

Optimization of Quadrature Formulas in the Sobolev Space $W_2^{(m)}(0, 1)$

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Abstract. It is known that in many scientific and engineering problems the integration of functions constitutes an essential part of solving the complete problem. The construction of optimal quadrature formulas for the approximate evaluation of definite integrals on classes of functions is an essential part of this problem. In the present work, optimal quadrature formulas in the Sobolev space $W_2^{(m)}(0, 1)$ are constructed. Here the optimal coefficients of the quadrature formulas are found in the cases $m = 1$, $m = 2$, and $m = 3$. In addition, in the space $W_2^{(1)}(0, 1)$ the square of the norm of the error functional of the optimal quadrature formulas is computed.

1. INTRODUCTION AND STATEMENT OF THE PROBLEM

In many scientific and engineering problems, the integration of functions constitutes an essential part of solving the complete problem. The computation of areas and volumes, the determination of the center of mass and moments of inertia of bodies, the evaluation of the work performed by given forces, and many other tasks lead to the integration of functions. It is known that not all integrals can be evaluated analytically. For this reason, quadrature and cubature formulas are employed for the numerical evaluation of definite integrals.

Methods of functional analysis have found wide application in the construction of cubature and quadrature formulas for the approximate evaluation of multidimensional and one-dimensional regular and singular integrals, as well as integrals of fractional order.

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The main results and a more detailed bibliography on cubature and quadrature formulas based on methods of functional analysis have been published in monographs (see [1–4]).

Optimal quadrature formulas were first studied by Sard A., Meyers L.F., and Schoenberg I.J. (see [5–10]).

Furthermore, a number of works have been devoted to optimal quadrature formulas for arbitrary weighted integrals, in particular for Fourier integrals (see [11–14]).

It should be noted that many results on optimal quadrature formulas are presented in the monograph by Brass and Petras [15]. In addition, the computation of oscillatory integrals was considered in the work of Huybrechs and Olver [16]. In [17], optimal quadrature formulas for curvilinear integrals were considered.

In [18], the construction of quadrature formulas that are exact for solutions of differential equations is discussed.

Further, in [19–22], quadrature and cubature formulas and their applications to integral equations are investigated.

It should be noted that in recent years a number of works [23–30] on optimal formulas of approximate integration in the Hilbert and the Sobolev spaces have been published.

In the present work, functional methods are applied to obtain the optimal coefficients of quadrature formulas in the Sobolev space $W_2^{(m)}(0, 1)$ in the cases $m = 1$, $m = 2$, and $m = 3$.

2. STATEMENT OF THE PROBLEM ON OPTIMAL QUADRATURE FORMULAS IN $W_2^{(m)}(0, 1)$

The Sobolev space $W_2^{(m)}(0, 1)$ is the Hilbert space of classes of real-valued functions $\varphi(x)$ that differ by at most a polynomial of degree $m - 2$, having derivatives (in the sense of generalized functions) of order m that are square integrable on the interval $(0, 1)$, with the inner product

$$(\varphi, \psi) = \int_0^1 \left(\frac{d^m \varphi(x)}{dx^m} \frac{d^m \psi(x)}{dx^m} + \frac{d^{m-1} \varphi(x)}{dx^{m-1}} \frac{d^{m-1} \psi(x)}{dx^{m-1}} \right) dx.$$

Consequently, the square of the norm of a function $\varphi(x)$ in the space $W_2^{(m)}(0, 1)$ is defined by the equality

$$\|\varphi\|_{W_2^{(m)}(0, 1)}^2 = \int_0^1 \left(\left(\frac{d^m \varphi(x)}{dx^m} \right)^2 + \left(\frac{d^{m-1} \varphi(x)}{dx^{m-1}} \right)^2 \right) dx.$$

Consider quadrature formulas with equally spaced nodes of the form

$$\int_0^1 \varphi(x) dx \cong \sum_{\beta=0}^N K[\beta] \varphi[\beta]. \quad (2.1)$$

Here $\varphi(x)$ is an element of the space $W_2^{(m)}(0, 1)$, $K[\beta]$ are the coefficients, and $[\beta] = h\beta$ are the nodes of the quadrature formula (2.1), $h = \frac{1}{N}$, $N = 2, 3, \dots$

The error of this quadrature formula is written as the equality

$$\int_0^1 \varphi(x) dx - \sum_{\beta=0}^N K[\beta] \varphi[\beta] = (\ell, \varphi), \quad (2.2)$$

where

$$\ell(x) = \varepsilon_{[0,1]}(x) - \sum_{\beta=0}^N K[\beta] \delta(x - h\beta)$$

is called the error functional of the quadrature formula (2.1),

$$\varepsilon_{[0,1]}(x) = \begin{cases} 1, & x \in [0, 1], \\ 0, & x \notin [0, 1], \end{cases}$$

- is the characteristic function of the segment $[0, 1]$, and $\delta(x)$ is the Dirac delta function.

It is known that the error (ℓ, φ) is defined by the equality

$$(\ell, \varphi) = \int_{-\infty}^{\infty} \ell(x) \varphi(x) dx.$$

The error functional $\ell(x)$ belongs to the space $W_2^{(m)*}(0, 1)$. For the error functional $\ell(x)$ to belong to the space $(W_2^{(m)})^*(0, 1)$, it is necessary and sufficient that

$$(\ell, x^\alpha) = 0, \quad \alpha = 0, 1, 2, \dots, m-2. \quad (2.3)$$

It is natural to assess the quality of the quadrature formula (2.1) by means of the maximum error of this formula, that is, by means of the norm of the error functional $\ell(x)$:

$$\|\ell|W_2^{(m)*}(0, 1)\| = \sup_{\|\varphi|W_2^{(m)}\|=1} |(\ell, \varphi)|.$$

Evidently, the norm of the error functional $\ell(x)$ depends on the coefficients $K[\beta]$ of the quadrature formula (2.1). If

$$\|\ell|W_2^{(m)*}(0, 1)\| = \inf_{K[\beta]} \|\ell|W_2^{(m)*}(0, 1)\|, \quad (2.4)$$

then the functional $\ell(x)$ is said to correspond to the optimal quadrature formula of the form (2.1) in the space $W_2^{(m)}(0, 1)$. The coefficients $K[\beta]$ yielding the infimum in (2.4) are the optimal coefficients of the quadrature formula. In what follows we denote them by $K[\beta]$.

In [31], the square of the norm of the error functional $\ell(x)$ in the space $W_2^{(m)}(0, 1)$ was computed, and by minimizing this norm under conditions (2.3), a system of linear algebraic equations for finding the optimal coefficients $K[\beta]$ was obtained.

The following theorem holds.

Theorem 2.1. *For the square of the norm of the error functional, the following formula is valid*

$$\|\ell|W_2^{(m)*}(0, 1)\|^2 = (-1)^m \left[\sum_{\beta=0}^N \sum_{\gamma=0}^N K[\beta] K[\gamma] \mu_m(h\beta - h\gamma) \right]$$

$$-2 \sum_{\beta=0}^N K[\beta] \int_0^1 \mu_m(x - h\beta) dx + \int_0^1 \int_0^1 \mu_m(x - y) dx dy \Bigg], \quad (2.5)$$

where

$$\mu_m(x) = \frac{\text{sign } x}{2} \left(\frac{e^x - e^{-x}}{2} - \sum_{n=1}^{m-1} \frac{x^{2n-1}}{(2n-1)!} \right). \quad (2.6)$$

By minimizing this norm with respect to the coefficients $K[\beta]$, a system is obtained for finding the optimal coefficients $K[\beta]$, $\beta = 0, 1, \dots, N$.

$$\sum_{\gamma=0}^N K[\gamma] \mu_m(h\beta - h\gamma) + P_{m-2}[\beta] = f_m[\beta], \quad \beta = 0, 1, \dots, N, \quad (2.7)$$

$$\sum_{\gamma=0}^N K[\gamma] [\gamma]^\alpha = \frac{1}{\alpha + 1}, \quad \alpha = \overline{0, m-2}, \quad (2.8)$$

$$f_m[\beta] = \int_0^1 \mu_m(x - h\beta) dx. \quad (2.9)$$

In the following sections we shall solve this system in the cases $m = 1$, $m = 2$, and $m = 3$. For this purpose we shall need the discrete analogue $D_{m,w}[\beta]$ of the differential operator $\frac{d^{2m}}{dx^{2m}} - \frac{d^{2m-2}}{dx^{2m-2}}$ for $m = 1$, $m = 2$, $m = 3$.

These operators were constructed in [32], and they have the form

$$D_{1,w}[\beta] = \frac{1}{1 - e^{2h}} \begin{cases} 0, & |\beta| \geq 2, \\ -2e^h, & |\beta| = 1, \\ 2(1 + e^{2h}), & \beta = 0. \end{cases} \quad (2.10)$$

$$D_{2,w}[\beta] = \frac{2}{e^{2h} - 2he^h - 1} \begin{cases} A\lambda^{|\beta|-1}, & |\beta| \geq 2, \\ e^h + A, & |\beta| = 1, \\ -(1 + e^h)^2 - pe^h + \frac{A}{\lambda}, & \beta = 0, \end{cases} \quad (2.11)$$

$$p = \frac{-2 + 2e^{2h} - 2h - 2he^{2h}}{-e^{2h} + 2he^h + 1},$$

$$\lambda = \frac{h(1 + e^{2h}) + 1 - e^{2h} - (e^h - 1) \sqrt{h^2(1 + e^h)^2 - 2h(e^{2h} - 1)}}{1 + 2he^h - e^{2h}}, \quad |\lambda| < 1.$$

$$D_{3,w}[\beta] = \frac{2}{p_4^{(4)}} \begin{cases} \sum_{k=1}^2 A_k \lambda_k^{|\beta|-1}, & |\beta| \geq 2, \\ -2 \cdot e^h + \sum_{k=1}^2 A_k, & |\beta| = 1, \\ 2C + \sum_{k=1}^2 \frac{A_k}{\lambda_k}, & \beta = 0. \end{cases} \quad (2.12)$$

Here $C = 1 + 4e^h + e^{2h} \cdot \frac{P_3^{(4)}}{P_4^{(4)'}}$,

$$A_\kappa = \frac{2(1 - \lambda_\kappa)^4 [\lambda_\kappa(e^{2h} + 1) - e^h(\lambda_\kappa^2 + 1)] P_4^{(4)}}{\lambda_\kappa P_4'(\lambda_\kappa)},$$

$$P_4(\lambda) = \sum_{s=0}^4 P_s^{(4)} \lambda^s = (1 - e^{2h})(1 - \lambda)^4 - 2 \left[\lambda(e^{2h} + 1) - e^h(\lambda^2 + 1) \right] \cdot \left[h(1 - \lambda)^2 + \frac{h^3}{6}(\lambda^2 + 4\lambda + 1) \right], \quad (2.13)$$

where λ_1 and λ_2 are the roots of the polynomial $P_4(\lambda)$ whose moduli are less than unity, i.e. $|\lambda_1| < 1$ and $|\lambda_2| < 1$.

The operator $D_{m,w}[\beta]$ possesses the following properties:

- (1) $D_{m,w}[\beta] * e^{h\beta} = 0$,
- (2) $D_{m,w}[\beta] * e^{-h\beta} = 0$,
- (3) $D_{m,w}[\beta] * (h\beta)^n = 0$ for $m = 2$, $n \leq 1$, and for $m = 3$, $n \leq 3$,
- (4) $D_{1,w}[\beta] * \mu_1[\beta] = \delta[\beta]$,
- (5) $D_{2,w}[\beta] * \mu_2[\beta] = \delta[\beta]$,
- (6) $D_{3,w}[\beta] * \mu_3[\beta] = \delta[\beta]$,
- (7) $D_{3,w}[\beta] * (h\beta)^4 = 4! \cdot h$.

In the next section we present the main results of this work, that is, the analytic form of the optimal coefficients $K[\beta]$ of quadrature formulas of the form (2.1) in the spaces $W_2^{(1)}(0,1)$, $W_2^{(2)}(0,1)$, $W_2^{(3)}(0,1)$.

3. THE MAIN RESULTS

In this section we present the main results of the present work. Here we state the theorems on optimal quadrature formulas with equally spaced nodes in the spaces $W_2^{(1)}(0,1)$, $W_2^{(2)}(0,1)$, $W_2^{(3)}(0,1)$.

Theorem 3.1. *Among the quadrature formulas of the form (2.1), the following formula is optimal in the Sobolev space $W_2^{(1)}(0,1)$:*

$$\int_0^1 \varphi(x) dx \cong \frac{e^h - 1}{e^h + 1} \left[2 \sum_{\beta=1}^{N-1} \varphi[\beta] + \varphi[0] + \varphi[N] \right], \quad (3.1)$$

where $h = \frac{1}{N}$, $N = 2, 3, \dots$. In this case, the square of the norm of the error functional $\ell(x)$ takes the form

$$\left\| \ell^\circ |W_2^{(1)*} \right\|^2 = 1 - \frac{2(e^h - 1)}{h(e^h + 1)} = \frac{h^2}{12} - \frac{h^4}{5!} + \frac{h^6}{4 \cdot 7!} - \dots$$

Theorem 3.2. The optimal coefficients of the quadrature formula of the form (2.1) in the space $W_2^{(2)}(0, 1)$ are determined by the formulas

$$\overset{\circ}{K}[0] = \overset{\circ}{K}[N] = \frac{h}{2} + a \frac{\lambda - \lambda^N}{\lambda - 1},$$

$$\overset{\circ}{K}[\beta] = h + a(\lambda^\beta + \lambda^{N-\beta}), \quad \beta = \overline{1, N-1}.$$

Here the parameters a and λ are determined, respectively, by the equalities

$$a = \frac{(e+1) \left[2(e^h - 1) - h(e^h + 1) \right] (\lambda - 1)(e^h - \lambda)(\lambda e^h - 1)}{2\lambda(e^h - 1)^2 [(\lambda^N + e)(\lambda e^h - 1) - (\lambda^N e + 1)(e^h - \lambda)]},$$

$$\lambda = \frac{h(1 + e^{2h}) + 1 - e^{2h} - (e^h - 1) \sqrt{h^2(1 + e^h)^2 - 2h(e^{2h} - 1)}}{1 - 2he^h - e^{2h}}.$$

Theorem 3.3. The coefficients of the optimal quadrature formulas of the form (2.1) in the space $W_2^{(3)}(0, 1)$ are given by the equalities

$$\overset{\circ}{K}[0] = \overset{\circ}{K}[N] = \frac{h}{2} + h \sum_{k=1}^2 a_k \frac{\lambda_k - \lambda_k^N}{\lambda_k - 1},$$

$$\overset{\circ}{K}(\beta) = h + h \sum_{k=1}^2 a_k (\lambda_k^\beta + \lambda_k^{N-\beta}), \quad \beta = 1, 2, \dots, N-1.$$

Here $a_k, k = 1, 2$, are determined from the system

$$\sum_{k=1}^2 a_k \frac{\lambda_k + \lambda_k^{N+1}}{(1 - \lambda_k)^2} = \frac{1}{12},$$

$$\sum_{k=1}^2 a_k \frac{\lambda_k(e^h - 1) \left[(\lambda_k + e)(\lambda_k e^h - 1) + (\lambda_k^N e + 1)(\lambda_k - e^h) \right]}{(\lambda_k - 1)(\lambda_k - e^h)(1 - \lambda_k e^h)} = \frac{e+1}{h} \left[1 - \frac{h(e^h + 1)}{2(e^h - 1)} \right],$$

and λ_1 and λ_2 are the roots of the polynomial (2.13).

Remark 3.1. The optimal quadrature formula (3.1) in the space $W_2^{(1)}(0, 1)$ coincides with the optimal quadrature formula in the space $W_2^{(1,0)}(0, 1)$. Since the norms of functions in these spaces are defined by different formulas, the quadrature formula (3.1) is a universally optimal quadrature formula.

Remark 3.2. The proofs of Theorem 3.1 are given in [31]. For completeness, we have reproduced them here.

In the next section we present the proofs of Theorem 3.1 and Theorem 3.3.

4. PROOF OF THE THEOREMS

First we prove Theorem 2.1.

Proof. In this case, the system (2.7) for finding the optimal coefficients takes the form

$$\sum_{\gamma=0}^N \overset{\circ}{K}[\gamma] \mu_1(h\beta - h\gamma) = \frac{e+1}{4} [e^{h\beta-1} + e^{-h\beta}] - 1, \quad \beta = 0, 1, 2, \dots, N, \quad (4.1)$$

where

$$\mu_1(x) = \frac{\text{sign } x}{2} \left(\frac{e^x - e^{-x}}{2} \right). \quad (4.2)$$

Since $m = 1$, the system (2.8) does not participate here.

Let us denote by $U[\beta]$ the left-hand side of (4.1), that is,

$$U[\beta] = \sum_{\gamma=0}^N \overset{\circ}{K}[\gamma] \mu_1(h\beta - h\gamma). \quad (4.3)$$

From (4.1) it is seen that $U[\beta]$ for $\beta = 0, 1, \dots, N$ takes the following form

$$U[\beta] = \frac{e+1}{4} [e^{h\beta-1} + e^{-h\beta}] - 1. \quad (4.4)$$

We now proceed to determine $U[\beta]$ for $\beta < 0$ and $\beta > N$. Let $\beta < 0$; then

$$\begin{aligned} U[\beta] &= \frac{1}{4} \sum_{\gamma=0}^N \overset{\circ}{K}[\gamma] \text{sign}(h\beta - h\gamma) (e^{h\beta-h\gamma} - e^{-h\beta+h\gamma}) = -\frac{1}{4} \sum_{\gamma=0}^N \overset{\circ}{K}[\gamma] (e^{h\beta-h\gamma} - e^{-h\beta+h\gamma}) \\ &= -\frac{1}{4} \left[e^{h\beta} \sum_{\gamma=0}^N \overset{\circ}{K}[\gamma] e^{-h\gamma} - e^{-h\beta} \sum_{\gamma=0}^N \overset{\circ}{K}[\gamma] e^{h\gamma} \right] = a_1^- e^{h\beta} + a_2^- e^{-h\beta}, \end{aligned}$$

where

$$a_1^- = -\frac{1}{4} \sum_{\gamma=0}^N \overset{\circ}{K}[\gamma] e^{-h\gamma}, \quad a_2^- = \frac{1}{4} \sum_{\gamma=0}^N \overset{\circ}{K}[\gamma] e^{h\gamma}.$$

Let $\beta > N$; then, analogously,

$$U[\beta] = \frac{1}{4} \left[e^{h\beta} \sum_{\gamma=0}^N \overset{\circ}{K}[\gamma] e^{-h\gamma} - e^{-h\beta} \sum_{\gamma=0}^N \overset{\circ}{K}[\gamma] e^{h\gamma} \right].$$

Hence $U[\beta] = -a_1^- e^{h\beta} - a_2^- e^{-h\beta}$.

Thus,

$$U[\beta] = \begin{cases} a_1^- e^{h\beta} + a_2^- e^{-h\beta}, & \beta < 0, \\ \frac{e+1}{4} [e^{h\beta-1} + e^{-h\beta}] - 1, & 0 \leq \beta \leq N, \\ -a_1^- e^{h\beta} - a_2^- e^{-h\beta}, & \beta > N, \end{cases} \quad (4.5)$$

where the unknowns are a_1^- and a_2^- .

It is known [1] that the convolution of two functions of a discrete argument is defined by the formula

$$\varphi[\beta] * \psi[\beta] = \sum_{\gamma=-\infty}^{\infty} \varphi[\beta - \gamma] \psi[\gamma] = \sum_{\gamma=-\infty}^{\infty} \varphi[\gamma] \psi[\beta - \gamma]. \quad (4.6)$$

Assuming that $K[\beta] = 0$ for $\beta < 0$, $\beta > N$ and taking (4.6) into account, we rewrite the system (4.1) in the form of a convolution of two functions

$$K[\beta] * \mu_1[\beta] = \frac{e+1}{4} [e^{h\beta-1} + e^{-h\beta}] - 1. \quad (4.7)$$

Using the property of the operator $D_{1,w}[\beta] * \mu_1[\beta] = \delta[\beta]$, we have

$$D_{1,w}[\beta] * U[\beta] = \overset{\circ}{K}[\beta]. \quad (4.8)$$

By virtue of the fact that $\overset{\circ}{K}[\beta] = 0$ for $\beta < 0$ and $\beta > N$, and using (4.4), we have

$$\begin{aligned} D_{1,w}[\beta] * U[\beta] &= \sum_{\gamma=-\infty}^{\infty} D_{1,w}[\beta - \gamma] U[\gamma] = \\ &= \sum_{\gamma=0}^N D_{1,w}[\beta - \gamma] \left[\frac{e-1}{4} (e^{h\gamma-1} + e^{-h\gamma}) - 1 \right] + \sum_{\gamma=-\infty}^{-1} D_{1,w}[\beta - \gamma] (a_1^- e^{h\gamma} + a_2^- e^{-h\gamma}) - \\ &\quad - \sum_{\gamma=N+1}^{\infty} D_{1,w}[\beta - \gamma] (a_1^- e^{h\gamma} + a_2^- e^{-h\gamma}) = 0. \end{aligned}$$

Now, using formula (2.10) for $\beta = -1$ and $\beta = N+1$, we obtain the system for finding a_1^- and a_2^- :

$$a_1^- + a_2^- = \frac{(e+1)^2}{4e} - 1, \quad a_1^- e + a_2^- e^{-1} = 1 - \frac{(e+1)^2}{4e}.$$

Hence, solving this system, we find

$$\begin{aligned} a_1^- &= \frac{1}{4} (e^{-1} - 1), \\ a_2^- &= \frac{e-1}{4}. \end{aligned}$$

Thus $U[\beta]$ is defined on $\mathbb{Z}$, the set of integers, that is, $\beta \in \mathbb{Z}$:

$$U[\beta] = \begin{cases} \frac{e^{-1}-1}{4} e^{h\beta} + \frac{e-1}{4} e^{-h\beta}, & \beta < 0, \\ \frac{e+1}{4} [e^{h\beta-1} + e^{-h\beta}] - 1, & 0 \leq \beta \leq N, \\ \frac{1-e^{-1}}{4} e^{h\beta} + \frac{1-e}{4} e^{-h\beta}, & \beta > N. \end{cases} \quad (4.9)$$

Using formulas (2.10), (4.8) and (4.9), we find the optimal coefficients $\overset{\circ}{K}[\beta]$ for $\beta = 0, 1, \dots, N$.

Let $\beta = 0$; then

$$\overset{\circ}{K}[0] = D_{1,w}[\beta] * U[\beta]|_{\beta=0} = \sum_{\gamma=-\infty}^{\infty} D_{1,w}[-\gamma] U[\gamma].$$

Hence, by virtue of formulas (2.10) and (4.9), after some simplifications we have

$$\overset{\circ}{K}[0] = \frac{e^h - 1}{e^h + 1}. \quad (4.10)$$

In exactly the same way we obtain

$$\overset{\circ}{K}[N] = \frac{e^h - 1}{e^h + 1}. \quad (4.11)$$

Let $\beta = 1, 2, \dots, N-1$; then

$$\overset{\circ}{K}[0] = D_{1,w}[\beta] * U[\beta] = \sum_{\gamma=-\infty}^{\infty} D_{1,w}[\beta - \gamma] U[\gamma].$$

Hence, by virtue of (2.10) and (4.9), after some simplifications we have

$$\overset{\circ}{K}[\beta] = \frac{2(e^h - 1)}{e^h + 1}, \quad \beta = 1, 2, \dots, N-1. \quad (4.12)$$

In the case $m = 1$, the equality (2.5) takes the form

$$\begin{aligned} \left\| \ell |W_2^{(1)*}(0, 1) \right\|^2 &= - \sum_{\beta=0}^N \sum_{\gamma=0}^N K[\beta] K[\gamma] \mu_1(h\beta - h\gamma) \\ &\quad + 2 \sum_{\beta=0}^N K[\beta] \int_0^1 \mu_1(x - h\beta) dx - \int_0^1 \int_0^1 \mu_1(x - y) dx dy. \end{aligned}$$

Hence, having evaluated the integrals, we have

$$\begin{aligned} \left\| \ell |W_2^{(1)*}(0, 1) \right\|^2 &= - \sum_{\beta=0}^N \sum_{\gamma=0}^N K[\beta] K[\gamma] \mu_1(h\beta - h\gamma) \\ &\quad + 2 \sum_{\beta=0}^N K[\beta] \left[\frac{e+1}{4} (e^{h\beta-1} + e^{-h\beta}) - 1 \right] - \frac{e^2 - 2e - 1}{2e}. \end{aligned}$$

We now proceed to the computation of the square of the norm of the error functional of the optimal quadrature formula. In this case, taking (4.1) into account and substituting the values (4.10), (4.11), (4.12) found in place of $K[\beta]$, after some simplifications we obtain

$$\begin{aligned} \left\| \overset{\circ}{\ell} |W_2^{(1)*}(0, 1) \right\|^2 &= - \sum_{\beta=0}^N \overset{\circ}{K}[\beta] \left(\sum_{\gamma=0}^N \overset{\circ}{K}[\gamma] \mu_1(h\beta - h\gamma) - \frac{e+1}{4} (e^{h\beta-1} + e^{-h\beta}) + 1 \right) + \\ &\quad + \sum_{\beta=0}^N \overset{\circ}{K}[\beta] \left[\frac{e+1}{4} (e^{h\beta-1} + e^{-h\beta}) - 1 \right] - \frac{e^2 - 2e - 1}{2e} = \\ &= \frac{1}{4} \sum_{\beta=0}^N \overset{\circ}{K}[\beta] \left[(e+1) (e^{h\beta-1} + e^{-h\beta}) - 4 \right] - \frac{e^2 - 2e - 1}{2e} = 1 - \frac{2(e^h - 1)}{h(e^h + 1)}. \end{aligned}$$

Thus, Theorem 2.1 is completely proved. $\square$

In what follows we shall need the following known formulas from [33,34]:

$$\sum_{\gamma=0}^{n-1} q^{\gamma} \gamma^k = \frac{1}{1-q} \sum_{i=0}^k \left(\frac{q}{1-q} \right)^i \Delta^i 0^k - \frac{q^n}{1-q} \sum_{i=0}^k \left(\frac{q}{1-q} \right)^i \Delta^i \gamma^k|_{\gamma=n}, \quad (4.13)$$

where $\Delta^i \gamma^k$ is the finite difference of order i of γ^k , $\Delta^i 0^k = \Delta^i \gamma^k|_{\gamma=0}$, and $\Delta^i 0^k = \sum_{j=1}^i (-1)^{i-j} C_i^j j^k$,

$$\sum_{\gamma=0}^{\beta-1} \gamma^k = \sum_{j=1}^{\kappa+1} \frac{\kappa! B_{\kappa+1-j}}{j! (\kappa+1-j)!} \beta^j, \quad (4.14)$$

where $B_{\kappa+1-j}$ are the Bernoulli numbers.

Proof of Theorem 3.2. In this case, the systems (2.6) and (2.7) take the following form

$$\sum_{\gamma=0}^N \overset{\circ}{K}[\gamma] \mu_3(h\beta - h\gamma) + d_0 + d_1(h\beta) = f_3[\beta], \quad \beta = 0, 1, \dots, N, \quad (4.15)$$

$$\sum_{\gamma=0}^N \overset{\circ}{K}[\gamma] [\gamma]^\alpha = \frac{1}{\alpha+1}, \quad \alpha = 0, 1. \quad (4.16)$$

Here the unknowns are the coefficients $\overset{\circ}{K}[\gamma]$, $\gamma = 0, 1, \dots, N$, d_0 , d_1 . The functions $\mu_3(x)$ and $f_3[\beta]$ are given by the formulas

$$\mu_3(x) = \frac{\text{sign } x}{2} \left(\frac{e^x - e^{-x}}{2} - x - \frac{x^3}{6} \right), \quad (4.17)$$

$$f_3[\beta] = e^{h\beta} \frac{e+1}{4e} + e^{-h\beta} \frac{e+1}{4} - \frac{(h\beta)^4}{24} + \frac{(h\beta)^3}{12} - \frac{5}{8}(h\beta)^2 + \frac{7}{12}h\beta - \frac{61}{48}. \quad (4.18)$$

Here too, assuming that $\overset{\circ}{K}[\beta] = 0$ for $\beta < 0$ and $\beta > N$, we write the sum in (4.15) in the form of a convolution of two functions

$$\overset{\circ}{K}[\beta] * \mu_3[\beta] + d_0 + d_1(h\beta) = f_3[\beta], \quad \beta = 0, 1, \dots, N. \quad (4.19)$$

We denote the left-hand side by

$$U[\beta] = \overset{\circ}{K}[\beta] * \mu_3[\beta] + d_0 + d_1(h\beta). \quad (4.20)$$

From (32) it is seen that

$$U[\beta] = f_3[\beta], \quad \beta = 0, 1, \dots, N. \quad (4.21)$$

We now proceed to find $U[\beta]$ for $\beta < 0$ and $\beta > N$.

Let $\beta < 0$; then, taking (4.16) into account, after some simplifications we have

$$\begin{aligned} U[\beta] &= \overset{\circ}{K}[\beta] * \mu_3[\beta] + d_0 + d_1(h\beta) \\ &= a_0^- e^{h\beta} + a_1^- e^{-h\beta} + a_2^-(h\beta) + a_3^- + \frac{1}{12}(h\beta)^3 - \frac{1}{8}(h\beta)^2 + \frac{1}{2}h\beta - \frac{1}{4}. \end{aligned}$$

Here

$$\begin{aligned} a_0^- &= -\frac{1}{4} \sum_{\gamma=0}^N \overset{\circ}{K}[\gamma] e^{-h\gamma}, & a_1^- &= \frac{1}{4} \sum_{\gamma=0}^N \overset{\circ}{K}[\gamma] e^{h\gamma}, \\ a_2^- &= \frac{1}{4} \sum_{\gamma=0}^N \overset{\circ}{K}[\gamma] h\gamma + d_1, & a_3^- &= d_0 - \frac{1}{12} \sum_{\gamma=0}^N \overset{\circ}{K}[\gamma] (h\gamma)^3. \end{aligned}$$

In exactly the same way we determine $U[\beta]$ for $\beta > N$:

$$U[\beta] = a_0^+ e^{h\beta} + a_1^+ e^{-h\beta} + a_2^+ (h\beta) + a_3^+ - \frac{1}{12} (h\beta)^3 + \frac{1}{8} (h\beta)^2 - \frac{1}{2} h\beta + \frac{1}{4},$$

where

$$a_0^+ = -a_0^-, \quad a_1^+ = -a_1^-, \quad a_2^+ = -\frac{1}{4} \sum_{\gamma=0}^N \overset{\circ}{K}[\gamma] [\gamma]^2 + d_1, \quad a_3^+ = d_0 + \frac{1}{12} \sum_{\gamma=0}^N \overset{\circ}{K}[\gamma] [\gamma]^3.$$

Thus we have determined the form of the function $U[\beta]$ for $\beta \in \mathbb{Z}$:

$$U[\beta] = \begin{cases} a_0^- e^{h\beta} + a_1^- e^{-h\beta} + a_2^- (h\beta) + a_3^- + \frac{1}{12} (h\beta)^3 - \frac{1}{8} (h\beta)^2 + \frac{1}{2} h\beta - \frac{1}{4}, & \beta < 0, \\ f_3[\beta], & \beta = 0, 1, \dots, N, \\ a_0^+ e^{h\beta} + a_1^+ e^{-h\beta} + a_2^+ (h\beta) + a_3^+ - \frac{1}{12} (h\beta)^3 + \frac{1}{8} (h\beta)^2 - \frac{1}{2} h\beta + \frac{1}{4}, & \beta > N. \end{cases} \quad (4.22)$$

Here we shall choose a different route for finding $\overset{\circ}{K}[\beta]$.

Using the operator $D_{3,w}[\beta]$ and its properties, that is, formulas (2.12), (4.22), after some computations we obtain the following representation for the optimal coefficients $\overset{\circ}{K}[\beta]$ for $\beta = 1, 2, \dots, N-1$:

$$\overset{\circ}{K}[\beta] = h + h \sum_{k=1}^2 (a_k \lambda_k^\beta + b_k \lambda_k^{N-\beta}). \quad (4.23)$$

Here a_k and b_k are unknown coefficients; they are expressed through the coefficients of the operator $D_{3,w}[\beta]$ and the function $U[\beta]$.

Substituting the representation $\overset{\circ}{K}[\beta]$, found by formula (4.23), into the system (4.16), and solving this system with respect to $\overset{\circ}{K}[0]$ and $\overset{\circ}{K}[N]$, we have

$$\overset{\circ}{K}[0] = \frac{h}{2} + h \sum_{\kappa=1}^2 \left[\frac{a_\kappa \lambda_\kappa - b_\kappa \lambda_\kappa^N}{\lambda_\kappa - 1} - \frac{h(\lambda_\kappa^{N+1} - \lambda_\kappa)(a_\kappa - b_\kappa)}{(1 - \lambda_\kappa)^2} \right], \quad (4.24)$$

$$\overset{\circ}{K}[N] = \frac{h}{2} + h \sum_{\kappa=1}^2 \left[\frac{b_\kappa \lambda_\kappa - a_\kappa \lambda_\kappa^N}{\lambda_\kappa - 1} + \frac{h(\lambda_\kappa^{N+1} - \lambda_\kappa)(a_\kappa - b_\kappa)}{(1 - \lambda_\kappa)^2} \right]. \quad (4.25)$$

Now, substituting (4.23) into the left-hand side of (4.15) and taking the function sign x into account, we have

$$\sum_{\gamma=0}^N \overset{\circ}{K}[\gamma] \mu_3(h\beta - h\gamma) + d_0 + d_1(h\beta) =$$

$$\begin{aligned}
&= \sum_{\gamma=0}^N \overset{\circ}{K}[\gamma] \frac{\text{sign}(h\beta - h\gamma)}{2} \left[\frac{e^{h\beta-h\gamma} - e^{-h\beta+h\gamma}}{2} - (h\beta - h\gamma) - \frac{(h\beta - h\gamma)^3}{6} \right] + d_0 + d_1(h\beta) = \\
&= \sum_{\gamma=0}^{\beta} \overset{\circ}{K}[\gamma] \left[\frac{e^{h\beta-h\gamma} - e^{-h\beta+h\gamma}}{2} - (h\beta - h\gamma) - \frac{(h\beta - h\gamma)^3}{6} \right] - \\
&\quad - \frac{1}{2} \sum_{\gamma=0}^N \overset{\circ}{K}[\gamma] \left[\frac{e^{h\beta-h\gamma} - e^{-h\beta+h\gamma}}{2} - (h\beta - h\gamma) - \frac{(h\beta - h\gamma)^3}{6} \right] + d_0 + d_1(h\beta) = \\
&= \frac{\overset{\circ}{K}}{2} \left[\frac{e^{h\beta} - e^{-h\beta}}{2} - h\beta - \frac{(h\beta)^3}{6} \right] + \\
&\quad + \sum_{\gamma=1}^{\beta} h \left(1 + \sum_{\kappa=1}^2 (a_{\kappa} \lambda_{\kappa}^{\beta} + b_{\kappa} \lambda_{\kappa}^{N-\beta}) \right) \cdot \left[\frac{e^{h\beta-h\gamma} - e^{-h\beta+h\gamma}}{2} - (h\beta - h\gamma) - \frac{(h\beta - h\gamma)^3}{6} \right] + \\
&\quad + \frac{\overset{\circ}{K}(N)}{2} \left[\frac{e^{h\beta-1} - e^{-h\beta+1}}{2} - (h\beta - 1) - \frac{(h\beta - 1)^3}{6} \right] - \\
&\quad - \frac{1}{2} \sum_{\gamma=1}^N h \left(1 + \sum_{\kappa=1}^2 (a_{\kappa} \lambda_{\kappa}^N + b_{\kappa} \lambda_{\kappa}^{\alpha-\beta}) \right) \left[\frac{e^{h\beta-h\gamma} - e^{-h\beta+h\gamma}}{2} - (h\beta - h\gamma) - \frac{(h\beta - h\gamma)^3}{6} \right] + d_0 + d_1(h\beta) \quad (4.26)
\end{aligned}$$

Computing the sums in expression (4.26) by means of the known formulas (4.13), (4.14), we bring this expression to the form

$$\begin{aligned}
&\frac{e^{h\beta}}{4} \left[\overset{\circ}{K}[0] - e^{-1} \overset{\circ}{K}[N] + \frac{h(1 + e^{h-1})}{e^h - 1} - h \sum_{\kappa=1}^2 \left(a_{\kappa} \frac{\lambda_{\kappa} + e^{h-1} \lambda_{\kappa}^N}{\lambda_{\kappa} - e^h} + b_{\kappa} \frac{\lambda_{\kappa}^N + e^{h-1} \lambda_{\kappa}}{1 - \lambda_{\kappa} e^h} \right) \right] + \\
&\quad + \frac{e^{-h\beta}}{4} \left[-\overset{\circ}{K}[0] + e \overset{\circ}{K}[N] + \frac{h(e + e^h)}{e^h - 1} + h \sum_{\kappa=1}^2 \left(a_{\kappa} \frac{e^h \lambda_{\kappa} + \lambda_{\kappa}^N e}{\lambda_{\kappa} e^h - 1} + b_{\kappa} \frac{\lambda_{\kappa}^N e^h + \lambda_{\kappa} e}{e^h - \lambda_{\kappa}} \right) \right] - \\
&\quad - \frac{(h\beta)^4}{24} + (h\beta)^3 \left[-\frac{\overset{\circ}{K}[0]}{6} + \frac{h}{12} + \frac{h}{6} \sum_{\kappa=1}^2 \frac{a_{\kappa} \lambda_{\kappa} - b_{\kappa} \lambda_{\kappa}^N}{\lambda_{\kappa} - 1} + \frac{1}{12} \right] + \\
&\quad + (h\beta)^2 \left[\frac{h^2}{2} \sum_{\kappa=1}^2 \frac{\lambda_{\kappa} (a_{\kappa} + b_{\kappa} \lambda_{\kappa}^N)}{(1 - \lambda_{\kappa})^2} - \frac{h^2}{24} - \frac{5}{8} \right] + \\
&\quad + (h\beta) \left[-\overset{\circ}{K}[0] + \frac{h}{2} - h \sum_{\kappa=1}^2 \frac{1}{(1 - \lambda_{\kappa})^2} \left(a_{\kappa} \lambda_{\kappa} \left((1 - \lambda_{\kappa})^2 + \frac{h}{2} (1 + \lambda_{\kappa}) \right) - \right. \right. \\
&\quad \left. \left. - b_{\kappa} \lambda_{\kappa}^N \left((1 - \lambda_{\kappa})^2 + \frac{h^2 \lambda_{\kappa}^2 (1 + \lambda_{\kappa})}{2} \right) \right) \right] + \frac{1}{4} \sum_{\gamma=0}^N \overset{\circ}{K}[\gamma] [\gamma]^2 + \frac{1}{4} + d_1 \Bigg] + \\
&\quad + \frac{h(1 + e^{2h})}{2(1 - e^h)} + h \sum_{\kappa=1}^2 \frac{a_{\kappa} \lambda_{\kappa} + b_{\kappa} \lambda_{\kappa}^{N+1}}{(1 - \lambda_{\kappa})^4} \cdot \left[h(1 - \lambda_{\kappa})^2 + \frac{h^3}{6} (\lambda_{\kappa}^2 + 4\lambda_{\kappa} + 1) \right] - \\
&\quad - \frac{1}{8} - \frac{1}{12} \sum_{\gamma=0}^N \overset{\circ}{K}[\gamma] [\gamma]^3 + d_0. \quad (4.27)
\end{aligned}$$

It is seen that the resulting expression (4.27) is the left-hand side of the equality (4.15); further, taking (4.18) into account and equating the expressions at $e^{h\beta}$, $e^{-h\beta}$, $(h\beta)^3$, $(h\beta)^2$, we obtain a system of equations for finding the unknown coefficients a_κ and b_κ , $\kappa = 1, 2$:

$$\overset{\circ}{K}[0] - e^{-1}\overset{\circ}{K}[N] + \frac{h(1+e^{h-1})}{e^h-1} - h \sum_{\kappa=1}^2 \left[a_\kappa \frac{\lambda_\kappa + e^{h-1}\lambda_\kappa^N}{\lambda_\kappa - e^h} + b_\kappa \frac{\lambda_\kappa^N + \lambda_\kappa e^{h-1}}{1 - \lambda_\kappa e^h} \right] = \frac{e+1}{e}, \quad (4.28)$$

$$-\overset{\circ}{K}[0] + e\overset{\circ}{K}[N] + \frac{h(e+e^h)}{e^h-1} + h \sum_{\kappa=1}^2 \left[a_\kappa \frac{e^h\lambda_\kappa + \lambda_\kappa^N e}{\lambda_\kappa e^h - 1} + b_\kappa \frac{\lambda_\kappa^N e^h + \lambda_\kappa e}{e^h - \lambda_\kappa} \right] = e+1, \quad (4.29)$$

$$-\overset{\circ}{K}[0] + \frac{h}{12} + \frac{h}{6} \sum_{\kappa=1}^2 \frac{a_\kappa - b_\kappa \lambda_\kappa^N}{\lambda_\kappa - 1} + \frac{1}{12} = \frac{1}{12}, \quad (4.30)$$

$$\frac{h^2}{2} \sum_{\kappa=1}^2 \frac{\lambda_\kappa(a_\kappa + b_\kappa \lambda_\kappa^N)}{(1 - \lambda_\kappa)^2} - \frac{h^2}{24} - \frac{5}{8} = -\frac{5}{8}. \quad (4.31)$$

Hence, after some simplifications, we bring the equalities (4.28)–(4.31) to the form

$$e\overset{\circ}{K}[0] - \overset{\circ}{K}[N] + \frac{h(e+e^{h-1})}{e^h-1} - h \sum_{\kappa=1}^2 \left[a_\kappa \frac{\lambda_\kappa e - e^h \lambda_\kappa^N}{\lambda_\kappa - e^h} + b_\kappa \frac{e\lambda_\kappa^N - e^h \lambda_\kappa}{1 - \lambda_\kappa e^h} \right] = e+1, \quad (4.32)$$

$$-\overset{\circ}{K}[0] + e\overset{\circ}{K}[N] + \frac{h(e+e^h)}{e^h-1} + h \sum_{\kappa=1}^2 \left[a_\kappa \frac{e^h \lambda_\kappa - e \lambda_\kappa^N}{\lambda_\kappa e^h - 1} + b_\kappa \frac{e^h \lambda_\kappa^N + e \lambda_\kappa}{e^h - \lambda_\kappa} \right] = e+1, \quad (4.33)$$

$$-\overset{\circ}{K}[0] + \frac{h}{2} + h \sum_{\kappa=1}^2 \frac{a_\kappa \lambda_\kappa - b_\kappa \lambda_\kappa^N}{\lambda_\kappa - 1} = 0, \quad (4.34)$$

$$\sum_{\kappa=1}^2 \frac{a_\kappa \lambda_\kappa + b_\kappa \lambda_\kappa^{N+1}}{(1 - \lambda_\kappa)^2} = \frac{1}{12}. \quad (4.35)$$

Now, eliminating $\overset{\circ}{K}[0]$ from (4.32) and the equality (4.33), and substituting the expressions (4.24), (4.25) in place of $\overset{\circ}{K}[0]$, $\overset{\circ}{K}[N]$, after simplifications we obtain

$$\sum_{\kappa=1}^2 (a_\kappa - b_\kappa) \left[(e+1) \frac{\lambda_\kappa + \lambda_\kappa^N}{\lambda_\kappa - 1} + \frac{h(\lambda_\kappa - \lambda_\kappa^{N+1})}{(1 - \lambda_\kappa)^2} - \frac{\lambda_\kappa e + e^h \lambda_\kappa^N}{\lambda_\kappa - e^h} - \frac{e^h \lambda_\kappa + \lambda_\kappa^N e}{\lambda_\kappa e^h - 1} \right] = 0. \quad (4.36)$$

Further, substituting (4.24) in place of $\overset{\circ}{K}[0]$ in (4.34), we have

$$\sum_{\kappa=1}^2 (a_\kappa - b_\kappa) \frac{\lambda_\kappa^{N+1} - \lambda_\kappa}{(1 - \lambda_\kappa)^2} = 0. \quad (4.37)$$

From (4.36) and (4.37) it follows that

$$a_\kappa = b_\kappa, \quad \kappa = 1, 2. \quad (4.38)$$

By virtue of (4.38), the expressions (4.23), (4.24), (4.25) take the following final form:

$$\overset{\circ}{K}[\beta] = h + h \sum_{\kappa=1}^2 a_\kappa (\lambda_\kappa^\beta + \lambda_\kappa^{N-\beta}), \quad \beta = 1, 2, \dots, N-1,$$

$$\mathring{K}[0] = \frac{h}{2} + h \sum_{k=1}^2 a_k \frac{\lambda_k - \lambda_k^N}{\lambda_k - 1}, \quad \mathring{K}[N] = \frac{h}{2} + \sum_{k=1}^2 a_k \frac{\lambda_k - \lambda_k^N}{\lambda_k - 1}.$$

Substituting the found $\mathring{K}[0]$ and $\mathring{K}[N]$ into (4.32), we obtain

$$h \sum_{\kappa=1}^2 a_{\kappa} \frac{\lambda_{\kappa}(e^h - 1) [(\lambda_{\kappa}^N + e)(\lambda_{\kappa} e^h - 1) + (\lambda_{\kappa}^N e + 1)(\lambda_{\kappa} - e^h)]}{(\lambda_{\kappa} - 1)(\lambda_{\kappa} - e^h)(1 - \lambda_{\kappa} e^h)} = (e + 1) \left[1 - \frac{h(e^h + 1)}{2(e^h - 1)} \right]. \quad (4.39)$$

If (4.38) is taken into account, then (4.35) takes the following form:

$$\sum_{\kappa=1}^2 a_{\kappa} \frac{\lambda_{\kappa} + \lambda_{\kappa}^{N+1}}{(1 - \lambda_{\kappa})^2} = \frac{1}{12}. \quad (4.40)$$

Thus the unknown coefficients a_{κ} , $\kappa = 1, 2$, are determined from the system (4.39) and (4.40).

Theorem 3.2 is completely proved. $\square$

For convenience the absolute value of the error (2.2) of the optimal quadrature formula (2.1) we denote by $R(\varphi)$. Then by the Cauchy–Schwarz inequality we have

$$|R(\varphi)| \leq \|\varphi\| W_2^{(m)}(0, 1) \cdot \|\ell\| W_2^{(m)*}(0, 1). \quad (4.41)$$

In the space $W_2^{(m)}(0, 1)$ using Theorems 3.1–3.3 and (4.41), for the error of the optimal quadrature formula (2.1) we obtain the following values of $\|\ell\| W_2^{(m)*}(0, 1)$ for $m = 1, m = 2, m = 3$ and various values of N :

TABLE 1. Values of $\|\ell\| W_2^{(m)*}(0, 1)$ for $m = 1, m = 2, m = 3$ and various N .

N	$m = 1$	$m = 2$	$m = 3$
10	$2.8862 \cdot 10^{-2}$	$4.2308 \cdot 10^{-4}$	$1.1136 \cdot 10^{-5}$
50	$5.7706 \cdot 10^{-3}$	$1.5330 \cdot 10^{-5}$	$5.6569 \cdot 10^{-8}$
100	$2.8862 \cdot 10^{-3}$	$3.7683 \cdot 10^{-6}$	$6.4343 \cdot 10^{-9}$
150	$1.9261 \cdot 10^{-3}$	$1.6703 \cdot 10^{-6}$	$1.8412 \cdot 10^{-9}$

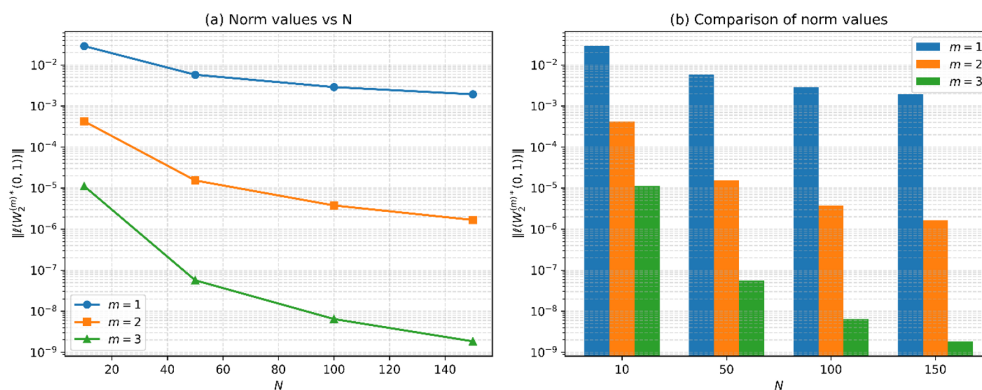


FIGURE 1. The values of $\|\ell\| W_2^{(m)*}(0, 1)$ for $m = 1, m = 2, m = 3$ versus N : (a) line plot; (b) bar chart.

The numerical results show that the error of the optimal quadrature formula decreases as N increases, and for a fixed N the accuracy improves significantly with growing m . All calculations were performed in Maple.

5. CONCLUSION

In this work we have investigated the construction of optimal quadrature formulas by means of the discrete analogue $D_{m,w}[\beta]$ of the differential operator $\frac{d^{2m}}{dx^{2m}} - \frac{d^{2m-2}}{dx^{2m-2}}$.

Further, by solving a system of convolution type, we have found the optimal coefficients of the quadrature formulas in the Sobolev spaces $W_2^{(1)}(0, 1)$, $W_2^{(3)}(0, 1)$, and $W_2^{(2)}(0, 1)$.

In addition, we have computed the square of the norm of the error functional of the optimal quadrature formulas in the space $W_2^{(1)}(0, 1)$.

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