Int. J. Anal. Appl. (2026), 24:300

International Journal of Analysis and Applications

# Complex-Order $q$-Bi-Bazilevič-Type Bi-Univalent Functions Associated with a $q$-Poisson Convolution Operator and Leaf-Like Domains

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**Abstract.** A new subclass of bi-univalent functions of complex order is formulated by combining a $q$-Poisson convolution operator with a Ma–Minda subordination associated with a leaf-shaped image domain. The defining conditions are imposed simultaneously on a function and its inverse and involve a weighted interaction between a Bazilevič-type expression and the corresponding $q$-derivative term. This construction produces the class $\mathsf{QPL}_{\Sigma}^{r,q}(\epsilon, \pounds, \gamma; \Omega)$, whose parameters provide considerable flexibility in controlling its analytic and geometric structure. The non-emptiness of the proposed family is verified, and several relevant specializations are identified, including leaf-like $q$-Poisson bi-starlike and derivative-type subclasses. By expanding the subordinating functions and comparing the resulting coefficient identities for the function and its inverse, upper bounds for the initial Taylor–Maclaurin coefficients $|a_2|$ and

Received: Jul. 28, 2026.

2020 *Mathematics Subject Classification.* 30A36; 30C45; 81P68; 11B37.

*Key words and phrases.* bi-univalent functions; $q$-bi-Bazilevič-type functions; $q$-Poisson convolution operator; leaf-like domains; Ma–Minda subordination; coefficient estimates; Fekete–Szegö inequalities; $q$-calculus.

https://doi.org/10.28924/2291-8639-24-2026-300

ISSN: 2291-8639

$|a_3|$ are established. A Fekete–Szegö inequality for $\left|a_3 - \varkappa a_2^2\right|$, $\varkappa \in \mathbb{C}$, is also derived under appropriate nondegeneracy restrictions on the parameters. The results exhibit how the $q$-Poisson distribution, complex-order Bazilevič structure, and leaf-like geometry interact in the coefficient theory of bi-univalent functions.

## 1. Introduction

The theory of quantum calculus, commonly known as $q$-calculus, extends ordinary calculus by replacing the classical differential and integral operators with their $q$-counterparts. Its formulation is distinguished by the absence of the usual limiting process, and thus offers a convenient setting in which discrete structures can be analyzed alongside continuous ones. The parameter $q$ governs the degree of deformation, while the classical operators are recovered as $q \to 1^-$. This feature has made $q$-calculus a useful analytical tool in several areas, including basic hypergeometric functions, combinatorial analysis, orthogonal polynomials, quantum theory, and mathematical physics. The standard foundations of the subject, together with a systematic treatment of basic hypergeometric series, are presented in the monograph of Gasper and Rahman [23].

Within geometric function theory, the introduction of $q$-difference operators has opened a broad avenue for constructing parameter-dependent analogues of classical families of analytic functions. By replacing an ordinary derivative with an appropriate $q$-derivative, one obtains deformed versions of starlike, convex, close-to-convex, Bazilevič, univalent, and bi-univalent function classes. These formulations allow the geometric influence of the parameter $q$ to be observed directly through coefficient estimates and subordination conditions. They have also stimulated investigations of Fekete–Szegö functionals, Hankel determinants, radius problems, inclusion properties, growth estimates, and various extremal questions.

A related line of development originates in probability theory. For a discrete random variable, the probability mass function can be encoded in the coefficients of a generating series. This transformation converts distributional information into an analytic power series and thereby creates a useful interface between probabilistic modelling and complex analysis. In analytic combinatorics, generating functions are employed to extract asymptotic and enumerative information, whereas in geometric function theory their coefficients can be incorporated into linear operators acting on normalized analytic functions. Standard background on these analytic and combinatorial viewpoints may be found in [3–5].

The Hadamard product provides a particularly effective mechanism for transferring a probability-generating sequence into geometric function theory. If the coefficients of a probability distribution are used as the multipliers of the Taylor coefficients of an analytic function, the resulting convolution operator retains the algebraic structure of the original distribution while permitting geometric analysis through subordination. This principle has been applied to generating functions associated with the Poisson, Pascal, binomial, logarithmic, and other discrete laws. The corresponding operators have been used to establish coefficient inequalities, inclusion relations, growth and covering properties, and Fekete–Szegö-type bounds for analytic, univalent, bi-univalent, and harmonic function classes; see [9–17, 23, 25].

From a statistical perspective, the Poisson distribution is one of the most fundamental models for count data. Its probability mass function has a simple coefficient structure, and its generating function admits a particularly transparent power-series representation. These properties make the Poisson law well suited to convolution constructions. In addition to its familiar role in modelling event counts, Poisson-type structures arise in branching processes, enumeration theory, and non-linear functional relations involving the Lambert $W$-function; see [1, 2]. The analytic simplicity of its coefficients, together with its wide probabilistic relevance, explains its frequent appearance in operator-based studies of geometric function classes.

The $q$-Poisson law provides a natural deformation of this classical model. Its coefficients depend simultaneously on the distributional parameter and the quantum parameter $q$, and hence furnish a family of kernels whose probabilistic and analytic behavior varies continuously with $q$. The admissibility condition imposed on the parameters guarantees that the associated generating series is well defined, while the limiting transition $q \to 1^-$ restores the ordinary Poisson distribution. Thus, the $q$-Poisson model supplies a suitable bridge between discrete probability, quantum calculus, and convolution methods in geometric function theory.

Another essential ingredient in the present study is the Ma–Minda subordination framework. In this approach, the geometry of a function class is determined by an analytic dominant whose image of the open unit disk is a prescribed region. Choosing a leaf-like dominant introduces a specific geometric constraint on the corresponding logarithmic or Bazilevič-type expressions. When this geometric condition is combined with a $q$-difference operator and a probability-generated convolution kernel, the resulting class reflects three different forms of structure: the geometry of the dominant domain, the deformation induced by $q$-calculus, and the coefficient weighting inherited from the underlying probability distribution.

Motivated by this interaction, we define a convolution operator generated by the $q$-Poisson distribution and apply it to normalized analytic functions in $\mathbb{U}$. The operator is incorporated into two Ma–Minda-type subordination conditions, one for a bi-univalent function and the other for its analytic inverse. In this manner, we introduce a new family of complex-order $q$-Bi-Bazilevič-type functions associated with a leaf-shaped image domain. The proposed class unifies the probabilistic coefficient structure of the $q$-Poisson law, the analytic flexibility of complex-order Bazilevič expressions, and the geometric control provided by leaf-like subordination.

The principal objective of the paper is to investigate the initial coefficient behavior of this new class. In particular, estimates are obtained for the first nontrivial Taylor–Maclaurin coefficients, and a Fekete–Szegö-type inequality is derived. The dependence of these bounds on the deformation and distributional parameters reveals how the $q$-Poisson kernel modifies the analytic behavior of the associated bi-univalent functions. Several classical and previously studied situations can be recovered through suitable parameter choices, especially in the limiting regime $q \to 1^-$.

## 2. Preliminaries and Fundamental Function Classes

This section gathers the analytic notation and the basic function classes needed in the sequel. We begin with the standard normalized family, recall the Carathéodory class and its coefficient estimate, and then describe the inverse expansion associated with bi-univalent functions.

2.1. **Basic Analytic Classes.** Let $\mathcal{A}$ denote the class of functions $f$ that are analytic in the open unit disk

$$\mathbb{U} = \left\{ z \in \mathbb{C} : |z| < 1 \right\}$$

and normalized by

$$f(0) = 0 \qquad \text{and} \qquad f'(0) = 1.$$

Every function $f \in \mathcal{A}$ therefore possesses a Taylor–Maclaurin expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad z \in \mathbb{U}. \tag{2.1}$$

The coefficients $a_n$ encode the local analytic behavior of $f$ near the origin and constitute the principal objects in the coefficient problems considered throughout this work.

We also require the classical Carathéodory class $\mathcal{P}$, consisting of all functions $\varphi$ analytic in $\mathbb{U}$ that satisfy

$$\operatorname{Re} \varphi(z) > 0, \qquad z \in \mathbb{U},$$

together with the normalization

$$\varphi(0) = 1.$$

Accordingly, each $\varphi \in \mathcal{P}$ has a power-series representation

$$\varphi(z) = 1 + \sum_{n=1}^{\infty} \varphi_n z^n, \qquad z \in \mathbb{U}. \tag{2.2}$$

A fundamental coefficient estimate for functions with positive real part is provided by the Carathéodory lemma, namely,

$$\left| \varphi_n \right| \leq 2, \qquad n \geq 1. \tag{2.3}$$

This bound will be used repeatedly when the subordination conditions are converted into coefficient identities.

Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ whose members are univalent in $\mathbb{U}$. If $f \in \mathcal{S}$, then the Koebe one-quarter theorem guarantees that $f(\mathbb{U})$ contains the disk

$$\left\{ \xi \in \mathbb{C} : |\xi| < \frac{1}{4} \right\}.$$

Consequently, the inverse function

$$f^{-1} = f^{-1}$$

is analytic in a disk

$$|\xi| < r_0(f), \qquad r_0(f) \geq \frac{1}{4}.$$

On their respective domains of definition, $f$ and $f^{-1}$ satisfy the inverse relations

$$f^{-1}\Big(f(z)\Big) = z \qquad \text{and} \qquad f\Big(f^{-1}(\xi)\Big) = \xi.$$

Reverting the series in (2.1) about the origin gives the local expansion of the inverse function:

$$f^{-1}(\xi) = \xi - a_2\xi^2 + \Big(2a_2^2 - a_3\Big)\xi^3 - \Big(5a_2^3 - 5a_2a_3 + a_4\Big)\xi^4 + O\Big(\xi^5\Big). \tag{2.4}$$

Thus, the coefficients of $f^{-1}$ are expressed entirely in terms of the Taylor coefficients of $f$, a fact that is essential in the analysis of bi-univalent function classes.

A function $f \in \mathcal{A}$ is called *bi-univalent* in $\mathbb{U}$ whenever both $f$ and its analytic inverse $f^{-1}$ are univalent in $\mathbb{U}$. The family of all such functions is denoted by $\Sigma$. Hence,

$$f \in \Sigma \quad \Longleftrightarrow \quad f \text{ and } f^{-1} \text{ are univalent in } \mathbb{U}.$$

The simultaneous univalence of a function and its inverse makes the coefficient theory of $\Sigma$ considerably more delicate than that of the classical class $\mathcal{S}$.

2.2. **Ma–Minda Subordination and Geometrically Prescribed Domains.** Subordination is one of the principal tools for encoding geometric properties of analytic functions. For two functions $p$ and $h$ analytic in $\mathbb{U}$, we write

$$p(z) \prec h(z), \qquad z \in \mathbb{U},$$

whenever there exists a Schwarz function $\omega$, analytic in $\mathbb{U}$, such that

$$\omega(0) = 0, \qquad |\omega(z)| < 1,$$

and

$$p(z) = h\Big(\omega(z)\Big).$$

When $h$ is univalent in $\mathbb{U}$, this relation is equivalently expressed as

$$p(0) = h(0) \qquad \text{and} \qquad p(\mathbb{U}) \subset h(\mathbb{U}).$$

Thus, subordination converts an analytic relation into a geometric inclusion between image domains.

A particularly influential application of this principle is the framework introduced by Ma and Minda [24]. Let $\Omega$ be analytic and univalent in $\mathbb{U}$, normalized by

$$\Omega(0) = 1 \qquad \text{and} \qquad \Omega'(0) > 0.$$

Assume further that $\Omega(\mathbb{U})$ is symmetric with respect to the real axis and starlike with respect to the point 1. The corresponding Ma–Minda-type starlike family is defined by

$$K(\Omega) = \left\{ f \in \mathcal{A} : \frac{z f'(z)}{f(z)} \prec \Omega(z), \quad z \in \mathbb{U} \right\}. \tag{2.5}$$

Equivalently, for each $f \in K(\Omega)$, there exists a Schwarz function $\omega$ such that

$$\frac{z f'(z)}{f(z)} = \Omega\Big(\omega(z)\Big).$$

The defining relation in (2.5) shows that the geometry of the class is inherited directly from the dominant function $\Omega$. Indeed, the logarithmic derivative

$$\frac{z f'(z)}{f(z)}$$

is confined to the region $\Omega(\mathbb{U})$, and hence the shape of this image domain determines the analytic and geometric restrictions imposed on $f$. By changing the dominant $\Omega$, one may therefore construct a wide variety of subclasses within a single unified scheme.

This flexibility has led to the study of dominant functions whose images possess distinctive geometric forms. Among the most prominent examples are mappings onto lemniscate-shaped, lune-shaped, leaf-like, sine-associated, cosine-associated, and petal-shaped regions. Each choice produces a different admissible range for the logarithmic derivative and consequently a different family of starlike functions, with its own coefficient, distortion, growth, and extremal properties.

Of particular relevance to the present work are leaf-like image domains. When $\Omega(\mathbb{U})$ has a leaf-shaped geometry, the subordination condition does more than impose a positive-real-part restriction: it confines the relevant analytic expression to a nonclassical region whose boundary reflects the finer geometric features of the chosen dominant. Such domains have proved effective in defining refined subclasses of univalent and bi-univalent functions and in deriving coefficient estimates that depend explicitly on the Taylor coefficients of $\Omega$.

For comparison, Table 1 presents several representative Ma–Minda-type dominants together with the geometry of their image regions, the corresponding references, and numerical illustrations of the domains $\Omega(\mathbb{U})$. The table emphasizes that the dominant function is not merely an auxiliary component of the definition; rather, it is the geometric mechanism that shapes the resulting function class and governs the form of the associated coefficient inequalities.

Table 1. Selected analytic functions associated with geometrically distinguished image domains in geometric function theory.

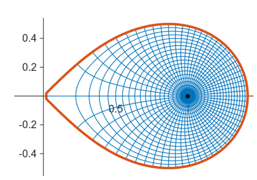
| **Generating function** | **Associated domain and class** | **Authors** | **Ref.** | **Image of $\mathbb{U}$** |
|---|---|---|---|---|
| $\Omega(z) = \sqrt{1+z}$ | The function maps $\mathbb{U}$ conformally onto the right-hand part of the Bernoulli lemniscate and generates the lemniscate starlike class. | Sokół and Stankiewicz | [18] |  |

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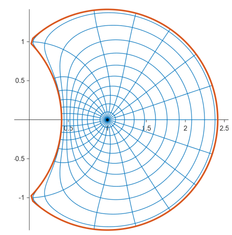
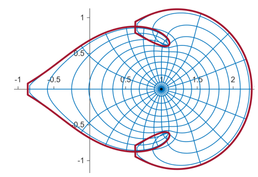
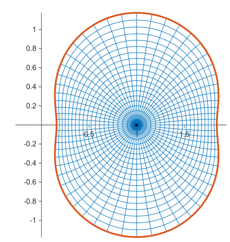
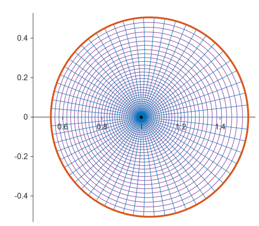
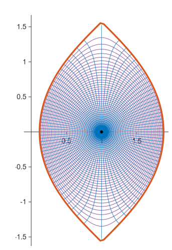
Table 1 – *continued from the previous page*

| **Generating function** | **Associated domain and class** | **Authors** | **Ref.** | **Image of $\mathbb{U}$** |
|---|---|---|---|---|
| $\Omega(z) = z + \sqrt{1+z^2}$ | The function maps $\mathbb{U}$ onto a lune-shaped domain and gives rise to the corresponding lune-associated starlike class. | Priya and Sharma | [20] |  |
| $\Omega(z) = z + \sqrt[3]{1+z^3}$ | The mapping generates a symmetric leaf-shaped image domain and is used to define the corresponding leaf-associated class. | Singh and Kaur | [21] |  |
| $\Omega(z) = 1 + \sin z$ | The function maps $\mathbb{U}$ onto an eight-shaped region contained in the right half-plane and generates the sine-associated starlike class. | Cho et al. | [7] |  |
| $\Omega(z) = \cos z$ | The function is associated with the cosine-generated class defined through $\dfrac{zf'(z)}{f(z)} \prec \cos z$. | Bano and Raza | [8] |  |
| $\Omega(z) = 1 + \operatorname{arcsinh}(z)$ | The function maps $\mathbb{U}$ onto a symmetric petal-shaped domain and generates the corresponding petal-shaped starlike class. | Arora and Kumar | [6] |  |

To describe the $q$-deformation of the leaf-like geometry used in this paper, we recall the following family.

**Definition 2.1.** *Let $f \in \Sigma$ be represented by* (2.1). *Then $f$ is said to belong to the class*

$$\Sigma_q\Big(\epsilon, \Omega(z;q)\Big)$$

*if*

$$\frac{z^{1-\epsilon}\, \eth_q\langle f(z)\rangle}{\big(f(z)\big)^{1-\epsilon}} \prec \Omega(z;q), \qquad z \in \mathbb{U}, \tag{2.6}$$

*where $\epsilon \in \mathbb{R}$. The dominant function $\Omega(z;q)$ is defined by*

$$\Omega(z;q) = \frac{[2]_q z}{2+(1-q)z} + \sqrt[3]{1+\left(\frac{[2]_q z}{2+(1-q)z}\right)^3}, \qquad 0 < q < 1, \tag{2.7}$$

*where the branch of the cubic root is chosen so that $\sqrt[3]{1} = 1$, and $\eth_q\langle f(z)\rangle$ denotes the $q$-derivative of $f$.*

The function in (2.7) may be expressed as

$$\Omega(z;q) = \Psi_q(z) + \sqrt[3]{1+\Psi_q^3(z)},$$

where

$$\Psi_q(z) = \frac{[2]_q z}{2+(1-q)z}.$$

The Möbius transformation $\Psi_q$ introduces the $q$-deformation, while the outer mapping generates the leaf-shaped geometry. In particular,

$$\Omega(0;q) = 1.$$

Moreover, the initial expansion of $\Omega$ is

$$\Omega(z;q) = 1 + \frac{[2]_q}{2} z - \frac{(1-q)[2]_q}{4} z^2 + O(z^3). \tag{2.8}$$

Consequently,

$$\eth_q\langle \Omega(0;q)\rangle = \frac{[2]_q}{2}.$$

In the limiting case $q \to 1^-$, one has

$$\Psi_q(z) \longrightarrow z$$

and hence

$$\Omega(z;q) \longrightarrow z + \sqrt[3]{1+z^3}.$$

Thus, the classical leaf-like mapping is recovered as the limiting member of the family (2.7).

For each fixed $0 < q < 1$, the image $\Omega(\mathbb{U};q)$ describes a $q$-deformed leaf-like region symmetric about the real axis. The subordination condition (2.6) therefore imposes a geometric restriction whose shape varies continuously with the quantum parameter $q$. The effect of this parameter on the corresponding image domains is illustrated in Figure 2.

TABLE 2. Geometric deformation of the leaf-like image domain $\Omega(\mathbb{U};q)$ for selected values of $q$.

**Fixed class-defining subordination condition**

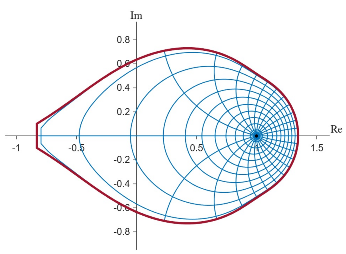
$$\frac{z^{1-\epsilon}\,\eth_q\langle f(z)\rangle}{\big(f(z)\big)^{1-\epsilon}} \prec \Omega(z;q), \qquad z \in \mathbb{U}$$

**$q$-deformed leaf-like dominant function**

$$\Omega(z;q) = \frac{[2]_q z}{2+(1-q)z} + \sqrt[3]{1+\left(\frac{[2]_q z}{2+(1-q)z}\right)^3}$$

**(a) Strongly deformed regime**

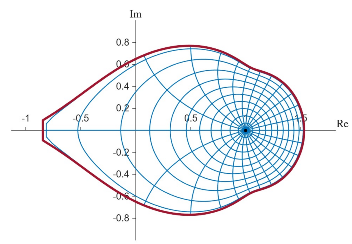
$q = 10^{-6}$




The image is relatively contracted and exhibits the strongest deformation from the classical leaf-like domain.

**(b) Lower intermediate regime**

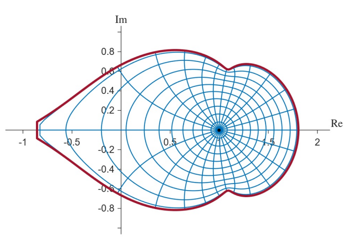
$q = 0.33$




The image begins to expand and gradually develops a more distinct leaf-like boundary.

**(c) Upper intermediate regime**

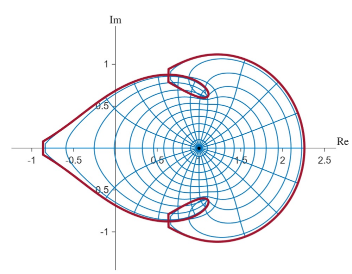
$q = 0.65$




The symmetric leaf-like profile becomes broader and more pronounced as the value of $q$ increases.

**(d) Near-classical regime**

$q = 0.9999$




The image approaches the classical leaf-like domain generated by $z + \sqrt[3]{1+z^3}$.

Table 2 illustrates the geometric influence of the deformation parameter $q$ while the defining subordination condition remains fixed. As $q$ increases from a value close to zero toward 1, the image domain gradually expands and approaches the classical leaf-like region. Indeed,

$$\lim_{q\to 1^-} \frac{[2]_q z}{2+(1-q)z} = z,$$

and consequently,

$$\lim_{q\to 1^-} \Omega(z;q) = z + \sqrt[3]{1+z^3}.$$

2.3. **The $q$-Poisson distribution and its convolution operator.** We begin with a probabilistic kernel whose coefficients will subsequently generate an analytic convolution operator. For $0 < q < 1$, define the $q$-exponential function by

$$\mathcal{E}_q(\chi) = \sum_{n=0}^{\infty} \frac{\chi^n}{[n]_q!}, \tag{2.9}$$

where the $q$-factorial is given by

$$[n]_q! = \begin{cases} 1, & n = 0, \\ [1]_q [2]_q \cdots [n]_q, & n \geq 1. \end{cases} \tag{2.10}$$

A discrete random variable $\mathcal{Y}$ is said to follow the $q$-Poisson distribution with intensity parameter $r > 0$ if

$$\mathbb{P}_q(\mathcal{Y} = n) = \frac{r^n}{[n]_q!\,\mathcal{E}_q(r)}, \qquad n \in \mathbb{N} \cup \{0\}. \tag{2.11}$$

Indeed, whenever $\mathcal{E}_q(r)$ converges, one has

$$\sum_{n=0}^{\infty} \mathbb{P}_q(\mathcal{Y} = n) = \frac{1}{\mathcal{E}_q(r)} \sum_{n=0}^{\infty} \frac{r^n}{[n]_q!} = 1.$$

*Admissible parameter region.* The parameters $r$ and $q$ cannot be selected independently. To determine the convergence region, let

$$u_n = \frac{r^n}{[n]_q!}.$$

Then

$$\frac{u_{n+1}}{u_n} = \frac{r}{[n+1]_q},$$

and, since

$$\lim_{n\to\infty} [n+1]_q = \frac{1}{1-q},$$

we obtain

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = r(1-q).$$

Consequently, the normalizing factor $\mathcal{E}_q(r)$ is finite precisely under the condition

$$r(1-q) < 1. \tag{2.12}$$

Thus, the admissible parameter set can be written as

$$\mathscr{D} = \{(r, q) \in (0, \infty) \times (0, 1) : r(1 - q) < 1\},$$

or, equivalently,

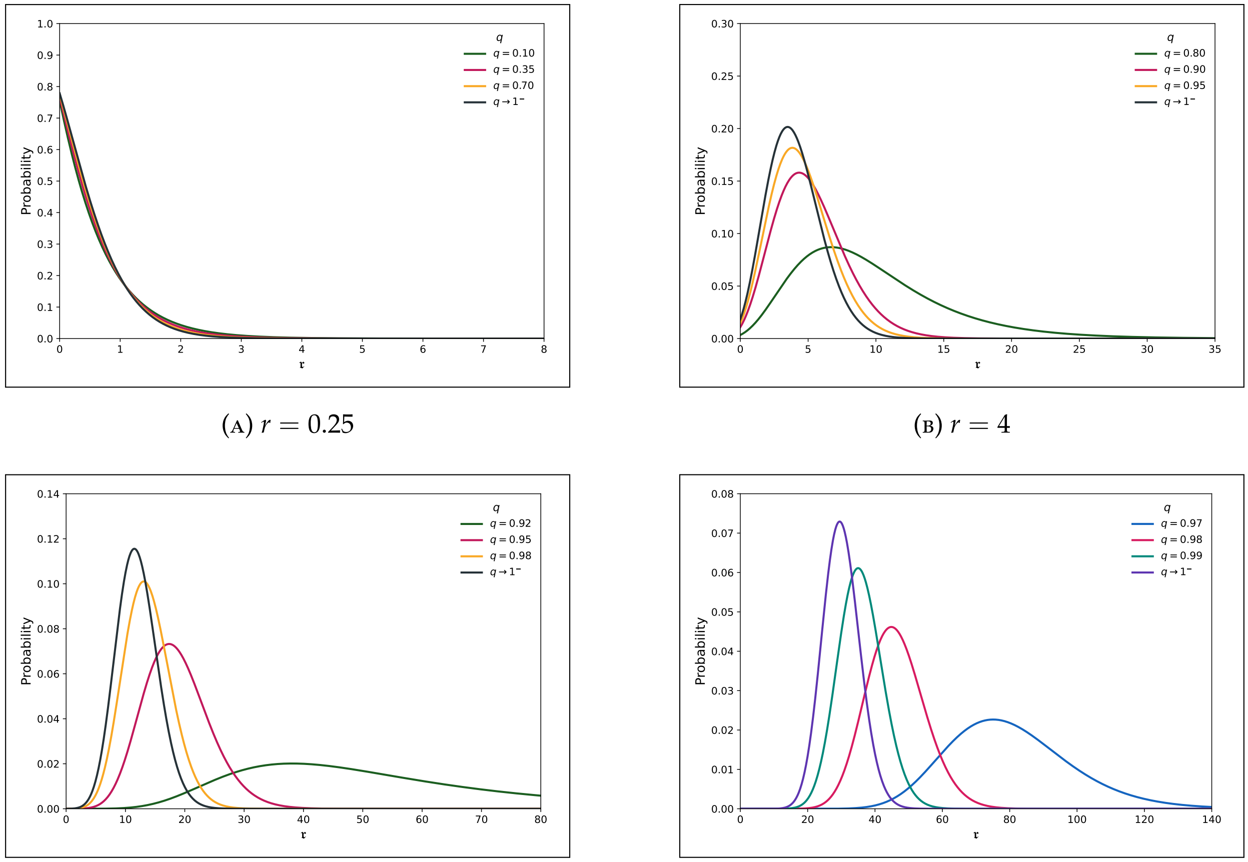
$$\begin{cases} 0 < q < 1, & 0 < r \leq 1, \\ 1 - \dfrac{1}{r} < q < 1, & r > 1. \end{cases} \tag{2.13}$$

Table 3 records the admissible $q$-intervals corresponding to the values of $r$ used in Figure 1.

TABLE 3. Admissible $q$-ranges for the selected intensity parameters.

| **Panel** | **Intensity** $r$ | **Admissible range of** $q$ | **Geometric interpretation** |
|---|---|---|---|
| (a) | $r = 0.25$ | $0 < q < 1$ | The complete interval $(0, 1)$ is admissible because $r \leq 1$. |
| (b) | $r = 4$ | $\frac{3}{4} < q < 1$ | The deformation parameter must remain above 0.75. |
| (c) | $r = 12$ | $\frac{11}{12} < q < 1$ | Only values sufficiently close to unity are admissible. |
| (d) | $r = 30$ | $\frac{29}{30} < q < 1$ | The allowed interval is concentrated near the classical limit $q = 1$. |

*Behavior of the probability profiles.* Figure 1 illustrates the effect of the deformation parameter $q$ on the probability mass function for representative values of $r$.




(A) $r = 0.25$ (B) $r = 4$

(C) $r = 12$ (D) $r = 30$

Figure 1. Profiles of the $q$-Poisson probability mass function for selected values of the intensity parameter $r$. In each panel, the parameter $q$ is chosen from the admissible interval determined by $r(1-q) < 1$. The graphs demonstrate how the $q$-deformation changes the position of the mode, the peak height, and the dispersion of the distribution. As $q \to 1^-$, the corresponding profiles approach the classical Poisson distribution.

For small values of $r$, the admissible interval for $q$ is relatively broad, allowing the deformation effect to be observed over a large range of parameters. As $r$ increases, condition (2.12) forces $q$ to move closer to unity. This explains why the profiles associated with larger values of $r$ display a more restricted range of admissible deformations.

*The associated analytic kernel.* Motivated by the probability weights in (2.11), we introduce the normalized analytic function

$$\mathrm{P}_q(z) = z + \sum_{n=2}^{\infty} \frac{r^{n-1}}{[n-1]_q!\,\mathcal{E}_q(r)} z^n, \qquad z \in \mathbb{U}. \tag{2.14}$$

The dependence of $\mathrm{P}_q$ on $r$ and $q$ is suppressed from the notation whenever no ambiguity can arise.

For the coefficients in (2.14), we have

$$\lim_{n\to\infty}\left|\frac{r^n/[n]_q!}{r^{n-1}/[n-1]_q!}\right| = r(1-q).$$

Therefore, under (2.12), the radius of convergence of (2.14) is

$$R = \frac{1}{r(1-q)} > 1.$$

Hence, $\mathrm{P}_q$ is analytic in a disk containing $\overline{\mathbb{U}}$. Moreover,

$$\mathrm{P}_q(0) = 0 \qquad \text{and} \qquad \mathrm{P}_q'(0) = 1,$$

so that

$$\mathrm{P}_q \in \mathcal{A}.$$

*The q-Poisson convolution operator.* Let

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}.$$

We define the $q$-Poisson convolution operator

$$\mathrm{P}_q^r : \mathcal{A} \longrightarrow \mathcal{A}$$

by the Hadamard product

$$\mathrm{P}_q^r(f(z)) = \mathrm{P}_q(z) * f(z) = z + \sum_{n=2}^{\infty} \frac{r^{n-1}}{[n-1]_q!\,\mathcal{E}_q(r)} a_n z^n, \qquad z \in \mathbb{U}. \tag{2.15}$$

In particular, the initial terms of the transformed function are

$$\mathrm{P}_q^r(f(z)) = z + \frac{r}{\mathcal{E}_q(r)} a_2 z^2 + \frac{r^2}{[2]_q!\,\mathcal{E}_q(r)} a_3 z^3 + \frac{r^3}{[3]_q!\,\mathcal{E}_q(r)} a_4 z^4 + \cdots. \tag{2.16}$$

Thus, $\mathrm{P}_q^r$ acts diagonally on the Taylor–Maclaurin coefficients of $f$, with multipliers determined by the $q$-Poisson probabilities.

The operator is linear, and its normalization is preserved:

$$\mathrm{P}_q^r(af_1 + bf_2) = a\,\mathrm{P}_q^r(f_1) + b\,\mathrm{P}_q^r(f_2),$$

whenever the linear combination is normalized appropriately.

Finally, since

$$[n]_q \longrightarrow n \qquad \text{and} \qquad \mathcal{E}_q(r) \longrightarrow e^r \quad \text{as } q \to 1^-,$$

the operator $\mathrm{P}_q^r$ reduces to the classical Poisson-distribution convolution operator

$$\mathrm{P}_q^r(f(z)) \longrightarrow z + \sum_{n=2}^{\infty} \frac{r^{n-1}}{(n-1)!\,e^r} a_n z^n.$$

This limiting relation confirms that the proposed operator constitutes a natural $q$-extension of its classical counterpart.

## 3. A $q$-Poisson Leaf-Like Bi-Bazilevič Class

The purpose of this section is to merge two independent structures: the coefficient-weighting action of the $q$-Poisson convolution operator $\mathrm{P}_q^r$, introduced in (2.15), and the geometric constraint generated by the $q$-deformed leaf-like function $\Omega(z;q)$ given in (2.7). Their combination leads to a flexible family of bi-univalent functions in which the analytic behavior of both a function and its inverse is governed by the same leaf-like image domain.

Let $f$ and its analytic inverse $f^{-1} = f^{-1}$ be defined by (2.1) and (2.4), respectively. Throughout this section, we assume that

$$0 < q < 1, \qquad r > 0, \qquad r(1-q) < 1.$$

The last condition ensures the convergence of the underlying $q$-Poisson series and, consequently, guarantees that the coefficients of the associated $q$-Poisson convolution operator are well defined.

For subsequent use, we introduce the following auxiliary functional:

$$\mathcal{B}_{\epsilon,\alpha,\gamma}^{r,q}[f](z) := 1 + \frac{1}{\gamma}\left[(1-£)\left(\frac{\mathrm{P}_q^r(f(z))}{z}\right)^{\epsilon} + £\,\frac{z^{1-\epsilon}\,\eth_q\left\langle \mathrm{P}_q^r(f(z))\right\rangle}{\left(\mathrm{P}_q^r(f(z))\right)^{1-\epsilon}} - 1\right], \qquad z \in \mathbb{U}, \tag{3.1}$$

where

$$\epsilon, £ \in \mathbb{C}, \qquad \gamma \in \mathbb{C}^*.$$

The functional in (3.1) combines a Bazilevič-type power term with a normalized $q$-derivative expression associated with the $q$-Poisson transform of $f$. The parameter $\epsilon$ controls the power structure, £ determines the relative contribution of the two constituent terms, and the nonzero complex parameter $\gamma$ provides an additional scaling and rotational effect.

Whenever nonintegral complex powers occur, they are interpreted through analytic branches selected in a neighborhood of the origin. This choice is well defined because

$$\frac{\mathrm{P}_q^r(f(z))}{z} = 1 + O(z) \qquad (z \to 0),$$

and the branches are normalized so that

$$\left(\frac{\mathrm{P}_q^r(f(z))}{z}\right)^{\epsilon}\Bigg|_{z=0} = 1.$$

Consequently,

$$\mathcal{B}_{\epsilon,\alpha,\gamma}^{r,q}[f](0) = 1,$$

which is consistent with the normalization of the Ma–Minda dominant function used in the defining subordination conditions.

The parameter $\epsilon$ controls the Bazilevič-type power structure, whereas £ determines the balance between the quotient term and the $q$-derivative term. The nonzero complex parameter $\gamma$ provides an additional scaling and rotational effect.

**Definition 3.1.** *A function $f \in \Sigma$ is said to belong to the class* $\mathsf{QPL}_{\Sigma}^{r,q}(\epsilon, \pounds, \gamma; \Omega)$, *if*

$$\mathcal{B}_{\epsilon,\alpha,\gamma}^{r,q}[f](z) \prec \Omega(z;q), \qquad z \in \mathbb{U}, \tag{3.2}$$

*and*

$$\mathcal{B}_{\epsilon,\alpha,\gamma}^{r,q}[f^{-1}](\xi) \prec \Omega(\xi;q), \qquad \xi \in \mathbb{U}. \tag{3.3}$$

*Here, in the second condition,*

$$\mathcal{B}_{\epsilon,\alpha,\gamma}^{r,q}[f^{-1}](\xi) = 1 + \frac{1}{\gamma}\left[(1-\pounds)\left(\frac{\mathrm{P}_q^r(f^{-1}(\xi))}{\xi}\right)^{\epsilon} + \pounds\frac{\xi^{1-\epsilon}\,\eth_q\left\langle \mathrm{P}_q^r(f^{-1}(\xi))\right\rangle}{\left(\mathrm{P}_q^r(f^{-1}(\xi))\right)^{1-\epsilon}} - 1\right].$$

Definition 3.1 imposes the same geometric restriction on the $q$-Poisson transforms of both $f$ and $f^{-1}$. Thus, the image domains

$$\mathcal{B}_{\epsilon,\alpha,\gamma}^{r,q}[f](\mathbb{U}) \quad \text{and} \quad \mathcal{B}_{\epsilon,\alpha,\gamma}^{r,q}[f^{-1}](\mathbb{U})$$

are simultaneously contained in the leaf-like region $\Omega(\mathbb{U};q)$. This two-sided requirement is the essential feature distinguishing the present class from its corresponding univalent counterpart.

3.1. **Structural reductions.** The parameters $\epsilon$, £, and $\gamma$ generate several natural subclasses. To avoid repeating nearly identical subordinations for the function and its inverse, Table 4 records the reduced functional for $f$; in each case, the corresponding inverse condition is obtained by replacing

$$f(z),\ z \quad \text{with} \quad f^{-1}(\xi),\ \xi,$$

respectively.

TABLE 4. Principal reductions of $\mathsf{QPL}_{\Sigma}^{r,q}(\epsilon, \pounds, \gamma; \Omega)$.

| **No.** | **Parameter choice** | **Resulting subclass** | **Reduced defining functional** |
|---|---|---|---|
| **1** | $\epsilon = 0$ | Generalized $q$-Poisson leaf-like bi-starlike-type class | $1 + \frac{\pounds}{\gamma}\left[\frac{z\,\eth_q\left\langle \mathrm{P}_q^r(f(z))\right\rangle}{\mathrm{P}_q^r(f(z))} - 1\right]$ |
| **2** | $\epsilon = 0,\ \pounds = \gamma \neq 0$ | Normalized $q$-Poisson leaf-like bi-starlike class | $\frac{z\,\eth_q\left\langle \mathrm{P}_q^r(f(z))\right\rangle}{\mathrm{P}_q^r(f(z))}$ |
| **3** | $\epsilon = 1$ | Weighted quotient–derivative-type bi-univalent class | $1 + \frac{1}{\gamma}\left[(1-\pounds)\frac{\mathrm{P}_q^r(f(z))}{z} + \pounds\,\eth_q\left\langle \mathrm{P}_q^r(f(z))\right\rangle - 1\right]$ |
| **4** | $\epsilon = 1,\ \pounds = \gamma = 1$ | $q$-Poisson leaf-like derivative-type bi-univalent class | $\eth_q\left\langle \mathrm{P}_q^r(f(z))\right\rangle$ |

In each row of Table 4, the displayed functional is required to be subordinate to $\Omega(z;q)$. The corresponding condition for the inverse function is obtained by replacing $f(z)$ and $z$ with $f^{-1}(\xi)$ and $\xi$, respectively.

3.2. **The classical limiting configuration.** The limiting process $q \to 1^-$ simultaneously transforms the $q$-difference operator, the $q$-Poisson kernel, and the leaf-like dominant function. More precisely,

$$ð_q\langle f(z)\rangle \longrightarrow f'(z), \qquad [n]_q \longrightarrow n,$$

and

$$\Omega(z;q) \longrightarrow \Omega_0(z) := z + \sqrt[3]{1+z^3}. \tag{3.4}$$

Moreover, the operator $\mathrm{P}_q^r$ approaches the classical Poisson convolution operator

$$\mathcal{P}_r(f)(z) = z + \sum_{n=2}^{\infty} \frac{r^{n-1}}{(n-1)!\,e^r} a_n z^n. \tag{3.5}$$

Accordingly, Definition 3.1 reduces to the classical Poisson leaf-like bi-Bazilevič-type class determined by

$$1 + \frac{1}{\gamma}\left[(1-\pounds)\left(\frac{\mathcal{P}_r(f)(z)}{z}\right)^{\epsilon} + \pounds\frac{z^{1-\epsilon}\,(\mathcal{P}_r(f)(z))'}{(\mathcal{P}_r(f)(z))^{1-\epsilon}} - 1\right] \prec \Omega_0(z), \tag{3.6}$$

together with

$$1 + \frac{1}{\gamma}\left[(1-\pounds)\left(\frac{\mathcal{P}_r(f^{-1})(\xi)}{\xi}\right)^{\epsilon} + \pounds\frac{\xi^{1-\epsilon}\left(\mathcal{P}_r(f^{-1})(\xi)\right)'}{(\mathcal{P}_r(f^{-1})(\xi))^{1-\epsilon}} - 1\right] \prec \Omega_0(\xi). \tag{3.7}$$

Two familiar limiting subclasses follow immediately.

**Corollary 3.1.** *If $\epsilon = 0$ and $q \to 1^-$, then the defining conditions become*

$$1 + \frac{\pounds}{\gamma}\left[\frac{z\,(\mathcal{P}_r(f)(z))'}{\mathcal{P}_r(f)(z)} - 1\right] \prec z + \sqrt[3]{1+z^3},$$

*and*

$$1 + \frac{\pounds}{\gamma}\left[\frac{\xi\left(\mathcal{P}_r(f^{-1})(\xi)\right)'}{\mathcal{P}_r(f^{-1})(\xi)} - 1\right] \prec \xi + \sqrt[3]{1+\xi^3}.$$

*In particular, the choice $\pounds = \gamma \neq 0$ yields the classical Poisson leaf-like bi-starlike-type conditions.*

**Corollary 3.2.** *If $\epsilon = 1$ and $q \to 1^-$, then*

$$1 + \frac{1}{\gamma}\left[(1-\pounds)\frac{\mathcal{P}_r(f)(z)}{z} + \pounds\,(\mathcal{P}_r(f)(z))' - 1\right] \prec z + \sqrt[3]{1+z^3},$$

*with the analogous condition for $f^{-1}$. When $\pounds = \gamma = 1$, this reduces to*

$$(\mathcal{P}_r(f)(z))' \prec z + \sqrt[3]{1+z^3}$$

*and*

$$\left(\mathcal{P}_r(f^{-1})(\xi)\right)' \prec \xi + \sqrt[3]{1+\xi^3}.$$

These limiting relations show that the newly introduced family is not an isolated construction; rather, it provides a genuine $q$-deformation of several Poisson leaf-like bi-univalent subclasses.

3.3. **Nonemptiness of the class.** The following observation confirms that the defining conditions are consistent.

**Proposition 3.1.** *For every admissible pair $(r, q)$ and every $\epsilon, £ \in \mathbb{C},\ \gamma \in \mathbb{C}^*$, the class $\mathsf{QPL}_{\Sigma}^{r,q}(\epsilon, £, \gamma; \Omega)$ is nonempty.*

*Proof.* Consider the identity function

$$f_0(z) = z.$$

Clearly,

$$f_0 \in \Sigma \qquad \text{and} \qquad f_0^{-1}(\xi) = \xi.$$

Since all Taylor–Maclaurin coefficients of order at least two vanish, the definition of the $q$-Poisson convolution operator gives

$$\mathrm{P}_q^r(f_0)(z) = z \qquad \text{and} \qquad \mathrm{P}_q^r(f_0^{-1})(\xi) = \xi.$$

Consequently,

$$ð_q\Big\langle \mathrm{P}_q^r(f_0)(z)\Big\rangle = 1, \qquad ð_q\Big\langle \mathrm{P}_q^r(f_0^{-1})(\xi)\Big\rangle = 1.$$

Therefore,

$$\mathcal{B}_{\epsilon,\alpha,\gamma}^{r,q}[f_0](z) = 1 + \frac{1}{\gamma}\Big[(1 - £) + £ - 1\Big] = 1,$$

and similarly,

$$\mathcal{B}_{\epsilon,\alpha,\gamma}^{r,q}[f_0^{-1}](\xi) = 1.$$

Since

$$\Omega(0; q) = 1,$$

the constant function 1 can be written as

$$1 = \Omega(\omega(z); q), \qquad \omega(z) \equiv 0,$$

where $\omega$ is a Schwarz function. Hence,

$$1 \prec \Omega(z; q).$$

Both subordinations in Definition 3.1 are therefore satisfied, and thus

$$f_0 \in \mathsf{QPL}_{\Sigma}^{r,q}(\epsilon, £, \gamma; \Omega).$$

This proves the assertion. $\square$

Coefficient problems occupy a central position in geometric function theory, since the Taylor–Maclaurin coefficients $\left|a_j\right|$, $j \in \mathbb{N}$, often reflect the principal analytic and geometric features of a function class. In the bi-univalent setting, particular attention is devoted to estimates for $|a_2|$ and $|a_3|$, together with the associated Fekete–Szegö functional and Hankel determinants [29,31–33,37]. Motivated by these developments, the next section derives the corresponding coefficient bounds and Fekete–Szegö estimate for $\mathsf{QPL}_{\Sigma}^{r,q}(\epsilon, £, \gamma; \Omega)$.

## 4. Coefficient Estimates and the Fekete–Szegö Problem

This section is devoted to the initial coefficient problem for the class

$$\mathsf{QPL}_{\Sigma}^{r,q}(\epsilon, £, \gamma; \Omega).$$

By combining the coefficient structure of the $q$-Poisson convolution operator with the Ma–Minda subordination generated by the leaf-like dominant function $\Omega(z; q)$, we derive explicit estimates for $|a_2|$ and $|a_3|$. The same coefficient relations are then employed to investigate the Fekete–Szegö functional

$$\left|a_3 - \varkappa a_2^2\right|.$$

For notational convenience, throughout this section we put

$$\mathcal{A}_1 = \epsilon + £q, \qquad \mathcal{A}_2 = \epsilon + £q[2]_q, \tag{4.1}$$

and

$$\Delta_{\epsilon,£,\gamma}^{r,q} := 2\gamma\mathcal{E}_q^r\mathcal{A}_2 + \gamma[2]_q(\epsilon-1)\Big(\epsilon + 2£q\Big) + 2(3-q)\mathcal{A}_1^2. \tag{4.2}$$

The quantities in (4.1) and (4.2) arise naturally from the first- and second-order coefficient comparisons.

### 4.1. **Bounds for the initial coefficients.**

**Theorem 4.1.** *Let $0 < q < 1,\ r > 0,\ r(1-q) < 1$, and suppose that $\epsilon, £ \in \mathbb{C},\ \gamma \in \mathbb{C}^*$. Assume further that $\mathcal{A}_1 \neq 0,\ \mathcal{A}_2 \neq 0,\ \Delta_{\epsilon,£,\gamma}^{r,q} \neq 0$. If $f \in \mathsf{QPL}_{\Sigma}^{r,q}(\epsilon, £, \gamma; \Omega)$, then*

$$|a_2| \leq \min\left\{\frac{\left|\gamma\right|\mathcal{E}_q^r[2]_q}{2r\,|\mathcal{A}_1|}, \frac{\left|\gamma\right|\mathcal{E}_q^r[2]_q}{r\sqrt{\left|\Delta_{\epsilon,£,\gamma}^{r,q}\right|}}\right\}. \tag{4.3}$$

*Moreover,*

$$|a_3| \leq \min\left\{\frac{\left|\gamma\right|^2\mathcal{E}_q^{2r}[2]_q{}^2}{4r^2\,|\mathcal{A}_1|^2}, \frac{\left|\gamma\right|^2\mathcal{E}_q^{2r}[2]_q{}^2}{r^2\left|\Delta_{\epsilon,£,\gamma}^{r,q}\right|}\right\} + \frac{\left|\gamma\right|\mathcal{E}_q^r[2]_q{}^2}{2r^2\,|\mathcal{A}_2|}. \tag{4.4}$$

*Proof.* Let $f \in \mathsf{QPL}_{\Sigma}^{r,q}(\epsilon, £, \gamma; \Omega)$. By Definition 3.1, there exist Schwarz functions $\tau_1$ and $\tau_2$, analytic in $\mathbb{U}$, such that

$$\tau_1(0) = \tau_2(0) = 0, \qquad |\tau_1(z)| < 1, \qquad |\tau_2(\xi)| < 1,$$

and

$$\mathcal{B}_{\epsilon,\alpha,\gamma}^{r,q}[f](z) = \Omega(\tau_1(z);q), \tag{4.5}$$

whereas

$$\mathcal{B}_{\epsilon,\alpha,\gamma}^{r,q}[f^{-1}](\xi) = \Omega(\tau_2(\xi);q). \tag{4.6}$$

Introduce the associated Carathéodory functions

$$\mathcal{T}_1(z) = \frac{1+\tau_1(z)}{1-\tau_1(z)} = 1 + \omega_1 z + \omega_2 z^2 + \cdots \tag{4.7}$$

and

$$\mathcal{T}_2(\xi) = \frac{1+\tau_2(\xi)}{1-\tau_2(\xi)} = 1 + \imath_1 \xi + \imath_2 \xi^2 + \cdots . \tag{4.8}$$

Since

$$\Re\{\mathcal{T}_1(z)\} > 0 \qquad \text{and} \qquad \Re\{\mathcal{T}_2(\xi)\} > 0,$$

the Carathéodory lemma gives

$$|\omega_j| \le 2, \qquad |\imath_j| \le 2, \qquad j \in \mathbb{N}. \tag{4.9}$$

Solving (4.7) and (4.8) for the corresponding Schwarz functions yields

$$\tau_1(z) = \frac{\omega_1}{2} z + \left(\frac{\omega_2}{2} - \frac{\omega_1^2}{4}\right) z^2 + O(z^3), \tag{4.10}$$

and

$$\tau_2(\xi) = \frac{\imath_1}{2} \xi + \left(\frac{\imath_2}{2} - \frac{\imath_1^2}{4}\right) \xi^2 + O(\xi^3). \tag{4.11}$$

The expansion of the leaf-like dominant function at the origin is

$$\Omega(z;q) = 1 + \frac{[2]_q}{2} z - \frac{(1-q)[2]_q}{4} z^2 + O(z^3). \tag{4.12}$$

Combining (4.12) with (4.10) and (4.11), we obtain

$$\Omega(\tau_1(z);q) = 1 + \frac{[2]_q}{4} \omega_1 z + \frac{[2]_q}{4} \left(\omega_2 - \frac{3-q}{4} \omega_1^2\right) z^2 + O(z^3), \tag{4.13}$$

and

$$\Omega(\tau_2(\xi);q) = 1 + \frac{[2]_q}{4} \imath_1 \xi + \frac{[2]_q}{4} \left(\imath_2 - \frac{3-q}{4} \imath_1^2\right) \xi^2 + O(\xi^3). \tag{4.14}$$

By the definition of the $q$-Poisson convolution operator,

$$\mathrm{P}_q^r(f(z)) = z + \frac{r}{\mathcal{E}_q^r} a_2 z^2 + \frac{r^2}{[2]_q!\, \mathcal{E}_q^r} a_3 z^3 + O(z^4). \tag{4.15}$$

Using the inverse expansion $f^{-1}(\xi)$ given by (2.4) we similarly have

$$\mathrm{P}_q^r(f^{-1}(\xi)) = \xi - \frac{r}{\mathcal{E}_q^r} a_2 \xi^2 + \frac{r^2}{[2]_q!\, \mathcal{E}_q^r} \left(2a_2^2 - a_3\right) \xi^3 + O(\xi^4). \tag{4.16}$$

Substitution into the functional $\mathcal{B}^{r,q}_{\epsilon,\alpha,\gamma}$ gives

$$\mathcal{B}^{r,q}_{\epsilon,\alpha,\gamma}[f](z) = 1 + \frac{r\mathcal{A}_1}{\gamma\mathcal{E}_q^r}a_2 z + \left[\frac{r^2\mathcal{A}_2}{\gamma[2]_q!\,\mathcal{E}_q^r}a_3 + \frac{r^2(\epsilon-1)(\epsilon+2\pounds q)}{2\gamma\mathcal{E}_q^{2r}}a_2^2\right]z^2 + O(z^3), \tag{4.17}$$

whereas

$$\mathcal{B}^{r,q}_{\epsilon,\alpha,\gamma}[f^{-1}](\xi) = 1 - \frac{r\mathcal{A}_1}{\gamma\mathcal{E}_q^r}a_2 \xi + \left[\frac{r^2\mathcal{A}_2}{\gamma[2]_q!\,\mathcal{E}_q^r}\left(2a_2^2 - a_3\right) + \frac{r^2(\epsilon-1)(\epsilon+2\pounds q)}{2\gamma\mathcal{E}_q^{2r}}a_2^2\right]\xi^2 + O(\xi^3). \tag{4.18}$$

Comparing the coefficients in (4.5)–(4.18), we obtain

$$\frac{r\mathcal{A}_1}{\gamma\mathcal{E}_q^r}a_2 = \frac{[2]_q}{4}\omega_1, \tag{4.19}$$

$$\frac{r^2\mathcal{A}_2}{\gamma[2]_q!\,\mathcal{E}_q^r}a_3 + \frac{r^2(\epsilon-1)(\epsilon+2\pounds q)}{2\gamma\mathcal{E}_q^{2r}}a_2^2 = \frac{[2]_q}{4}\left(\omega_2 - \frac{3-q}{4}\omega_1^2\right), \tag{4.20}$$

$$-\frac{r\mathcal{A}_1}{\gamma\mathcal{E}_q^r}a_2 = \frac{[2]_q}{4}\imath_1, \tag{4.21}$$

and

$$\frac{r^2\mathcal{A}_2}{\gamma[2]_q!\,\mathcal{E}_q^r}\left(2a_2^2 - a_3\right) + \frac{r^2(\epsilon-1)(\epsilon+2\pounds q)}{2\gamma\mathcal{E}_q^{2r}}a_2^2 = \frac{[2]_q}{4}\left(\imath_2 - \frac{3-q}{4}\imath_1^2\right). \tag{4.22}$$

Equations (4.19) and (4.21) give

$$\omega_1 = -\imath_1. \tag{4.23}$$

In particular,

$$\omega_1^2 = \imath_1^2.$$

From (4.19), together with $|\omega_1| \le 2$, we immediately obtain

$$|a_2| \le \frac{|\gamma|\,\mathcal{E}_q^r[2]_q}{2r\,|\mathcal{A}_1|}. \tag{4.24}$$

A second estimate follows by adding (4.20) and (4.22). After using (4.23) and eliminating $\omega_1^2$ by means of (4.19), we arrive at

$$a_2^2 = \frac{\gamma^2\mathcal{E}_q^{2r}{[2]_q}^2\,(\omega_2 + \imath_2)}{4r^2\Delta^{r,q}_{\epsilon,\pounds,\gamma}}. \tag{4.25}$$

Since

$$|\omega_2 + \imath_2| \le |\omega_2| + |\imath_2| \le 4,$$

we obtain

$$|a_2| \le \frac{|\gamma|\,\mathcal{E}_q^r[2]_q}{r\sqrt{\left|\Delta^{r,q}_{\epsilon,\pounds,\gamma}\right|}}. \tag{4.26}$$

Combining (4.24) and (4.26) proves (4.3).

Next, subtracting (4.22) from (4.20) and using $\omega_1^2 = \imath_1^2$, we find

$$\frac{2r^2\mathcal{A}_2}{\gamma[2]_q!\,\mathcal{E}_q^r}\left(a_3 - a_2^2\right) = \frac{[2]_q}{4}\left(\omega_2 - \imath_2\right). \tag{4.27}$$

Since

$$[2]_q! = [2]_q,$$

equation (4.27) becomes

$$a_3 = a_2^2 + \frac{\gamma\mathcal{E}_q^r[2]_q{}^2}{8r^2\mathcal{A}_2}\left(\omega_2 - \imath_2\right). \tag{4.28}$$

Therefore,

$$|a_3| \le |a_2|^2 + \frac{\left|\gamma\right|\mathcal{E}_q^r[2]_q{}^2}{8r^2\left|\mathcal{A}_2\right|}\left(|\omega_2| + |\imath_2|\right),$$

and hence

$$|a_3| \le |a_2|^2 + \frac{\left|\gamma\right|\mathcal{E}_q^r[2]_q{}^2}{2r^2\left|\mathcal{A}_2\right|}. \tag{4.29}$$

Applying the two estimates obtained for $|a_2|$ gives (4.4). □

4.2. **The Fekete–Szegö functional.** We now turn to the Fekete–Szegö problem for the class

$$\mathsf{QPL}_\Sigma^{r,q}(\epsilon, £, \gamma; \Omega).$$

The argument is based on expressing $a_3 - \varkappa a_2^2$ as a linear combination of the second coefficients of two Carathéodory functions. This approach remains valid for complex values of the parameters and leads naturally to a two-case estimate.

**Theorem 4.2.** *Let $0 < q < 1$, $r > 0$, $r(1-q) < 1$, and suppose that $\epsilon, £ \in \mathbb{C}$, $\gamma \in \mathbb{C}^*$. Let $f \in \mathsf{QPL}_\Sigma^{r,q}(\epsilon, £, \gamma; \Omega)$. Assume that*

$$\epsilon + £q[2]_q \ne 0$$

*and*

$$D_3(\epsilon, £, \gamma, q; r) \ne 0,$$

*where*

$$D_3(\epsilon, £, \gamma, q; r) := 2\gamma\mathcal{E}_q^r\left(\epsilon + £q[2]_q\right) + \gamma[2]_q(\epsilon - 1)\left(\epsilon + 2£q\right) + 2(3-q)\left(\epsilon + £q\right)^2. \tag{4.30}$$

*Then, for every $\varkappa \in \mathbb{C}$,*

$$\left|a_3 - \varkappa a_2^2\right| \le \begin{cases} \dfrac{\left|\gamma\right|\mathcal{E}_q^r[2]_q{}^2}{\sqrt{2}\,r^2\left|\epsilon + £q[2]_q\right|}, & |1-\varkappa| \le \dfrac{\left|D_3(\epsilon, £, \gamma, q; r)\right|}{2\left|\gamma\right|\mathcal{E}_q^r\left|\epsilon + £q[2]_q\right|}, \\[3ex] \dfrac{\sqrt{2}\left|\gamma\right|^2\mathcal{E}_q^{2r}[2]_q{}^2|1-\varkappa|}{r^2\left|D_3(\epsilon, £, \gamma, q; r)\right|}, & |1-\varkappa| > \dfrac{\left|D_3(\epsilon, £, \gamma, q; r)\right|}{2\left|\gamma\right|\mathcal{E}_q^r\left|\epsilon + £q[2]_q\right|}. \end{cases} \tag{4.31}$$

*Proof.* From the coefficient relations established in the proof of Theorem 4.1, we have

$$a_2^2 = \frac{\gamma^2 \mathcal{E}_q^{2r} [2]_q^{\,2} (\omega_2 + \imath_2)}{4r^2 D_3(\epsilon, £, \gamma, q; r)}. \tag{4.32}$$

Moreover, subtracting (4.22) from (4.20) and using

$$\omega_1^2 = \imath_1^2,$$

we obtain

$$a_3 = a_2^2 + \frac{\gamma \mathcal{E}_q^r [2]_q^{\,2}}{8r^2 \Big(\epsilon + £q[2]_q\Big)} (\omega_2 - \imath_2). \tag{4.33}$$

For brevity, define

$$\mathcal{D}_{\epsilon,£,\gamma}(\varkappa, r; q) := \frac{2(1-\varkappa)\gamma \mathcal{E}_q^r}{D_3(\epsilon, £, \gamma, q; r)}. \tag{4.34}$$

It follows from (4.32) and (4.33) that

$$\begin{aligned} a_3 - \varkappa a_2^2 &= (1-\varkappa) a_2^2 + \frac{\gamma \mathcal{E}_q^r [2]_q^{\,2}}{8r^2 \Big(\epsilon + £q[2]_q\Big)} (\omega_2 - \imath_2) \\ &= \frac{\gamma \mathcal{E}_q^r [2]_q^{\,2}}{8r^2} \left[ \left( \mathcal{D}_{\epsilon,£,\gamma}(\varkappa, r; q) + \frac{1}{\epsilon + £q[2]_q} \right) \omega_2 + \left( \mathcal{D}_{\epsilon,£,\gamma}(\varkappa, r; q) - \frac{1}{\epsilon + £q[2]_q} \right) \imath_2 \right]. \end{aligned} \tag{4.35}$$

Since the functions $\mathcal{T}_1$ and $\mathcal{T}_2$ belong to the Carathéodory class, the classical coefficient estimate gives

$$|\omega_2| \le 2, \qquad |\imath_2| \le 2.$$

Therefore, taking absolute values in (4.35) and applying the triangle inequality, we obtain

$$\left| a_3 - \varkappa a_2^2 \right| \le \frac{|\gamma| \, \mathcal{E}_q^r [2]_q^{\,2}}{4r^2} \left[ \left| \mathcal{D}_{\epsilon,£,\gamma}(\varkappa, r; q) + \frac{1}{\epsilon + £q[2]_q} \right| + \left| \mathcal{D}_{\epsilon,£,\gamma}(\varkappa, r; q) - \frac{1}{\epsilon + £q[2]_q} \right| \right]. \tag{4.36}$$

For arbitrary $u, v \in \mathbb{C}$, the parallelogram identity and the Cauchy–Schwarz inequality yield

$$\begin{aligned} \Big(|u+v| + |u-v|\Big)^2 &\le 2\left(|u+v|^2 + |u-v|^2\right) \\ &= 4\left(|u|^2 + |v|^2\right) \\ &\le 8 \max\{|u|, |v|\}^2. \end{aligned}$$

Consequently,

$$|u+v| + |u-v| \le 2\sqrt{2} \max\{|u|, |v|\}. \tag{4.37}$$

Applying (4.37) to (4.36), with

$$u = \mathcal{D}_{\epsilon,£,\gamma}(\varkappa, r; q) \qquad \text{and} \qquad v = \frac{1}{\epsilon + £q[2]_q},$$

gives

$$\left| a_3 - \varkappa a_2^2 \right| \le \frac{|\gamma| \, \mathcal{E}_q^r [2]_q^{\,2}}{\sqrt{2}\, r^2} \max\left\{ \left| \mathcal{D}_{\epsilon,£,\gamma}(\varkappa, r; q) \right|, \frac{1}{\left| \epsilon + £q[2]_q \right|} \right\}. \tag{4.38}$$

By (4.34), the inequality

$$\left|\mathcal{D}_{\epsilon,£,\gamma}(\varkappa, r; q)\right| \leq \frac{1}{\left|\epsilon + £q[2]_q\right|}$$

is equivalent to

$$|1-\varkappa| \leq \frac{\left|D_3(\epsilon, £, \gamma, q; r)\right|}{2\left|\gamma\right| \mathcal{E}_q^r \left|\epsilon + £q[2]_q\right|}. \tag{4.39}$$

In this case, the second quantity in the maximum appearing in (4.38) dominates, and hence

$$\left|a_3 - \varkappa a_2^2\right| \leq \frac{\left|\gamma\right| \mathcal{E}_q^r [2]_q{}^2}{\sqrt{2}\, r^2 \left|\epsilon + £q[2]_q\right|}.$$

On the other hand,

$$\left|\mathcal{D}_{\epsilon,£,\gamma}(\varkappa, r; q)\right| > \frac{1}{\left|\epsilon + £q[2]_q\right|}$$

is equivalent to

$$|1-\varkappa| > \frac{\left|D_3(\epsilon, £, \gamma, q; r)\right|}{2\left|\gamma\right| \mathcal{E}_q^r \left|\epsilon + £q[2]_q\right|}. \tag{4.40}$$

Thus, the first quantity in the maximum dominates, and

$$\left|a_3 - \varkappa a_2^2\right| \leq \frac{\sqrt{2}\, \left|\gamma\right|^2 \mathcal{E}_q^{2r} [2]_q{}^2 |1-\varkappa|}{r^2 \left|D_3(\epsilon, £, \gamma, q; r)\right|}.$$

The two alternatives yield (4.31), completing the proof. □

4.3. **Numerical inspection of the nonvanishing condition.** The coefficient estimates and the Fekete–Szegö inequality involve the quantity

$$D_3(\varepsilon, £, \gamma, q; r) := 2\gamma \mathcal{E}_q^r\left(\varepsilon + £q[2]_q\right) + \gamma[2]_q(\varepsilon - 1)\left(\varepsilon + 2£q\right) + 2(3-q)\left(\varepsilon + £q\right)^2. \tag{4.41}$$

To illustrate the nonvanishing requirement appearing in Theorem 4.2, we examine the modulus

$$\left|D_3(\varepsilon, £, \gamma, q; r)\right|$$

over two representative real parameter regimes. In each experiment, the surface is evaluated on a uniform $250 \times 250$ grid within the corresponding admissible region.

First parameter regime. We first take

$$r = 1, \qquad £ = 1, \qquad \gamma = 1.$$

In this case, the admissibility condition $r(1-q) < 1$ is satisfied for every $0 < q < 1$.

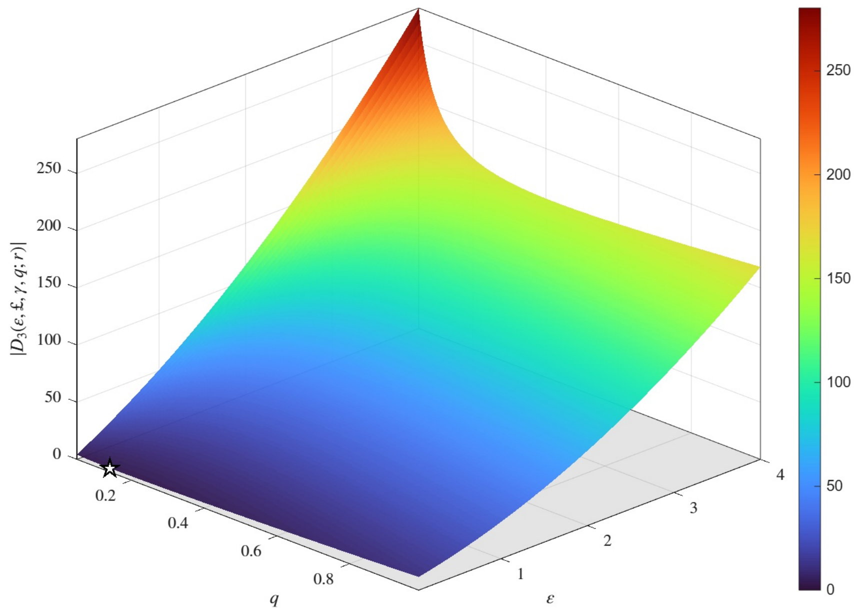



FIGURE 2. Numerical surface of $\left|D_3(\varepsilon,1,1,q;1)\right|$ over $0.05 \le \varepsilon \le 4$ and $0.05 \le q \le 0.99$. At every sampled point, the surface remains above the plane $|D_3| = 0$.

The numerical search associated with Figure 2 gives

$$\min\left|D_3(\varepsilon,1,1,q;1)\right| \approx 3.321062117516, \tag{4.42}$$

with the smallest sampled value occurring near

$$(q,\varepsilon) \approx (0.14060241, 0.05).$$

Thus, no zero is detected within the displayed region, and the modulus of $D_3$ remains numerically separated from zero throughout the selected grid.
Second parameter regime. For a second illustration, we choose

$$r = 2, \qquad \pounds = \frac{1}{2}, \qquad \gamma = \frac{3}{2}.$$

The admissibility condition now becomes

$$2(1-q) < 1,$$

and hence the deformation parameter must satisfy

$$q > \frac{1}{2}.$$

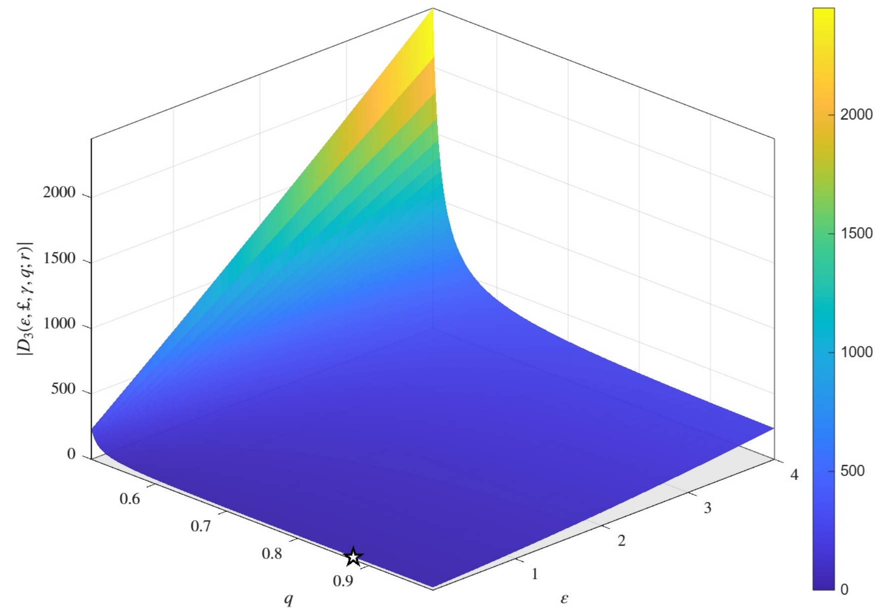



FIGURE 3. Numerical surface of $\left|D_3\left(\varepsilon, \frac{1}{2}, \frac{3}{2}, q; 2\right)\right|$ over the admissible region $0.05 \leq \varepsilon \leq 4$ and $0.51 \leq q \leq 0.99$. The sampled surface remains well above the zero plane.

For this parameter configuration, the numerical computation yields

$$\min\left|D_3\left(\varepsilon, \frac{1}{2}, \frac{3}{2}, q; 2\right)\right| \approx 20.99680939631, \tag{4.43}$$

attained on the selected grid near

$$(q, \varepsilon) \approx (0.87819277, 0.05).$$

The substantially positive minimum displayed in (4.43) indicates that the denominator occurring in the coefficient estimates remains numerically far from zero over the prescribed parameter region. Comparison of the two experiments. The principal numerical outcomes of the two parameter regimes are collected in Table 5.

TABLE 5. Comparison of the sampled minima of $\left|D_3(\varepsilon, £, \gamma, q; r)\right|$.

| **No.** | $(r, £, \gamma)$ | **Sampled region** | **Minimum modulus** | **Approximate minimizer** |
|---|---|---|---|---|
| 1 | $(1, 1, 1)$ | $0.05 \leq \varepsilon \leq 4$, $0.05 \leq q \leq 0.99$ | 3.321062117516 | $(q, \varepsilon) \approx (0.14060241, 0.05)$ |
| 2 | $\left(2, \frac{1}{2}, \frac{3}{2}\right)$ | $0.05 \leq \varepsilon \leq 4$, $0.51 \leq q \leq 0.99$ | 20.99680939631 | $(q, \varepsilon) \approx (0.87819277, 0.05)$ |

For a more detailed view of the second regime, Table 6 lists the cross-sectional minima obtained for several representative values of $q$.

Table 6. Cross-sectional minima of $\left|D_3\left(\varepsilon, \frac{1}{2}, \frac{3}{2}, q; 2\right)\right|$ for selected values of $q$.

| **No.** | $q$ | $\varepsilon$ **at the minimum** | $\min_{0.05\le\varepsilon\le4} \lvert D_3\rvert$ |
|---|---|---|---|
| 1 | 0.51000000 | 0.05000000 | 230.386822 |
| 2 | 0.60000000 | 0.05000000 | 33.045940 |
| 3 | 0.70000000 | 0.05000000 | 23.498483 |
| 4 | 0.80000000 | 0.05000000 | 21.321910 |
| **5** | **0.87819277** | **0.05000000** | **20.996809** |
| 6 | 0.90000000 | 0.05000000 | 21.016749 |
| 7 | 0.99000000 | 0.05000000 | 21.422850 |

The values in Table 6 show that the modulus initially decreases as $q$ moves away from the admissibility boundary, reaches its smallest sampled value near $q = 0.87819277$, and then begins to increase slightly as $q$ approaches 1. For every listed value of $q$, the cross-sectional minimum occurs at the lower boundary $\varepsilon = 0.05$.

**Remark 4.1.** *The computations displayed in Figures 2 and 3 provide numerical evidence for the nonvanishing of $D_3$ only over the specified real parameter regions. They do not establish global nonvanishing for arbitrary complex values of $\varepsilon$, £, and $\gamma$. Consequently, the assumption*

$$D_3(\varepsilon, £, \gamma, q; r) \neq 0$$

*must remain among the hypotheses of Theorem 4.2.*

**Remark 4.2.** *At the transition point*

$$|1-\varkappa| = \frac{\left|D_3(\varepsilon, £, \gamma, q; r)\right|}{2|\gamma|\mathcal{E}_q^r \left|\varepsilon + £q[2]_q\right|},$$

*the two expressions on the right-hand side of* (4.31) *agree. Therefore, assigning the equality case to the first branch and using a strict inequality in the second branch produces a nonoverlapping piecewise formulation while preserving continuity at the threshold.*

## 5. Coefficient Consequences for the Principal Reductions

The unified class $\mathsf{QPL}_{\Sigma}^{r,q}(\epsilon, \pounds, \gamma; \Omega)$ contains several geometrically meaningful families as particular configurations of the parameters $\epsilon$, £, and $\gamma$. The purpose of this section is to translate the general estimates obtained in Theorems 4.1 and 4.2 into explicit coefficient bounds for the four principal reductions listed in Table 4.

The resulting formulas reveal that each reduction possesses a distinct denominator structure, reflecting the way in which the power term, the $q$-derivative term, and the $q$-Poisson convolution interact.

5.1. **The generalized $q$-Poisson bi-starlike-type reduction.** We begin with the choice $\epsilon = 0$. In this case, the defining functional reduces to

$$1 + \frac{\pounds}{\gamma}\left[\frac{z\,\eth_q\left\langle \mathrm{P}_q^r(f(z))\right\rangle}{\mathrm{P}_q^r(f(z))} - 1\right].$$

For later use, define

$$\Delta_0(\pounds, \gamma, q; r) := 2\pounds q\left[\gamma[2]_q\left(\mathcal{E}_q^r - 1\right) + (3-q)\pounds q\right]. \tag{5.1}$$

**Corollary 5.1.** *Let $0 < q < 1,\ r > 0,\ r(1-q) < 1$, and suppose that $\pounds \in \mathbb{C}^*,\ \gamma \in \mathbb{C}^*,\ \Delta_0(\pounds, \gamma, q; r) \neq 0$. If $f \in \mathsf{QPL}_{\Sigma}^{r,q}(0, \pounds, \gamma; \Omega)$, then*

$$|a_2| \leq \min\left\{\frac{|\gamma|\mathcal{E}_q^r[2]_q}{2rq|\pounds|}, \frac{|\gamma|\mathcal{E}_q^r[2]_q}{r\sqrt{\left|\Delta_0(\pounds, \gamma, q; r)\right|}}\right\}. \tag{5.2}$$

*Moreover,*

$$|a_3| \leq \min\left\{\frac{|\gamma|^2\mathcal{E}_q^{2r}[2]_q{}^2}{4r^2q^2|\pounds|^2}, \frac{|\gamma|^2\mathcal{E}_q^{2r}[2]_q{}^2}{r^2\left|\Delta_0(\pounds, \gamma, q; r)\right|}\right\} + \frac{|\gamma|\mathcal{E}_q^r[2]_q}{2r^2q|\pounds|}. \tag{5.3}$$

*For every $\varkappa \in \mathbb{C}$,*

$$\left|a_3 - \varkappa a_2^2\right| \leq \begin{cases} \dfrac{|\gamma|\mathcal{E}_q^r[2]_q}{\sqrt{2}\,r^2q|\pounds|}, & |1-\varkappa| \leq \dfrac{\left|\Delta_0(\pounds, \gamma, q; r)\right|}{2|\gamma|\mathcal{E}_q^r q[2]_q|\pounds|}, \\[2ex] \dfrac{\sqrt{2}\,|\gamma|^2\mathcal{E}_q^{2r}[2]_q{}^2|1-\varkappa|}{r^2\left|\Delta_0(\pounds, \gamma, q; r)\right|}, & |1-\varkappa| > \dfrac{\left|\Delta_0(\pounds, \gamma, q; r)\right|}{2|\gamma|\mathcal{E}_q^r q[2]_q|\pounds|}. \end{cases} \tag{5.4}$$

*Proof.* Setting $\epsilon = 0$ gives

$$\epsilon + \pounds q = \pounds q, \qquad \epsilon + \pounds q[2]_q = \pounds q[2]_q,$$

while

$$D_3(0, \pounds, \gamma, q; r) = \Delta_0(\pounds, \gamma, q; r).$$

The conclusions follow directly from Theorems 4.1 and 4.2. ∎

5.2. **The normalized $q$-Poisson bi-starlike reduction.** We next impose $\epsilon = 0$, $£ = \gamma \neq 0$. The defining functional then becomes

$$\frac{z\,ð_q\Big\langle \mathrm{P}_q^r(f(z))\Big\rangle}{\mathrm{P}_q^r(f(z))}.$$

Introduce the positive quantity

$$\Lambda_q^r := [2]_q\Big(\mathcal{E}_q^r - 1\Big) + (3-q)q. \tag{5.5}$$

Since $r > 0$, one has $\mathcal{E}_q^r > 1$, and therefore $\Lambda_q^r > 0$.

**Corollary 5.2.** *Let $0 < q < 1,\ r > 0,\ r(1-q) < 1$. If $f \in \mathsf{QPL}_\Sigma^{r,q}(0,\gamma,\gamma;\Omega),\ \gamma \in \mathbb{C}^*$, then*

$$|a_2| \leq \min\left\{\frac{\mathcal{E}_q^r[2]_q}{2rq}, \frac{\mathcal{E}_q^r[2]_q}{r\sqrt{2q\Lambda_q^r}}\right\}. \tag{5.6}$$

*Furthermore,*

$$|a_3| \leq \min\left\{\frac{\mathcal{E}_q^{2r}[2]_q{}^2}{4r^2q^2}, \frac{\mathcal{E}_q^{2r}[2]_q{}^2}{2r^2q\Lambda_q^r}\right\} + \frac{\mathcal{E}_q^r[2]_q}{2r^2q}. \tag{5.7}$$

*Moreover, for every $\varkappa \in \mathbb{C}$,*

$$\left|a_3 - \varkappa a_2^2\right| \leq \begin{cases} \dfrac{\mathcal{E}_q^r[2]_q}{\sqrt{2}\,r^2q}, & |1-\varkappa| \leq \dfrac{\Lambda_q^r}{\mathcal{E}_q^r[2]_q}, \\[2ex] \dfrac{\mathcal{E}_q^{2r}[2]_q{}^2|1-\varkappa|}{\sqrt{2}\,r^2q\Lambda_q^r}, & |1-\varkappa| > \dfrac{\Lambda_q^r}{\mathcal{E}_q^r[2]_q}. \end{cases} \tag{5.8}$$

*Proof.* For $\epsilon = 0$ and $£ = \gamma$, we have

$$\epsilon + £q = \gamma q, \qquad \epsilon + £q[2]_q = \gamma q[2]_q,$$

and

$$\begin{aligned} D_3(0,\gamma,\gamma,q;r) &= 2\gamma^2 q\left[[2]_q\Big(\mathcal{E}_q^r - 1\Big) + (3-q)q\right] \\ &= 2\gamma^2 q\Lambda_q^r. \end{aligned} \tag{5.9}$$

The positivity of $\Lambda_q^r$, together with $\gamma \neq 0$, guarantees that the denominator in (5.9) does not vanish. Substitution into Theorems 4.1 and 4.2 yields the stated estimates. □

5.3. **The weighted quotient–derivative reduction.** We now take $\epsilon = 1$. The defining functional assumes the form

$$1 + \frac{1}{\gamma}\left[(1-£)\frac{\mathrm{P}_q^r(f(z))}{z} + £\,ð_q\Big\langle \mathrm{P}_q^r(f(z))\Big\rangle - 1\right].$$

Set

$$\Delta_1(£,\gamma,q;r) := 2\gamma\mathcal{E}_q^r\Big(1 + £q[2]_q\Big) + 2(3-q)\Big(1 + £q\Big)^2. \tag{5.10}$$

**Corollary 5.3.** *Let $0 < q < 1,\ r > 0,\ r(1-q) < 1$, and suppose that $£ \in \mathbb{C},\ \gamma \in \mathbb{C}^*$. Assume further that*

$$1 + £q \neq 0, \qquad 1 + £q[2]_q \neq 0, \qquad \Delta_1(£, \gamma, q; r) \neq 0.$$

*If $f \in \mathsf{QPL}_{\Sigma}^{r,q}(1, £, \gamma; \Omega)$, then*

$$|a_2| \leq \min\left\{ \frac{|\gamma| \mathcal{E}_q^r [2]_q}{2r|1 + £q|}, \frac{|\gamma| \mathcal{E}_q^r [2]_q}{r\sqrt{\left|\Delta_1(£, \gamma, q; r)\right|}} \right\}. \tag{5.11}$$

*Moreover,*

$$|a_3| \leq \min\left\{ \frac{|\gamma|^2 \mathcal{E}_q^{2r} [2]_q{}^2}{4r^2|1 + £q|^2}, \frac{|\gamma|^2 \mathcal{E}_q^{2r} [2]_q{}^2}{r^2\left|\Delta_1(£, \gamma, q; r)\right|} \right\} + \frac{|\gamma| \mathcal{E}_q^r [2]_q{}^2}{2r^2|1 + £q[2]_q|}. \tag{5.12}$$

*For every $\varkappa \in \mathbb{C}$,*

$$\left|a_3 - \varkappa a_2^2\right| \leq \begin{cases} \dfrac{|\gamma| \mathcal{E}_q^r [2]_q{}^2}{\sqrt{2}\, r^2 |1 + £q[2]_q|}, & |1 - \varkappa| \leq \dfrac{\left|\Delta_1(£, \gamma, q; r)\right|}{2|\gamma| \mathcal{E}_q^r |1 + £q[2]_q|}, \\[2ex] \dfrac{\sqrt{2}\, |\gamma|^2 \mathcal{E}_q^{2r} [2]_q{}^2 |1 - \varkappa|}{r^2\left|\Delta_1(£, \gamma, q; r)\right|}, & |1 - \varkappa| > \dfrac{\left|\Delta_1(£, \gamma, q; r)\right|}{2|\gamma| \mathcal{E}_q^r |1 + £q[2]_q|}. \end{cases} \tag{5.13}$$

*Proof.* For $\epsilon = 1$,

$$\epsilon + £q = 1 + £q, \qquad \epsilon + £q[2]_q = 1 + £q[2]_q.$$

Furthermore, the factor $\epsilon - 1$ vanishes, and therefore

$$D_3(1, £, \gamma, q; r) = \Delta_1(£, \gamma, q; r).$$

The assertions now follow by applying Theorems 4.1 and 4.2. $\square$

5.4. **The normalized $q$-Poisson derivative reduction.** Finally, consider $\epsilon = 1,\ £ = \gamma = 1$ The defining functional reduces simply to

$$ð_q \left\langle \mathrm{P}_q^r (f(z)) \right\rangle.$$

Define

$$\Omega_q^r := \mathcal{E}_q^r [3]_q + (3 - q)[2]_q{}^2. \tag{5.14}$$

Because $0 < q < 1,\ r > 0$, we have $\Omega_q^r > 0$.

**Corollary 5.4.** *Let $0 < q < 1,\ r > 0,\ r(1-q) < 1$. If $f \in \mathsf{QPL}_{\Sigma}^{r,q}(1, 1, 1; \Omega)$, then*

$$|a_2| \leq \min\left\{ \frac{\mathcal{E}_q^r}{2r}, \frac{\mathcal{E}_q^r [2]_q}{r\sqrt{2\Omega_q^r}} \right\}. \tag{5.15}$$

*Moreover,*

$$|a_3| \leq \min\left\{ \frac{\mathcal{E}_q^{2r}}{4r^2}, \frac{\mathcal{E}_q^{2r} [2]_q{}^2}{2r^2 \Omega_q^r} \right\} + \frac{\mathcal{E}_q^r [2]_q{}^2}{2r^2 [3]_q}. \tag{5.16}$$

*For every $\varkappa \in \mathbb{C}$,*

$$\left|a_3 - \varkappa a_2^2\right| \leq \begin{cases} \dfrac{\mathcal{E}_q^r [2]_q^{\ 2}}{\sqrt{2}\, r^2 [3]_q}, & |1-\varkappa| \leq \dfrac{\Omega_q^r}{\mathcal{E}_q^r [3]_q}, \\ \dfrac{\mathcal{E}_q^{2r} [2]_q^{\ 2} |1-\varkappa|}{\sqrt{2}\, r^2 \Omega_q^r}, & |1-\varkappa| > \dfrac{\Omega_q^r}{\mathcal{E}_q^r [3]_q}. \end{cases} \tag{5.17}$$

*Proof.* Under the choice

$$\epsilon = £ = \gamma = 1,$$

we have

$$\epsilon + £q = 1 + q = [2]_q,$$

and

$$\epsilon + £q[2]_q = 1 + q + q^2 = [3]_q.$$

Moreover,

$$\begin{aligned} D_3(1,1,1,q;r) &= 2\mathcal{E}_q^r [3]_q + 2(3-q)[2]_q^{\ 2} \\ &= 2\Omega_q^r. \end{aligned} \tag{5.18}$$

Since $\Omega_q^r > 0$, the nonvanishing condition is automatically satisfied. The conclusions follow by substituting these relations into Theorems 4.1 and 4.2. □

5.5. **Concluding observation.** The preceding corollaries demonstrate that the general coefficient inequalities specialize coherently to the principal subclasses generated by the choices $\epsilon = 0$ and $\epsilon = 1$. In particular, the normalized reductions

$$(\epsilon, £, \gamma) = (0, \gamma, \gamma) \qquad \text{and} \qquad (\epsilon, £, \gamma) = (1,1,1)$$

lead to estimates independent of the complex scaling parameter $\gamma$. This cancellation is consistent with the fact that the corresponding defining functionals no longer contain an external scaling factor.

The quantities

$$\Lambda_q^r \qquad \text{and} \qquad \Omega_q^r$$

are strictly positive throughout the admissible parameter region. Consequently, the normalized starlike and derivative reductions do not require an additional nonvanishing hypothesis for their denominator expressions.

## 6. Conclusion

In this paper, we introduced and investigated the new family

$$\mathsf{QPL}_{\Sigma}^{r,q}(\epsilon, £, \gamma; \Omega)$$

of bi-univalent functions associated with a $q$-Poisson convolution operator and a Ma–Minda-type subordination. The proposed formulation provides a unified framework that incorporates several parameters and consequently contains a number of previously studied subclasses as particular cases.

By comparing the coefficients arising from the defining subordinations for a function (f) and its inverse $f^{-1}$, we obtained explicit estimates for the initial Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$. We also established a Fekete–Szegö-type inequality for the functional

$$\left|a_3 - \eta a_2^2\right|,$$

where $\eta \in \mathbb{R}$. The positivity of the quantities generated by the $q$-Poisson operator, together with the assumption $\gamma \neq 0$, ensures that the denominators appearing in the derived bounds are well defined.

Several consequences of the principal theorems were obtained by assigning suitable values to the the complex parameters $\epsilon$, £, $\gamma$, $r$, and $q$. These reductions demonstrate that the results of the present work extend and unify a variety of known coefficient estimates for bi-starlike, bi-convex, and related Ma–Minda-type bi-univalent function classes. The developed approach may also be applied to other convolution operators and image domains, providing a useful basis for further investigations of higher-order coefficients, Hankel determinants, and related extremal problems in geometric function theory. To explore other future studies, these concepts can be combined with other concepts. See [35–37].

**Conflicts of Interest:** The authors declare that there are no conflicts of interest regarding the publication of this paper.

## References

[1] R. Otter, The Multiplicative Process, Ann. Math. Stat. 20 (1949), 206–224. https://doi.org/10.1214/aoms/1177730031.

[2] T.E. Harris, The Theory of Branching Processes, Springer Berlin Heidelberg, 1963. https://doi.org/10.1007/978-3-642-51866-9.

[3] P. Flajolet, R. Sedgewick, Analytic Combinatorics, Cambridge University Press, 2009. https://doi.org/10.1017/CBO9780511801655.

[4] J.B. Conway, Functions of One Complex Variable I, 2nd ed., Springer New York, 1978. https://doi.org/10.1007/978-1-4612-6313-5.

[5] P.L. Duren, Univalent Functions, Springer-Verlag New York, 1983.

[6] K. Arora, S.S. Kumar, Starlike Functions Associated with a Petal Shaped Domain, Bull. Korean Math. Soc. 59 (2022), 993–1010. https://doi.org/10.4134/BKMS.b210602.

[7] N.E. Cho, V. Kumar, S.S. Kumar, V. Ravichandran, Radius Problems for Starlike Functions Associated with the Sine Function, Bull. Iran. Math. Soc. 45 (2019), 213–232. https://doi.org/10.1007/s41980-018-0127-5.

[8] K. Bano, M. Raza, Starlike Functions Associated with Cosine Functions, Bull. Iran. Math. Soc. 47 (2021), 1513–1532. https://doi.org/10.1007/s41980-020-00456-9.

[9] E.E. Ali, R.M. El-Ashwah, W.Y. Kota, A.M. Albalahi, T. Bulboacă, A Study of Generalized Distribution Series and Their Mapping Properties in Univalent Function Theory, AIMS Math. 10 (2025), 13296–13318. https://doi.org/10.3934/math.2025596.

[10] A. Alsoboh, A. Amourah, M. Darus, C.A. Rudder, Investigating New Subclasses of Bi-Univalent Functions Associated with $q$-Pascal Distribution Series Using the Subordination Principle, Symmetry 15 (2023), 1109. https://doi.org/10.3390/sym15051109.

[11] A. Alsoboh, A. Amourah, M. Darus, R.I. Al Sharefeen, Applications of Neutrosophic $q$-Poisson Distribution Series for Subclass of Analytic Functions and Bi-Univalent Functions, Mathematics 11 (2023), 868. https://doi.org/10.3390/math11040868.

[12] A. Hussen, An Application of the Mittag-Leffler-Type Borel Distribution and Gegenbauer Polynomials on a Certain Subclass of Bi-Univalent Functions, Heliyon 10 (2024), e31469. https://doi.org/10.1016/j.heliyon.2024.e31469.

[13] A.M. Gbolagade, I.T. Awolere, O. Adeyemo, A.T. Oladipo, Application of the Neutrosophic Poisson Distribution Series on the Harmonic Subclass of Analytic Functions using the Salagean Derivative Operator, Neutrosoph. Syst. Appl. 23 (2024), 33–46. https://doi.org/10.61356/j.nswa.2024.23390.

[14] B.M. Algethami, A.Y. Lashin, F.Z. El-Emam, On a Subclass of Starlike Functions Related to Pascal and Poisson Distributions, Eur. J. Pure Appl. Math. 18 (2025), 7014. https://doi.org/10.29020/nybg.ejpam.v18i4.7014.

[15] B.S. Jubeir, M. El-Ityan, R.H. Buti, M.H. Hamza, On Class of Bi-Univalent Functions Involving Neutrosophic $q$-Poisson Distribution Series, Int. J. Neutrosophic Sci. 26 (2025), 359–365. https://doi.org/10.54216/IJNS.260326.

[16] E.A. Hussein, E.M. Hameed, R.H. Buti, Coefficient Estimates for a Subclass of Bi-Univalent Function Associated with Borel Distributions Using the Subordination Principle, J. Al-Qadisiyah Comput. Sci. Math. 17 (2025), 16–24. https://doi.org/10.29304/jqcsm.2025.17.11986.

[17] T. Al-Hawary, A. Alsoboh, A. Amourah, O. Ogilat, I. Harny, M. Darus, Applications of $q$-Borel Distribution Series Involving $q$-Gegenbauer Polynomials to Subclasses of Bi-Univalent Functions, Heliyon 10 (2024), e34187. https://doi.org/10.1016/j.heliyon.2024.e34187.

[18] J. Sokół, J. Stankiewicz, Radius of Convexity of Some Subclasses of Strongly Starlike Functions, Zesz. Nauk. Politech. Rzesz. Mat. 19 (1996), 101–105.

[19] K. Piejko, J. Sokół, On the Convolution and Subordination of Convex Functions, Appl. Math. Lett. 25 (2012), 448–453. https://doi.org/10.1016/j.aml.2011.09.034.

[20] M.H. Priya, R.B. Sharma, On a Class of Bounded Turning Functions Subordinate to a Leaf-Like Domain, J. Phys.: Conf. Ser. 1000 (2018), 012056. https://doi.org/10.1088/1742-6596/1000/1/012056.

[21] G. Singh, C. Kaur, Starlike and Convex Functions Subordinate to Leaf-Like Domain, Turk. J. Comput. Math. Educ. 12 (2021), 6098–6102.

[22] A. Alsoboh, G.I. Oros, A Class of Bi-Univalent Functions in a Leaf-Like Domain Defined through Subordination via $q$-Calculus, Mathematics 12 (2024), 1594. https://doi.org/10.3390/math12101594.

[23] G. Gasper, M. Rahman, Basic Hypergeometric Series, 2nd ed., Cambridge University Press, 2004. https://doi.org/10.1017/CBO9780511526251.

[24] W.C. Ma, D. Minda, A Unified Treatment of Some Special Classes of Univalent Functions, in: Z. Li, F. Ren, L. Yang, S. Zhang (Eds.), Proceedings of the Conference on Complex Analysis, Conference Proceedings and Lecture Notes in Analysis, Vol. 1, International Press, 1994, 157–169.

[25] S.S. Miller, Differential Inequalities and Carathéodory Functions, Bull. Am. Math. Soc. 81 (1975), 79–81. https://doi.org/10.1090/S0002-9904-1975-13643-3.

[26] M. El-Ityan, A. Amourah, S. Hammad, R. Buti, A. Alsoboh, New Subclass of Bi-Univalent Functions Involving the Wright Function Associated with the Jung–Kim–Srivastava Operator, Gulf J. Math. 19 (2025), 451–462. https://doi.org/10.56947/gjom.v19i2.2817.

[27] M. Ahmed, A. Alsoboh, A. Amourah, J. Salah, On the Fractional $q$-Differintegral Operator for Subclasses of Bi-Univalent Functions Subordinate to $q$-Ultraspherical Polynomials, Eur. J. Pure Appl. Math. 18 (2025), 6586. https://doi.org/10.29020/nybg.ejpam.v18i3.6586.

[28] A. Amourah, A. Alsoboh, J. Salah, K. Al Kalbani, Bounds on Initial Coefficients for Bi-Univalent Functions Linked to $q$-Analog of Le Roy-Type Mittag-Leffler Function, WSEAS Trans. Math. 23 (2024), 714–722. https://doi.org/10.37394/23206.2024.23.73.

[29] M. Rasheed, A.H. Majeed, E.N. Abdulwahab, D.S. Ali, Z. Mohammed, Certain Results of Riemann-Liouville Fractional Calculus Involving Seven-Parametric Mittag-Leffler Function, Gulf J. Math. 21 (2025), 625–632. https://doi.org/10.56947/gjom.v21i1.3585.

[30] N.M. Hammad, F. Al-Sharqi, Z.M. Rodzi, Similarity Measures of Bipolar Interval Valued-Fuzzy Soft Sets and Their Application in Multi-Criteria Decision-Making Method, J. Appl. Math. Inform. 43 (2025), 821–837. https://doi.org/10.14317/jami.2025.821.

[31] A. Alsoboh, A.S. Tayyah, A. Amourah, A.A. Al-Maqbali, K. Al Mashrafi, T. Sasa, Hankel Determinant Estimates for Bi-Bazilevič-Type Functions Involving $q$-Fibonacci Numbers, Eur. J. Pure Appl. Math. 18 (2025), 6698. https://doi.org/10.29020/nybg.ejpam.v18i3.6698.

[32] A. Almalkawi, A. Amourah, A. Alsoboh, J. Salah, K. Al Mashrafi, A.A.-R. Malkawi, T. Sasa, Analytic Estimates for Bi-Univalent Functions Associated with a New Operator Involving the $q$-Rabotnov Function, Int. J. Anal. Appl. 24 (2026), 1. https://doi.org/10.28924/2291-8639-24-2026-1.

[33] A. Alsoboh, A. Amourah, K. Al Mashrafi, T. Sasa, Bi-Starlike and Bi-Convex Function Classes Connected to Shell-Like Curves and the $q$-Analogue of Fibonacci Numbers, Int. J. Anal. Appl. 23 (2025), 201. https://doi.org/10.28924/2291-8639-23-2025-201.

[34] P. Zaprawa, On the Fekete-Szegö Problem for Classes of Bi-Univalent Functions, Bull. Belg. Math. Soc. Simon Stevin 21 (2014), 169–178. https://doi.org/10.36045/bbms/1394544302.

[35] M.M. Abed, F.G. Al-Sharqi, Classical Artinian Module and Related Topics, J. Phys.: Conf. Ser. 1003 (2018), 012065. https://doi.org/10.1088/1742-6596/1003/1/012065.

[36] Y. Al-Qudah, A.O. Hamadameen, N.A. Kh, F.A. Al-Sharqi, A New Generalization of Interval-Valued Q-Neutrosophic Soft Matrix and Its Applications, Int. J. Neutrosophic Sci. 25 (2025), 242–257. https://doi.org/10.54216/IJNS.250322.

[37] Y. Al-Qudah, N. Hassan, Complex Multi-Fuzzy Relation for Decision Making Using Uncertain Periodic Data, Int. J. Eng. Technol. 7 (2018), 2437–2445. https://doi.org/10.14419/ijet.v7i4.16976.