

Power Dominator Equitable Coloring of Graphs: Bounds and Exact Values for Special Graph Classes

G. Navamani¹, L. Jacqueline^{1,*}, N. Revathi², Lambodharan Velappan³, V. Kolappan⁴

¹*Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Saveetha University, Thandalam, Chennai - 602 105, Tamil Nadu, India*

²*Department of Computer Applications, Faculty of Liberal Arts and Business Studies, SRM Institute of Science and Technology, Ramapuram Campus, Chennai - 600 089, Tamil Nadu, India*

³*Department of Mathematics, Er. Perumal Manimekalai College of Engineering, Hosur - 635 117, Tamil Nadu, India*

⁴*Department of Mathematics, R.M.K. Engineering College, Kavaraipettai - 601 206, Gummidipoondi, Thiruvallur, Tamil Nadu, India*

*Corresponding author: jackinfanci@gmail.com

Abstract. This paper introduces the concept of the power dominator equitable coloring, which combines the monitoring capability of power dominator coloring with the balanced partition requirement of equitable coloring. A proper coloring is called a power dominator equitable coloring if every vertex power dominates at least one color class and the sizes of any two color classes differ by at most one. The minimum number of colors required for such a coloring is called the power dominator equitable chromatic number and is denoted by $\chi_{pde}(G)$. We establish several fundamental bounds for this new parameter and characterize graphs attaining equality. Exact values are obtained for complete multipartite graphs, disconnected graphs, forests and circulant graphs $C_n(1, 2)$. In addition, relationships between the power dominator equitable chromatic number and the equitable chromatic number are investigated, together with sufficient conditions under which they coincide. These results provide a theoretical foundation for studying graph coloring problems involving both monitoring capability and balanced resource allocation.

1. INTRODUCTION

Several fundamental concepts have been developed from the study of interactions among entities interconnected through a network. One of the most basic conceptions is that of domination, wherein the objective is to find the least number of nodes such that all the other nodes in the

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network can be “reached” by these nodes. This concept, initially posed by Haynes [9], describes a dominating set as a subset of vertices such that every vertex not in the subset is adjacent to at least one vertex in the subset. The minimum number of vertices in a dominating set is called the domination number, denoted by $\gamma(G)$.

Building on this, Haynes et al. [8] introduced the concept of power domination. This model is especially applicable when modeling electric power systems, looking for a minimum number of monitoring devices required, known as PMUs, or Phasor Measurement Units, needed to observe the whole system. In this setting, the graph is used to model an electrical power system, where each vertex corresponds to a network node and every edge represents a transmission connection between them.

- (1) **Initialization:** Begin by forming the set $M(Z)$, which includes the vertices in Z together with all vertices directly adjacent to them.
- (2) **Propagation:** Continue the process as follows; if there exists a vertex $z \in M(Z)$ whose neighbors are already contained in $M(Z)$ except for one vertex Y , then this remaining neighbor is added to $M(Z)$.

A set Z is a power dominating set if $M(Z)$ encompasses all vertices of the graph, and the minimum set of this kind defines the power domination number, $\gamma_p(G)$. Extensive studies have been carried out in this field, as referenced in [4,5,8,12].

A vertex p is said to power dominate the vertices in a set $M(p)$ when the following criteria are fulfilled:

- (1) The set $M(p)$ is initialized by including p and all its neighbors, that is,

$$N(p) \cup \{p\} \subseteq M(p).$$

- (2) Add the vertex $c \in V(G)$ to $M(p)$ whenever $\exists b \in M(p)$ such that b has exactly one neighbor outside $M(p)$, namely c . Equivalently, every neighbor of b except c already belongs to $M(p)$.
- (3) The previous step is repeated until no further vertices can be included.

Further enriching graph analysis, vertex coloring assigns colors to vertices such that adjacent vertices always have different colors. A sophisticated variant, dominator coloring, requires that every vertex in the graph “dominates” all vertices belonging to at least one color class. The smallest number of colors needed for such coloring is the dominator chromatic number $\chi_d(G)$ [2,6,18]. Taking this a step further, power dominator coloring introduces a dynamic aspect: each vertex must power dominate at least one color class within a proper coloring. The minimum number of colors required for this is the power dominator chromatic number, $\chi_{pd}(G)$ [17,19]. A related notion is equitable coloring, where the sizes of any two color classes differ by at most one, ensuring a balanced distribution of vertices among colors. The minimum number of colors for an equitable coloring is the equitable chromatic number, $\chi_e(G)$ [3,15].

This paper introduces the concept of power dominator equitable coloring, which combines the principles of power dominator coloring with equitable distribution. In this coloring scheme, each

vertex power dominates at least one color class, and the color classes themselves are equitable in size. The smallest number of colors required for a power dominator equitable coloring of a graph G is called the power dominator equitable chromatic number of G , and is denoted by $\chi_{pde}(G)$. This work focuses on establishing bounds for this newly defined parameter, $\chi_{pde}(G)$.

The significance of Power Dominator Equitable Coloring lies in combining the monitoring capability of power domination with the balancing requirement of equitable coloring. The equitable condition ensures that the sizes of the color classes differ by at most one, leading to a balanced distribution of vertices among the classes. Such a balance is desirable in many practical situations where monitoring responsibilities, service regions, or network resources must be distributed uniformly. Therefore, the proposed parameter provides a useful model for studying monitoring and allocation problems in networks while maintaining the structural properties of both power domination and equitable coloring.

Example illustration:

In Fig. 1, consider the vertex v_1 . Initially, the monitoring set is the closed neighborhood of v_1 , that is, $M(v_1) = N[v_1] = \{v_1, v_2\}$. To continue the monitoring process, we examine the unmonitored vertex v_3 . The only neighbor of v_3 that belongs to $M(v_1)$ is v_2 . However, v_2 has another neighbor that is not yet monitored. Therefore, v_2 does not satisfy the propagation condition required to force v_3 into the monitored set. Consequently, no further vertices can be monitored, and the monitoring process terminates. Hence,

$$M(v_1) = \{v_1, v_2\}.$$

Therefore, the vertex v_1 power dominates only the vertices v_1 and v_2 . Likewise,

$$M(v_2) = \{v_1, v_2, v_3, v_4, v_5, v_6\},$$

$$M(v_3) = \{v_2, v_3, v_5\},$$

$$M(v_4) = \{v_2, v_4, v_5\},$$

$$M(v_5) = \{v_1, v_2, v_3, v_4, v_5, v_6\},$$

and

$$M(v_6) = \{v_5, v_6\}.$$

Each monitoring set should contain all the vertices of at least one color class, so that the color classes are as follows:

$$C^1 = \{v_2\}, \quad C^2 = \{v_5\}, \quad C^3 = \{v_1, v_6\}, \quad C^4 = \{v_3, v_4\}.$$

Vertex v_1 power dominates the color class C^1 , v_2 power dominates all the color classes, v_3 power dominates the color classes C^1 and C^2 , v_4 power dominates the color classes C^1 and C^2 , v_5 power dominates all the color classes, and v_6 power dominates the color class C^2 .

Therefore, each vertex power dominates at least one color class, and the sizes of all color classes differ by at most one. Hence,

$$\chi_{pde}(G) = 4.$$

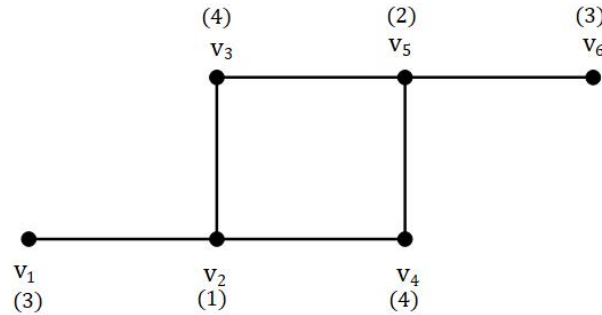


FIGURE 1. Power dominator equitable coloring of G within braces and $\chi_{pde}(G) = 4$

2. PRELIMINARIES

A graph G is basically determined by two non-empty sets, where L describes its vertices and M its edges. A graph is connected when a path exists between any two vertices s and t . For any vertex t , its open neighborhood $N(t)$ contains all the vertices directly incident with t , whereas the closed neighborhood $N[t]$ adds t itself to the set, i.e., $N[t] = N(t) \cup \{t\}$. Among specific classes of graphs, W_n denotes a wheel graph with $n + 1$ vertices. A caterpillar is a special form of tree such that, when all degree-one vertices and their incident edges are removed from it, the remaining graph forms a simple path [7]. A color class C^i is the set of vertices receiving the same color i [14]. An acyclic graph is a graph that contains no cycles. A forest is defined as an acyclic graph. The connected components of a forest are trees [1].

3. SOME BOUNDS

Theorem 3.1. For any graph G with n vertices, the power dominator equitable chromatic number satisfies $2 \leq \chi_{pde}(G) \leq n$.

Proof. The lower bound is attained by several graph families admitting a power dominator equitable 2-coloring, while the upper bound follows by assigning a distinct color to each vertex. Hence both bounds are sharp. $\square$

Theorem 3.2. Consider a graph G with $|V(G)| = n$. If the maximum degree of G is $\Delta(G) = n - 1$, then $\left\lceil \frac{n+1}{2} \right\rceil \leq \chi_{pde}(G) \leq n$.

Proof. By Theorem 3.1, it suffices to prove the lower bound. Consider a graph G of n vertices with $\Delta(G) = n - 1$, and let $w \in V(G)$ be a node of degree $n - 1$. Clearly, w is incident with every other vertex in G , and each vertex of G power dominates the color class $C^1 = \{w\}$. In equitable coloring,

the remaining color classes are either singleton sets or 2-element sets, since $|C^1| = 1$. We get the lower bound in the case when, under the corresponding power dominator equitable coloring c , all color classes C^i , $i \geq 2$, possibly except for one, are 2-element sets. Therefore, the remaining $n - 1$ vertices are partitioned into $\left\lceil \frac{n-1}{2} \right\rceil$ color classes. Consequently, the total number of color classes is

$$1 + \left\lceil \frac{n-1}{2} \right\rceil = \left\lceil \frac{n+1}{2} \right\rceil.$$

Hence,

$$\chi_{pde}(G) \geq \left\lceil \frac{n+1}{2} \right\rceil.$$

□

Both bounds are tight in Theorem 3.2. We get the upper bound for complete graphs K_n and the lower bound is exact for complete bipartite graphs $K_{1,n-1}$.

A classical result by Hajnal and Szemerédi [11] states that any graph G whose largest vertex degree is $\Delta(G)$ can be colored so that the color classes differ in size by at most one, using no more than $\Delta(G) + 1$ colors; in other words,

$$\chi_e(G) \leq \Delta(G) + 1.$$

Motivated by this result, Meyer [15] introduced the associated conjecture presented below.

Conjecture 3.1. [15] *For any graph G which is not an odd cycle and not a complete graph, and whose largest vertex degree is $\Delta(G)$, Meyer's claim states that $\chi_e(G) \leq \Delta(G)$.*

Theorem 3.3. *For every tree T with n vertices, $2 \leq \chi_{pde}(T) \leq \omega + \left\lceil \frac{n-\omega}{2} \right\rceil$, where $\omega = |\{N(q) \mid q \text{ is a pendant vertex and } \deg(N(q)) > 2\}|$.*

Proof. To justify the lower bound, take the tree T formed by a single edge, which is the graph P_2 . According to Corollary 3.1, it follows immediately that $\chi_{pde}(P_2) = 2$. Therefore,

$$2 \leq \chi_{pde}(T).$$

For the upper bound, let q be an arbitrary pendant vertex of T and let $u = N(q)$. If $\deg(u) > 2$, then q power dominates its closed neighborhood $N[q] = \{q, u\}$. Since u has at least two neighbors other than q , the propagation rule cannot be applied at u . Hence, the monitoring process initiated at q stops at $N[q]$.

Let $S = \{N(q) \mid q \text{ is a pendant vertex and } \deg(N(q)) > 2\}$, and let $|S| = \omega$. Assign a unique color to each vertex in S so that the adjacent pendant vertex q power dominates the color class containing $N(q)$. Thus, each such color class satisfies the power dominator equitable coloring requirement.

The remaining $(n - \omega)$ vertices of T receive $\left\lceil \frac{n-\omega}{2} \right\rceil$ colors equitably, so that $||C^i| - |C^j|| \leq 1, \forall i, j$.

Then

$$\chi_{pde}(T) = \omega + \left\lceil \frac{n-\omega}{2} \right\rceil.$$

Hence,

$$\chi_{pde}(T) \leq \omega + \left\lceil \frac{n-\omega}{2} \right\rceil.$$

Combining the lower and upper bounds,

$$2 \leq \chi_{pde}(T) \leq \omega + \left\lceil \frac{n - \omega}{2} \right\rceil.$$

The upper bound is sharp for a caterpillar. Therefore, the bound is sharp. $\square$

Theorem 3.4. *If every vertex of a graph G can power dominate the entire vertex set $V(G)$, then $\chi_{pde}(G) = \chi_e(G)$.*

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of G . By assumption, each v_i can initiate a power domination process that reaches all vertices of the graph. Consequently, every color class $C^1, C^2, \dots, C^k$ is also power dominated by every vertex of G . Since the coloring already satisfies the equitable condition

$$||C^i| - |C^j|| \leq 1, \forall i, j,$$

the additional requirement for power domination imposes no further restriction. Thus, a power dominator equitable coloring coincides with an equitable coloring, establishing that

$$\chi_{pde}(G) = \chi_e(G).$$

$\square$

Using the graph classes characterized in Reference [16] and Theorem 3.4, we obtain the following corollaries. Since each vertex of P_n, C_n, K_n , and W_n power dominates all vertices of P_n, C_n, K_n , and W_n , the conclusions follow immediately.

Corollary 3.1. *For a path P_n with $n \geq 2$, $\chi_{pde}(P_n) = 2$.*

Corollary 3.2. *For a cycle C_n with $n \geq 3$,*

$$\chi_{pde}(C_n) = \begin{cases} 2, & \text{if } n \text{ is even,} \\ 3, & \text{if } n \text{ is odd.} \end{cases}$$

Corollary 3.3. *For a complete graph K_n with $n \geq 1$, $\chi_{pde}(K_n) = n$.*

Corollary 3.4. *For a wheel W_n on $n + 1$ vertices with $n \geq 4$,*

$$\chi_{pde}(W_n) = \left\lceil \frac{n}{2} \right\rceil + 1.$$

Observation 3.1. *The converse of Theorem 3.4 is false. That is, $\chi_{pde}(G) = \chi_e(G)$ does not imply that each vertex of G power dominates all the vertices of the graph G .*

Proof. As a counterexample, consider the star graph $K_{1,n}$ with central vertex u and pendant vertices $v_1, v_2, \dots, v_n$. The vertex u power dominates all the vertices of $K_{1,n}$, but each pendant vertex power dominates only its closed neighborhood. Thus, not every vertex of $K_{1,n}$ power dominates all the vertices of the graph.

Assign color 1 to the central vertex and assign $\left\lceil \frac{n}{2} \right\rceil$ additional color classes to the pendant vertices equitably. Hence,

$$\chi_{pde}(K_{1,n}) = 1 + \left\lceil \frac{n}{2} \right\rceil.$$

For the star graph $K_{1,n}$, it is known that

$$\chi_e(K_{1,n}) = 1 + \left\lceil \frac{n}{2} \right\rceil.$$

Therefore,

$$\chi_{pde}(K_{1,n}) = \chi_e(K_{1,n}),$$

even though not every vertex of $K_{1,n}$ power dominates all the vertices of the graph. Hence, the converse of Theorem 3.4 does not hold. $\square$

4. COMPLETE MULTIPARTITE GRAPHS

Consider $K_{l_1, l_2, \dots, l_t}$ as a complete multipartite graph with t independent sets of sizes $l_1, l_2, \dots, l_t$. lam, Shiu, Tong, and Zhang [13] proved an explicit formula for the equitable chromatic number of a complete t -partite graph $K_{l_1, l_2, \dots, l_t}$. Namely, if M is the largest integer such that

$$l_r \pmod{M} < \left\lceil \frac{l_r}{M} \right\rceil, \quad (r = 1, 2, \dots, t).$$

then

$$\chi_e(K_{l_1, l_2, \dots, l_t}) = \sum_{r=1}^t \left\lceil \frac{l_r}{M+1} \right\rceil$$

Theorem 4.1. Let $K_{l_1, l_2, \dots, l_t}$ be a complete t -partite graph with partite sets of sizes $l_1, l_2, \dots, l_t$. Let M be the largest positive integer satisfying $l_r \pmod{M} < \left\lceil \frac{l_r}{M} \right\rceil$, $r = 1, 2, \dots, t$. Then $\chi_{pde}(K_{l_1, l_2, \dots, l_t}) = \sum_{r=1}^t \left\lceil \frac{l_r}{M+1} \right\rceil$.

Proof. Since the inequality $\chi_e(G) \leq \chi_{pde}(G)$ holds for all graphs G , it follows that $\chi_e(K_{l_1, l_2, \dots, l_t})$ serves as a lower bound for $\chi_{pde}(K_{l_1, l_2, \dots, l_t})$. Therefore,

$$\chi_e(K_{l_1, l_2, \dots, l_t}) \leq \chi_{pde}(K_{l_1, l_2, \dots, l_t}).$$

Now we prove that

$$\chi_e(K_{l_1, l_2, \dots, l_t}) \geq \chi_{pde}(K_{l_1, l_2, \dots, l_t}).$$

Consider an equitable coloring c of $K_{l_1, l_2, \dots, l_t}$. Since c is a proper coloring, every color class must be contained entirely within a single partite set. Indeed, vertices belonging to the same partite set are pairwise nonadjacent, whereas every vertex in one partite set is adjacent to every vertex in every other partite set. Therefore, no color class can contain vertices from two different partite sets.

Let v be any vertex belonging to the partite set V_i . By the definition of a complete multipartite graph, v is adjacent to every vertex in each partite set V_j , where $j \neq i$. Consequently, every color class contained in a partite set different from V_i lies entirely in the closed neighborhood $N[v]$. Hence, each such color class is power dominated by v during the initialization step itself.

Therefore, every vertex of a partite set power dominates all color classes contained in the remaining partite sets. Thus, every equitable coloring of $K_{l_1, l_2, \dots, l_t}$ is also a power dominator equitable coloring. Hence,

$$\chi_e(K_{l_1, l_2, \dots, l_t}) \geq \chi_{pde}(K_{l_1, l_2, \dots, l_t}).$$

Since every power dominator equitable coloring is, by definition, an equitable coloring, we also have

$$\chi_{pde}(K_{l_1, l_2, \dots, l_t}) \geq \chi_e(K_{l_1, l_2, \dots, l_t}).$$

Therefore,

$$\chi_{pde}(K_{l_1, l_2, \dots, l_t}) = \chi_e(K_{l_1, l_2, \dots, l_t}).$$

□

5. DISCONNECTED GRAPHS

Let G be an n -vertex graph whose smallest vertex degree equals zero. Suppose $w \in V(G)$ is a vertex with degree 0.

Theorem 5.1. *Let G be an n -vertex graph which is not connected. Then $\chi_{pde}(G) = \gamma_p(G)$ if and only if $G \cong \overline{K_n}$.*

Proof. If $G \cong \overline{K_n}$, then every vertex is isolated. Hence, every vertex must belong to a power dominating set and must form its own color class. Therefore,

$$\gamma_p(G) = n = \chi_{pde}(G).$$

Conversely, if $G \not\cong \overline{K_n}$, then G contains at least one edge. Thus, some vertex observes more than one vertex, implying $\gamma_p(G) < n$. Therefore, $\chi_{pde}(G) \neq \gamma_p(G)$. Hence,

$$\chi_{pde}(G) = \gamma_p(G) \quad \text{if and only if} \quad G \cong \overline{K_n}.$$

□

Theorem 5.2. *Let F be a forest with m components, where each component of F is a path $P_{n_1}, P_{n_2}, \dots, P_{n_m}$ satisfying $|n_r - n_s| \leq 3, \forall 1 \leq r, s \leq m$. Then*

$$\chi_{pde}(F) = m + \left\lceil \frac{\sum_{i=1}^m \left\lfloor \frac{n_i}{2} \right\rfloor}{\left\lfloor \frac{n_j}{2} \right\rfloor + 1} \right\rceil.$$

Proof. Let $F = \bigcup_{r=1}^m P_{n_r}$. By Corollary 3.1, each vertex of the path $P_{n_r}, 1 \leq r \leq m$, power dominates every vertex of its component, and $\left\lceil \frac{n_r}{2} \right\rceil$ vertices of each component P_{n_r} receive distinct colors. Thus, m colors are sufficient to power dominate the entire forest F .

To obtain an equitable coloring of the remaining vertices, let P_{n_j} be the component having the least number of vertices among all components of F . Then $|C^j| = \lceil \frac{n_j}{2} \rceil$. The remaining $\sum_{i=1}^m \lfloor \frac{n_i}{2} \rfloor$

vertices are distributed equitably among $\left\lceil \frac{\sum_{i=1}^m \lfloor \frac{n_i}{2} \rfloor}{\lceil \frac{n_j}{2} \rceil + 1} \right\rceil$ additional color classes.

Since the smallest color class has cardinality $\lceil \frac{n_j}{2} \rceil$, the largest color class has size at most $\lceil \frac{n_j}{2} \rceil + 1$. Hence, $||C^i| - |C^j|| \leq 1, \forall i, j$. Therefore,

$$\chi_{pde}(F) = m + \left\lceil \frac{\sum_{i=1}^m \lfloor \frac{n_i}{2} \rfloor}{\lceil \frac{n_j}{2} \rceil + 1} \right\rceil.$$

□

6. CIRCULANT GRAPH $C_n(1, 2)$

Let $C_n(1, 2)$ be the specific class of 4-regular circulant graphs on n vertices, where each vertex v_i is adjacent to the vertices $v_{i \pm 1}$ and $v_{i \pm 2}$, with indices taken modulo n . Let $V(C_n(1, 2)) = \{v_1, v_2, \dots, v_n\}$ be the vertex set. The edge set is $E(C_n(1, 2)) = \{(v_i, v_j) \mid \min(|i - j|, n - |i - j|) \in \{1, 2\}\}$.

Theorem 6.1. [10] Let $G = C_n(1, 2)$ be a connected circulant graph. Then

$$\chi(G) = \begin{cases} 5, & n = 5, \\ 3, & n \equiv 0 \pmod{3}, n \geq 6, \\ 4, & n \not\equiv 0 \pmod{3}, n \geq 6. \end{cases}$$

Theorem 6.2. Let $C_n(1, 2)$ be a circulant graph. Then every vertex of $C_n(1, 2)$ power dominates $V(C_n(1, 2))$. Consequently,

$$\chi_{pd}(C_n(1, 2)) = \chi(C_n(1, 2)).$$

Therefore,

$$\chi_{pd}(C_n(1, 2)) = \begin{cases} 5, & n = 5, \\ 3, & n \equiv 0 \pmod{3}, n \geq 6, \\ 4, & n \not\equiv 0 \pmod{3}, n \geq 6. \end{cases}$$

Proof. Consider an arbitrary vertex v_1 of $C_n(1, 2)$. Its closed neighborhood is $N[v_1] = \{v_{n-1}, v_n, v_1, v_2, v_3\}$. Let $M(v_1) = N[v_1]$ be the initial monitored set.

Observe that $N(v_2) = \{v_n, v_1, v_3, v_4\}$. Since v_n, v_1 , and v_3 already belong to $M(v_1)$, the only unmonitored neighbor of v_2 is v_4 . By the power domination propagation rule, v_2 forces v_4 , and hence $v_4 \in M(v_1)$.

Similarly, after v_4 is monitored, the vertex v_3 has exactly one unmonitored neighbor, namely v_5 . Therefore, v_3 forces v_5 , and thus $v_5 \in M(v_1)$.

Continuing inductively, once $v_4, v_5, \dots, v_k$ are monitored, the vertex v_{k-1} has a unique unmonitored neighbor v_{k+1} , which is then forced.

Hence, the monitoring process propagates successively around the cycle and eventually monitors every vertex of $C_n(1,2)$. Therefore,

$$M(v_1) = V(C_n(1,2)).$$

Since v_1 was chosen arbitrarily, the same argument holds for every vertex v_i , $1 \leq i \leq n$. Thus,

$$M(v_i) = V(C_n(1,2)), \quad 1 \leq i \leq n.$$

Hence, every vertex of $C_n(1,2)$ power dominates $V(C_n(1,2))$. Therefore,

$$\chi_{pd}(C_n(1,2)) = \chi(C_n(1,2)).$$

The result now follows from Theorem 6.1.

Figure 2 demonstrates the power dominator coloring of $C_{13}(1,2)$

□

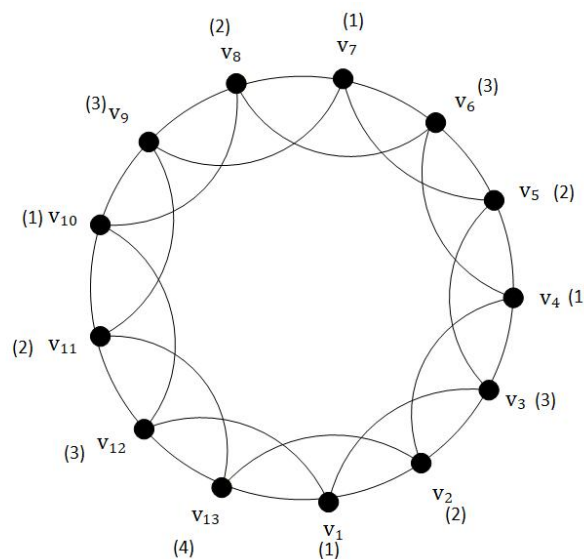


FIGURE 2. $\chi_{pd}(C_{13}(1,2)) = 4$

Theorem 6.3. Let $C_n(1,2)$ be the circulant graph. Then, for $n \geq 5$,

$$\chi_{pde}(C_n(1,2)) = \begin{cases} 3, & n \equiv 0 \pmod{3}, \\ 4, & n \equiv 0, 2, 3 \pmod{4}, n \not\equiv 0 \pmod{3}, \\ 5, & n \equiv 1 \pmod{4}, n \not\equiv 0 \pmod{3}. \end{cases}$$

Proof. The structure of $C_n(1,2)$ follows from Theorem 6.2. Each vertex of $C_n(1,2)$ power dominates all the vertices of $C_n(1,2)$.

By Theorem 3.4,

$$\chi_{pde}(C_n(1,2)) = \chi_e(C_n(1,2)).$$

Therefore, it remains to determine the equitable chromatic number of $C_n(1,2)$.

Case (i): $n \equiv 0 \pmod{3}$

Define the color classes by

$$C^1 = \{v_{3i+1} ; 0 \leq i \leq \frac{n-3}{3}\},$$

$$C^2 = \{v_{3i-1} ; 1 \leq i \leq \left\lceil \frac{n-3}{3} \right\rceil\},$$

and

$$C^3 = \{v_{3i} ; 1 \leq i \leq \frac{n}{3}\}.$$

Each color class has exactly $\frac{n}{3}$ vertices. Thus, the coloring is proper and equitable. Therefore,

$$\chi_e(C_n(1,2)) = 3.$$

Hence,

$$\chi_{pde}(C_n(1,2)) = 3.$$

Figure 3 demonstrates the power dominator equitable coloring of $C_{12}(1,2)$.

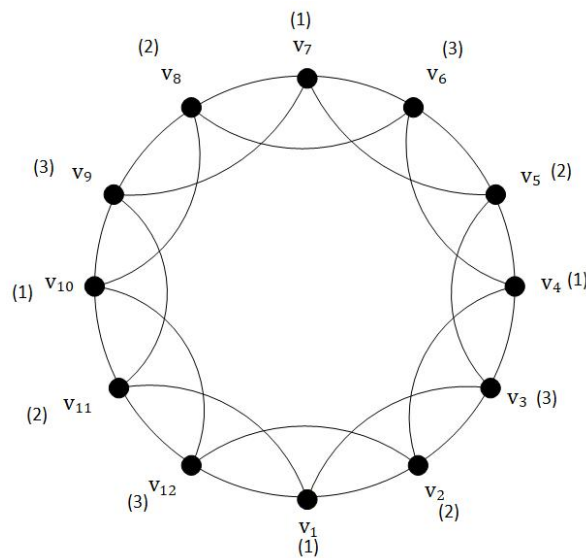


FIGURE 3. $\chi_{pde}(C_{12}(1,2)) = 3$

Case (ii): $n \equiv 0 \pmod{4}$, $n \not\equiv 0 \pmod{3}$

The color classes are as follows:

$$C^1 = \{v_{4i+1} ; 0 \leq i \leq \frac{n-4}{4}\},$$

$$C^2 = \{v_{4i-2} ; 1 \leq i \leq \frac{n}{4}\},$$

$$C^3 = \{v_{4i-1} ; 1 \leq i \leq \frac{n}{4}\},$$

and

$$C^4 = \{v_{4i} ; 1 \leq i \leq \frac{n}{4}\}.$$

Each color class has exactly $\frac{n}{4}$ vertices. Thus, the coloring is proper and equitable. Therefore,

$$\chi_e(C_n(1,2)) = 4.$$

Hence,

$$\chi_{pde}(C_n(1,2)) = 4.$$

Case (iii): $n \equiv 2 \pmod{4}$, $n \not\equiv 0 \pmod{3}$

The color classes are as follows:

$$C^1 = \{v_{4i+1} ; 0 \leq i \leq \frac{n-6}{4}\},$$

$$C^2 = \{v_{4i-2} ; 1 \leq i \leq \frac{n-2}{4}\} \cup \{v_{n-1}\},$$

$$C^3 = \{v_{4i-1} ; 1 \leq i \leq \frac{n-2}{4}\} \cup \{v_n\},$$

and

$$C^4 = \{v_{4i} ; 1 \leq i \leq \frac{n-2}{4}\}.$$

The color classes differ in size by at most one. Hence, the coloring is equitable. Therefore,

$$\chi_e(C_n(1,2)) = 4.$$

Hence,

$$\chi_{pde}(C_n(1,2)) = 4.$$

Case (iv): $n \equiv 3 \pmod{4}$, $n \not\equiv 0 \pmod{3}$

The color classes are as follows:

$$C^1 = \{v_{4i+1} ; 0 \leq i \leq \frac{n-3}{4}\},$$

$$C^2 = \{v_{4i-2} ; 1 \leq i \leq \frac{n+1}{4}\},$$

$$C^3 = \{v_{4i-1} ; 1 \leq i \leq \frac{n+1}{4}\},$$

and

$$C^4 = \{v_{4i} ; 1 \leq i \leq \frac{n-3}{4}\}.$$

The color classes differ in size by at most one. Hence, the coloring is equitable. Therefore,

$$\chi_e(C_n(1,2)) = 4.$$

Hence,

$$\chi_{pde}(C_n(1,2)) = 4.$$

Case (v): $n \equiv 1 \pmod{4}$, $n \not\equiv 0 \pmod{3}$

We construct an equitable 5-coloring according to the residue class of n modulo 5.

Subcase (i): $n \equiv 1 \pmod{5}$

$$\begin{aligned}
C^1 &= \{v_{5i+1} ; 0 \leq i \leq \frac{n-6}{5}\}, \\
C^2 &= \{v_{5i+2} ; 0 \leq i \leq \frac{n-6}{5}\}, \\
C^3 &= \{v_{5i+3} ; 0 \leq i \leq \frac{n-6}{5}\} \cup \{v_n\}, \\
C^4 &= \{v_{5i+4} ; 0 \leq i \leq \frac{n-6}{5}\},
\end{aligned}$$

and

$$C^5 = \{v_{5i+5} ; 0 \leq i \leq \frac{n-6}{5}\}.$$

Subcase (ii): $n \equiv 2 \pmod{5}$

$$\begin{aligned}
C^1 &= \{v_{5i+1} ; 0 \leq i \leq \frac{n-7}{5}\}, \\
C^2 &= \{v_{5i+2} ; 0 \leq i \leq \frac{n-7}{5}\} \cup \{v_{n-1}\}, \\
C^3 &= \{v_{5i+3} ; 0 \leq i \leq \frac{n-7}{5}\} \cup \{v_n\}, \\
C^4 &= \{v_{5i+4} ; 0 \leq i \leq \frac{n-7}{5}\},
\end{aligned}$$

and

$$C^5 = \{v_{5i+5} ; 0 \leq i \leq \frac{n-7}{5}\}.$$

Subcase (iii): $n \equiv 3 \pmod{5}$

$$\begin{aligned}
C^1 &= \{v_{5i+1} ; 0 \leq i \leq \frac{n-3}{5}\}, \\
C^2 &= \{v_{5i+2} ; 0 \leq i \leq \frac{n-3}{5}\}, \\
C^3 &= \{v_{5i+3} ; 0 \leq i \leq \frac{n-3}{5}\}, \\
C^4 &= \{v_{5i+4} ; 0 \leq i \leq \frac{n-8}{5}\},
\end{aligned}$$

and

$$C^5 = \{v_{5i+5} ; 0 \leq i \leq \frac{n-8}{5}\}.$$

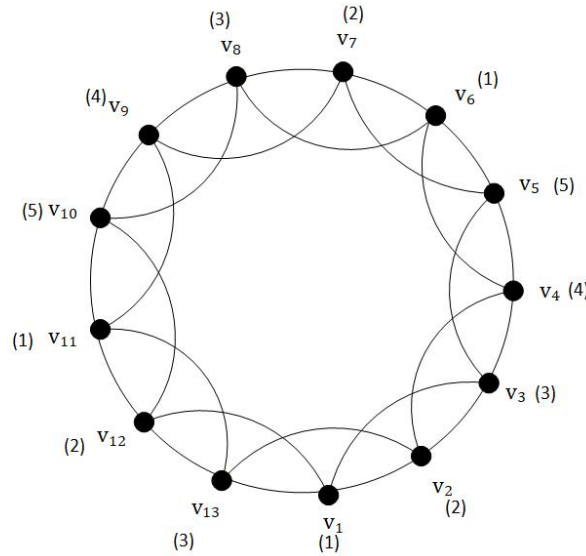
Figure 4 demonstrates the power dominator equitable coloring of $C_{13}(1, 2)$.

Subcase (iv): $n \equiv 4 \pmod{5}$

$$\begin{aligned}
C^1 &= \{v_{5i+1} ; 0 \leq i \leq \frac{n-4}{5}\}, \\
C^2 &= \{v_{5i+2} ; 0 \leq i \leq \frac{n-4}{5}\}, \\
C^3 &= \{v_{5i+3} ; 0 \leq i \leq \frac{n-4}{5}\}, \\
C^4 &= \{v_{5i+4} ; 0 \leq i \leq \frac{n-4}{5}\},
\end{aligned}$$

and

$$C^5 = \{v_{5i+5} ; 0 \leq i \leq \frac{n-9}{5}\}.$$

FIGURE 4. $\chi_{pde}(C_{13}(1,2)) = 5$

In each subcase, all vertices with the same color are pairwise nonadjacent; therefore, the coloring is proper. In addition, since the sizes of the color classes differ by at most 1, the coloring is equitable. Hence,

$$\chi_e(C_n(1,2)) = 5.$$

Each vertex of $C_n(1,2)$ power dominates all the color classes, since every vertex of $C_n(1,2)$ power dominates all the vertices. Hence, the above equitable coloring is a power dominator equitable coloring. Therefore,

$$\chi_{pde}(C_n(1,2)) = 5.$$

Combining all the cases,

$$\chi_{pde}(C_n(1,2)) = \begin{cases} 3, & n \equiv 0 \pmod{3}, \\ 4, & n \equiv 0, 2, 3 \pmod{4}, n \not\equiv 0 \pmod{3}, \\ 5, & n \equiv 1 \pmod{4}, n \not\equiv 0 \pmod{3}. \end{cases}$$

□

7. CONCLUSION

This paper has presented bounds and characterization theorems for power dominator equitable coloring, offering valuable insights into this area of graph theory. Future research may focus on determining the power dominator equitable chromatic number for other graph families, such as graph products, interval graphs, chordal graphs, and planar graphs. Another promising direction is the development of efficient algorithms and complexity analyses for computing $\chi_{pde}(G)$, particularly for rooted trees and large-scale network structures. These avenues of future research promise

to deepen our understanding and advance practical applications of power dominator equitable coloring.

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