

Convergence Theorems for Two ϕ -Nonexpansive Mappings in Banach Spaces

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Abstract. Our purpose in this work is to introduce a new iterative scheme for finding the set of common fixed points of two ϕ -nonexpansive mappings in uniformly smooth and uniformly convex Banach spaces by using the concept of sunny generalized nonexpansive retractions. Moreover, weak and strong convergence theorems of the proposed iterative scheme are established under suitable conditions.

1. INTRODUCTION

Let E be a Banach space with norm $\|\cdot\|$ and E^* be the dual space of E . Let C be a nonempty closed subset of E and T be a mapping from C into itself. A point $x \in C$ is called a *fixed point* of T if $Tx = x$. We denote the set of all fixed points of T by $F(T)$, that is, $F(T) := \{x \in C : Tx = x\}$. Several methods have been proposed to approximate fixed points of T ; see, for instance, [1, 2, 11, 15, 21, 23].

A mapping $T : C \rightarrow C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

The *normalized duality mapping* $J : E \rightarrow 2^{E^*}$ is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2, \|x^*\| = \|x\|\},$$

where $\langle \cdot, \cdot \rangle$ denote the duality pairing of E^* and E .

Let E be a smooth, strictly convex and reflexive Banach spaces and C be a nonempty closed convex subset of E . The *Lyapunov functional* $\phi : E \times E \rightarrow \mathbb{R}$ is defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2, \quad \forall x, y \in E, \quad (1.1)$$

Received: Jul. 31, 2026.

2020 *Mathematics Subject Classification.* 47H10.

Key words and phrases. ϕ -nonexpansive mappings; sunny generalized nonexpansive retractions; Banach spaces.

where J is the normalized duality mapping from E to E^* .

A mapping $T : C \rightarrow C$ is said to be ϕ -nonexpansive [22,33] if

$$\phi(Tx, Ty) \leq \phi(x, y), \quad \forall x, y \in C.$$

Let C be a nonempty closed subset of a smooth Banach space E . A mapping $T : C \rightarrow C$ is said to be *generalized nonexpansive* [12] if $F(T)$ is nonempty and

$$\phi(Tx, p) \leq \phi(x, p), \quad \forall x \in C, p \in F(T).$$

A mapping $R : E \rightarrow C$ is said to be:

(1) a *retraction* [12] if $R^2 = R$;

(2) *sunny* [12] if $R(Rx + t(x - Rx)) = Rx$ for all $x \in E$ and $t > 0$.

A nonempty closed subset C of a smooth Banach space E is called a *sunny generalized nonexpansive retract* [12] of E if there exists a sunny generalized nonexpansive retraction R from E onto C .

Many researchers have studied sunny generalized nonexpansive retractions; for instance, see the following works. Aleyner and Reich [5] analyzed an explicit scheme for constructing the unique sunny nonexpansive retraction onto the common fixed point set of a nonlinear semigroup of nonexpansive mappings in a Banach space. Ibaraki and Takahashi [13] introduced a new projection in a Banach space and showed an example of the projections. Kohsaka and Takahashi [17] studied the relation between generalized nonexpansive retractions in Banach spaces and generalized projections in dual Banach spaces. Using their results, the authors investigated a proximal-type algorithm in Banach spaces. Honda et al. [12], investigates the existence and properties of nonexpansive retractions onto closed convex cones in Banach spaces. Chen et al. [7] introduced and studied shrinking projection algorithms for finding a solution of an equilibrium problem with a bifunction defined on the dual space of a Banach space. Cai and Bu [6] studied a new iterative algorithm by a modified extragradient method for finding a common element of the set of solutions of variational inequalities for two inverse strongly accretive mappings and the set of common fixed points of an infinite family of nonexpansive mappings in a real 2-uniformly smooth and uniformly convex Banach space. They proved some strong convergence theorems under suitable conditions. Wang and Pan [30] studied the problem of solving a more general variational inequality system for an asymptotically nonexpansive mapping. They introduced a new viscosity approximation scheme to find a common element. Some strong convergence theorems of the proposed iterative method were obtained. Khuangsatung and Kangtunyakarn [16] modified Halpern iteration's process to find a common element of the set of solutions to a variational inequality problem and the set of fixed points of a strictly pseudo-contractive mapping in a q -uniformly smooth Banach space. They also introduced a new technique to establish a strong convergence theorem for a finite family of strictly pseudo-contractive mappings in a q -uniformly smooth Banach space. Yambangwai et al. [32] using sunny nonexpansive retractions in Banach spaces, they developed the CR-iteration algorithm in view of two quasi-nonexpansive nonself mappings and also provided the convergence analysis for the proposed method in the setting

of uniformly convex Banach spaces. Agarwalb and O'Reganc [24] presented an implicit scheme for a representation of nonexpansive mappings on a closed convex subset of a smooth uniformly convex Banach space. Sunthrayuth et al. [25] proposed two Halpern-type forward-backward splitting methods with two-step inertial extrapolation for finding a zero of the sum of two accretive operators in Banach spaces. They established two strong convergence results for the proposed methods under different conditions on the inertial parameters by using a sunny nonexpansive retraction.

Motivated by previous research, in this paper, we introduce a new iterative scheme for finding a common fixed point of two ϕ -nonexpansive mappings in uniformly smooth and uniformly convex Banach spaces by using sunny generalized nonexpansive retractions. We also prove strong and weak convergence theorems for the sequence generated by the proposed iterative scheme under suitable conditions.

2. PRELIMINARIES AND LEMMAS

Let E be a Banach space with norm $\|\cdot\|$ and $B(0) = \{x \in E : \|x\| = 1\}$ be the unit sphere of E . A Banach space E is said to be *smooth* if the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \quad (2.1)$$

exists for any $x, y \in B(0)$. A Banach space E is said to be *uniformly smooth* if the limit (2.1) is attained uniformly for (x, y) in $B(0) \times B(0)$.

A Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$ ([26]). The *modulus of convexity* of E is the function $\delta : [0, 2] \rightarrow [0, 1]$ defined by

$$\delta(\varepsilon) = \inf\{1 - \|\frac{x+y}{2}\| : x, y \in E, \|x\| = \|y\| = 1, \|x - y\| \geq \varepsilon\}.$$

A Banach space E is *uniformly convex* if and only if $\delta(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. A Banach space E is uniformly convex if and only if the dual space E^* is uniformly smooth [7].

Lemma 2.1. ([31]). Let E be a uniformly convex Banach space and $B_r(0) = \{x \in E : \|x\| \leq r\}$ be a closed ball of E . Then, for any $r > 0$, there exists a continuous, strictly increasing and convex function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ such that

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|),$$

for all $x, y \in B_r(0)$ and $\lambda \in [0, 1]$.

Lemma 2.2. ([8]) Let E be a uniformly convex Banach space and $B_r(0)$ a closed ball of E . Then there exists a continuous strictly increasing convex function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ such that

$$\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \lambda\mu g\|x - y\|,$$

for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0, 1]$ with $\lambda + \mu + \gamma = 1$.

The normalized duality mapping J is said to be *weakly sequentially continuous* if the strong convergence of a sequence $\{x_n\}$ to x in E implies the weak* convergence of $\{J(x_n)\}$ to $J(x)$ in E^* .

The following properties of the normalized duality mapping J can be found in [3,4,7,9,26].

- (1) $J(x)$ is nonempty for any $x \in E$.
- (2) J is a monotone and bounded operator in Banach spaces.
- (3) J is a strictly monotone operator in strictly convex Banach spaces.
- (4) If E is smooth, then J is single valued.
- (5) If E is reflexive, then J is onto.
- (6) If E is strictly convex, then J is one-to-one.
- (7) If E is uniformly smooth, then J is uniformly norm-to-norm continuous on each subset of E .
- (8) If E be a reflexive, strictly convex and smooth Banach space then J^{-1} is also single valued, one-to-one, surjective and it is also the duality mapping from E^* into E .
- (9) If E is uniformly convex and uniformly smooth Banach space, then J^{-1} is also uniformly norm-to-norm continuous on bounded subsets of E^* .

From the definition of the function ϕ , it follows that (see [7,18,27])

- (1) $(\|y\| - \|x\|)^2 \leq \phi(y, x) \leq (\|y\| + \|x\|)^2$ for all $x, y \in E$;
- (2) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle$ for all $x, y, z \in E$;
- (3) $\phi(x, y) = \langle x, Jx - Jy \rangle + \langle y - x, Jy \rangle \leq \|x\| \|Jx - Jy\| + \|y - x\| \|y\|$ for all $x, y \in E$;
- (4) If E is a reflexive, strictly convex and smooth Banach space, then, for all $x, y \in E$,

$$\phi(x, y) = 0 \iff x = y.$$

If E is a Hilbert space, then $J = I$, where I is the identity mapping and $\langle \cdot, \cdot \rangle$ denotes an inner product on E . In this case, the Lyapunov functional reduces to $\phi(y, x) = \|x - y\|^2$ for all $x, y \in H$.

Lemma 2.3. [14] Let E be a smooth and uniformly convex Banach space and $r > 0$. Then there exists a strictly increasing, continuous and convex function $h : [0, 2r] \rightarrow \mathbb{R}$ such that $h(0) = 0$ and

$$h(\|x - y\|) \leq \phi(x, y), \quad \forall x, y \in B_r(0),$$

where $B_r(0) = \{x \in E : \|x\| \leq r\}$.

Lemma 2.4. [14] Let E be a uniformly convex and smooth Banach space and let $\{x_n\}, \{y_n\}$ be two sequences of E . If $\phi(x_n, y_n) \rightarrow 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \rightarrow 0$.

Lemma 2.5. [22] Let E be a strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $T : C \rightarrow C$ be a closed ϕ -nonexpansive mapping. Then $F(T)$ is a closed convex subset of C .

The mapping T is said to be *closed* if, for any sequence $\{x_n\} \subset C$ such that $\lim_{n \rightarrow \infty} x_n = x_0$ and $\lim_{n \rightarrow \infty} Tx_n = y_0$, we have $Tx_0 = y_0$.

Lemma 2.6. [29] Let C be a nonempty closed subset of a smooth, strictly convex and reflexive Banach space E and $R : E \rightarrow C$ be a retraction. Then the following statements are equivalent:

- (1) R is sunny generalized nonexpansive;
- (2) $\langle x - Rx, J(y) - J(Rx) \rangle \leq 0$ for all $(x, y) \in E \times C$.

Lemma 2.7. [10] Let C be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space E . Then the sunny generalized nonexpansive retraction from E onto C is uniquely determined.

Lemma 2.8. [12] Let C be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C . Then, for any $x \in E$ and $z \in C$, the following statements hold:

- (1) $z = Rx$ if and only if $\langle x - z, Jy - Jz \rangle \leq 0$ for all $y \in C$;
- (2) $\phi(x, Rx) + \phi(Rx, z) \leq \phi(x, z)$.

Lemma 2.9. [12] Let C be a nonempty closed subset of a smooth, strictly convex and reflexive Banach space E . Then the following statements are equivalent:

- (1) C is a sunny generalized nonexpansive retract of E ;
- (2) $J(C)$ is closed and convex.

Lemma 2.10. [12] Let C be a nonempty closed sunny generalized nonexpansive retract subset of a smooth, strictly convex and reflexive Banach space E . Let R be the sunny generalized nonexpansive retraction from E onto C . Then, for any $x \in E$ and $z \in C$,

$$z = Rx \iff \phi(x, z) = \min_{y \in C} \phi(x, y).$$

Lemma 2.11. [28] Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers satisfying the inequality

$$a_{n+1} \leq a_n + b_n, \quad \forall n \geq 1.$$

If $\sum_{n=0}^{\infty} b_n < \infty$, then $\lim_{n \rightarrow \infty} a_n$ exists.

3. MAIN RESULTS

In this section, we introduce the new iterative scheme for approximating a common element of fixed points of two ϕ -nonexpansive mappings in a Banach space by using a sunny generalized nonexpansive retraction.

Theorem 3.1. Let E be a uniformly smooth and uniformly convex Banach space and let C be a nonempty closed and convex subset of E . Assume that S and T are closed and ϕ -nonexpansive mappings on C and $F = F(S) \cup F(T)$ is nonempty and bounded. For an initial point $x_1 \in C$, generate a sequence $\{x_n\}$ by

$$\begin{cases} y_n = a_n x_n + b_n T x_n + c_n S x_n, \\ x_{n+1} = \alpha_n x_1 + (1 - \alpha_n)(\beta_n x_n + (1 - \beta_n) y_n), \quad \forall n \geq 1, \end{cases} \quad (3.1)$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are real sequences in $[0, 1]$. If the following conditions are satisfied:

- (1) $a_n + b_n + c_n = 1$;
 (2) $\sum_{n=0}^{\infty} \alpha_n < \infty$.

Then the sequence $\{R_F x_n\}$ converges strongly to a point $q \in F$, where R_F is the sunny generalized nonexpansive retraction of E onto F .

Proof. Since S and T are ϕ -nonexpansive mappings, from Lemma 2.5 that $F(S)$ and $F(T)$ are closed convex subset of C . It follows that F is closed convex subset of E . From the assumption that F is nonempty, we obtain that $q \in F$. Since $y_n = a_n x_n + b_n T x_n + c_n S x_n$, we have

$$\begin{aligned}
 \phi(y_n, q) &= \phi(a_n x_n + b_n T x_n + c_n S x_n, q) \\
 &= \|a_n x_n + b_n T x_n + c_n S x_n\|^2 \\
 &\quad - 2\langle a_n x_n + b_n T x_n + c_n S x_n, Jq \rangle + \|q\|^2 \\
 &\leq a_n \|x_n\|^2 + b_n \|T x_n\|^2 + c_n \|S x_n\|^2 - 2a_n \langle x_n, Jq \rangle \\
 &\quad - 2b_n \langle T x_n, Jq \rangle - 2c_n \langle S x_n, Jq \rangle + \|q\|^2 \\
 &= a_n \phi(x_n, q) + b_n \phi(T x_n, q) + c_n \phi(S x_n, q) \\
 &\leq a_n \phi(x_n, q) + b_n \phi(x_n, q) + c_n \phi(x_n, q) \\
 &= \phi(x_n, q).
 \end{aligned} \tag{3.2}$$

Let $u_n = \beta_n x_n + (1 - \beta_n) y_n$, we have

$$\begin{aligned}
 \phi(u_n, q) &= \phi(\beta_n x_n + (1 - \beta_n) y_n, q) \\
 &= \|\beta_n x_n + (1 - \beta_n) y_n\|^2 - 2\langle \beta_n x_n + (1 - \beta_n) y_n, Jq \rangle \\
 &\quad + \|q\|^2 \\
 &\leq \beta_n \|x_n\|^2 + (1 - \beta_n) \|y_n\|^2 - 2\beta_n \langle x_n, Jq \rangle \\
 &\quad - 2(1 - \beta_n) \langle y_n, Jq \rangle + \|q\|^2 \\
 &= \beta_n \phi(x_n, q) + (1 - \beta_n) \phi(y_n, q) \\
 &\leq \beta_n \phi(x_n, q) + (1 - \beta_n) \phi(x_n, q) \\
 &= \phi(x_n, q).
 \end{aligned} \tag{3.3}$$

From the definition of x_{n+1} , it follows that

$$\begin{aligned}
 \phi(x_{n+1}, q) &= \phi(\alpha_n x_1 + (1 - \alpha_n) u_n, q) \\
 &= \|\alpha_n x_1 + (1 - \alpha_n) u_n\|^2 - 2\langle \alpha_n x_1 + (1 - \alpha_n) u_n, Jq \rangle \\
 &\quad + \|q\|^2 \\
 &\leq \alpha_n \|x_1\|^2 + (1 - \alpha_n) \|u_n\|^2 - 2\alpha_n \langle x_1, Jq \rangle \\
 &\quad - 2(1 - \alpha_n) \langle u_n, Jq \rangle + \|q\|^2 \\
 &= \alpha_n \phi(x_1, q) + (1 - \alpha_n) \phi(u_n, q)
 \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\leq \alpha_n \phi(x_1, q) + (1 - \alpha_n) \phi(x_n, q) \\ &\leq \alpha_n \phi(x_1, q) + \phi(x_n, q). \end{aligned}$$

Since $\{\phi(x_1, q)\}$ is constant and hence bounded, and from $\sum_{n=1}^{\infty} \alpha_n < \infty$, it follows that $\sum_{n=1}^{\infty} \alpha_n \phi(x_1, q) < \infty$. Hence, by Lemma 2.11, the limit of $\{\phi(x_n, q)\}$ exists. This implies that $\{\phi(x_n, q)\}$ is bounded and so are $\{x_n\}$ and $\{y_n\}$. Letting $w_n = R_F x_n$, it follows from Lemma 2.8 that

$$\begin{aligned} \phi(x_n, w_n) &= \phi(x_n, R_F x_n) \\ &\leq \phi(x_n, q) - \phi(R_F x_n, q) \\ &\leq \phi(x_n, q). \end{aligned}$$

Since limit of $\{\phi(x_n, q)\}$ exists, it follows that $\{w_n\}$ is bounded and also $\phi(x_1, w_n)$ is bounded. By apply (3.4), we have

$$\phi(x_{n+1}, w_n) \leq \alpha_n \phi(x_1, w_n) + \phi(x_n, w_n). \quad (3.5)$$

From Lemma 2.8 and (3.5), we have

$$\begin{aligned} \phi(x_{n+1}, w_{n+1}) &= \phi(x_{n+1}, R_F x_{n+1}) \\ &\leq \phi(x_{n+1}, w_n) - \phi(R_F x_{n+1}, w_n) \\ &\leq \phi(x_{n+1}, w_n) \\ &\leq \alpha_n \phi(x_1, w_n) + \phi(x_n, w_n). \end{aligned}$$

Again, from $\sum_{n=0}^{\infty} \alpha_n < \infty$ and the boundedness of $\{\phi(x_1, w_n)\}$, we have $\sum_{n=0}^{\infty} \alpha_n \phi(x_1, w_n) < \infty$. From Lemma 2.11, it follows that $\lim_{n \rightarrow \infty} \{\phi(x_n, w_n)\}$ exists. For any $m \geq 1$, apply from (3.4) that

$$\phi(x_{n+m}, q) \leq \phi(x_n, q) + \sum_{j=0}^{m-1} \alpha_{n+j} \phi(x_1, q) \quad (3.6)$$

and for $w_n = R_F x_n$, we have

$$\phi(x_{n+m}, w_n) \leq \phi(x_n, w_n) + \sum_{j=0}^{m-1} \alpha_{n+j} \phi(x_1, w_n).$$

From $w_{n+m} = R_F x_{n+m}$ and Lemma 2.8, it follows that

$$\begin{aligned} \phi(x_{n+m}, w_{n+m}) + \phi(w_{n+m}, w_n) &\leq \phi(x_{n+m}, w_n) \\ &\leq \phi(x_n, w_n) + \sum_{j=0}^{m-1} \alpha_{n+j} \phi(x_1, w_n). \end{aligned}$$

As a result,

$$\phi(w_{n+m}, w_n) \leq \phi(x_n, w_n) - \phi(x_{n+m}, w_{n+m}) + \sum_{j=0}^{m-1} \alpha_{n+j} \phi(x_1, w_n). \quad (3.7)$$

Since $\{\phi(x_1, w_n)\}$ is bounded and $\sum_{n=0}^{\infty} \alpha_n < \infty$, we have $\sum_{j=0}^{m-1} \alpha_{n+j} \phi(x_1, w_n) < \infty$. It follows that $\lim_{n \rightarrow \infty} \{\phi(x_1, w_n)\} = 0$. Since $\{\phi(x_n, w_n)\}$ is a convergent sequence and we have

$$\lim_{n \rightarrow \infty} \{\phi(w_{n+m}, w_n)\} = 0.$$

Let $r = \sup\{\|w_n\| : n \in \mathbb{Z}^+\}$. Then, from Lemma 2.3, it follows that there exists a strictly increasing, continuous and convex function $h : [0, 2r] \rightarrow \mathbb{R}$ such that $h(0) = 0$ and

$$h\|w_n - w_{n+m}\| \leq \phi(w_{n+m}, w_n).$$

Since $\lim_{n \rightarrow \infty} \{\phi(x_n, w_n)\}$ exists, from the property of h , letting $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} \|w_n - w_{n+m}\| = 0.$$

This implies that $\{w_n\}$ is a Cauchy sequence. Since F is closed, there exists $q \in F$ such that $w_n \rightarrow q$. Therefore, the sequence $\{R_F x_n\}$ converges strongly to a point $q \in F$. The proof is complete. $\square$

Theorem 3.2. Let E be a uniformly smooth and uniformly convex Banach space and let C be a nonempty closed convex subset of E . Assume that S and T are closed and ϕ -nonexpansive mappings on C and $F = F(S) \cup F(T)$ is nonempty and bounded. For an initial point $x_1 \in C$, generate a sequence $\{x_n\}$ by

$$\begin{cases} y_n = a_n x_n + b_n T x_n + c_n S x_n, \\ x_{n+1} = \alpha_n x_1 + (1 - \alpha_n)(\beta_n x_n + (1 - \beta_n) y_n), \quad \forall n \geq 1, \end{cases} \quad (3.8)$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are real sequences in $[0, 1]$ satisfying the following conditions:

- (1) $a_n + b_n + c_n = 1$;
- (2) $\liminf_{n \rightarrow \infty} a_n b_n > 0$;
- (3) $\liminf_{n \rightarrow \infty} a_n c_n > 0$;
- (4) $\liminf_{n \rightarrow \infty} (1 - \beta_n) > 0$;
- (5) $\sum_{n=0}^{\infty} \alpha_n < \infty$.

If J is weakly sequentially continuous, then the sequence $\{x_n\}$ converges weakly to a point $q \in F$, where $q = \lim_{n \rightarrow \infty} R_F x_n$ is the sunny generalized nonexpansive retraction from E onto F .

Proof. Let $y_n = a_n x_n + b_n T x_n + c_n S x_n$ for all $n \geq 1$. Since S and T are closed and ϕ -nonexpansive mappings on C , it follows from (3.3) and from Lemma 2.2 that, for $r = \sup_{n \geq 1} \{\|x_n\|, \|T x_n\|, \|S x_n\|\}$, there exists a strictly increasing, continuous and convex function $g : [0, \infty) \rightarrow [0, \infty)$ such that $g(0) = 0$. For any $q \in F$, we obtain that

$$\begin{aligned} \phi(y_n, q) &= \phi(a_n x_n + b_n T x_n + c_n S x_n, q) \\ &= \|a_n x_n + b_n T x_n + c_n S x_n\|^2 \\ &\quad - 2\langle a_n x_n + b_n T x_n + c_n S x_n, Jq \rangle + \|q\|^2 \\ &\leq a_n \|x_n\|^2 + b_n \|T x_n\|^2 + c_n \|S x_n\|^2 - 2a_n \langle x_n, Jq \rangle \\ &\quad - 2b_n \langle T x_n, Jq \rangle \\ &\quad - 2c_n \langle S x_n, Jq \rangle + \|q\|^2 - a_n b_n g\|x_n - T x_n\| \end{aligned} \quad (3.9)$$

$$\begin{aligned}
&= a_n\phi(x_n, q) + b_n\phi(Tx_n, q) + c_n\phi(Sx_n, q) \\
&\quad - a_nb_ng\|x_n - Tx_n\| \\
&\leq a_n\phi(x_n, q) + b_n\phi(x_n, q) + c_n\phi(x_n, q) \\
&\quad - a_nb_ng\|x_n - Tx_n\| \\
&= \phi(x_n, q) - a_nb_ng\|x_n - Tx_n\|.
\end{aligned}$$

Let $u_n = \beta_n x_n + (1 - \beta_n)y_n$, we have

$$\begin{aligned}
\phi(u_n, q) &= \phi(\beta_n x_n + (1 - \beta_n)y_n, q) \\
&\leq \beta_n\|x_n\|^2 + (1 - \beta_n)\|y_n\|^2 - 2\beta_n\langle x_n, Jq \rangle \\
&\quad - 2\beta_n\langle y_n, Jq \rangle + \|q\|^2 \\
&= \beta_n\phi(x_n, q) + (1 - \beta_n)\phi(y_n, q) \\
&\leq \beta_n\phi(x_n, q) + (1 - \beta_n)[\phi(x_n, q) \\
&\quad - a_nb_ng\|x_n - Tx_n\|] \\
&\leq \beta_n\phi(x_n, q) + (1 - \beta_n)\phi(x_n, q) \\
&\quad - (1 - \beta_n)a_nb_ng\|x_n - Tx_n\| \\
&= \phi(x_n, q) - (1 - \beta_n)a_nb_ng\|x_n - Tx_n\|.
\end{aligned} \tag{3.10}$$

Let $x_{n+1} = \alpha_n x_1 + (1 - \alpha_n)u_n$, it follows that

$$\begin{aligned}
\phi(x_{n+1}, q) &= \phi(\alpha_n x_1 + (1 - \alpha_n)u_n, q) \\
&= \|\alpha_n x_1 + (1 - \alpha_n)u_n\|^2 - 2\langle \alpha_n x_1 + (1 - \alpha_n)u_n, Jq \rangle \\
&\quad + \|q\|^2 \\
&\leq \alpha_n\|x_1\|^2 + (1 - \alpha_n)\|u_n\|^2 - 2\alpha_n\langle x_1, Jq \rangle \\
&\quad - 2(1 - \alpha_n)\langle u_n, Jq \rangle + \|q\|^2 \\
&= \alpha_n\phi(x_1, q) + (1 - \alpha_n)\phi(u_n, q) \\
&\leq \alpha_n\phi(x_1, q) + \phi(u_n, q) \\
&\leq \alpha_n\phi(x_1, q) + \phi(x_n, q) - (1 - \beta_n)a_nb_ng\|x_n - Tx_n\|.
\end{aligned} \tag{3.11}$$

Consequently,

$$(1 - \beta_n)a_nb_ng\|x_n - Tx_n\| \leq \alpha_n\phi(x_1, q) + \phi(x_n, q) - \phi(x_{n+1}, q). \tag{3.12}$$

From $\lim_{n \rightarrow \infty} \phi(x_n, q)$ exists and $\lim_{n \rightarrow \infty} \alpha_n = 0$, letting $n \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} [\alpha_n\phi(x_1, q) + \phi(x_n, q) - \phi(x_{n+1}, q)] = 0. \tag{3.13}$$

From (3.18), it follows that

$$\lim_{n \rightarrow \infty} (1 - \beta_n)a_nb_ng(\|x_n - Tx_n\|) = 0. \tag{3.14}$$

Since $\liminf_{n \rightarrow \infty} a_n b_n > 0$ $\liminf_{n \rightarrow \infty} (1 - \beta_n) > 0$, we have

$$\lim_{n \rightarrow \infty} g(\|x_n - Tx_n\|) = 0. \quad (3.15)$$

From the property of g that

$$\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0. \quad (3.16)$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that x_{n_k} converges weakly to $\hat{x} \in C$ where $\hat{x} = J^{-1}q^*$ for some $q^* \in J(C)$. Since T is closed and $Tx_n \rightarrow x_n$ as $n \rightarrow \infty$, it follows that $\hat{x} \in F(T)$.

Again since $y_n = a_n x_n + b_n Tx_n + c_n Sx_n$, we have

$$\begin{aligned} \phi(y_n, q) &= \phi(a_n x_n + b_n Tx_n + c_n Sx_n, q) \\ &= \|a_n x_n + b_n Tx_n + c_n Sx_n\|^2 - 2\langle a_n x_n + b_n Tx_n + c_n Sx_n, Jq \rangle + \|q\|^2 \\ &\leq a_n \|x_n\|^2 + b_n \|Tx_n\|^2 + c_n \|Sx_n\|^2 - 2a_n \langle x_n, Jq \rangle - 2b_n \langle Tx_n, Jq \rangle \\ &\quad - 2c_n \langle Sx_n, Jq \rangle + \|q\|^2 - a_n c_n g\|x_n - Sx_n\| \\ &= a_n \phi(x_n, q) + b_n \phi(Tx_n, q) + c_n \phi(Sx_n, q) - a_n c_n g\|x_n - Sx_n\| \\ &\leq a_n \phi(x_n, q) + b_n \phi(x_n, q) + c_n \phi(x_n, q) - a_n c_n g\|x_n - Sx_n\| \\ &= \phi(x_n, q) - a_n c_n g\|x_n - Sx_n\|. \end{aligned}$$

It follows that

$$\begin{aligned} \phi(u_n, q) &= \phi(\beta_n x_n + (1 - \beta_n)y_n, q) \\ &\leq \beta_n \|x_n\|^2 + (1 - \beta_n) \|y_n\|^2 - 2\beta_n \langle x_n, Jq \rangle - 2\beta_n \langle y_n, Jq \rangle + \|q\|^2 \\ &= \beta_n \phi(x_n, q) + (1 - \beta_n) \phi(y_n, q) \\ &\leq \beta_n \phi(x_n, q) + (1 - \beta_n) [\phi(x_n, q) - a_n b_n g\|x_n - Sx_n\|] \\ &\leq \beta_n \phi(x_n, q) + (1 - \beta_n) \phi(x_n, q) - (1 - \beta_n) a_n c_n g\|x_n - Sx_n\| \\ &= \phi(x_n, q) - (1 - \beta_n) a_n c_n g\|x_n - Sx_n\|. \end{aligned}$$

Let $x_{n+1} = \alpha_n x_1 + (1 - \alpha_n)u_n$, we have

$$\begin{aligned} \phi(x_{n+1}, q) &= \phi(\alpha_n x_1 + (1 - \alpha_n)u_n, q) \\ &\leq \alpha_n \phi(x_1, q) + (1 - \alpha_n) \phi(u_n, q) \\ &\leq \alpha_n \phi(x_1, q) + \phi(u_n, q) \\ &\leq \alpha_n \phi(x_1, q) + \phi(x_n, q) - (1 - \beta_n) a_n c_n g\|x_n - Sx_n\|. \end{aligned} \quad (3.17)$$

The result is

$$(1 - \beta_n) a_n c_n g\|x_n - Sx_n\| \leq \alpha_n \phi(x_1, q) + \phi(x_n, q) - \phi(x_{n+1}, q). \quad (3.18)$$

From $\lim_{n \rightarrow \infty} \phi(x_n, q)$ exists and $\lim_{n \rightarrow \infty} \alpha_n = 0$, letting $n \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} [\alpha_n \phi(x_1, q) + \phi(x_n, q) - \phi(x_{n+1}, q)] = 0. \quad (3.19)$$

From (3.18), it follows that

$$\lim_{n \rightarrow \infty} (1 - \beta_n) a_n c_n g(\|x_n - Sx_n\|) = 0. \quad (3.20)$$

Since $\liminf_{n \rightarrow \infty} a_n c_n > 0$ $\liminf_{n \rightarrow \infty} (1 - \beta_n) > 0$, we have

$$\lim_{n \rightarrow \infty} g(\|x_n - Sx_n\|) = 0. \quad (3.21)$$

From the property of g that

$$\lim_{n \rightarrow \infty} \|x_n - Sx_n\| = 0. \quad (3.22)$$

Since $\{x_n\}$ is bounded, it has a subsequence $\{x_{n_k}\}$ that converges weakly to $\hat{x} \in C$, where $\hat{x} = J^{-1}q^*$ for some $q^* \in J(C)$. Furthermore, since S is closed and $Sx_n \rightarrow x_n$ as $n \rightarrow \infty$, we conclude that $\hat{x} \in F(S)$.

Let $w_n = R_F x_n$ for all $n \geq 1$, from Lemma 2.8 that

$$\langle x_n - w_n, q^* - Jw_n \rangle \leq 0.$$

It follows from Theorem 3.1 that $w_n \rightarrow q$, we obtain

$$\langle x_{n_k} - w_{n_k}, q^* - Jw_{n_k} \rangle \leq 0.$$

Since J is weakly sequentially continuous, it follows that

$$\langle J^{-1}(q^*) - J^{-1}(Jq), Jq - q^* \rangle \leq 0. \quad (3.23)$$

From the monotonicity of J^{-1} , we have

$$\langle J^{-1}(q^*) - J^{-1}(Jq), Jq - q^* \rangle \geq 0. \quad (3.24)$$

Hence, by (3.24) and (3.23), it follows that

$$\langle J^{-1}(q^*) - J^{-1}(Jq), J(q) - J(J^{-1}(q^*)) \rangle = 0.$$

Hence, $J^{-1}(q^*) = q$. Consequently, the sequence $\{x_n\}$ converges weakly to the point $q \in F$. The proof is complete. $\square$

An example of computing x_{n+1} by the proposed algorithm for two ϕ -nonexpansive mappings under the assumptions of the theorem is given as follows.

Example 3.1. Let $E = \mathbb{R}$, $C = [-100, 100]$. Define a mapping $T : C \rightarrow C$ by $Sx = \frac{x}{3}$ and $Tx = \frac{x}{2}$ for all $x \in C$. Then we have

$$\begin{aligned} \phi(Sx, Sy) &= \phi\left(\frac{x}{3}, \frac{y}{3}\right) \\ &= \left|\frac{x}{3}\right|^2 - 2\left\langle \frac{x}{3}, J\frac{y}{3} \right\rangle + \left|\frac{y}{3}\right|^2 \\ &= \left|\frac{x}{3}\right|^2 - 2\left\langle \frac{x}{3}, \frac{y}{3} \right\rangle + \left|\frac{y}{3}\right|^2 \\ &= \left|\frac{x}{3}\right|^2 - 2\left(\frac{x}{3} \frac{y}{3}\right) + \left|\frac{y}{3}\right|^2 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{9}(|x|^2 - 2xy + |y|^2) \\
&\leq |x|^2 - 2(xy) + |y|^2 \\
&= |x|^2 - 2\langle x, y \rangle + |y|^2 \\
&= \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \\
&= \phi(x, y).
\end{aligned}$$

This shows that S is ϕ -nonexpansive mapping. Similarly, we can show that T is ϕ -nonexpansive mapping, that is,

$$\phi(Tx, Ty) = \phi\left(\frac{x}{2}, \frac{y}{2}\right) \leq \phi(x, y).$$

Since $F(S) = \{0\}$ and $F(T) = \{0\}$ so, $F(S) \cap F(T) = \{0\} \neq \emptyset$. Therefore, all the assumptions in Theorems 3.1 and 3.2 are satisfied. We obtain the following algorithms:

Algorithm. Let $\alpha_n = \frac{1}{2n^2}$, $\beta_n = \frac{1}{n^2}$, $a_n = \frac{1}{3} + \frac{1}{n}$, $b_n = \frac{1}{3}$ and $c_n = \frac{1}{3} - \frac{1}{n}$. For the initial values $x_1 = 1$, $x_1 = 0.1$, and $x_1 = -0.5$, the sequence $\{x_n\}$ generated by the proposed algorithm is as follows:

$$x_{n+1} = \frac{1}{2n^2}x_1 + \frac{22n^5 + 24n^4 + 3n^3 - 36n^2 - 7n + 12}{36n^5}x_n, \quad \forall n \geq 1.$$

Corollary 3.1. Let E be a uniformly smooth and uniformly convex Banach space and let C be a nonempty closed convex subset of E . Assume that T is a closed and ϕ -nonexpansive mapping on C and $F(T)$ is nonempty and bounded. For an initial point $x_1 \in C$, generate a sequence $\{x_n\}$ by

$$\begin{cases} y_n = a_n x_n + (1 - a_n)Tx_n, \\ x_{n+1} = \alpha_n x_1 + (1 - \alpha_n)(\beta_n x_n + (1 - \beta_n)y_n), \end{cases} \quad \forall n \geq 1, \quad (3.25)$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{a_n\}$ are real sequences in $[0, 1]$. If $\sum_{n=0}^{\infty} \alpha_n < \infty$, then the sequence $\{R_F x_n\}$ converges strongly to a point $q \in F$, where R_F is the sunny generalized nonexpansive retraction of E onto $F(T)$.

Corollary 3.2. Let E be a uniformly smooth and uniformly convex Banach space and let C be a nonempty closed convex subset of E . Assume that T is a closed and ϕ -nonexpansive mapping on C and $F(T)$ is nonempty and bounded. For an initial point $x_1 \in C$, generate a sequence $\{x_n\}$ by

$$\begin{cases} y_n = a_n x_n + (1 - a_n)Tx_n, \\ x_{n+1} = \alpha_n x_1 + (1 - \alpha_n)(\beta_n x_n + (1 - \beta_n)y_n), \end{cases} \quad \forall n \geq 1, \quad (3.26)$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{a_n\}$ are real sequences in $[0, 1]$ satisfying the following conditions:

- (1) $\liminf_{n \rightarrow \infty} (1 - a_n) > 0$;
- (2) $\liminf_{n \rightarrow \infty} (1 - \beta_n) > 0$;
- (3) $\sum_{n=0}^{\infty} \alpha_n < \infty$.

If J is weakly sequentially continuous, then the sequence $\{x_n\}$ converges weakly to the point $q \in F(T)$, where $q = \lim_{n \rightarrow \infty} R_F x_n$ is the sunny generalized nonexpansive retraction from E onto $F(T)$.

n	x_n		
	$x_1 = 1$	$x_1 = 0.1$	$x_1 = -0.5$
1	1.000000	0.100000	-0.500000
2	1.000000	0.100000	-0.500000
3	0.963542	0.096354	-0.481771
4	0.830728	0.083073	-0.415364
5	0.668242	0.066824	-0.334121
6	0.514174	0.051417	-0.257087
.	.	.	.
.	.	.	.
.	.	.	.
32	0.000047	0.000005	-0.000024
33	0.000027	0.000003	-0.000014
34	0.000014	0.000001	-0.000007
35	0.000006	0.000001	-0.000003
36	0.000001	0.000000	-0.000001
37	0.000000	0.000000	0.000000
38	0.000000	0.000000	0.000000

TABLE 1. The numerical results for initial x_1

Nonlinear parabolic partial differential equation

Consider the nonlinear parabolic partial differential equation presented in [19] as follows:

$$\frac{\partial z(t, y)}{\partial t} = \frac{\partial^2 z(t, y)}{\partial y^2} + f(t, z(t, y)), \quad 0 < y < \pi, t > 0, \quad (3.27)$$

$$z(t, 0) = z(t, \pi) = 0, z'_y(t, 0) = z'_y(t, \pi), z(0, y) = z_0(y), \quad 0 \leq y \leq \pi.$$

Problem (3.27) can be rewritten as a Cauchy problem by setting, $x(\cdot) = z(\cdot, y)$, $E = L_2(0, \pi)$.

For a given $b > 0$, let $J = [0, b]$ and define the set

$$C = \{x \in E : \|x(t)\| \leq b, \forall t \in J\}.$$

Then, C is a bounded, closed, and convex subset of E . Let us denote the operator $A = \frac{\partial^2}{\partial y^2}$ with

$$D(A) = \left\{ z \in E \text{ with } \frac{\partial^2 z}{\partial y^2} \in E \text{ and } z(0) = z(\pi) = 0 \right\},$$

see [19, 20] for more details. Problem (3.27) can be written as:

$$Dx(t) = Ax(t) + f(t, x(t)), t \in J, x(0) = x_0, \quad (3.28)$$

where $f : J \times E \rightarrow E$ is a nonlinear continuous mapping satisfying the Lipschitz condition with Lipschitz constant M . That is,

$$\|f(t, x_1(t)) - f(t, x_2(t))\| \leq M\|x_1(t) - x_2(t)\|,$$

for all $x_1, x_2 \in C$. The solution of (3.28) is

$$x(t) = S(t)x_0 + \int_0^t S(t-s)f(s, x(s))ds, \quad \text{for } t \in J,$$

where $S(t)$ is the semigroup generated by the infinitesimal operator A on E . We assume that $S(t)$ is a compact operator for $t > 0$, and its norm satisfies $\|S(t)\| \leq M$ for some constant $0 < M \leq 1$.

Let the norm on E be defined as $\|x\| = \sup\{|x(t)| < \infty : t \in [0, b]\}$. Define the operator $T : C \rightarrow E$ by:

$$(Tx)(t) = S(t)x_0 + \int_0^t S(t-s)f(s, x(s))ds \text{ on } t \in J, x \in C. \quad (3.29)$$

Based on the results in Narayanan and Marudai [19], setting $J = I$ implies that T is ϕ -nonexpansive mapping. The fixed point solution of (3.29) is also a solution to (3.28). Consequently, it is also a solution to the differential equation (3.27).

Let T be defined by (3.27) as in Corollary 3.1 and let $J = I$. Then, the sequence $\{x_n\}$ converge to $F(T)$, which is a solution to (3.27).

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

REFERENCES

- [1] A. Adamu, S. Reich, D. Viet Thong, H. Van Thang, N. Thi An, New Convergence Results for Algorithms for Solving Split Common Fixed Point Problems, *Numer. Algorithms* (2026). <https://doi.org/10.1007/s11075-026-02429-1>.
- [2] F. Ahmad, M. Samraiz, J. Ahmad, S. Etemad, M. De la Sen, AK-Iterative Scheme for Fixed Point Approximation in Hyperbolic Metric Spaces and Applications in Volterra Integral Equation, *Fixed Point Theory Algorithms Sci. Eng.* 2026 (2026), 7. <https://doi.org/10.1186/s13663-025-00822-4>.
- [3] Ya.I. Alber, Metric and Generalized Projection Operators in Banach Spaces: Properties and Applications, in: A.G. Kartsatos (Ed.), *Theory and Applications of Nonlinear Operators of Accretive and Monotone Type*, Marcel Dekker, 1996, pp. 15–50.
- [4] Ya.I. Alber, Generalized Projection Operators in Banach Spaces: Properties and Applications, *Funct. Differ. Equ.* 1 (1993), 1–21.
- [5] A. Aleyner, S. Reich, An Explicit Construction of Sunny Nonexpansive Retractions in Banach Spaces, *Fixed Point Theory Appl.* 2005 (2005), 675365. <https://doi.org/10.1155/fpta.2005.295>.
- [6] G. Cai, S. Bu, Convergence Analysis for Variational Inequality Problems and Fixed Point Problems in 2-Uniformly Smooth and Uniformly Convex Banach Spaces, *Math. Comput. Model.* 55 (2012), 538–546. <https://doi.org/10.1016/j.mcm.2011.08.031>.
- [7] J.-W. Chen, Y.J. Cho, Z. Wan, Shrinking Projection Algorithms for Equilibrium Problems with a Bifunction Defined on the Dual Space of a Banach Space, *Fixed Point Theory Appl.* 2011 (2011), 91. <https://doi.org/10.1186/1687-1812-2011-91>.

- [8] Y.J. Cho, H. Zhou, G. Guo, Weak and Strong Convergence Theorems for Three-Step Iterations with Errors for Asymptotically Nonexpansive Mappings, *Comput. Math. Appl.* 47 (2004), 707–717. [https://doi.org/10.1016/s0898-1221\(04\)90058-2](https://doi.org/10.1016/s0898-1221(04)90058-2).
- [9] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Springer Netherlands, 1990. <https://doi.org/10.1007/978-94-009-2121-4>.
- [10] K. Deimling, *Nonlinear Functional Analysis*, Springer Berlin Heidelberg, 1985. <https://doi.org/10.1007/978-3-662-00547-7>.
- [11] K. Gdawiec, B. Nawaz, M. Noorwali, M. Aphane, Fixed-Point Approximation for Generalized Nonexpansive Mappings Using a Novel Iteration Scheme and Visual Insights via Polynomiography, *J. Comput. Appl. Math.* 490 (2027), 117978. <https://doi.org/10.1016/j.cam.2026.117978>.
- [12] T. Honda, W. Takahashi, J.-C. Yao, Nonexpansive Retractions onto Closed Convex Cones in Banach Spaces, *Taiwan. J. Math.* 14 (2010), 1023–1046. <https://doi.org/10.11650/twjm/1500405903>.
- [13] T. Ibaraki, W. Takahashi, A New Projection and Convergence Theorems for the Projections in Banach Spaces, *J. Approx. Theory* 149 (2007), 1–14. <https://doi.org/10.1016/j.jat.2007.04.003>.
- [14] S. Kamimura, W. Takahashi, Strong Convergence of a Proximal-Type Algorithm in a Banach Space, *SIAM J. Optim.* 13 (2002), 938–945. <https://doi.org/10.1137/s105262340139611x>.
- [15] P. Kanjanasamranwong, K. Kongkul, C. Noisri, S. Saewan, Common Fixed Point Theorems for Asymptotically Quasi G - ϕ -Nonexpansive Mappings, *Int. J. Anal. Appl.* 24 (2026), 187. <https://doi.org/10.28924/2291-8639-24-2026-187>.
- [16] W. Khuangsatsung, A. Kangtunyakarn, A Method for Solving the Variational Inequality Problem and Fixed Point Problems in Banach Spaces, *Tamkang J. Math.* 53 (2022), 23–36. <https://doi.org/10.5556/j.tkm.53.2022.3307>.
- [17] F. Kohsaka, W. Takahashi, Generalized Nonexpansive Retractions and a Proximal-Type Algorithm in Banach Spaces, *J. Nonlinear Convex Anal.* 8 (2007), 197–209.
- [18] S.-Y. Matsushita, W. Takahashi, A Strong Convergence Theorem for Relatively Nonexpansive Mappings in a Banach Space, *J. Approx. Theory* 134 (2005), 257–266. <https://doi.org/10.1016/j.jat.2005.02.007>.
- [19] M.S. Narayanan, M. Marudai, Fixed Points of Nonexpansive and Quasi-Nonexpansive Mappings, *J. Anal.* 27 (2019), 75–87. <https://doi.org/10.1007/s41478-018-0104-7>.
- [20] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer New York, 1983. <https://doi.org/10.1007/978-1-4612-5561-1>.
- [21] V. Pragadeeswarar, V. Vishnu KS, M. De la Sen, Common Fixed Point Theorems in Orthogonal Sets, *Int. J. Anal. Appl.* 23 (2025), 196. <https://doi.org/10.28924/2291-8639-23-2025-196>.
- [22] X. Qin, Y.J. Cho, S.M. Kang, Convergence Theorems of Common Elements for Equilibrium Problems and Fixed Point Problems in Banach Spaces, *J. Comput. Appl. Math.* 225 (2009), 20–30. <https://doi.org/10.1016/j.cam.2008.06.011>.
- [23] G.S. Saluja, On Weak and Strong Convergence Theorems of Modified SP -Iteration Scheme for Total Asymptotically Nonexpansive Mappings, *Int. J. Anal. Appl.* 10 (2016), 24–39. <https://etamaths.com/index.php/ijaa/article/view/625>.
- [24] E. Soori, R.P. Agarwal, D. O'Regan, Existence of Sunny Nonexpansive Retractions and Approximation of Fixed Points of a Representation of Nonexpansive Mappings, *Nonlinear Anal. Model. Control* 29 (2024), 878–889. <https://doi.org/10.15388/namc.2024.29.35559>.
- [25] P. Sunthrayuth, R. Promkam, K. Muangchoo, W. Nithiarayaphaks, Forward–Backward Splitting Methods with Two Inertial Parameters for Solving Inclusion Problems in Banach Spaces, *Comput. Appl. Math.* 45 (2026), 213. <https://doi.org/10.1007/s40314-025-03613-0>.
- [26] W. Takahashi, *Nonlinear Functional Analysis: Fixed Point Theory and Its Applications*, Yokohama Publishers, 2000.
- [27] W. Takahashi, T. Honda, M. Hojo, Strong Convergence Theorems for Commutative Linear Contractive Mappings in Banach Spaces Based on Nonlinear Analytic Methods, *Linear Nonlinear Anal.* 7 (2021), 157–171.

- [28] K.K. Tan, H.K. Xu, Approximating Fixed Points of Nonexpansive Mappings by the Ishikawa Iteration Process, *J. Math. Anal. Appl.* 178 (1993), 301–308. <https://doi.org/10.1006/jmaa.1993.1309>.
- [29] W. Takahashi, K. Zembayashi, Strong and Weak Convergence Theorems for Equilibrium Problems and Relatively Nonexpansive Mappings in Banach Spaces, *Nonlinear Anal. Theory Methods Appl.* 70 (2009), 45–57. <https://doi.org/10.1016/j.na.2007.11.031>.
- [30] Y. Wang, C. Pan, Viscosity Approximation Methods for a General Variational Inequality System and Fixed Point Problems in Banach Spaces, *Symmetry* 12 (2020), 36. <https://doi.org/10.3390/sym12010036>.
- [31] H.-K. Xu, Inequalities in Banach Spaces with Applications, *Nonlinear Anal. Theory Methods Appl.* 16 (1991), 1127–1138. [https://doi.org/10.1016/0362-546x\(91\)90200-k](https://doi.org/10.1016/0362-546x(91)90200-k).
- [32] D. Yambangwai, C. Chairatsiripong, T. Thianwan, Iterative Manner Involving Sunny Nonexpansive Retractions for Nonlinear Operators from the Perspective of Convex Programming as Applicable to Differential Problems, Image Restoration and Signal Recovery, *AIMS Math.* 8 (2023), 7163–7195. <https://doi.org/10.3934/math.2023361>.
- [33] H. Zhou, G. Gao, B. Tan, Convergence Theorems of a Modified Hybrid Algorithm for a Family of Quasi- ϕ -Asymptotically Nonexpansive Mappings, *J. Appl. Math. Comput.* 32 (2010), 453–464. <https://doi.org/10.1007/s12190-009-0263-4>.