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Simpson's Type Estimates for Multiplicatively Strongly Convex Function via Non-Newtonian Calculus

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Abstract. In this paper, we initially establish a novel integral identity for multiplicatively differentiable functions within the framework of non-Newtonian calculus. By judiciously exploiting this identity, we derive a new Simpson's-type integral inequality for multiplicatively strongly convex functions. Several significant special cases and particular instances of the principal result are subsequently examined, thereby demonstrating the breadth and versatility of the proposed framework. Furthermore, illustrative applications involving various special means are presented to substantiate the analytical significance of the obtained results. The findings contribute to and substantially extend the existing theory of multiplicative integral inequalities by providing a unified analytical framework for the investigation of Simpson's estimates in non-Newtonian calculus. The newly established identity constitutes a powerful analytical instrument that may serve as a foundation for deriving further integral inequalities associated with broader classes of multiplicatively strongly convex functions and related generalized convexity structures.

1. Introduction

Inequality theory constitutes one of the most profound and rapidly evolving disciplines within mathematical analysis, furnishing indispensable analytical instruments for establishing quantitative estimates, deriving sharp bounds, and investigating intrinsic relationships among diverse mathematical entities. Owing to its remarkable versatility and far-reaching applicability, it has become an indispensable component of numerous branches of pure and applied mathematics, including optimization theory, functional analysis, approximation theory, probability theory, numerical analysis, differential equations, information theory, mathematical physics, engineering sciences, economics, and data science. The remarkable proliferation of inequality theory has been intrinsically intertwined with the evolution of convexity theory, which has long served as one of the most influential paradigms for the derivation of fundamental analytical inequalities. The geometric structure and analytical regularity exhibited by convex functions have facilitated the establishment of a vast spectrum of classical inequalities, thereby positioning convexity at the forefront of contemporary mathematical research. Nevertheless, the increasing complexity of modern mathematical models has demonstrated that the classical notion of convexity is frequently inadequate for describing numerous nonlinear phenomena encountered in theoretical investigations and practical applications. This limitation has stimulated an extensive body of research devoted to the formulation and systematic investigation of generalized convexity classes that preserve the essential characteristics of convexity while significantly enlarging its analytical framework [1–10].

The notion of strongly convex functions can be viewed as an essential strengthening of the classical notion of convexity. As mentioned by Karamardian [11], the basis of strongly convex functions was initiated in the field of mathematical analysis. Later, Polyak [12] made an important contribution to the theoretical foundation and rigorous study of the above notion. Unlike ordinary convex functions, strongly convex functions have an extra quadratic curvature requirement that makes the structure of such functions more stringent. The notion of strongly convex functions has been of great importance in various fields like optimization, variational analysis, mathematical economics, and the theory of integral inequalities, among others.

This theory was further generalized by Lin and Fukushima [13] by generalizing the classical theory of convexity with the use of higher order highly convex functions. In their studies, they were able to demonstrate the link between the higher order convexity of functions and the higher order monotonicity properties of the gradient. Specifically, for continuously differentiable functions, they were able to show that the strong higher order convexity of a function could be formulated with the help of the strong monotonicity of the gradient of the function. Consequently, their work provides a bridge between the geometrical properties of higher order convex functions and the analytic properties of their derivatives.

Another important development in the theory of mathematical inequalities came in 2011, when Srivastava et al. [14] contributed to the theory by improving and generalizing two of its basic inequalities, the Jensen inequality and the H-H inequality. The research was performed in the framework of the functions of n variables, thus expanding the classical inequalities into the multi-dimensional environment. These generalizations became a valuable tool in studying convex functions and showed the importance of Jensen- and H-H-type inequalities.

Later, Mishra and Sharma [15] proposed the concept of strongly generalized convex functions of higher order. This is indeed a great generalization of the classical theory of convexity as it brings together the idea of higher-order structural features within the theory of generalized convex functions. The advantage of this generalization lies in the fact that it helps us study functions which display complex geometrical or analytical properties beyond what is possible for ordinary convex functions or strong convexity. Using this generalization of convex functions, they derived some new H-H-type integral inequalities. For the literature, related to inequalities, see the references [16–22].

One such contribution can be attributed to Noor and Noor [23], who presented and studied the idea of strongly exponentially convex functions. Such a type of functions generalizes the classic notion of convexity by including an exponential structure in the convexity condition. Thus, the obtained class of functions can be used to describe a wider family of functions and allows for greater application flexibility where the exponential behavior is inherent. As a result, strongly exponentially convex functions have become part of generalized convexity theory and opened up new opportunities for deriving integral inequalities related to them.

Strong convexity function theory has also proved itself fruitful in connection with the recently arisen area of fractal analysis. In this regard, the author Butt [24] has studied inequalities of the Jensen–H-H–Mercer type in fractal sets using the notion of strong convexity. This is a significant intersection between generalized convex function theory and fractal structure analysis. Through application of the idea of strong convexity in a non-classical geometric setting, the author was able to create new opportunities in the area of studying integral inequalities in such spaces.

Moreover, the theory of strongly convex functions gained much from the works of Merentes and Nikodem [25], who studied the basic properties and analysis of such functions. This research helped to obtain much information regarding the behaviour of strongly convex functions and provided additional support for the theoretical bases of this class of generalized convex functions. In general, the above contributions can be considered an example of gradual development of the theory of strong convexity from the classical version up to high-order, exponential, fractal and other generalized forms.

In view of the pioneering study conducted by Ali et al. [26], multiplicative calculus has received widespread interest among mathematicians and has developed into a highly effective tool for the analysis of integral inequalities. The unique property of multiplicative calculus has enabled researchers to derive a number of inequalities of integer order related to many generalized convexity classes. Among others, major progress has been made in the field of multiplicative harmonic convexity [27] and multiplicative preinvex P-convexity [28].

Further studies in this area have extended the domain of multiplicative integral inequalities by dealing with various types of multiplicatively differentiable functions and inequality constructions. For example, Khan and Budak [29] derived H-H type inequalities for *differentiable functions via multiplicative integration, while Xie et al. [30] derived similar inequalities for **differentiable functions. The aforementioned works show how multiplicative calculus has been gradually developing from its initial concepts to a more advanced theory of generalized inequalities. Other important results on multiplicative Ostrowski, Simpson, and Maclaurin-type inequalities may be found in [31, 32].

In a more recent development, the application of multiplicative calculus has been extended from the analysis of integer order integral inequalities to that of fractional order analysis. In this context, some authors have studied the interplay between multiplicative calculus and the theory of fractional integrals, which has resulted in a new set of fractional inequalities. For an overview on recent advances in the field of fractional calculus along with its application in other areas apart from the theory of integral inequalities, one can refer to [33–41]. The mentioned studies collectively illustrate the rapidly increasing role of multiplicative calculus in contemporary mathematical theory.

Motivated by recent developments in multiplicative calculus and multiplicative integral inequalities, we establish a new identity for multiplicatively differentiable functions. Using this identity, we derive a novel tarpezoid-type inequality for multiplicatively strong convex functions, thereby extending the existing theory of multiplicative integral inequalities.

2. Material and Methods

In this section, we throw light on the basic notions and properties associated with multiplicative calculus.

Definition 2.1. *[25] A function Φ is termed as strongly convex on* I*, if*

$$\Phi\left(\varepsilon_1\mathfrak{y} + (1-\mathfrak{y})\varepsilon_1\right) \le \mathfrak{y}\Phi(\varepsilon_1) + (1-\mathfrak{y})\Phi(\varepsilon_2) - c\mathfrak{y}(1-\mathfrak{y})(\varepsilon_1-\varepsilon_2)^2,$$

holds $\forall\ \varepsilon_1, \varepsilon_2 \in \mathrm{I}$, $\mathfrak{y} \in [0,1]$ *and* $c > 0$. *Where* I *is a convex set.*

Definition 2.2. *[23] A function Φ is termed as multiplicatively strongly convex on* I*, if*

$$\log\Phi\left(\varepsilon_1\mathfrak{y} + (1-\mathfrak{y})\varepsilon_2\right) \le \mathfrak{y}\log\Phi(\varepsilon_1) + (1-\mathfrak{y})\log\Phi(\varepsilon_2) - c\mathfrak{y}(1-\mathfrak{y})(\varepsilon_1-\varepsilon_2)^2, \tag{2.1}$$

holds $\forall\ \varepsilon_1, \varepsilon_2 \in \mathrm{I}$, $\mathfrak{y} \in [0,1]$ *and* $c > 0$.

An alternate way of expression 2.1 is given by

$$\Phi\left(\varepsilon_1\mathfrak{y} + (1-\mathfrak{y})\varepsilon_2\right) \leq \frac{[\Phi(\varepsilon_1)]^{\mathfrak{y}}[\Phi(\varepsilon_2)]^{1-\mathfrak{y}}}{exp\{c\mathfrak{y}(1-\mathfrak{y})(\varepsilon_1-\varepsilon_2)^2\}},$$

$\forall\ \varepsilon_1, \varepsilon_2 \in \mathrm{I}$ *and* $\mathfrak{y} \in [0,1]$.

Next consider the properties associated with multiplicative calculus

Definition 2.3. *[34] The *derivative for a positive valued function* $\Phi : \mathbb{R} \to \mathbb{R}$ *is defined as in the subsequent*

$$\frac{d^*\Phi}{d\mathfrak{x}} = \Phi^*(\mathfrak{x}) = \lim_{h\to 0}\left(\frac{\Phi(\mathfrak{x}+h)}{\Phi(\mathfrak{x})}\right)^{\frac{1}{h}}.$$

Remark 2.1. *The relationship between the classical and *derivatives of a positive function is as follows:*

$$\Phi^*(\mathfrak{x}) = \exp\{(\ln\Phi(\mathfrak{x}))'\} = \exp\left\{\frac{\Phi'(\mathfrak{x})}{\Phi(\mathfrak{x})}\right\}.$$

While the relationship between *integral and the Riemann integral is as follows [34]:

Proposition 2.1. *Let the function* Φ *is both Riemann and *integrable on the* $[\varepsilon_1, \varepsilon_2]$ *then.*

$$\int_{\varepsilon_2}^{\varepsilon_1}(\Phi(\mathfrak{x}))^{d\mathfrak{x}} = \exp\left\{\int_{\varepsilon_2}^{\varepsilon_1}\ln(\Phi(\mathfrak{x}))d\mathfrak{x}\right\}.$$

Moreover, Bashirov et al. [34] established several properties and related results concerning *integrable functions, including the following:

Proposition 2.2. *Let the function* Φ *is *integrable on* $[\mu o, \nu o]$, *then*

(1) $\int\limits_{\varepsilon_1}^{\varepsilon_2}\left((\Phi(\mathfrak{x}))^p\right)^{d\mathfrak{x}} = \int\limits_{\varepsilon_2}^{\varepsilon_1}\left((\Phi(\mathfrak{x}))^{d\mathfrak{x}}\right)^p$

(2) $\int\limits_{\varepsilon_1}^{\varepsilon_2}(\Phi(\mathfrak{x})\,\Upsilon(\mathfrak{x}))^{d\mathfrak{x}} = \int\limits_{\varepsilon_1}^{\varepsilon_2}(\Phi(\mathfrak{x}))^{d\mathfrak{x}} \cdot \int\limits_{\varepsilon_1}^{\varepsilon_2}(\Upsilon(\mathfrak{x}))^{d\mathfrak{x}}$

(3) $\int\limits_{\varepsilon_1}^{\varepsilon_2}\left(\frac{\Phi(\mathfrak{x})}{\Upsilon(\mathfrak{x})}\right)^{d\mathfrak{x}} = \frac{\int_{\varepsilon_1}^{\varepsilon_2}(\Phi(\mathfrak{x}))^{d\mathfrak{x}}}{\int_{\varepsilon_1}^{\varepsilon_2}(\Upsilon(\mathfrak{x}))^{d\mathfrak{x}}}$

(4) $\int\limits_{\varepsilon_1}^{\varepsilon_2}(\Phi(\mathfrak{x}))^{d\mathfrak{x}} = \int\limits_{\varepsilon_1}^{c}(\Phi(\mathfrak{x}))^{d\mathfrak{x}} \cdot \int\limits_{c}^{\varepsilon_2}(\Phi(\mathfrak{x}))^{d\mathfrak{x}}, \quad \varepsilon_1 \leq c \leq \varepsilon_2$

(5) $\int\limits_{\varepsilon_1}^{\varepsilon_1}(\Phi(\mathfrak{x}))^{d\mathfrak{x}} = 1, \quad \int\limits_{\varepsilon_1}^{\varepsilon_2}(\Phi(\mathfrak{x}))^{d\mathfrak{x}} = \left(\int\limits_{\varepsilon_2}^{\varepsilon_1}(\Phi(\mathfrak{x}))^{d\mathfrak{x}}\right)^{-1}.$

Theorem 2.1. *(Multiplicative version of the integration by parts formula [34]) Let the functions* $\Phi : [\varepsilon_1, \varepsilon_2] \to \mathbb{R}$ *and* $\Upsilon : [\varepsilon_1, \varepsilon_2] \to \mathbb{R}$ *be *differentiable and differentiable respectively and the function* Φ^{Υ} *be a *integrable then*

$$\int_{\varepsilon_1}^{\varepsilon_2}\left(\Phi^*(\mathfrak{x})^{\Upsilon(\mathfrak{x})}\right)^{d\mathfrak{x}} = \frac{\Phi(\varepsilon_2)^{\Upsilon(\varepsilon_2)}}{\Phi(\varepsilon_1)^{\Upsilon(\varepsilon_1)}} \times \frac{1}{\int_{\varepsilon_1}^{\varepsilon_2}\left(\Phi(\mathfrak{x})^{\Upsilon'(\mathfrak{x})}\right)^{d\mathfrak{x}}}.$$

Lemma 2.1. *[34] Let $\Phi : [\varepsilon_1, \varepsilon_2] \to \mathbb{R}$ and $\Upsilon : [\varepsilon_1, \varepsilon_2] \to \mathbb{R}$ be *differentiable and differentiable respectively, and Φ^{Υ} is *integrable then*

$$\int_{\varepsilon_1}^{\varepsilon_2} \left(\Phi^*(\mathrm{h}(\mathfrak{x}))^{\mathrm{h}'(\mathfrak{x})\Upsilon(\mathfrak{x})} \right)^{d\mathfrak{x}} = \frac{\Phi(\varepsilon_2)^{\Upsilon(\varepsilon_2)}}{\Phi(\varepsilon_1)^{\Upsilon(\varepsilon_1)}} \times \frac{1}{\int_{\varepsilon_1}^{\varepsilon_2} \left(\Phi(\mathrm{h}(\mathfrak{x}))^{\Upsilon'(\mathfrak{x})} \right)^{d\mathfrak{x}}}.$$

3. Main Results

In this section, we first develop the identity, then utilizing that identity, we establish the Simpson's type inequality.

Lemma 3.1. *Let $\Phi : [\varepsilon_1, \varepsilon_2] \to \mathbb{R}^+$ be a *differentiable function on $[\varepsilon_1, \varepsilon_2]$, where $\varepsilon_1 < \varepsilon_2$. If Φ^* is *integrable on $[\varepsilon_1, \varepsilon_2]$, then*

$$\begin{aligned} &\left(\Phi(\varepsilon_1) \left(\Phi\left(\frac{\varepsilon_1 + \varepsilon_2}{2} \right) \right)^4 \Phi(\varepsilon_2) \right)^{\frac{1}{6}} \left(\int_{\varepsilon_1}^{\varepsilon_2} (\Phi(\mathfrak{x}))^{d\mathfrak{x}} \right)^{\frac{1}{\varepsilon_1 - \varepsilon_2}} \\ &= \left[\int_0^1 \left(\Phi^*\left(\left(\frac{1+\mathfrak{y}}{2} \right) \varepsilon_2 + \left(\frac{1-\mathfrak{y}}{2} \right) \varepsilon_1 \right) \right)^{\left(\frac{\mathfrak{y}}{2} - \frac{1}{3} \right) d\mathfrak{y}} \right]^{\frac{\varepsilon_2 - \varepsilon_1}{2}} \\ &\times \left[\int_0^1 \left(\Phi^*\left(\left(\frac{1+\mathfrak{y}}{2} \right) \varepsilon_1 + \left(\frac{1-\mathfrak{y}}{2} \right) \varepsilon_2 \right) \right)^{\left(\frac{1}{3} - \frac{\mathfrak{y}}{2} \right) d\mathfrak{y}} \right]^{\frac{\varepsilon_2 - \varepsilon_1}{2}}. \end{aligned} \tag{3.1}$$

Proof. For convenience, introduce the auxiliary quantities

$$\mathrm{I}_1 = \left[\int_0^1 \left(\Phi^*\left(\frac{1+\mathfrak{y}}{2} \varepsilon_2 + \frac{1-\mathfrak{y}}{2} \varepsilon_1 \right) \right)^{\left(\frac{\mathfrak{y}}{2} - \frac{1}{3} \right) d\mathfrak{y}} \right]^{\frac{\varepsilon_2 - \varepsilon_1}{2}}, \tag{3.2}$$

$$\mathrm{I}_2 = \left[\int_0^1 \left(\Phi^*\left(\frac{1+\mathfrak{y}}{2} \varepsilon_1 + \frac{1-\mathfrak{y}}{2} \varepsilon_2 \right) \right)^{\left(\frac{1}{3} - \frac{\mathfrak{y}}{2} \right) d\mathfrak{y}} \right]^{\frac{\varepsilon_2 - \varepsilon_1}{2}}. \tag{3.3}$$

We first evaluate I_1. By incorporating the factor $\frac{\varepsilon_2 - \varepsilon_1}{2}$ into the exponent, we obtain

$$\mathrm{I}_1 = \int_0^1 \left(\Phi^*\left(\frac{1+\mathfrak{y}}{2} \varepsilon_2 + \frac{1-\mathfrak{y}}{2} \varepsilon_1 \right) \right)^{\frac{\varepsilon_2 - \varepsilon_1}{2} \left(\frac{\mathfrak{y}}{2} - \frac{1}{3} \right) d\mathfrak{y}}. \tag{3.4}$$

Applying integration by parts in the framework of *calculus and using the fact that

$$\frac{\varepsilon_2 - \varepsilon_1}{2} \left(\frac{\mathfrak{y}}{2} - \frac{1}{3} \right) = \frac{\varepsilon_2 - \varepsilon_1}{4} \mathfrak{y} - \frac{\varepsilon_2 - \varepsilon_1}{6},$$

we obtain

$$\mathrm{I}_1 = \frac{(\Phi(\varepsilon_2))^{\frac{1}{6}}}{\left(\Phi\left(\frac{\varepsilon_1 + \varepsilon_2}{2} \right) \right)^{-\frac{1}{3}}} \left[\int_0^1 \left(\Phi\left(\frac{1+\mathfrak{y}}{2} \varepsilon_2 + \frac{1-\mathfrak{y}}{2} \varepsilon_1 \right) \right)^{\frac{1}{2} d\mathfrak{y}} \right]^{-\frac{1}{2}}. \tag{3.5}$$

Now, making the change of variables

$$\mathfrak{x} = \frac{1+\mathfrak{y}}{2} \varepsilon_2 + \frac{1-\mathfrak{y}}{2} \varepsilon_1,$$

we have

$$\mathrm{d}\mathfrak{x} = \frac{\varepsilon_2 - \varepsilon_1}{2}\mathrm{d}\mathfrak{y},$$

and, as $\mathfrak{y}$ varies from 0 to 1, the variable $\mathfrak{x}$ varies from $\frac{\varepsilon_1+\varepsilon_2}{2}$ to ε_2. Consequently,

$$\mathrm{I}_1 = (\Phi(\varepsilon_2))^{\frac{1}{6}} \left(\Phi\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\right)^{\frac{1}{3}} \left[\int_{\frac{\varepsilon_1+\varepsilon_2}{2}}^{\varepsilon_2} (\Phi(\mathfrak{x}))^{\mathrm{d}\mathfrak{x}}\right]^{-\frac{1}{\varepsilon_2-\varepsilon_1}}. \tag{3.6}$$

Similarly, for I_2, we have

$$\mathrm{I}_2 = \left[\int_0^1 \left(\Phi^*\left(\frac{1+\mathfrak{y}}{2}\varepsilon_1 + \frac{1-\mathfrak{y}}{2}\varepsilon_2\right)\right)^{\left(\frac{1}{3}-\frac{\mathfrak{y}}{2}\right)\mathrm{d}\mathfrak{y}}\right]^{\frac{\varepsilon_2-\varepsilon_1}{2}}. \tag{3.7}$$

Employing the same integration-by-parts formula for *integrals, followed by the substitution

$$\mathfrak{x} = \frac{1+\mathfrak{y}}{2}\varepsilon_1 + \frac{1-\mathfrak{y}}{2}\varepsilon_2,$$

we obtain

$$\mathrm{I}_2 = (\Phi(\varepsilon_1))^{\frac{1}{6}} \left(\Phi\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\right)^{\frac{1}{3}} \left[\int_{\varepsilon_1}^{\frac{\varepsilon_1+\varepsilon_2}{2}} (\Phi(\mathfrak{x}))^{\mathrm{d}\mathfrak{x}}\right]^{-\frac{1}{\varepsilon_2-\varepsilon_1}}. \tag{3.8}$$

Multiplying (3.6) and (3.8), we consequently obtain

$$\begin{aligned}\mathrm{I}_1\mathrm{I}_2 = {} & (\Phi(\varepsilon_1))^{\frac{1}{6}} (\Phi(\varepsilon_2))^{\frac{1}{6}} \left(\Phi\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\right)^{\frac{2}{3}} \\ & \times \left[\int_{\varepsilon_1}^{\frac{\varepsilon_1+\varepsilon_2}{2}} (\Phi(\mathfrak{x}))^{\mathrm{d}\mathfrak{x}}\right]^{-\frac{1}{\varepsilon_2-\varepsilon_1}} \left[\int_{\frac{\varepsilon_1+\varepsilon_2}{2}}^{\varepsilon_2} (\Phi(\mathfrak{x}))^{\mathrm{d}\mathfrak{x}}\right]^{-\frac{1}{\varepsilon_2-\varepsilon_1}}.\end{aligned} \tag{3.9}$$

Using the multiplicative additivity property of the *integral,

$$\left(\int_{\varepsilon_1}^{\frac{\varepsilon_1+\varepsilon_2}{2}} (\Phi(\mathfrak{x}))^{\mathrm{d}\mathfrak{x}}\right)\left(\int_{\frac{\varepsilon_1+\varepsilon_2}{2}}^{\varepsilon_2} (\Phi(\mathfrak{x}))^{\mathrm{d}\mathfrak{x}}\right) = \int_{\varepsilon_1}^{\varepsilon_2} (\Phi(\mathfrak{x}))^{\mathrm{d}\mathfrak{x}},$$

we arrive at

$$\mathrm{I}_1\mathrm{I}_2 = (\Phi(\varepsilon_1))^{\frac{1}{6}} \left(\Phi\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\right)^{\frac{2}{3}} (\Phi(\varepsilon_2))^{\frac{1}{6}} \left(\int_{\varepsilon_1}^{\varepsilon_2} (\Phi(\mathfrak{x}))^{\mathrm{d}\mathfrak{x}}\right)^{-\frac{1}{\varepsilon_2-\varepsilon_1}}. \tag{3.10}$$

Rearranging the last equality yields

$$(\Phi(\varepsilon_1)\Phi(\varepsilon_2))^{\frac{1}{6}} \left(\Phi\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\right)^{\frac{2}{3}} \left(\int_{\varepsilon_1}^{\varepsilon_2} (\Phi(\mathfrak{x}))^{\mathrm{d}\mathfrak{x}}\right)^{\frac{1}{\varepsilon_1-\varepsilon_2}} = \mathrm{I}_1\mathrm{I}_2. \tag{3.11}$$

Substituting the definitions of I_1 and I_2 from (3.2)–(3.3) completes the proof of (3.1). □

The preceding result enable us to establish novel Simpson's type inequality for multiplicatively strongly convex functions associated with multiplicative integrals.

Theorem 3.1. *Let the mapping $\Phi : [\varepsilon_1, \varepsilon_2] \to \mathbb{R}^+$ be multiplicatively differentiable and Φ^* be a multiplicatively strongly convex function on $[\varepsilon_1, \varepsilon_2]$. Then*

$$\left|\left(\Phi(\varepsilon_1)\left(\Phi\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\right)^4 \Phi(\varepsilon_2)\right)^{\frac{1}{6}} \left(\int_{\varepsilon_1}^{\varepsilon_2} (\Phi(\mathfrak{x}))^{d\mathfrak{x}}\right)^{\frac{1}{\varepsilon_1-\varepsilon_2}}\right| \leq \frac{[\Phi^*(\varepsilon_1)\Phi^*(\varepsilon_2)]^{\frac{5(\varepsilon_2-\varepsilon_1)}{72}}}{\exp\left\{\frac{211\,c(\varepsilon_2-\varepsilon_1)^3}{7776}\right\}}, \tag{3.12}$$

holds.

Proof. From Lemma 3.1 and the properties of the *integral, we obtain

$$\begin{aligned}
&\left|\left(\Phi(\varepsilon_1)\left(\Phi\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\right)^4 \Phi(\varepsilon_2)\right)^{\frac{1}{6}} \left(\int_{\varepsilon_1}^{\varepsilon_2} (\Phi(\mathfrak{x}))^{d\mathfrak{x}}\right)^{\frac{1}{\varepsilon_1-\varepsilon_2}}\right| \\
&\leq \left(\exp\left\{\frac{\varepsilon_2-\varepsilon_1}{2}\int_0^1 \left|\ln\left(\Phi^*\left(\frac{1+\mathfrak{y}}{2}\varepsilon_2+\frac{1-\mathfrak{y}}{2}\varepsilon_1\right)\right)\right|^{\left(\frac{\mathfrak{y}}{2}-\frac{1}{3}\right)} d\mathfrak{y}\right\}\right) \\
&\times\left(\exp\left\{\frac{\varepsilon_2-\varepsilon_1}{2}\int_0^1 \left|\ln\left(\Phi^*\left(\frac{1+\mathfrak{y}}{2}\varepsilon_1+\frac{1-\mathfrak{y}}{2}\varepsilon_2\right)\right)\right|^{\left(\frac{1}{3}-\frac{\mathfrak{y}}{2}\right)} d\mathfrak{y}\right\}\right) \\
&= \left(\exp\left\{\frac{\varepsilon_2-\varepsilon_1}{2}\int_0^1 \left|\frac{\mathfrak{y}}{2}-\frac{1}{3}\right|\left|\ln\left(\Phi^*\left(\frac{1+\mathfrak{y}}{2}\varepsilon_2+\frac{1-\mathfrak{y}}{2}\varepsilon_1\right)\right)\right| d\mathfrak{y}\right\}\right) \\
&\times\left(\exp\left\{\frac{\varepsilon_2-\varepsilon_1}{2}\int_0^1 \left|\frac{1}{3}-\frac{\mathfrak{y}}{2}\right|\left|\ln\left(\Phi^*\left(\frac{1+\mathfrak{y}}{2}\varepsilon_1+\frac{1-\mathfrak{y}}{2}\varepsilon_2\right)\right)\right| d\mathfrak{y}\right\}\right).
\end{aligned} \tag{3.13}$$

By using the multiplicatively strongly convexity of Φ^*, we have

$$\begin{aligned}
&\left|\left(\Phi(\varepsilon_1)\left(\Phi\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\right)^4 \Phi(\varepsilon_2)\right)^{\frac{1}{6}} \left(\int_{\varepsilon_1}^{\varepsilon_2} (\Phi(\mathfrak{x}))^{d\mathfrak{x}}\right)^{\frac{1}{\varepsilon_1-\varepsilon_2}}\right| \\
&\leq \left(\exp\left\{\frac{\varepsilon_2-\varepsilon_1}{2}\int_0^1 \left|\frac{\mathfrak{y}}{2}-\frac{1}{3}\right|\left|\ln\left(\frac{(\Phi^*(\varepsilon_2))^{\left(\frac{1+\mathfrak{y}}{2}\right)}(\Phi^*(\varepsilon_1))^{\left(\frac{1-\mathfrak{y}}{2}\right)}}{exp\{\frac{c(1-\mathfrak{y}^2)(\varepsilon_2-\varepsilon_1)^2}{4}\}}\right)\right| d\mathfrak{y}\right\}\right) \\
&\times\left(\exp\left\{\frac{\varepsilon_2-\varepsilon_1}{2}\int_0^1 \left|\frac{1}{3}-\frac{\mathfrak{y}}{2}\right|\left|\ln\left(\frac{(\Phi^*(\varepsilon_1))^{\left(\frac{1+\mathfrak{y}}{2}\right)}(\Phi^*(\varepsilon_2))^{\left(\frac{1-\mathfrak{y}}{2}\right)}}{exp\{\frac{c(1-\mathfrak{y}^2)(\varepsilon_2-\varepsilon_1)^2}{4}\}}\right)\right| d\mathfrak{y}\right\}\right) \\
&= \left(\exp\left\{\frac{\varepsilon_2-\varepsilon_1}{2}\int_0^1 \left|\frac{\mathfrak{y}}{2}-\frac{1}{3}\right|\left[\left(\frac{1+\mathfrak{y}}{2}\right)\ln\left(\Phi^*(\varepsilon_2)\right)+\left(\frac{1-\mathfrak{y}}{2}\right)\ln\left(\Phi^*(\varepsilon_1)\right)-\frac{c(1-\mathfrak{y}^2)(\varepsilon_2-\varepsilon_1)^2}{4}\right] d\mathfrak{y}\right\}\right) \\
&\times\left(\exp\left\{\frac{\varepsilon_2-\varepsilon_1}{2}\int_0^1 \left|\frac{1}{3}-\frac{\mathfrak{y}}{2}\right|\left[\left(\frac{1+\mathfrak{y}}{2}\right)\ln\left(\Phi^*(\varepsilon_1)\right)+\left(\frac{1-\mathfrak{y}}{2}\right)\ln\left(\Phi^*(\varepsilon_2)\right)-\frac{c(1-\mathfrak{y}^2)(\varepsilon_2-\varepsilon_1)^2}{4}\right] d\mathfrak{y}\right\}\right) \\
&= \exp\left\{\frac{\varepsilon_2-\varepsilon_1}{2}\int_0^1 \left|\frac{\mathfrak{y}}{2}-\frac{1}{3}\right|\left[\ln(\Phi^*(\varepsilon_1))+\ln(\Phi^*(\varepsilon_2))-\frac{c(1-\mathfrak{y}^2)(\varepsilon_2-\varepsilon_1)^2}{2}\right] d\mathfrak{y}\right\} \\
&= \exp\left\{\frac{\varepsilon_2-\varepsilon_1}{2}\int_0^1 \left|\frac{\mathfrak{y}}{2}-\frac{1}{3}\right|\left[\ln(\Phi^*(\varepsilon_1)\Phi^*(\varepsilon_2))-\frac{c(1-\mathfrak{y}^2)(\varepsilon_2-\varepsilon_1)^2}{2}\right] d\mathfrak{y}\right\} \\
&= \exp\left\{\frac{\varepsilon_2-\varepsilon_1}{4}\int_0^1 \left|\frac{\mathfrak{y}}{2}-\frac{1}{3}\right|\left[2\ln(\Phi^*(\varepsilon_1)\Phi^*(\varepsilon_2))-c(1-\mathfrak{y}^2)(\varepsilon_2-\varepsilon_1)^2\right] d\mathfrak{y}\right\}
\end{aligned}$$

$$= \exp\left\{\frac{\varepsilon_2-\varepsilon_1}{4}\left(\frac{5\ln(\Phi^*(\varepsilon_1)\Phi^*(\varepsilon_2))}{18} - \frac{211\,c(\varepsilon_2-\varepsilon_1)^2}{1944}\right)\right\}$$

$$= \exp\left\{\ln(\Phi^*(\varepsilon_1)\Phi^*(\varepsilon_2))^{\frac{5(\varepsilon_2-\varepsilon_1)}{72}} - \frac{211\,c(\varepsilon_2-\varepsilon_1)^3}{7776}\right\}$$

$$= \frac{[\Phi^*(\varepsilon_1)\Phi^*(\varepsilon_2)]^{\frac{5(\varepsilon_2-\varepsilon_1)}{72}}}{\exp\left\{\frac{211\,c(\varepsilon_2-\varepsilon_1)^3}{7776}\right\}}.$$

□

Corollary 3.1. *If we set c=0 in the Theorem 3.1, we get the Simpson's type inequality for multiplicatively convex function.*

Corollary 3.2. *In Theorem 3.1, if we set* $\Phi(\varepsilon_1) = \Phi\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right) = \Phi(\varepsilon_2)$, *then we obtain*

$$\left|\left(\Phi\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\right)\left(\int_{\varepsilon_1}^{\varepsilon_2}(\Phi(\mathfrak{x}))^{d\mathfrak{x}}\right)^{\frac{1}{\varepsilon_1-\varepsilon_2}}\right| \leq \frac{[\Phi^*(\varepsilon_1)\Phi^*(\varepsilon_2)]^{\frac{5(\varepsilon_2-\varepsilon_1)}{72}}}{\exp\left\{\frac{211\,c(\varepsilon_2-\varepsilon_1)^3}{7776}\right\}}, \tag{3.14}$$

Corollary 3.3. *In Theorem 3.1, if we set* $\Phi^* \leq M$, *then we have*

$$\left|\left(\Phi(\varepsilon_1)\left(\Phi\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\right)^4\Phi(\varepsilon_2)\right)^{\frac{1}{6}}\left(\int_{\varepsilon_1}^{\varepsilon_2}(\Phi(\mathfrak{x}))^{d\mathfrak{x}}\right)^{\frac{1}{\varepsilon_1-\varepsilon_2}}\right| \exp\left\{\frac{211\,c(\varepsilon_2-\varepsilon_1)^3}{7776}\right\} \leq [M]^{\frac{5(\varepsilon_2-\varepsilon_1)}{36}},$$

4. Applications

This section is devoted to the applications of the obtained results in terms of special means. Let us consider some means for the numbers ε_1 and ε_2.

The Arithmetic mean: $A(\varepsilon_1,\varepsilon_2) = \frac{\varepsilon_1+\varepsilon_2}{2}$.

Geometric mean: $G(\varepsilon_1,\varepsilon_2) = \sqrt{\varepsilon_1\varepsilon_2}$.

The logarithmic means: $L(\varepsilon_1,\varepsilon_2) = \frac{\varepsilon_2-\varepsilon_1}{\ln\varepsilon_2 - \ln\varepsilon_1}$, $\varepsilon_1,\varepsilon_2 > 0$ and $\varepsilon_1 \neq \varepsilon_2$.

The m-Logarithmic mean: $L_m(\varepsilon_1,\varepsilon_2) = \left(\frac{\varepsilon_2^{m+1}-\varepsilon_1^{m+1}}{(m+1)(\varepsilon_2-\varepsilon_1)}\right)^{\frac{1}{m}}$,

$\varepsilon_1,\varepsilon_2 > 0$, $\varepsilon_1 \neq \varepsilon_2$ and $m \in \mathbb{R}\backslash\{-1,0\}$.

The function $\Phi(\mathfrak{x}) = \mathfrak{x}^{\mathfrak{m}}$, is strongly convex with modulus c=1 and $\mathfrak{m} \geq 2$. Therefore the function $\Phi(\mathfrak{x}) = \exp\{\mathfrak{x}^{\mathfrak{m}}\}$ is then multiplicatively strong convex. Also the function $\Phi^*(\mathfrak{x}) = \exp\{\mathfrak{m}\mathfrak{x}^{\mathfrak{m}-1}\}$ is multiplicatively strong convex with modulus c=1 and $\mathfrak{m} \geq 3$. Setting these functions in Theorem 3.1, we achieve the following recurrence relation.

Proposition 4.1. *Let* $\varepsilon_1,\varepsilon_2 \in \mathbb{R}^+$ *with* $0 < \varepsilon_1 < \varepsilon_2$, *then*

$$\frac{{\varepsilon_1}^m + 4A^m(\varepsilon_1,\varepsilon_2) + {\varepsilon_2}^m}{6} - L_m^m(\varepsilon_1,\varepsilon_2) \leq \frac{5m(\varepsilon_2-\varepsilon_1)}{36}A\left({\varepsilon_1}^{m-1},{\varepsilon_2}^{m-1}\right) - \frac{211c(\varepsilon_2-\varepsilon_1)^3}{7776}.$$

Proof. Let $\Phi(\mathfrak{x}) = \exp\{\mathfrak{x}^{m}\}$, and $\Phi^{*}(\mathfrak{x}) = \exp\left\{m\mathfrak{x}^{m-1}\right\}$. Further, recall the following classical means:

$$
\begin{aligned}
A(\varepsilon_1,\varepsilon_2) &= \frac{\varepsilon_1+\varepsilon_2}{2},\\
G(\varepsilon_1,\varepsilon_2) &= \sqrt{\varepsilon_1\varepsilon_2},\\
L(\varepsilon_1,\varepsilon_2) &= \frac{\varepsilon_2-\varepsilon_1}{\ln\varepsilon_2-\ln\varepsilon_1},\\
L_m(\varepsilon_1,\varepsilon_2) &= \left(\frac{\varepsilon_2^{m+1}-\varepsilon_1^{m+1}}{(m+1)(\varepsilon_2-\varepsilon_1)}\right)^{\frac{1}{m}}, \qquad m\neq -1,0.
\end{aligned}
$$

Substituting $\Phi(\mathfrak{x}) = \exp\{\mathfrak{x}^{m}\}$, into the left-hand side of inequality (3.12), we obtain

$$
\begin{aligned}
&\left(\Phi(\varepsilon_1)\left(\Phi\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\right)^4\Phi(\varepsilon_2)\right)^{\frac{1}{6}}\\
&= \left(e^{\varepsilon_1{}^{m}} e^{4\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)^{m}} e^{\varepsilon_2{}^{m}}\right)^{\frac{1}{6}}\\
&= \exp\left\{\frac{\varepsilon_1{}^{m}+4\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)^{m}+\varepsilon_2{}^{m}}{6}\right\}\\
&= \exp\left\{\frac{\varepsilon_1{}^{m}+4A^{m}(\varepsilon_1,\varepsilon_2)+\varepsilon_2{}^{m}}{6}\right\},
\end{aligned}
$$

since $A(\varepsilon_1,\varepsilon_2) = \frac{\varepsilon_1+\varepsilon_2}{2}$. Next, using the definition of the multiplicative integral,

$$
\begin{aligned}
\int_{\varepsilon_1}^{\varepsilon_2}(\Phi(\mathfrak{x}))^{d\mathfrak{x}} &= \exp\left\{\int_{\varepsilon_1}^{\varepsilon_2}\ln(\Phi(\mathfrak{x}))\,d\mathfrak{x}\right\}\\
&= \exp\left\{\int_{\varepsilon_1}^{\varepsilon_2}\mathfrak{x}^{m}\,d\mathfrak{x}\right\}\\
&= \exp\left\{\frac{\varepsilon_2{}^{m+1}-\varepsilon_1{}^{m+1}}{m+1}\right\}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\left(\int_{\varepsilon_1}^{\varepsilon_2}(\Phi(\mathfrak{x}))^{d\mathfrak{x}}\right)^{\frac{1}{\varepsilon_1-\varepsilon_2}} &= \exp\left\{\frac{\varepsilon_2{}^{m+1}-\varepsilon_1{}^{m+1}}{(m+1)(\varepsilon_1-\varepsilon_2)}\right\}\\
&= \exp\left\{-\frac{\varepsilon_2{}^{m+1}-\varepsilon_1{}^{m+1}}{(m+1)(\varepsilon_2-\varepsilon_1)}\right\}.
\end{aligned}
$$

Using the definition of the m-logarithmic mean, $L_m^m(\varepsilon_1,\varepsilon_2) = \frac{\varepsilon_2{}^{m+1}-\varepsilon_1{}^{m+1}}{(m+1)(\varepsilon_2-\varepsilon_1)}$, we have

$$
\left(\int_{\varepsilon_1}^{\varepsilon_2}(\Phi(\mathfrak{x}))^{d\mathfrak{x}}\right)^{\frac{1}{\varepsilon_1-\varepsilon_2}} = \exp\left\{-L_m^m(\varepsilon_1,\varepsilon_2)\right\}.
$$

Therefore,

$$
\left(\Phi(\varepsilon_1)\left(\Phi\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\right)^4\Phi(\varepsilon_2)\right)^{\frac{1}{6}}\left(\int_{\varepsilon_1}^{\varepsilon_2}(\Phi(\mathfrak{x}))^{d\mathfrak{x}}\right)^{\frac{1}{\varepsilon_1-\varepsilon_2}}
$$

$$= \exp\left\{\frac{\varepsilon_1{}^{m} + 4A^{m}(\varepsilon_1, \varepsilon_2) + \varepsilon_2{}^{m}}{6} - L_m^m(\varepsilon_1, \varepsilon_2)\right\}.$$

Now, $\Phi^*(\mathfrak{x}) = \exp\left\{m\mathfrak{x}^{m-1}\right\}$, implies $\Phi^*(\varepsilon_1) = \exp\{m\varepsilon_1{}^{m-1}\}$, and $\Phi^*(\varepsilon_2) = \exp\{m\varepsilon_2{}^{m-1}\}$. Hence,

$$\begin{aligned}
&[\Phi^*(\varepsilon_1)\Phi^*(\varepsilon_2)]^{\frac{5(\varepsilon_2-\varepsilon_1)}{72}} \\
&= \exp\left\{\frac{5m(\varepsilon_2 - \varepsilon_1)}{72}\left(\varepsilon_1{}^{m-1} + \varepsilon_2{}^{m-1}\right)\right\} \\
&= \exp\left\{\frac{5m(\varepsilon_2 - \varepsilon_1)}{36} A\left(\varepsilon_1{}^{m-1}, \varepsilon_2{}^{m-1}\right)\right\}.
\end{aligned}$$

Therefore, the right-hand side becomes

$$\exp\left\{\frac{5m(\varepsilon_2 - \varepsilon_1)}{36} A\left(\varepsilon_1{}^{m-1}, \varepsilon_2{}^{m-1}\right) - \frac{211c(\varepsilon_2 - \varepsilon_1)^3}{7776}\right\}.$$

Consequently, inequality (3.1) reduces to

$$\begin{aligned}
&\exp\left\{\frac{\varepsilon_1{}^{m} + 4A^{m}(\varepsilon_1, \varepsilon_2) + \varepsilon_2{}^{m}}{6} - L_m^m(\varepsilon_1, \varepsilon_2)\right\} \\
&\le \exp\left\{\frac{5m(\varepsilon_2 - \varepsilon_1)}{36} A\left(\varepsilon_1{}^{m-1}, \varepsilon_2{}^{m-1}\right) - \frac{211c(\varepsilon_2 - \varepsilon_1)^3}{7776}\right\}.
\end{aligned}$$

Taking the natural logarithm on both sides yields

$$\frac{\varepsilon_1{}^{m} + 4A^{m}(\varepsilon_1, \varepsilon_2) + \varepsilon_2{}^{m}}{6} - L_m^m(\varepsilon_1, \varepsilon_2) \le \frac{5m(\varepsilon_2 - \varepsilon_1)}{36} A\left(\varepsilon_1{}^{m-1}, \varepsilon_2{}^{m-1}\right) - \frac{211c(\varepsilon_2 - \varepsilon_1)^3}{7776}.$$

This completes the proof. □

5. Conclusion and Future Research Directions

In this paper, a new identity for the class of multiplicatively differentiable functions in the context of non-Newtonian calculus is constructed and used to obtain a new trapezoid-type integral inequality for multiplicatively strongly convex functions. The derived inequalities are an addition to the existing theory of multiplicative integral inequalities, since they give an effective tool for the study of Simpson's type inequalities. Moreover, the developed identity is an extremely useful analytical tool that can be applied for deriving new inequalities for more general classes of generalized convex functions.

The present work also opens several promising directions for future research, which remain interesting open problems:

- To establish corresponding Newton-, Ostrowski-, Fejér-, and Bullen-type inequalities for multiplicatively strongly convex functions.
- To extend the proposed methodology to broader classes of generalized multiplicatively convex functions and other generalized convexity structures.

- To develop analogous inequalities involving various fractional integral operators, including multiplicative Riemann–Liouville, Caputo, Katugampola, Hadamard, and other generalized fractional operators within multiplicative calculus.
- To investigate the corresponding Simpson's-type and related integral inequalities in interval-valued, fuzzy-valued, and set-valued frameworks.
- To explore further applications of the obtained inequalities to special means, special functions, numerical integration, approximation theory, optimization, and other areas of applied mathematics.

The solution to these unsolved questions will increase the field of applicability of multiplicative calculus and will help make further progress in generalized convexity, inequalities of multiplicative integrals and their applications in mathematics.

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