

Oscillation in Advanced Neutral Differential Equations with Mixed Nonlinear Structures

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Abstract. This paper investigates the oscillatory behavior of solutions to a class of second-order quasilinear advanced neutral differential equations with mixed nonlinearities. By applying the arithmetic–geometric mean inequality, we reduce the original equation to an associated half-linear advanced differential equation, providing an effective framework for analyzing its oscillatory properties. Through comparison principles and an iterative approach, several new sufficient conditions for oscillation are established, explicitly reflecting the effects of the advanced arguments and the neutral term. The obtained criteria describe the interplay between the neutral structure, advanced arguments, and mixed nonlinearities and extend several existing oscillation results in the literature. Illustrative examples demonstrate the applicability of the theoretical results and the effectiveness of the proposed criteria.

1. INTRODUCTION

Functional differential equations (FDEs) represent an important class of mathematical models that describe systems whose evolution is not determined solely by the current state but is also influenced by values of the dependent variable outside the present moment. In this context, advanced differential equations (ADEs) occupy a special place, as they represent cases in which

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the current behavior of a system is affected by anticipated future influences or interventions. This characteristic naturally arises in many population models, where anticipated environmental factors or future intervention strategies can affect current rates of change, making advanced equations a suitable tool for describing the dynamics of ecosystems, disease spread, and other complex population processes. Therefore, analyzing the qualitative properties of the solutions to these equations, particularly their oscillatory behavior, is not merely a mathematical problem but a crucial step toward understanding the long-term dynamical patterns of the systems under study; see [1] and the references therein.

Understanding the qualitative behavior of the solutions to these equations is a central topic in the qualitative theory of functional differential equations, with the study of oscillations receiving particular attention because of the information they provide about the long-term behavior of solutions. The origins of this topic can be traced back to the classic work of Fite [2], which paved the way for the development of various criteria for oscillation and non-oscillation. Since then, this field has witnessed significant progress, reflected in a large number of specialized studies and references, most notably the works cited in [3–6].

In this context, the present paper is concerned with the oscillatory behavior of solutions to the quasilinear second-order advanced neutral differential equation with mixed nonlinearities of the form

$$\left(a(v) \left((x(v) + p(v)x(\tau(v)))'\right)^{\gamma}\right)' + \sum_{i=1}^n q_i(v)x^{\gamma_i}(\sigma_i(v)) = 0, \quad v \geq v_0. \quad (1.1)$$

Throughout the paper, the following assumptions are imposed:

(H1): $\gamma, \gamma_1, \gamma_2, \dots, \gamma_n$ are ratios of positive odd integers satisfying

$$\gamma_1 > \gamma_2 > \dots > \gamma_i > \gamma > \gamma_{i+1} > \dots > \gamma_n, \quad n > i > 1;$$

(H2): $a \in C^1([v_0, \infty), (0, \infty))$, $p \in C([v_0, \infty), [0, \infty))$, $0 \leq p(v) < 1$, and $q_i \in C([v_0, \infty), [0, \infty))$, $i = 1, 2, \dots, n$;

(H3): $\tau, \sigma_i \in C([v_0, \infty), \mathbb{R})$, $\tau(v) \leq v$, $\sigma_i(v) \geq v$, and $\lim_{v \rightarrow \infty} \tau(v) = \lim_{v \rightarrow \infty} \sigma_i(v) = \infty$, where

$$\sigma(v) = \min\{\sigma_1(v), \sigma_2(v), \dots, \sigma_n(v)\};$$

(H4): Equation (1.1) is assumed to be in canonical form, that is,

$$\pi(v) := \int_{v_0}^v a^{-1/\gamma}(\ell) d\ell \rightarrow \infty \text{ as } v \rightarrow \infty. \quad (1.2)$$

For convenience, we define

$$z(v) := x(v) + p(v)x(\tau(v)).$$

A solution of (1.1) is a function $x \in C([v_x, \infty), \mathbb{R})$ for some $v_x \geq v_0$ such that $a(v)(z'(v))^{\gamma} \in C^1([v_x, \infty), \mathbb{R})$ and satisfies equation (1.1) for all $v \geq v_x$. Throughout this paper, we consider only those solutions $x(v)$ of (1.1) satisfying the condition

$$\sup\{|x(v)| : v \geq V\} > 0, \text{ for all } V \geq v_x.$$

A solution of (1.1) is said to be oscillatory if it has a sequence of zeros tending to infinity; otherwise, it is called nonoscillatory. The equation (1.1) is said to be oscillatory if every solution is oscillatory.

Although extensive research has been conducted on the oscillation theory of advanced differential equations, many existing results focus on equations with delay arguments [7–11]. In contrast, equations with advanced arguments have received less attention. This is mainly due to the analytical difficulties arising from the dependence on future values. Furthermore, a significant portion of the results concerning advanced equations has relied on techniques that do not fully exploit the effect of the advanced arguments, leading to oscillation criteria that do not accurately reflect the intrinsic nature of this class of equations. Jadlovská [12] drew attention to this issue, noting that many of the criteria used in the literature, including those in [13–17], do not actually take full advantage of the presence of the advanced arguments.

This has stimulated a growing line of research devoted to establishing oscillation criteria that accurately reflect the intrinsic characteristics of advanced differential equations. In this direction, Jadlovská [12] and Baculíková [18] investigated the second-order linear advanced differential equation

$$x''(v) + q_1(v)x(\sigma_1(v)) = 0,$$

which is obtained from equation (1.1), when $a(v) = 1$, $p(v) = 0$, $n = 1$, and $\gamma_i = \gamma = 1$. By introducing iterative techniques, they derived sharper oscillation criteria that effectively captured the interaction between the second-order structure and the advanced argument.

Subsequently, Džurina [19] and Baculíková [20] studied the canonical advanced differential equation

$$(a(v)x'(v))' + q_1(v)x(\sigma_1(v)) = 0$$

which corresponds to equation (1.1) under the assumptions $n = 1$, $\gamma_1 = \gamma = 1$, and $p(v) = 0$. Employing refined monotonicity properties of nonoscillatory solutions, they established comparison principles that led to improved oscillation criteria and laid the groundwork for subsequent investigations.

Sun and Wong [21], followed by Thandapani et al. [22] considered ordinary differential equations with mixed nonlinearities arising from equation (1.1) when $\sigma_i(v) = v$ for $i = 1, 2, \dots, n$ and $p(v) = 0$. Their studies provided several oscillation results for this class and stimulated further developments concerning equations with mixed nonlinear terms.

Agarwal et al. [23], Hassan [24], Baculikova [25], Bohner et al. [26], Alqahtani et al. [27], and Bazighifan et al. [28] investigated the Emden-Fowler type advanced differential equation

$$(a(v)(x'(v))^{\gamma})' + q_1(v)x^{\gamma_1}(\sigma_1(v)) = 0$$

which is obtained from equation (1.1) when $n = 1$ and $p(v) = 0$. They established a variety of oscillation criteria and significantly enriched the theory of advanced equations of Emden–Fowler type.

Later, Shi and Bai [29] considered the neutral advanced differential equation

$$\left(a(v) \left((x(v) + p(v)x(\tau(v)))' \right)^{\gamma}\right)' + q_1(v)x^{\gamma_1}(\sigma_1(v)) = 0$$

which is related to equation (1.1) through a suitable choice of the neutral argument. They derived additional oscillation criteria and extended several existing results to neutral equations with advanced arguments.

More recently, Purushothaman et al. [30] investigated the oscillatory behavior of solutions to the second-order half-linear advanced differential equation (1.1), where $p(v) = 0$. By establishing new sufficient conditions for oscillation, they generalized and improved several earlier results available in the literature.

Despite significant progress in studying the oscillatory behavior of advanced neutral differential equations, many existing results are restricted to linear, quasi-linear, or single-nonlinear models. Quasi-linear advanced neutral equations with mixed nonlinearities have received relatively little attention, as the simultaneous presence of neutral and advanced arguments presents considerable analytical difficulties. Consequently, existing oscillation criteria are often inadequate for characterizing the oscillatory behavior of solutions to such general equations. Therefore, developing new oscillation tests for Eq. (1.1) remains an important and challenging problem in the qualitative theory of functional differential equations. Inspired by the results in [19, 21, 30], we address this problem by establishing new oscillation criteria for Eq. (1.1) through the development of an iterative analytical approach that effectively captures the interaction among the neutral term, advanced arguments, and mixed nonlinear structure. The obtained results extend, complement, and improve several related contributions available in the literature.

2. PRELIMINARY RESULTS

In the subsequent analysis, all functional inequalities are considered to hold eventually, meaning they are valid for all sufficiently large v . This section is devoted to presenting the preliminary results necessary for establishing the main findings of the study. Let us define

$$\Phi(v) := \prod_{i=1}^n \left(\eta_i^{-1} q_i(v) (1 - p(\sigma_i(v)))^{\gamma_i} \right)^{\eta_i}$$

Lemma 2.1. [31] *Let $x(v)$ be an eventually positive solution of (1.1). Then, the corresponding function $z(v)$ satisfies*

$$z(v) > 0, \quad z'(v) > 0, \quad \left(a(v) (z'(v))^{\gamma} \right)' < 0,$$

for all $v \geq v_1 \geq v_0$

Lemma 2.2. *Let $\{\gamma_i\}$, $i = 1, 2, \dots, n$, satisfy*

$$\gamma_1 > \gamma_2 > \dots > \gamma_i > \gamma > \gamma_{i+1} > \dots > \gamma_n > 0.$$

Then there exists an n -tuple $\{\eta_1, \eta_2, \dots, \eta_n\}$ such that

$$\sum_{i=1}^n \gamma_i \eta_i = \gamma, \quad (2.1)$$

and

$$\sum_{i=1}^n \eta_i = 1, \quad 0 < \eta_i < 1. \quad (2.2)$$

Proof. Set $\lambda_i = \frac{\gamma_i}{\gamma}$ for $i = 1, 2, \dots, n$. Then

$$\lambda_1 > \lambda_2 > \dots > \lambda_i > 1 > \lambda_{i+1} > \dots > \lambda_n > 0.$$

Hence, (2.1) is rewritten as

$$\sum_{i=1}^n \lambda_i \eta_i = 1.$$

It remains to show the existence of $\eta_i \in (0, 1)$, $i = 1, 2, \dots, n$, satisfying

$$\sum_{i=1}^n \eta_i = 1,$$

and

$$\sum_{i=1}^n \lambda_i \eta_i = 1.$$

Since

$$\lambda_j > 1 > \lambda_{j+1},$$

there exist positive numbers η_i , $i = 1, 2, \dots, n$, satisfying both conditions above. Therefore, the existence of such an n -tuple $(\eta_1, \eta_2, \dots, \eta_n)$ follows directly from [21, Lemma 1], which completes the proof. $\square$

The following theorem provides a reduction of the oscillation of equation (1.1) to an appropriate comparison half-linear equation.

Theorem 2.1. *Suppose that the second-order half-linear advanced differential equation*

$$\left(a(v) (z'(v))^\gamma \right)' + \Phi(v) z^\gamma(\sigma(v)) = 0 \quad (2.3)$$

is oscillatory, where

$$\Phi(v) := \prod_{i=1}^n \left(\frac{q_i(v) (1 - p(\sigma_i(v)))^{\gamma_i}}{\eta_i} \right)^{\eta_i},$$

and $\eta_1, \eta_2, \dots, \eta_n$ are positive constants satisfying conditions (2.1) and (2.2) of Lemma 2.2. Then equation (1.1) is oscillatory.

Proof. Assume, for the sake of contradiction, that x is an eventually positive solution of (1.1). Then there exists $v_1 \geq v_0$ such that $x(v) > 0$, $x(\tau(v)) > 0$, and $x(\sigma_i(v)) > 0$, $i = 1, 2, \dots, n$ for all $v > v_1$. In particular, $x(\sigma(v)) > 0$ for all $v \geq v_1$, since $\sigma(v) = \min_{1 \leq i \leq n} \{\sigma_i(v)\}$. By Lemma 2.1, the associated neutral function

$$z(v) = x(v) + p(v)x(\tau(v))$$

is positive and nondecreasing for all $v > v_1$. From the definition of z , we have $z(v) > x(v)$ and hence

$$x(v) = z(v) - p(v)x(\tau(v)) \geq z(v) - p(v)z(\tau(v)).$$

Since $z(v)$ is nondecreasing and $\tau(v) \leq v$, it follows that $z(\tau(v)) \leq z(v)$. Consequently,

$$x(v) \geq z(v) - p(v)z(\tau(v)) \geq (1 - p(v))z(v). \quad (2.4)$$

Applying (2.4) into (1.1) yields

$$(a(v)(z'(v))^\gamma)' + \sum_{i=1}^n q_i(v)(1 - p(\sigma_i(v)))^{\gamma_i} z^{\gamma_i}(\sigma_i(v)) \leq 0. \quad (2.5)$$

Since $v \leq \sigma(v) \leq \sigma_i(v)$, for each $i = 1, 2, \dots, n$, and z is nondecreasing, we have $z(\sigma(v)) \leq z(\sigma_i(v))$. Using this fact in (2.5), we obtain

$$(a(v)(z'(v))^\gamma)' + \sum_{i=1}^n q_i(v)(1 - p(\sigma_i(v)))^{\gamma_i} z^{\gamma_i}(\sigma(v)) \leq 0. \quad (2.6)$$

Let $\eta_1, \eta_2, \dots, \eta_n$ be positive constants satisfying the conditions of Lemma 2.2

$$\sum_{i=1}^n \eta_i = 1 \text{ and } \sum_{i=1}^n \gamma_i \eta_i = \gamma \quad (2.7)$$

We apply the arithmetic–geometric mean inequality (see [32]):

$$\sum_{i=1}^n \eta_i y_i \geq \prod_{i=1}^n \left(\frac{y_i}{\eta_i} \right)^{\eta_i}, \quad y_i > 0. \quad (2.8)$$

Setting

$$y_i(v) = \frac{q_i(v)(1 - p(\sigma_i(v)))^{\gamma_i} z^{\gamma_i}(\sigma(v))}{\eta_i}, \quad i = 1, 2, \dots, n.$$

Applying (2.8) to the summation term in (2.6), we get

$$\begin{aligned} \sum_{i=1}^n q_i(v)(1 - p(\sigma_i(v)))^{\gamma_i} z^{\gamma_i}(\sigma(v)) &= \sum_{i=1}^n \eta_i \left(\frac{q_i(v)(1 - p(\sigma_i(v)))^{\gamma_i} z^{\gamma_i}(\sigma(v))}{\eta_i} \right) \\ &\geq \prod_{i=1}^n \left(\frac{q_i(v)(1 - p(\sigma_i(v)))^{\gamma_i} z^{\gamma_i}(\sigma(v))}{\eta_i} \right)^{\eta_i} \\ &= \prod_{i=1}^n \left(\frac{q_i(v)(1 - p(\sigma_i(v)))^{\gamma_i}}{\eta_i} \right)^{\eta_i} \cdot \prod_{i=1}^n (z^{\eta_i \gamma_i}(\sigma(v))) \\ &= \prod_{i=1}^n \left(\frac{q_i(v)(1 - p(\sigma_i(v)))^{\gamma_i}}{\eta_i} \right)^{\eta_i} \cdot z^{\sum_{i=1}^n \eta_i \gamma_i}(\sigma(v)) \end{aligned}$$

$$= \Phi(v) z^\gamma(\sigma(v)).$$

Combining this together with (2.6), we deduce that

$$(a(v) (z'(v))^\gamma)' + \Phi(v) z^\gamma(\sigma(v)) \leq 0. \quad (2.9)$$

Thus, z is a positive solution of the corresponding differential inequality associated with equation (2.3). By the comparison result in [33, Lemma 3.2], equation (2.3) then admits an eventually positive solution, contradicting the assumed oscillation of equation (2.3). Therefore, equation (1.1) is oscillatory. $\square$

Theorem 2.2. *If*

$$\int_{v_0}^{\infty} \Phi(\ell) d\ell = \infty, \quad (2.10)$$

then Eq. (1.1) is oscillatory.

Proof. Assume, for the sake of contradiction, that x is an eventually positive solution of (1.1). Then there exists $v_1 \geq v_0$ such that $x(v) > 0$, $x(\tau(v)) > 0$, and $x(\sigma_i(v)) > 0$, $i = 1, 2, \dots, n$ for all $v \geq v_1$. In particular, $x(\sigma(v)) > 0$ for all $v \geq v_1$, since $\sigma(v) = \min\{\sigma_1(v), \sigma_2(v), \dots, \sigma_n(v)\}$. By Theorem 2.1, the function $z(v)$ satisfies inequality (2.9). Moreover, by Lemma 2.1, $z(v)$ is increasing for all $v > v_1$. Since $\sigma(v) \rightarrow \infty$ as $v \rightarrow \infty$, there exists $v_2 \geq v_1$ such that

$$z(\sigma(v)) \geq z(\sigma(v_2)), \text{ for all } v \geq v_2.$$

Substituting this into (2.9), we obtain

$$(a(v) (z'(v))^\gamma)' + \Phi(v) z^\gamma(\sigma(v_2)) \leq 0, \text{ for all } v \geq v_2.$$

Integrating over $[v_1, v]$, we get

$$z^\gamma(\sigma(v_2)) \int_{v_2}^v \Phi(\ell) d\ell \leq a(v_2) (z'(v_2))^\gamma.$$

Letting $v \rightarrow \infty$, we obtain

$$z^\gamma(\sigma(v_2)) \int_{v_2}^{\infty} \Phi(\ell) d\ell \leq \infty.$$

On the other hand, condition (2.10) implies that

$$z^\gamma(\sigma(v_2)) \int_{v_2}^{\infty} \Phi(\ell) d\ell = \infty.$$

Since $z^\gamma(\sigma(v_2)) > 0$, this yields a contradiction. Therefore, equation (1.1) cannot admit an eventually positive solution. An analogous argument excludes an eventually negative solution. Consequently, every solution of equation (1.1) is oscillatory. The proof is complete. $\square$

Theorem 2.3. *If there exists a function $\rho \in C^1([v_0, \infty), (0, \infty))$ such that*

$$\int_{v_1}^{\infty} \left(\rho(\ell) \Phi(\ell) - \frac{1}{(\gamma+1)^{\gamma+1}} \frac{a(\ell) [\rho'_+(\ell)]^{\gamma+1}}{\rho^\gamma(\ell)} \right) d\ell = \infty, \quad (2.11)$$

where $\rho'_+(v) := \max\{0, \rho'(v)\}$, then every solution of Eq. (1.1) is oscillatory.

Proof. Assume, for the sake of contradiction, that x is an eventually positive solution of (1.1). Then there exists $v_1 \geq v_0$ such that $x(v) > 0$, $x(\tau(v)) > 0$, and $x(\sigma_i(v)) > 0$, $i = 1, 2, \dots, n$ for all $v > v_1$. In particular, $x(\sigma(v)) > 0$ for all $v \geq v_1$, since $\sigma(v) = \min\{\sigma_1(v), \sigma_2(v), \dots, \sigma_n(v)\}$. Define

$$\omega(v) := \rho(v) \frac{a(v)(z'(v))^\gamma}{z^\gamma(v)}, \quad v \geq v_1. \quad (2.12)$$

Clearly, $\omega(v) > 0$. Differentiating $\omega(v)$ and using inequality (2.9), we obtain

$$\begin{aligned} \omega'(v) &= \rho'(v) \frac{a(v)(z'(v))^\gamma}{z^\gamma(v)} + \rho(v) \frac{(a(v)(z'(v))^\gamma)'}{z^\gamma(v)} - \frac{\gamma \rho(v) a(v)(z'(v))^{\gamma+1}}{z^{\gamma+1}(v)} \\ &\leq -\rho(v) \Phi(v) \left(\frac{z(\sigma(v))}{z(v)} \right)^\gamma + \frac{\rho'(v)}{\rho(v)} \omega(v) - \frac{\gamma}{(a(v)\rho(v))^{1/\gamma}} \omega^{1+\frac{1}{\gamma}}(v). \end{aligned}$$

Since $z'(v) > 0$ and $\sigma(v) \geq v$, and since $z(v)$ is increasing, we have

$$z(\sigma(v)) \geq z(v),$$

and

$$\frac{z(\sigma(v))}{z(v)} \geq 1.$$

Therefore,

$$\omega'(v) \leq -\rho(v) \Phi(v) + \frac{\rho'(v)}{\rho(v)} \omega(v) - \frac{\gamma}{(a(v)\rho(v))^{1/\gamma}} \omega^{1+\frac{1}{\gamma}}(v) \quad (2.13)$$

Using the following inequality [32],

$$Bv - Av^{\frac{\gamma+1}{\gamma}} \leq \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^\gamma}, \quad A > 0, \quad (2.14)$$

with

$$A = \frac{\gamma}{(a(v)\rho(v))^{1/\gamma}} \text{ and } B = \frac{\rho'(v)}{\rho(v)},$$

we obtain

$$\omega'(v) \leq -\rho(v) \Phi(v) + \frac{1}{(\gamma+1)^{\gamma+1}} \frac{a(v) [\rho'_+(v)]^{\gamma+1}}{\rho^\gamma(v)}.$$

Integrating over $[v_1, v]$, we get

$$0 < \omega(v) \leq \omega(v_1) - \int_{v_1}^v \left(\rho(\ell) \Phi(\ell) - \frac{1}{(\gamma+1)^{\gamma+1}} \frac{a(\ell) [\rho'_+(\ell)]^{\gamma+1}}{\rho^\gamma(\ell)} \right) d\ell.$$

Letting $v \rightarrow \infty$ and using condition (2.11), the right-hand side tends to $-\infty$, contradicting $\omega(v) > 0$. Thus, Eq. (1.1) admits no eventually positive solution. This completes the proof. $\square$

Choosing $\rho(v) = \pi^\gamma(v)$ in Theorem 2.3, we obtain the following corollary.

Corollary 2.1. *If*

$$\int_{v_1}^{\infty} \left(\pi^{\gamma}(\ell) \Phi(\ell) - \left(\frac{\gamma}{\gamma+1} \right)^{\gamma+1} \frac{1}{\pi(\ell) a^{1/\gamma}(\ell)} \right) d\ell = \infty, \quad (2.15)$$

for some $v_1 \geq v_0$, then every solution of Eq. (1.1) is oscillatory.

Corollary 2.2. *If*

$$\liminf_{v \rightarrow \infty} \frac{1}{\ln \pi(v)} \int_{v_0}^v \pi^{\gamma}(\ell) \Phi(\ell) d\ell > \left(\frac{\gamma}{\gamma+1} \right)^{\gamma+1}, \quad (2.16)$$

then every solution of Eq. (1.1) is oscillatory.

Proof. It follows from condition (2.16) that there exist constants $\varepsilon > 0$ and $0 v_1 \geq v_0$ such that

$$\frac{1}{\ln \pi(v)} \int_{v_0}^v \pi^{\gamma}(\ell) \Phi(\ell) d\ell \geq \left(\frac{\gamma}{\gamma+1} \right)^{\gamma+1} + \varepsilon, \text{ for all } v \geq v_1.$$

Consequently,

$$\int_{v_0}^v \pi^{\gamma}(\ell) \Phi(\ell) d\ell \geq \left(\left(\frac{\gamma}{\gamma+1} \right)^{\gamma+1} + \varepsilon \right) \ln \pi(v). \quad (2.17)$$

Using (2.17), we obtain

$$\int_{v_1}^v \left(\pi^{\gamma}(\ell) \Phi(\ell) - \left(\frac{\gamma}{\gamma+1} \right)^{\gamma+1} \frac{1}{\pi(\ell) a^{1/\gamma}(\ell)} \right) d\ell > \varepsilon \ln \pi(v) + \left(\frac{\gamma}{\gamma+1} \right)^{\gamma+1} \ln \pi(v_1).$$

Hence, the left-hand side tends to $+\infty$ as $v \rightarrow \infty$, so condition (2.15) is satisfied. Therefore, by Corollary 2.1, every solution of Eq. (1.1) is oscillatory. $\square$

Remark 2.1. *In the special case $\rho(v) = 1$, condition (2.11) in Theorem 2.3 reduces to condition (2.10) in Theorem 2.2.*

In the sequel, we investigate the oscillatory behavior of equation (1.1) under the complementary case to condition (2.10), namely, when

$$\int_{v_0}^{\infty} \Phi(\ell) d\ell < \infty \quad (2.18)$$

Theorem 2.4. *Assume that condition (2.18) is satisfied. If*

$$\liminf_{v \rightarrow \infty} \pi^{\gamma}(v) \int_v^{\infty} \Phi(\ell) d\ell > \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma}}, \quad (2.19)$$

then Eq. (1.1) is oscillatory.

Proof. Assume, for the sake of contradiction, that x is an eventually positive solution of (1.1). Then there exists $v_1 \geq v_0$ such that $x(v) > 0$, $x(\tau(v)) > 0$, and $x(\sigma_i(v)) > 0$, $i = 1, 2, \dots, n$ for all $v > v_1$. In particular, $x(\sigma(v)) > 0$ for all $v \geq v_1$, since $\sigma(v) = \{\sigma_1(v), \sigma_2(v), \dots, \sigma_n(v)\}$. Following the same argument used in the proof of Theorem 2.1, it can be shown that $z(v)$ also satisfies the reduced equation (2.3), namely

$$(a(v)(z'(v))^{\gamma})' + \Phi(v)z^{\gamma}(\sigma(v)) = 0.$$

By applying [34, Theorem 2'(i)], condition (2.19) guarantees that equation (2.3) is oscillatory. This contradicts the existence of an eventually positive solution. Therefore, every solution of Eq. (1.1) is oscillatory, and the proof is complete. $\square$

In the following lemma, we obtain several estimates based on the successive substitution method, which will be used in the subsequent analysis.

Lemma 2.3. *Let x be an eventually positive solution of equation (1.1), and assume that condition (2.18) is satisfied. Define the function*

$$\beta_1(y, v) := \exp \left[\int_v^y \left(\frac{1}{a(\kappa)} \int_{\kappa}^{\infty} \Phi(\ell) d\ell \right)^{1/\gamma} d\kappa \right].$$

Further, suppose that the sequence of functions

$$\beta_{n+1}(y, v) := \exp \left[\int_v^y \left(\frac{1}{a(\kappa)} \int_{\kappa}^{\infty} \Phi(\ell) \beta_n^{\gamma}(\sigma(\ell), \kappa) d\ell \right)^{1/\gamma} d\kappa \right]$$

is well-defined and uniformly equicontinuous on the domain

$$\{(y, v), v_0 \leq v \leq y\}.$$

Then, for sufficiently large t , we have

$$z(y) \geq z(v) \beta_n(y, v), \quad (2.20)$$

holds for every $n \in \mathbb{N}$.

Proof. We establish the lemma by the principle of mathematical induction. The proof proceeds in two main stages: the base case $n = 1$, followed by the inductive step.

Stage 1: Base Case ($n = 1$). Since $x(v)$ is an eventually positive solution of (2.3), by Lemma 2.1, there exists $v_1 \geq v_0$ such that

$$z(v) > 0, z'(v) > 0, \left(a(v) (z'(v))^{\gamma} \right)' < 0, \text{ for all } v \geq v_1.$$

Furthermore, because z is increasing and $\sigma(v) \geq v$, we have

$$z(\sigma(v)) \geq z(v), v \geq v_1. \quad (2.21)$$

By Theorem 2.1, z satisfies inequality (2.3), namely

$$\left(a(v) (z'(v))^{\gamma} \right)' + \Phi(v) z^{\gamma}(\sigma(v)) \leq 0.$$

Integrating from v to l yields

$$a(l) (z'(l))^{\gamma} - a(v) (z'(v))^{\gamma} = - \int_v^l \Phi(v) z^{\gamma}(\sigma(v)) d\ell.$$

Since $a(l) (z'(l))^{\gamma} > 0$, we obtain

$$a(v) (z'(v))^{\gamma} \geq \int_v^l \Phi(v) z^{\gamma}(\sigma(v)) d\ell.$$

Letting $l \rightarrow \infty$ and using condition (2.18) (which ensures convergence of the integral), we get

$$a(v) (z'(v))^\gamma \geq \int_v^\infty \Phi(\ell) z'(\sigma(\ell)) d\ell. \quad (2.22)$$

Using inequality (2.21) in (2.22), we obtain

$$a(v) (z'(v))^\gamma \geq z'(v) \int_v^\infty \Phi(\ell) d\ell.$$

Therefore,

$$\frac{z'(v)}{z(v)} \geq \left(\frac{1}{a(v)} \int_v^\infty \Phi(\ell) d\ell \right)^{1/\gamma}. \quad (2.23)$$

Integrating (2.23) from v to y (with $y > v \geq v_1$), we get

$$z(y) \geq z(v) \exp \left[\int_v^y \left(\frac{1}{a(\kappa)} \int_\kappa^\infty \Phi(\ell) d\ell \right)^{1/\gamma} d\kappa \right].$$

By definition of $\beta_1(y; v)$, this is precisely

$$z(y) \geq z(v) \beta_1(y, v), \quad y > v \geq v_1. \quad (2.24)$$

Thus, the assertion holds for $n = 1$.

Induction Step: Assume that (2.20) holds for some fixed $n > 1$, i.e.,

$$z(y) \geq z(v) \beta_n(y, v), \quad y > v, \quad (2.25)$$

for all sufficiently large v . We aim to prove that the assertion holds for $n + 1$.

From (2.25), for $\sigma(y) > v$, we have

$$z(\sigma(y)) \geq z(v) \beta_n(\sigma(y), v), \quad \sigma(y) > v. \quad (2.26)$$

Substituting inequality (2.26) into the integral in (2.22), we have

$$a(v) (z'(v))^\gamma \geq z'(v) \int_v^\infty \Phi(\ell) \beta_n'(\sigma(\ell), v) d\ell.$$

Thus,

$$\frac{z'(v)}{z(v)} \geq \left(\frac{1}{a(v)} \int_v^\infty \Phi(\ell) \beta_n'(\sigma(\ell), v) d\ell \right)^{1/\gamma}.$$

Integrating from v to y , we have

$$z(y) \geq z(v) \exp \left[\int_v^y \left(\frac{1}{a(\kappa)} \int_\kappa^\infty \Phi(\ell) \beta_n'(\sigma(\ell), \kappa) d\ell \right)^{1/\gamma} d\kappa \right],$$

that is,

$$z(y) \geq z(v) \beta_{n+1}(y, v).$$

This completes the induction step, and hence the proof of the lemma. $\square$

Theorem 2.5. *Under the hypotheses of Lemma 2.3, suppose that for some $n \in \mathbb{N}$, the second-order half-linear differential equation*

$$\left(a(v)(z'(v))^\gamma\right)' + \Phi(v)\beta_n^\gamma(\sigma(v), v)z^\gamma(v) = 0 \quad (2.27)$$

is oscillatory. Then equation (2.3) is likewise oscillatory.

Proof. Assume, for the sake of contradiction, that x is an eventually positive solution of (1.1). Then there exists $v_1 \geq v_0$ such that $x(v) > 0$, $x(\tau(v)) > 0$, and $x(\sigma_i(v)) > 0$, $i = 1, 2, \dots, n$ for all $v > v_1$. In particular, $x(\sigma(v)) > 0$ for all $v \geq v_1$, since $\sigma(v) = \{\sigma_1(v), \sigma_2(v), \dots, \sigma_n(v)\}$. By Lemma 2.1, the corresponding neutral function $z(v)$ satisfies

$$z(v) > 0, z'(v) > 0, \left(a(v)(z'(v))^\gamma\right)' < 0, \text{ for all } v \geq v_1.$$

Consequently, $z(v)$ is positive and strictly increasing on $v \geq v_1$. Since all assumptions of Lemma 2.3 are satisfied, there exists $v_2 \geq v_1$ such that for all $y > v \geq v_2$,

$$z(y) \geq z(v)\beta_n(y, v), \quad (2.28)$$

for some $n \in \mathbb{N}$. Taking $y = \sigma(v)$ in (2.28), and noting that $\sigma(v) \geq v$ for all sufficiently large v , we deduce that

$$z(\sigma(v)) \geq z(v)\beta_n(\sigma(v), v), \quad v \geq v_2. \quad (2.29)$$

Substituting inequality (2.29) into equation (2.3), we obtain

$$\left(a(v)(z'(v))^\gamma\right)' + \Phi(v)\beta_n^\gamma(\sigma(v), v)z^\gamma(v) \leq 0.$$

Thus, z is an eventually positive solution of the differential inequality corresponding to equation (2.27). By [33, Lemma 3.2], equation (2.27) admits an eventually positive solution, which contradicts the assumption that equation (2.27) is oscillatory. Therefore, equation (1.1) is oscillatory. This completes the proof. $\square$

Theorem 2.6. *Suppose that the hypotheses of Lemma 2.3 are fulfilled. If there exists $n \in \mathbb{N}$ such that*

$$\liminf_{v \rightarrow \infty} \pi^\gamma(v) \int_v^\infty \Phi(\ell)\beta_n^\gamma(\sigma(\ell), \ell) d\ell > \frac{\gamma^\gamma}{(\gamma+1)^\gamma}, \quad (2.30)$$

then Eq. (1.1) is oscillatory.

Proof. Assume, for the sake of contradiction, that x is an eventually positive solution of (1.1). Then there exists $v_1 \geq v_0$ such that $x(v) > 0$, $x(\tau(v)) > 0$, and $x(\sigma_i(v)) > 0$, $i = 1, 2, \dots, n$ for all $v \geq v_1$. In particular, $x(\sigma(v)) > 0$ for all $v \geq v_1$, since $\sigma(v) = \min\{\sigma_1(v), \sigma_2(v), \dots, \sigma_n(v)\}$. Using the same reasoning as in Theorem 2.5, we deduce that $z(v)$ satisfies the differential inequality corresponding to the associated equation (2.27). Then, applying [34, Theorem 2'(i)], condition (2.30) implies that equation (2.27) is oscillatory. This contradicts the existence of an eventually positive solution of the differential inequality corresponding to (2.27). Therefore, every solution of Eq. (1.1) is oscillatory, and the proof is complete. $\square$

Theorem 2.7. Suppose that, for some $n \in \mathbb{N}$, the second-order differential equation

$$\left(a(v)(z'(v))^\gamma\right)' + \widetilde{\Phi}_n(v)z'(v) = 0 \quad (2.31)$$

is oscillatory. Then equation (2.3) is also oscillatory, where

$$\widetilde{\Phi}_n(v) = \Phi(v) \left(1 + \int_v^{\sigma(v)} \beta_n(\sigma(\kappa), v) \left(\frac{1}{a(\kappa)} \int_\kappa^\infty \Phi(\ell) \beta_n^\gamma(\sigma(\ell), \sigma(\kappa)) d\ell\right)^{1/\gamma} d\kappa\right), \quad (2.32)$$

where the function $\beta_n(y, v)$ is as introduced in Lemma 2.3.

Proof. Assume, for the sake of contradiction, that x is an eventually positive solution of (1.1). Then there exists $v_1 \geq v_0$ such that $x(v) > 0$, $x(\tau(v)) > 0$, and $x(\sigma_i(v)) > 0$, $i = 1, 2, \dots, n$ for all $v \geq v_1$. In particular, $x(\sigma(v)) > 0$ for all $v \geq v_1$, since $\sigma(v) = \min\{\sigma_1(v), \sigma_2(v), \dots, \sigma_n(v)\}$. By virtue of Lemma 2.3, for sufficiently large v and some fixed $n \in \mathbb{N}$, the inequality

$$z'(\sigma(\ell)) \geq z'(\sigma(v)) \beta_n^\gamma(\sigma(\ell), \sigma(v)), \quad \ell > v, \quad (2.33)$$

holds. By Theorem 2.1, z satisfies inequality (2.3). Integrating (2.3) from v to ∞ and using (2.33), we obtain

$$\begin{aligned} (z'(v))^\gamma &\geq \frac{1}{a(v)} \int_v^\infty \Phi(\ell) z'(\sigma(\ell)) d\ell \\ &\geq \frac{z'(\sigma(v))}{a(v)} \int_v^\infty \Phi(\ell) \beta_n^\gamma(\sigma(\ell), \sigma(v)) d\ell, \end{aligned}$$

which is equivalent to

$$z'(v) \geq z(\sigma(v)) \left(\frac{1}{a(v)} \int_v^\infty \Phi(\ell) \beta_n^\gamma(\sigma(\ell), \sigma(v)) d\ell\right)^{1/\gamma}. \quad (2.34)$$

Next, integrating (2.34) over the interval $[v, \sigma(v)]$, and invoking the auxiliary relation (2.20), namely

$$z(\sigma(\kappa)) \geq z(v) \beta_n(\sigma(\kappa), v), \quad \sigma(\kappa) > v,$$

we obtain

$$\begin{aligned} z(\sigma(v)) &> z(v) + \int_v^{\sigma(v)} z(\sigma(\kappa)) \left(\frac{1}{a(\kappa)} \int_\kappa^\infty \Phi(\ell) \beta_n^\gamma(\sigma(\ell), \sigma(\kappa)) d\ell\right)^{1/\gamma} d\kappa \\ &> z(v) \left[1 + \int_v^{\sigma(v)} \beta_n(\sigma(\kappa), v) \left(\frac{1}{a(\kappa)} \int_\kappa^\infty \Phi(\ell) \beta_n^\gamma(\sigma(\ell), \sigma(\kappa)) d\ell\right)^{1/\gamma} d\kappa\right]. \end{aligned}$$

Thus, by (2.32),

$$z(\sigma(v)) > \frac{z(v) \widetilde{\Phi}_n(v)}{\Phi(v)}.$$

Substituting this inequality into (2.3) reveals that $z(v)$ satisfies the differential inequality

$$\left(a(v)(z'(v))^\gamma\right)' + \widetilde{\Phi}_n(v)z'(v) \leq 0.$$

Thus, z is an eventually positive solution of the differential inequality corresponding to equation (2.31). By [33, Lemma 3.2], the existence of an eventually positive solution of this inequality implies

that equation (2.31) admits an eventually positive solution, which contradicts the assumption that equation (2.31) is oscillatory. Therefore, equation (2.3) is oscillatory. $\square$

Corollary 2.3. *Suppose that the hypotheses of Lemma 2.3 are fulfilled. If there exists $n \in \mathbb{N}$ such that*

$$\liminf_{v \rightarrow \infty} \pi^\gamma(v) \int_v^\infty \tilde{\Phi}_n(\ell) d\ell > \frac{\gamma^\gamma}{(\gamma+1)^\gamma}, \quad (2.35)$$

then Eq. (1.1) is oscillatory.

Proof. The proof is analogous to that of Theorem 2.6 and is therefore omitted for brevity. $\square$

Next, we extend the sharp oscillation criterion obtained in [35, 36] to the general equation (1.1).

Corollary 2.4. *Assume that condition (2.18) holds. Let*

$$\delta_* := \liminf_{v \rightarrow \infty} \frac{1}{\gamma} a^{1/\gamma}(v) \pi^{\gamma+1}(v) \Phi(v), \quad (2.36)$$

and

$$\mu_* = \liminf_{v \rightarrow \infty} \frac{\pi(\sigma(v))}{\pi(v)}. \quad (2.37)$$

If

$$\delta_* > \begin{cases} 0, & \text{for } \mu_* = \infty, \\ \frac{\gamma}{\gamma+1}, & \text{for } \mu_* = 1, \\ \max\{m^\gamma(1-m)\mu_*^{-m\gamma}\}, & 0 < m < 1, \text{ for } \mu_* < \infty, \end{cases} \quad (2.38)$$

then equation (1.1) is oscillatory.

Proof. Assume, for the sake of contradiction, that x is an eventually positive solution of (1.1). Then there exists $v_1 \geq v_0$ such that $x(v) > 0$, $x(\tau(v)) > 0$, and $x(\sigma_i(v)) > 0$, $i = 1, 2, \dots, n$ for all $v \geq v_1$. In particular, $x(\sigma(v)) > 0$ for all $v \geq v_1$, since $\sigma(v) = \min\{\sigma_1(v), \sigma_2(v), \dots, \sigma_n(v)\}$. Proceeding as in the proof of Theorem 2.1, we conclude that the associated differential equation (2.3) has an eventually positive solution.

If $\mu_* = 1$ then [35, Theorem 1] implies that equation (2.3) is oscillatory, which yields a contradiction. If $\mu_* < \infty$, then [35, Theorem 3] together with [16, Theorem 1] ensures that equation (2.3) is oscillatory, again leading to a contradiction. Finally, if $\mu_* = \infty$, then [35, Theorem 3], [36, Theorem 1], and [16, Theorem 2] imply that equation (2.3), which also contradicts the existence of such an eventually positive solution.

Therefore, equation (1.1) cannot possess an eventually positive solution. Hence, equation (1.1) is oscillatory. The proof is complete. $\square$

3. EXAMPLES

This section provides illustrative examples that highlight the applicability of the obtained results.

Example 3.1. Consider the neutral advanced differential equation

$$\left(v^{4/3} \left(\left(x(v) + \frac{1}{2} x(v/3) \right)' \right)^{7/5} \right)' + \frac{5}{v} x^3(v+1) + \frac{2}{v} x^{1/3}(v+2) = 0, \quad v \geq 1. \quad (3.1)$$

Comparing (3.1) with the general form (1.1), we identify that

$$\begin{aligned} n &= 2, \quad a(v) = v^{4/3}, \quad q_1(v) = \frac{5}{v}, \quad q_2(v) = \frac{2}{v}, \quad p(v) = \frac{1}{2}, \quad \tau(v) = v/3, \\ \gamma_1 &= 3 > \gamma = \frac{7}{5} > \gamma_2 = \frac{1}{3}, \quad \sigma_1(v) = v+1, \quad \sigma_2(v) = v+2. \end{aligned}$$

Choosing

$$\eta_1 = \frac{2}{5}, \quad \eta_2 = \frac{3}{5},$$

we get

$$\eta_1 + \eta_2 = \frac{2}{5} + \frac{3}{5} = 1,$$

and

$$\gamma_1 \eta_1 + \gamma_2 \eta_2 = 3 \cdot \frac{2}{5} + \frac{1}{3} \cdot \frac{3}{5} = \frac{7}{5} = \gamma.$$

Thus, conditions (2.1) and (2.2) of Lemma 2.2 are fulfilled. Consequently, we compute

Consequently, we compute

$$\Phi(v) = \prod_{i=1}^2 \left(\frac{q_i(v) (1 - p(\sigma_i(v)))^{\gamma_i}}{\eta_i} \right)^{\eta_i} = \left(\frac{(1/2)^3 5/v}{2/5} \right)^{2/5} \cdot \left(\frac{(1/2)^{1/3} 2/v}{3/5} \right)^{3/5} = \frac{2.143}{v}.$$

The corresponding transformed equation (2.3) takes the form

$$\left(v^{4/3} (z'(v))^{7/5} \right)' + \frac{2.143}{v} z^{7/5}(v+1) = 0.$$

Moreover, we have

$$\pi(v) \sim 21v^{1/21}, \quad \text{so that } \pi(v) \rightarrow \infty \text{ as } v \rightarrow \infty.$$

Applying condition (2.10), we obtain

$$\int_{v_0}^{\infty} \Phi(\ell) \, d\ell = \int_1^{\infty} \frac{2.15}{\ell} \, d\ell = \infty.$$

Therefore, by Theorem 2.2, equation (3.1) is oscillatory.

Finally, applying condition (2.15), we deduce

$$\begin{aligned} & \int_1^{\infty} \left(\pi^\gamma(\ell) \Phi(\ell) - \left(\frac{\gamma}{\gamma+1} \right)^{\gamma+1} \frac{1}{\pi(\ell) a^{1/\gamma}(\ell)} \right) d\ell \\ &= \int_1^{\infty} \left((21\ell^{1/21})^{5/7} \cdot \frac{2.143}{\ell} - \left(\frac{5/7}{5/7+1} \right)^{5/7+1} \frac{1}{21\ell^{1/21} \cdot \ell^{20/21}} \right) d\ell \\ &= \int_1^{\infty} \left(18.858 \cdot \ell^{-142/147} - \frac{0.0131}{\ell} \right) d\ell = \infty. \end{aligned}$$

Hence, by Corollary 2.1, equation (3.1) is also oscillatory.

Example 3.2. Consider the Emden–Fowler type advanced differential equation

$$\left(x(v) + \frac{1}{4}x(v/2)\right)'' + \frac{q_1}{v^4}x^3(3v) + \frac{q_2}{v^{4/3}}x^{1/3}(3v) = 0, \quad v \geq 1, \quad (3.2)$$

where $q_1, q_2 > 0$ and $0 \leq p_0 < 1$. The corresponding parameters are given by

$$\begin{aligned} n &= 2, \quad a(v) = 1, \quad q_1(v) = \frac{q_1}{v^4}, \quad q_2(v) = \frac{q_2}{v^{4/3}}, \quad p(v) = \frac{1}{4}, \quad \tau(v) = \frac{v}{2}, \\ \gamma_1 &= 3 > \gamma = 1 > \gamma_2 = \frac{1}{3}, \quad \sigma_1(v) = 3v, \quad \sigma_2(v) = 5v. \end{aligned}$$

We select

$$\eta_1 = \frac{1}{4}, \quad \eta_2 = \frac{3}{4},$$

so that

$$\eta_1 + \eta_2 = \frac{1}{4} + \frac{3}{4} = 1, \quad \gamma_1\eta_1 + \gamma_2\eta_2 = 1 \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{3}{4} = 1 = \gamma.$$

Hence, condition (2.1) and (2.2) of Lemma 2.2 are satisfied.

Consequently, we have

$$\pi(v) = v - 1, \quad \text{so that } \pi(v) \rightarrow \infty \text{ as } v \rightarrow \infty.$$

Moreover,

$$\begin{aligned} \Phi(v) &= \left(\frac{q_1(v)(1-p(\sigma_1(v)))^{\gamma_1}}{\eta_1} \right)^{\eta_1} \cdot \left(\frac{q_2(v)(1-p(\sigma_2(v)))^{\gamma_2}}{\eta_2} \right)^{\eta_2} \\ &= \left(\frac{\left(1 - \frac{1}{4}\right)^3 q_1/v^4}{1/4} \right)^{1/4} \cdot \left(\frac{\left(1 - \frac{1}{4}\right)^{1/3} q_2/v^{4/3}}{3/4} \right)^{3/4} \\ &= \frac{2^{3/4} 3^{1/4} q_1^{1/4} q_2^{3/4}}{v^2}. \end{aligned}$$

Thus, the transformed equation (2.3) becomes

$$z''(v) + \frac{2^{3/4} 3^{1/4} q_1^{1/4} q_2^{3/4}}{v^2} z(3v) = 0.$$

Next, we observe that

$$\int_{v_0}^{\infty} \Phi(\ell) \, d\ell = 2^{3/4} 3^{1/4} q_1^{1/4} q_2^{3/4} \int_1^{\infty} \frac{1}{\ell^2} \, d\ell = 2^{3/4} 3^{1/4} q_1^{1/4} q_2^{3/4} < \infty.$$

Therefore, neither Theorem 2.1 nor Theorem 2.2 is applicable.

We now apply condition (2.10), obtaining

$$\begin{aligned} \liminf_{v \rightarrow \infty} \pi^\gamma(v) \int_v^{\infty} \Phi(\ell) \, d\ell &= \liminf_{v \rightarrow \infty} (v-1) 2^{3/4} 3^{1/4} q_1^{1/4} q_2^{3/4} \int_v^{\infty} \frac{1}{\ell^2} \, d\ell \\ &= 2^{3/4} 3^{1/4} q_1^{1/4} q_2^{3/4} \liminf_{v \rightarrow \infty} \frac{v-1}{v} = 2^{3/4} 3^{1/4} q_1^{1/4} q_2^{3/4}. \end{aligned}$$

Hence, the required condition is satisfied whenever

$$2^{3/4} 3^{1/4} q_1^{1/4} q_2^{3/4} > \frac{1}{2},$$

or equivalently,

$$q_1^{1/4} q_2^{3/4} > 0.226.$$

Therefore, by Theorem 2.4, equation (3.2) is oscillatory provided that

$$q_1^{1/4} q_2^{3/4} > 0.226.$$

Example 3.3. Consider the second-order advanced neutral differential equation

$$\left(v^{1/5} \left(x(v) + \frac{1}{2} x(v/3) \right) \right)^{3/5} + \frac{a}{v^{4/3}} x^{7/5}(2v) + \frac{b}{v^{4/3}} x^{3/5}(3v) + \frac{c}{v^{4/3}} x^{1/5}(4v) = 0, \quad v \geq 1. \quad (3.3)$$

where a, b , and c are positive constants. For this equation, we identify

$$n = 3, \quad a(v) = v^{1/5}, \quad p(v) = \frac{1}{2}, \quad \tau(v) = v/3,$$

and

$$q_1(v) = \frac{a}{v^{4/3}}, \quad q_2(v) = \frac{b}{v^{4/3}}, \quad q_3(v) = \frac{c}{v^{4/3}},$$

with

$$\gamma_1 = 7/5, \quad \gamma_2 = \gamma = \frac{3}{5}, \quad \gamma_3 = \frac{1}{5},$$

and

$$\sigma_1(v) = 2v, \quad \sigma_2(v) = 3v, \quad \sigma_3(v) = 4v.$$

We now set

$$\eta_1 = \frac{1}{4}, \quad \eta_2 = \frac{1}{4}, \quad \eta_3 = \frac{1}{2}.$$

A direct computation yields

$$\eta_1 + \eta_2 + \eta_3 = \frac{1}{4} + \frac{1}{4} + \frac{1}{2} = 1,$$

and

$$\gamma_1 \eta_1 + \gamma_2 \eta_2 + \gamma_3 \eta_3 = \frac{7}{5} \cdot \frac{1}{4} + \frac{3}{5} \cdot \frac{1}{4} + \frac{1}{5} \cdot \frac{1}{2} = \frac{3}{5} = \gamma.$$

Thus, assumptions (2.1) and (2.2) of Lemma 2.2 are satisfied. Consequently, the function Φ defined in the lemma takes the form

$$\begin{aligned} \Phi(v) &= \prod_{i=1}^3 \left(\frac{q_i(v) (1 - p(\sigma_i(v)))^{\gamma_i}}{\eta_i} \right)^{\eta_i} \\ &= \left(\frac{(1/2)^{7/5} \cdot a/v^{4/3}}{1/4} \right)^{1/4} \cdot \left(\frac{(1/2)^{3/5} \cdot b/v^{4/3}}{1/4} \right)^{1/4} \cdot \left(\frac{(1/2)^{1/5} \cdot c/v^{4/3}}{1/2} \right)^{1/2} \\ &= \frac{1.87 a^{1/4} b^{1/4} c^{1/2}}{v^{4/3}}. \end{aligned}$$

The transformed equation (2.3) therefore becomes

$$\left(v^{1/5} (z'(v))^{3/5} \right)' + \frac{1.87 a^{1/4} b^{1/4} c^{1/2}}{v^{4/3}} z^{3/5}(2v) = 0.$$

Next, we have

$$\pi(v) = \frac{3}{2} (v^{2/3} - 1),$$

so that $\pi(v) \rightarrow \infty$ as $v \rightarrow \infty$. Moreover,

$$\int_{v_0}^{\infty} \Phi(\ell) d\ell = \int_1^{\infty} \frac{1.87a^{1/4}b^{1/4}c^{1/2}}{v^{4/3}} dv = 5.61a^{1/4}b^{1/4}c^{1/2} < \infty.$$

Hence condition (2.18) is fulfilled. Since $\pi(v) \sim \frac{3}{2}v^{2/3}$, we obtain

$$\mu_* = \liminf_{v \rightarrow \infty} \frac{\pi(\sigma(v))}{\pi(v)} = \liminf_{v \rightarrow \infty} \frac{\frac{3}{2}(2v)^{2/3}}{\frac{3}{2}v^{2/3}} = 2^{2/3} > 1.$$

Furthermore,

$$\begin{aligned} \delta_* &= \liminf_{v \rightarrow \infty} \frac{1}{\gamma} a^{1/\gamma}(v) \pi^{\gamma+1}(v) \Phi(v) \\ &= \liminf_{v \rightarrow \infty} \frac{5}{3} v^{1/3} \left(\frac{3}{2}\right)^{8/5} v^{16/15} \frac{1.87a^{1/4}b^{1/4}c^{1/2}}{v^{4/3}} \\ &= \frac{5}{3} \left(\frac{3}{2}\right)^{8/5} (1.87) a^{1/4}b^{1/4}c^{1/2} \liminf_{v \rightarrow \infty} v^{1/15} = \infty. \end{aligned} \quad (3.4)$$

It remains to verify condition (2.38). To this end, consider the auxiliary function

$$f(m) = m^\gamma (1-m) \mu_*^{-m\gamma} = m^{3/5} (1-m) 2^{-2m/5}, \quad 0 < m < 1.$$

Taking logarithms gives

$$\ln f(m) = \frac{3}{5} \ln m + \ln(1-m) - \frac{2m}{5} \ln 2$$

Differentiation yields

$$\frac{f'(m)}{f(m)} = \frac{3}{5m} - \frac{1}{1-m} - \frac{2}{5} \ln 2.$$

Setting $f'(m) = 0$ leads to

$$1.386m^2 - 9.386m + 3 = 0.$$

The admissible root lying in the interval $(0, 1)$ is

$$m^* \approx 0.337.$$

Consequently,

$$\begin{aligned} f(m^*) &= (m^*)^{3/5} (1-m^*) 2^{-2m^*/5} \\ &= (0.337)^{3/5} (1-0.337) 2^{-2(0.337)/5} = 0.3144. \end{aligned}$$

Therefore,

$$\max_{0 < m < 1} \{m^\gamma (1-m) \mu_*^{-m\gamma}\} = 0.3144. \quad (3.5)$$

Since $\delta_* = \infty$ by (3.4) and the maximum in (3.5) is finite, condition (2.38) is satisfied. Applying Corollary 2.4, we conclude that equation (3.3) is oscillatory.

Remark 3.1. *Examples 3.1-3.2 demonstrate the flexibility of the proposed approach in treating advanced neutral differential equations with mixed nonlinearities. Despite the differences in the coefficients, nonlinear exponents, and deviating arguments, the oscillatory behavior of each equation can be established through the same analytical framework. This indicates that the obtained criteria apply to a broad range of equations within the class considered in this paper.*

Remark 3.2. *It is worth noting that Examples 3.1-3.3 do not fall directly within the framework considered in [23–27, 29]. The equations studied therein are formulated under assumptions that differ from those underlying the present examples, which involve mixed nonlinearities with distinct exponents together with advanced and neutral terms. Consequently, the oscillation criteria established in the aforementioned studies cannot be applied directly to these examples. The applicability of the present results to such equations demonstrates that the proposed criteria complement and extend the existing theory to a broader class of advanced neutral differential equations with mixed nonlinearities.*

4. CONCLUSION

The results obtained in this manuscript contribute to the investigation of oscillation theory for functional differential equations by establishing novel criteria for second-order quasilinear neutral advanced differential equations with mixed nonlinearities. The proposed method integrates the effects of neutral terms, advanced arguments, and mixed nonlinearities within a unified analytical framework. By reducing the original equation to a related advanced quasilinear equation and combining comparison techniques with iterative substitution methods, sufficient conditions for oscillation are derived. The results expand upon many existing oscillation criteria found in the literature, and illustrative examples demonstrate their applicability to the class of equations under study. Several problems remain open for future research. In particular, it will be important to investigate the noncanonical form of equation (1.1) using the analytical method developed in this manuscript. Since the noncanonical case often involves additional technical difficulties, the extent to which similar oscillation criteria can be established remains an open question. It is also beneficial to extend this analysis to higher-order advanced neutral differential equations

$$\left(a(v) \left[(x(v) + p(v)x(\tau(v)))^{(n-1)} \right]' \right)' + \sum_{i=1}^n q_i(v)x^{\gamma_i}(\sigma_i(v)) = 0, \quad n \geq 4,$$

that exhibit mixed nonlinear properties. Further advances in these areas may contribute to a broader understanding of the oscillatory behavior of ADEs and offer further developments in the qualitative theory of functional differential equations.

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