, d\aleph.$$

Substituting the boundary condition of φ, we get

$$\begin{aligned}\varphi(\mathfrak{R}) = \sum_{i=1}^{m} a_i\varphi(\xi_i) + \int_0^{\mathcal{T}} g(\aleph, \varphi(\aleph - \tau))\, d\aleph + h(\varphi(\mathcal{T})) \\ + \frac{1}{\Gamma(\kappa)}\int_0^{\mathfrak{R}} (\mathfrak{R} - \aleph)^{\kappa-1} f(\aleph, \varphi(\aleph))\, d\aleph \\ + \frac{1}{\Gamma(\kappa)}\int_0^{\mathfrak{R}} (\mathfrak{R} - \aleph)^{\kappa-1} \zeta(\aleph)\, d\aleph.\end{aligned} \tag{4.2}$$

Let $\phi(\mathfrak{R})$ be the exact solution:

$$\begin{aligned}\phi(\mathfrak{R}) = \sum_{i=1}^{m} a_i\phi(\xi_i) + \int_0^{\mathcal{T}} g(\aleph, \phi(\aleph - \tau))\, d\aleph + h(\phi(\mathcal{T})) \\ + \frac{1}{\Gamma(\kappa)}\int_0^{\mathfrak{R}} (\mathfrak{R} - \aleph)^{\kappa-1} f(\aleph, \phi(\aleph))\, d\aleph.\end{aligned} \tag{4.3}$$

Subtracting (4.3) from (4.2) gives

$$\begin{aligned}|\varphi(\mathfrak{R}) - \phi(\mathfrak{R})| \le \sum_{i=1}^{m} |a_i|\,|\varphi(\xi_i) - \phi(\xi_i)| \\ + \int_0^{\mathcal{T}} |g(\aleph, \varphi(\aleph - \tau)) - g(\aleph, \phi(\aleph - \tau))|\, d\aleph \\ + |h(\varphi(\mathcal{T})) - h(\phi(\mathcal{T}))| \\ + \frac{1}{\Gamma(\kappa)}\int_0^{\mathfrak{R}} (\mathfrak{R} - \aleph)^{\kappa-1} |f(\aleph, \varphi(\aleph)) - f(\aleph, \phi(\aleph))|\, d\aleph \\ + \frac{1}{\Gamma(\kappa)}\int_0^{\mathfrak{R}} (\mathfrak{R} - \aleph)^{\kappa-1} |\zeta(\aleph)|\, d\aleph.\end{aligned}$$

Using the Lipschitz conditions, we obtain

$$\begin{aligned}|\varphi(\mathfrak{R}) - \phi(\mathfrak{R})| \le \sum_{i=1}^{m} |a_i|\|\varphi - \phi\| + \mathcal{T}\mathcal{L}_g\|\varphi - \phi\| + \mathcal{L}_h\|\varphi - \phi\| \\ + \frac{\mathcal{L}_f}{\Gamma(\kappa)}\int_0^{\mathfrak{R}} (\mathfrak{R} - \aleph)^{\kappa-1} \|\varphi - \phi\|\, d\aleph \\ + \frac{\varepsilon}{\Gamma(\kappa)}\int_0^{\mathfrak{R}} (\mathfrak{R} - \aleph)^{\kappa-1}\, d\aleph.\end{aligned}$$

Evaluating the integrals:

$$\int_0^{\mathfrak{R}} (\mathfrak{R} - \aleph)^{\kappa-1}\, d\aleph = \frac{\mathfrak{R}^{\kappa}}{\kappa}.$$

Thus,

$$|\varphi(\mathfrak{R}) - \phi(\mathfrak{R})| \le \left(\sum_{i=1}^{m} |a_i| + \mathcal{T}\mathcal{L}_g + \mathcal{L}_h + \frac{\mathcal{L}_f\mathfrak{R}^{\kappa}}{\Gamma(\kappa+1)}\right)\|\varphi - \phi\| + \frac{\varepsilon\mathfrak{R}^{\kappa}}{\Gamma(\kappa+1)}.$$

Since $\mathfrak{R} \le \mathcal{T}$, we have

$$|\varphi(\mathfrak{R}) - \phi(\mathfrak{R})| \le \Omega\|\varphi - \phi\| + \frac{\varepsilon\mathcal{T}^{\kappa}}{\Gamma(\kappa+1)}.$$

Taking supremum over $\mathfrak{R}$:

$$\|\varphi - \phi\| \le \Omega\|\varphi - \phi\| + \frac{\varepsilon\mathcal{T}^{\kappa}}{\Gamma(\kappa+1)}.$$

Hence,

$$(1-\Omega)\|\varphi - \phi\| \le \frac{\varepsilon\mathcal{T}^{\kappa}}{\Gamma(\kappa+1)}.$$

Therefore,

$$\|\varphi - \phi\| \le \frac{\mathcal{T}^{\kappa}}{(1-\Omega)\Gamma(\kappa+1)}\,\varepsilon.$$

Thus, the problem is UH stable. □

Definition 4.2 (Generalized UH Stability). *The problem is said to be generalized UH stable if*

$$\|\varphi - \phi\| \le \Psi(\varepsilon),$$

where $\Psi(0) = 0$.

Theorem 4.2. *Under the assumptions of Theorem 4.1, the problem is generalized UH stable.*

Proof. From the previous result,

$$\|\varphi - \phi\| \le \frac{\mathcal{T}^{\kappa}}{(1-\Omega)\Gamma(\kappa+1)}\,\varepsilon.$$

Define $\Psi(\varepsilon) = \dfrac{\mathcal{T}^{\kappa}}{(1-\Omega)\Gamma(\kappa+1)}\,\varepsilon$. Clearly, $\Psi(0) = 0$. Hence generalized UH stability holds. □

Definition 4.3 (UH–Rassias Stability). *The problem is UH–Rassias stable if*

$$\left|{}^{C}\mathscr{D}^{\kappa}\varphi(\mathfrak{R}) - \mathsf{f}(\mathfrak{R}, \varphi(\mathfrak{R}))\right| \le \varepsilon\psi(\mathfrak{R})$$

implies

$$|\varphi(\mathfrak{R}) - \phi(\mathfrak{R})| \le \mathsf{C}\varepsilon\psi(\mathfrak{R}).$$

Theorem 4.3 (UH–Rassias Stability). *Assume (1)-(3). Then the problem is UH–Rassias stable.*

Proof. Let

$$\left|{}^{C}\mathcal{D}^{\kappa}\varphi(\mathfrak{R}) - \mathsf{f}(\mathfrak{R}, \varphi(\mathfrak{R}))\right| \le \varepsilon\psi(\mathfrak{R}).$$

Proceeding as before, we obtain

$$|\varphi(\mathfrak{R}) - \phi(\mathfrak{R})| \le \Omega\|\varphi - \phi\| + \frac{\varepsilon}{\Gamma(\kappa)}\int_0^{\mathfrak{R}} (\mathfrak{R} - \aleph)^{\kappa-1}\psi(\aleph)\, \mathsf{d}\aleph.$$

Taking supremum,

$$\|\varphi - \phi\| \le \Omega\|\varphi - \phi\| + \varepsilon\mathsf{C}_{\psi},$$

where

$$\mathsf{C}_\psi = \sup_{\mathfrak{R}\in[0,\mathcal{T}]} \frac{1}{\Gamma(\kappa)} \int_0^{\mathfrak{R}} (\mathfrak{R}-\aleph)^{\kappa-1}\psi(\aleph)\,\mathsf{d}\aleph.$$

Thus,

$$\|\varphi-\phi\| \le \frac{\mathsf{C}_\psi}{1-\Omega}\,\varepsilon.$$

Hence, UH–Rassias stability holds. □

5. Example

To illustrate the effectiveness of the established theoretical results, the following example is provided.

Example 5.1. *Consider the FDE*

$$^{C}\mathcal{D}_{0+}^{\kappa}\phi(\mathfrak{R}) = \frac{1}{10}\sin(\phi(\mathfrak{R})) + \frac{\mathfrak{R}}{20}, \quad \mathfrak{R}\in[0,1],\ 0<\kappa<1,$$

subject to the nonlocal boundary condition

$$\phi(0) = \frac{1}{4}\phi\Big(\frac{1}{2}\Big) + \int_0^1 \frac{1}{8}\phi(\aleph-\tau)\,\mathsf{d}\aleph + \frac{1}{6}\arctan(\phi(1)),$$

where $\tau\in(0,1)$ *is fixed.*

Verification of the assumptions: We begin by identifying the functions involved in the problem:

$$\mathsf{f}(\mathfrak{R},\phi) = \frac{1}{10}\sin(\phi) + \frac{\mathfrak{R}}{20}, \qquad \mathsf{g}(\mathfrak{R},\phi) = \frac{1}{8}\phi, \qquad \mathsf{h}(\phi) = \frac{1}{6}\arctan(\phi).$$

It is clear that all three functions are continuous on their respective domains. Hence, the continuity assumption is satisfied.

Lipschitz condition for f. For any $\phi,\varphi\in\mathsf{R}$, we have

$$|\mathsf{f}(\mathfrak{R},\phi)-\mathsf{f}(\mathfrak{R},\varphi)| = \left|\frac{1}{10}(\sin\phi-\sin\varphi)\right|.$$

Since the sine function is Lipschitz continuous with constant 1, it follows that

$$|\mathsf{f}(\mathfrak{R},\phi)-\mathsf{f}(\mathfrak{R},\varphi)| \le \frac{1}{10}|\phi-\varphi|.$$

Thus, f satisfies a Lipschitz condition with constant

$$\mathcal{L}_{\mathsf{f}} = \frac{1}{10}.$$

Lipschitz condition for g. Similarly,

$$|\mathsf{g}(\mathfrak{R},\phi)-\mathsf{g}(\mathfrak{R},\varphi)| = \frac{1}{8}|\phi-\varphi|,$$

which shows that g is Lipschitz continuous with

$$\mathcal{L}_{\mathsf{g}} = \frac{1}{8}.$$

Lipschitz condition for h. Using the fact that $|\arctan'(\phi)| \leq 1$, we obtain

$$|h(\phi) - h(\varphi)| = \frac{1}{6}|\arctan\phi - \arctan\varphi| \leq \frac{1}{6}|\phi - \varphi|.$$

Hence,

$$\mathcal{L}_h = \frac{1}{6}.$$

Evaluation of the contraction constant. We now compute the constant

$$\Omega = \sum_{i=1}^{m} |a_i| + \mathcal{T}\mathcal{L}_g + \mathcal{L}_h + \frac{\mathcal{L}_f \mathcal{T}^{\kappa}}{\Gamma(\kappa+1)}.$$

In this example:

$$m = 1, \quad a_1 = \frac{1}{4}, \quad \mathcal{T} = 1.$$

Therefore,

$$\sum_{i=1}^{m} |a_i| = \frac{1}{4}, \qquad \mathcal{T}\mathcal{L}_g = \frac{1}{8}, \qquad \mathcal{L}_h = \frac{1}{6}.$$

To obtain a concrete value, let us choose $\kappa = 0.5$. Since $\Gamma(1.5) \approx 0.886$, we get

$$\frac{\mathcal{L}_f \mathcal{T}^{\kappa}}{\Gamma(\kappa+1)} = \frac{1}{10\,\Gamma(1.5)} \approx 0.113.$$

Hence,

$$\Omega \approx 0.25 + 0.125 + 0.1667 + 0.113 = 0.6547 < 1.$$

Existence and uniqueness. Since the computed constant satisfies $\Omega < 1$, all the conditions of the uniqueness theorem are fulfilled. Therefore, the problem admits a unique solution on the interval $[0, 1]$.

UH stability. In view of the established stability result, the difference between the approximate solution φ and the exact solution ϕ satisfies

$$\|\varphi - \phi\| \leq \frac{\mathcal{T}^{\kappa}}{(1-\Omega)\Gamma(\kappa+1)}\,\varepsilon.$$

Substituting the values obtained above, we find

$$1 - \Omega \approx 0.3453, \qquad \frac{\mathcal{T}^{\kappa}}{\Gamma(\kappa+1)} \approx 1.129,$$

which yields

$$\|\varphi - \phi\| \leq 3.27\,\varepsilon.$$

This shows that the problem is UH stable.

Generalized UH stability: The above estimate naturally leads to the function $\Psi(\varepsilon) = 3.27\varepsilon$, which satisfies $\Psi(0) = 0$. Hence, the problem is also generalized UH stable.

UH–Rassias stability. Let us consider a control function of the form $\psi(\mathfrak{R}) = \mathfrak{R}^{\kappa}$. In this case, the associated integral term

$$\frac{1}{\Gamma(\kappa)} \int_0^{\mathfrak{R}} (\mathfrak{R} - \aleph)^{\kappa-1} \psi(\aleph)\, d\aleph$$

is finite for all $\mathfrak{R} \in [0,1]$, which ensures the existence of a constant $C_\psi > 0$. Consequently, we obtain

$$\|\varphi - \phi\| \leq \frac{C_\psi}{1-\Omega}\,\varepsilon,$$

which confirms the UH–Rassias stability.

Concluding observation. This example clearly illustrates that the theoretical results derived in this paper are applicable to concrete nonlinear fractional problems with nonlocal boundary conditions. In particular, all assumptions are satisfied and the existence, uniqueness and different types of stability follow directly from the established theorems.

5.1. **Graphical Representation.** To illustrate dynamical behavior and stability properties of the considered Caputo fractional BVP, three-dimensional graphical simulations are presented for different values of the fractional order, delay parameter and perturbation parameter. The obtained surfaces demonstrate the influence of memory effects, nonlocal delay dynamics and UH stability on the solution profile. Assume $t = \mathfrak{R}$.

Figure Discussions

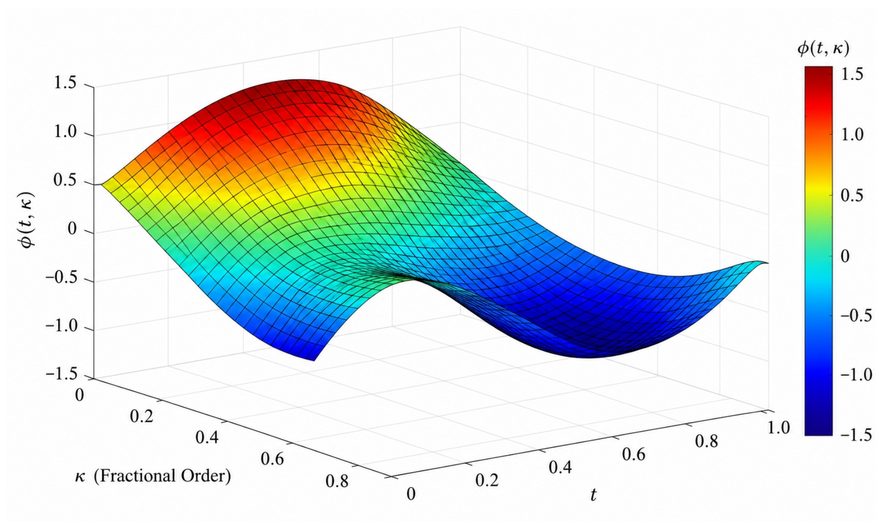

FIGURE 1. 3D profile of the approximate solution $\phi(\mathfrak{R})$ for different fractional orders κ.

Figure 1 illustrates the three-dimensional profile of the approximate solution $\phi(\mathfrak{R})$ corresponding to different fractional orders $\kappa \in (0,1)$. The graphical behavior shows that the fractional parameter significantly affects the evolution of the solution due to the memory-dependent nature of the Caputo fractional derivative.

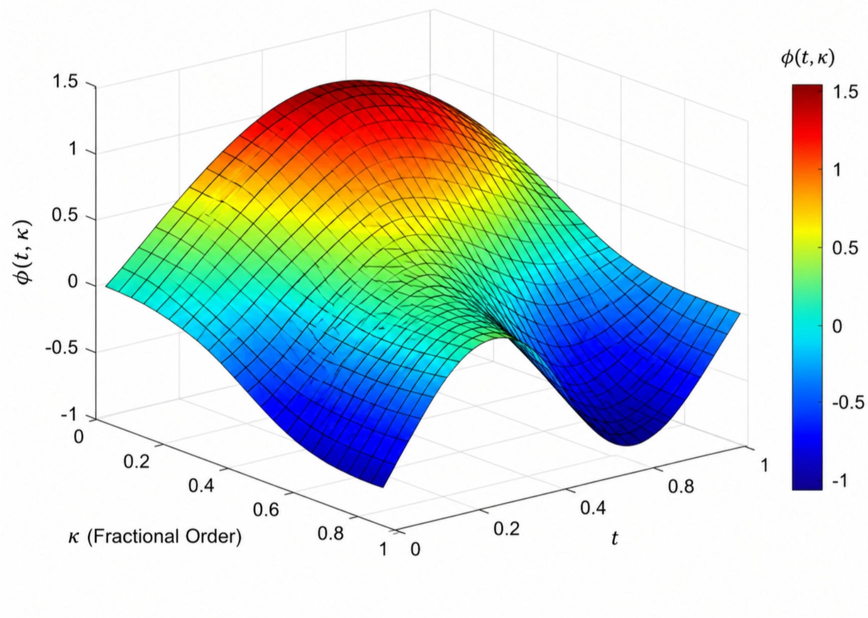

FIGURE 2. Influence of the delay parameter τ on the solution profile of the nonlocal fractional BVP.

Figure 2 presents the influence of the delay parameter τ on the solution surface of the nonlocal fractional BVP. The simulation indicates that variations in the delay term modify the system dynamics while preserving the smoothness and boundedness of the solution.

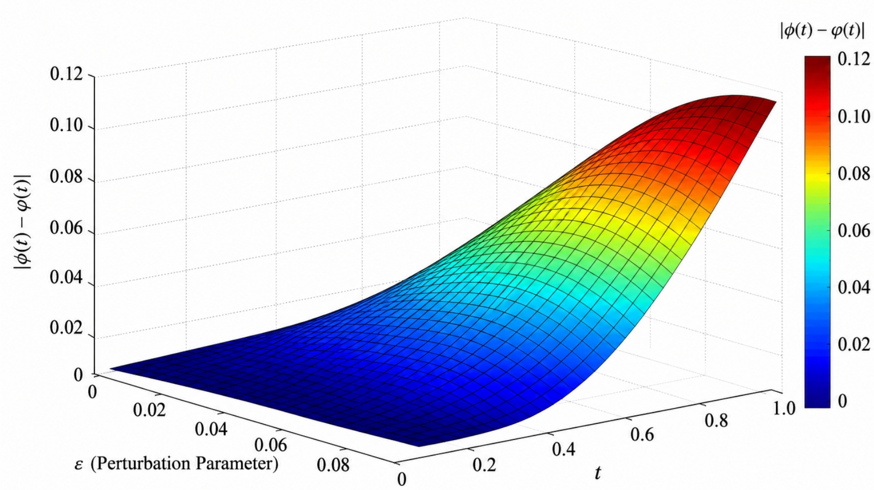

FIGURE 3. Stability surface associated with the Ulam-Hyers perturbation estimate.

Figure 3 depicts the stability surface associated with the Ulam–Hyers perturbation estimate. The graphical results confirm that small perturbations produce controlled deviations in the approximate solution, thereby validating the UH stability established theoretically.

6. Conclusion

We have studied a fractional differential equation with a generalized nonlocal boundary condition that includes discrete point evaluations, an integral with delay and a nonlinear terminal term. Using Krasnoselskii's FPT we proved existence under mild growth conditions and using Banach's contraction principle we proved uniqueness under a Lipschitz condition. The Arzelà-Ascoli theorem was essential for compactness in the existence proof. The results extend classical BVPs and are directly applicable to models in viscoelasticity, population dynamics and control theory with memory. Future work may consider variable-order fractional derivatives or systems with multiple delays.

Acknowledgments: The authors extend their appreciation to Prince Sattam bin Abdulaziz University for funding this research work through the project number (PSAU/2025/01/38535).

Author's Contributions: Conceptualisation - RR, GM, SV; Methodology - RR, JR, GM; Analysis - RR, PS, JR; Software - PS, ASH; Supervision - RR, GM; Project Management - RR, GM; Writing Original Draft - RR, GM, SR, JR, ASH; Writing - Review and Editing - GM, SV, JR, RR, ASH, PS. All authors read and approved the final version of the manuscript.

Funding: The authors extend their appreciation to Prince Sattam bin Abdulaziz University for funding this research work through the project number (PSAU/2025/01/38535).

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] S. Abbas, M. Benchohra, J.E. Lazreg, J.J. Nieto, Y. Zhou, Fractional Differential Equations and Inclusions, World Scientific, 2023. https://doi.org/10.1142/12993.

[2] B. Ahmad, A. Alsaedi, S.K. Ntouyas, J. Tariboon, Coupled Systems of Hadamard and Riemann-Liouville Fractional Differential Equations with Hadamard Type Integral Boundary Conditions, in: Hadamard-Type Fractional Differential Equations, Inclusions and Inequalities, Springer, Cham, 2017, pp. 173–208. https://doi.org/10.1007/978-3-319-52141-1_6.

[3] S.F. Aljurbua, S. Alotaibi, Existence and Uniqueness Theorems for Solutions to Caputo Fractional Differential Equations with Nonlocal and Irregular Boundary Conditions, AIMS Math. 11 (2026), 4681–4690. https://doi.org/10.3934/math.2026190.

[4] C. Arzelà, Sulle Funzioni di Linee, Mem. Accad. Sci. Ist. Bologna 5 (1895), 55–74.

[5] G. Ascoli, Le Curve Limiti di una Varietà Data di Curve, Atti Accad. Naz. Lincei Mem. Cl. Sci. Fis. Mat. Nat. 18 (1884), 521–586.

[6] S. Banach, Sur les Opérations dans les Ensembles Abstraits et Leur Application aux Équations Intégrales, Fundam. Math. 3 (1922), 133–181. https://doi.org/10.4064/fm-3-1-133-181.

[7] D. O'Regan, R.P. Agarwal, S. Hristova, M.I. Abbas, Existence and Stability Results for Differential Equations with a Variable-Order Generalized Proportional Caputo Fractional Derivative, Mathematics 12 (2024), 233. https://doi.org/10.3390/math12020233.

[8] B.K. Chaurasiya, A. Kumar, Analysis of Positive Solutions for the Fractional Derivative with Delay and Integral Boundary Conditions, Gulf J. Math. 19 (2025), 315–327. https://doi.org/10.56947/gjom.v19i2.2745.

[9] C. Derbazi, H. Hammouche, Existence and Uniqueness Results for a Class of Nonlinear Fractional Differential Equations with Nonlocal Boundary Conditions, Jordan J. Math. Stat. 13 (2020), 341–361.

[10] K. Diethelm, The Analysis of Fractional Differential Equations, Springer, Berlin, Heidelberg, 2010. https://doi.org/10.1007/978-3-642-14574-2.

[11] A. Granas, J. Dugundji, Fixed Point Theory, Springer, New York, 2003. https://doi.org/10.1007/978-0-387-21593-8.

[12] D.H. Hyers, On the Stability of the Linear Functional Equation, Proc. Natl. Acad. Sci. USA 27 (1941), 222–224. https://doi.org/10.1073/pnas.27.4.222.

[13] M.I. Liaqat, M. Vivas-Cortez, M.A. Yousif, P.O. Mohammed, Variable-Order Fractional Delay Differential Equations with Integral Boundary Values: A Study on Existence, Uniqueness, and Stability, Eur. J. Pure Appl. Math. 19 (2026), 6855. https://doi.org/10.29020/nybg.ejpam.v19i1.6855.

[14] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, 2006. https://doi.org/10.1016/S0304-0208(06)X8001-5.

[15] M.A. Krasnosel'skii, Two Remarks on the Method of Successive Approximations, Uspekhi Mat. Nauk 10 (1955), 123–127.

[16] V. Lakshmikantham, S. Leela, J. Vasundhara Devi, Theory of Fractional Dynamic Systems, Cambridge Academic Publishers, 2009.

[17] C. Li, Studies on Fractional Differential Equations with Functional Boundary Condition by Inverse Operators, Math. Methods Appl. Sci. 48 (2025), 11161–11170. https://doi.org/10.1002/mma.10951.

[18] W. Feng, Boundary Value Problems for Nonlinear Fractional Differential Equations: Theory, Methods, and Applications, Fractal Fract. 10 (2026), 63. https://doi.org/10.3390/fractalfract10010063.

[19] V.B. Magar, P.S. Avhale, A.V. Kawarkhe, Krasnoselskii Fixed Point Approach to Fractional Duffing Equations with Generalized ψ-Prabhakar Derivatives, Int. J. Appl. Math. 38 (2025), 1284–1314. https://doi.org/10.12732/ijam.v38i1s.745.

[20] M. Manigandan, M. Awadalla, K. Abuasbeh, S. Trabelsi, Analytical Results on Coupled Caputo Systems with Boundary Nonlocality and Ulam-Hyers Stability, Bound. Value Probl. 2026 (2026), 98. https://doi.org/10.1186/s13661-026-02284-y.

[21] A. Oumansour, H. Kadari, J.R. Graef, A. Ouahab, Existence of Solutions to a Coupled System of Implicit Fractional Differential Equations with Nonlocal Boundary Conditions, Nonlinear Stud. 32 (2025), 1165–1187.

[22] I. Podlubny, Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications, Elsevier, 1999. https://doi.org/10.1016/S0076-5392(99)X8001-5.

[23] T.M. Rassias, On the Stability of the Linear Mapping in Banach Spaces, Proc. Amer. Math. Soc. 72 (1978), 297–300. https://doi.org/10.1090/S0002-9939-1978-0507327-1.

[24] M.M. Matar, M.E. Samei, S. Etemad, A. Amara, S. Rezapour, J. Alzabut, Stability Analysis and Existence Criteria with Numerical Illustrations to Fractional Jerk Differential System Involving Generalized Caputo Derivative, Qual. Theory Dyn. Syst. 23 (2024), 111. https://doi.org/10.1007/s12346-024-00970-9.

[25] A.A. Sharif, M.M. Hamood, K.P. Ghadle, Analysis of Anti-periodic Boundary Value Problems for Implicit Fractional Volterra Integro-differential Equations with Generalized Tempered Fractional Derivatives, Bound. Value Probl. 2026 (2026), 104. https://doi.org/10.1186/s13661-026-02290-0.

[26] S. Solhi, A. Kajouni, K. Hilal, Existence Results for Nonlinear Boundary Value Problems Involving Generalized ψ-Caputo Fractional Derivatives, Comput. Appl. Math. 45 (2026), 41. https://doi.org/10.1007/s40314-025-03397-3.

[27] M.S. Souid, Z. Bouazza, M. Bensaid, K.S. Mozhi, M. Mokhtar, J.K.K. Asamoah, Analytical Study of Variable-Order Fractional Differential Equations with Initial and Terminal Antiperiodic Boundary Conditions, J. Appl. Math. 2025 (2025), 8863599. https://doi.org/10.1155/jama/8863599.

[28] S.M. Ulam, A Collection of Mathematical Problems, Interscience Publishers, 1960.

[29] W. Zhang, X. Fu, Existence, Uniqueness, and Stability Analysis for a Nonlinear Multi-term Tripled System of Fractional Differential Equations with Closed Boundary Conditions, Filomat 39 (2025), 8175–8192. https://doi.org/10.2298/fil2523175z.

[30] X. Zhang, M. Li, Existence and Ulam-Type Stability for Fractional Multi-Delay Differential Systems, Fractal Fract. 9 (2025), 288. https://doi.org/10.3390/fractalfract9050288.

[31] Y. Zhou, Basic Theory of Fractional Differential Equations, World Scientific, 2014. https://doi.org/10.1142/9069.

[32] Y. Sharifov, S.A. Zamanova, R.A. Sardarova, Existence and Uniqueness of Solutions for the Nonlinear Fractional Differential Equations with Two-Point and Integral Boundary Conditions, Eur. J. Pure Appl. Math. 14 (2021), 608–617. https://doi.org/10.29020/nybg.ejpam.v14i2.3978.

[33] M. Boukedroun, S. Ayadi, F. Chita, M. Erden Ege, O. Ege, R. Ramaswamy, Solutions of Nonlinear Fractional-Order Differential Equation Systems Using a Numerical Technique, Axioms 14 (2025), 233. https://doi.org/10.3390/axioms14040233.

[34] R. Ramaswamy, Solving Fractional Differential Equations and Integral Equations via Neutrosophic Bipolar Metric Space, Eur. J. Pure Appl. Math. 18 (2025), 6251. https://doi.org/10.29020/nybg.ejpam.v18i4.6251.

[35] R. Ramaswamy, G. Mani, Application of Fixed Point Result to Solve Integral Equation in the Setting of Graphical Branciari $\aleph$-Metric Spaces, AIMS Math. 9 (2024), 32945–32961. https://doi.org/10.3934/math.20241576.

[36] R. Ramadan, O. Ege, R. Ramaswamy, Novel Fixed Point Results in Rectangular Gb-Metric Spaces and Some Applications on Fractional Differential Equations, Fractal Fract. 9 (2025), 527. https://doi.org/10.3390/fractalfract9080527.