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UNIVALENT BIHARMONIC MAPPINGS AND LINEARLY CONNECTED DOMAINS

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Abstract. A four times continuously differentiable complex-valued function $F = u + iv$ in a simply connected domain Ω is biharmonic if the laplacian of F is harmonic. Every biharmonic mapping F in Ω has the representation $F = |z|^2 G + K$, where G and K are harmonic in Ω. This paper investigates the relationship between the univalence of F and of K using the concept of linearly connected domains.

1. Introduction

A planar harmonic mapping in a simply connected domain $\Omega \subset \mathbb{C}$ is a complex-valued harmonic function $f(z)$ defined on Ω, where $z = x + iy$. The mapping f has a canonical decomposition $f = h + \overline{g}$, where h and g are analytic (holomorphic) in Ω (see [13, 14]). We say that f is locally univalent and sense preserving if and only if its Jacobian $J_f(z)$ is positive, where $J_f(z)$ is given by

$$J_f\ (z) = |f_z(z)|^2 - |f_{\overline{z}}(z)|^2 = |h'(z)|^2 - |g'(z)|^2,$$

(See Lewy [11]).

Clunie and Sheil-Small made the following important observation : f is locally univalent and orientation -preserving in $\mathbb{D}$ if and only if $|g'(z)| < |h'(z)|$ in Ω; or equivalently if $h'(z) \neq 0$ and the dilatation $\omega(z) = \frac{g'(z)}{h'(z)}$ has the property $|\omega(z)| < 1$.

A four times continuously differentiable complex-valued function $F = u + iv$ in a simply connected domain Ω is biharmonic if the laplacian of F is harmonic. Note that $\triangle F$ is harmonic in Ω, if $\triangle F$ satisfies Laplace's equation $\triangle(\triangle F) = 0$, where

$$\triangle = 4\frac{\partial^2}{\partial z \partial \overline{z}} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Every harmonic function is biharmonic but not necessarily the converse. Moreover, it is easy to see that every biharmonic mapping F in Ω has the representation

$$F = |z|^2 G + K, \tag{1.1}$$

where G and K are harmonic in Ω and they can be expressed as,

$$\begin{aligned} G &= g_1 + \overline{g_2}, \\ K &= k_1 + \overline{k_2}, \end{aligned} \tag{1.2}$$

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where g_1, g_2, k_1 and k_2 are analytic in Ω (for details see [2]). Note that the composition $F \circ \phi$ of a harmonic function F with analytic function ϕ is harmonic, while this is not true when F is biharmonic.

Biharmonic mappings arise in a lot of physical situations, particularly in fluid dynamics and elasticity problems, and have many important applications in engineering and biology. Most important applications of the theory of functions of a complex variable were obtained in the plane theory of elasticity and in the approximate theory of plates subject to normal loading. That is, in cases when the solutions are biharmonic functions or functions associated with them (see [15, 16]). Moreover, biharmonic Mapping are closely related to the theory of Laguerre Minimal Surfaces (for details see [5, 7, 8, 9, 17, 18]). Investigation of biharmonic mappings in the context of geometric function theory started only recently (for details see[1, 2, 3, 4, 10]). For example, in [2], Abdulhadi, AbuMuhanna and Khuri analyze the univalence of the solutions of the biharmonic equations. Throughout we consider harmonic and biharmonic functions defined on the unit disk $\mathbb{D} = \{z : |z| < 1\}$.

Definition 1. *A domain $\Omega \subset \mathbb{C}$ is linearly connected if there exists a constant $M < \infty$ such that any two points $w_1, w_2 \in \Omega$ are joined by a path γ, $\gamma \subset \Omega$, of length $\ell(\gamma) \le M|w_1 - w_2|$.*

Such a domain is necessarily a Jordan domain, and for piecewise smoothly bounded domains, linear connectivity is equivalent to the boundary having no inward-pointing cusps.

In [12], Chuaqui and Hermandez, considered the relationship between the harmonic mapping $f = h + \overline{g}$ and its analytic factor h on linearly connected domains. They show that if h is an analytic univalent function, then every harmonic mapping $f = h + \overline{g}$ with dilatation $|\omega| < c$ is univalent if and only if $h(\mathbb{D})$ is linearly connected.

In this paper, we scrutinize the relationship between the univalence of the biharmonic function $F = |z|^2G + K$ and the univalence of the harmonic function K.

2. Main results

In our first results, we deduce the univalence of $F(z)$ from the univalence of $K(z)$. We first consider subclasses, where G, K are assumed to be analytic or antianalytic.

Theorem 1. *Let $F(z) = |z|^2G(z) + K(z)$ be a biharmonic function in the unit disk $\mathbb{D}$, where G, K are analytic. If K is univalent and $K(\mathbb{D})$ is a linearly connected domain with constant M, and if*

$$\frac{2|G| + |G'|}{|K'|} < \frac{1}{M},$$

then $F(z)$ is univalent.

Proof. Let $H(z) = |z|^2G(z)$. We define $\varphi = H \circ K^{-1}$. Given $w \in K(\mathbb{D})$, we claim $w + \varphi(w)$ is univalent. Assume $w + \varphi(w)$ is not univalent, then there exists $w_1 \neq w_2$ such that

$$\varphi(w_2) - \varphi(w_1) = w_1 - w_2.$$

Let γ be a path in $K(\mathbb{D})$ joining w_1, w_2 such that $l(\gamma) \le M|w_2 - w_1|$.

Then

$$|\varphi(w_2) - \varphi(w_1)| \le \left| \int_\gamma \varphi_w dw + \varphi_{\overline{w}} d\overline{w} \right|.$$

But

$$\varphi_w = H_z(K^{-1})_w + H_{\overline{z}}(\overline{K^{-1}})_w = \frac{H_z}{K'} = \frac{\overline{z}G + |z|^2 G'}{K'},$$

$$\varphi_{\overline{w}} = H_z(K^{-1})_{\overline{w}} + H_{\overline{z}}(\overline{K^{-1}})_{\overline{w}} = \frac{H_{\overline{z}}}{\overline{K'}} = \frac{zG}{\overline{K'}},$$

where $z = K^{-1}(w) \in \mathbb{D}$.

Therefore,

$$|\varphi(w_2) - \varphi(w_1)| \le \int_\gamma \sup_{\mathbb{D}} \frac{|2G| + |G'|}{|K'|} |dw| < \frac{1}{M} l(\gamma) < |w_2 - w_1|$$

which is a contradiction. Hence $F(z)$ is univalent. □

Remark 1. *In the above proof , if $K(\mathbb{D})$ is convex we may take $M = 1$, and thus F will be univalent as long as $\frac{2|G|+|G'|}{|K'|} < 1$.*

The special case $M = 1$ when K is convex, is an important special case and we will state it separately as a corollary.

Corollary 1. *Let $F(z) = |z|^2 G(z) + K(z)$ be a biharmonic function in the unit disk $\mathbb{D}$, where G, K are analytic. If K is univalent and convex with*

$$\frac{2|G| + |G'|}{|K'|} < 1,$$

then $F(z)$ is univalent.

As a consequence of Theorem 1, we have the following corollary :

Corollary 2. *Let $F(z) = |z|^2 G(z) + K(z)$ be a biharmonic function in the unit disk $\mathbb{D}$, where G, K are antianalytic. If K is univalent and $K(\mathbb{D})$ is a linearly connected domain with constant M, and*

$$\frac{2|G| + |G_{\overline{z}}|}{|K_{\overline{z}}|} < \frac{1}{M},$$

then $F(z)$ is univalent.

Our next result is the general case, where G , K are harmonic in the unit disk $\mathbb{D}$:

Theorem 2. *Let $F(z) = |z|^2 G(z) + K(z)$ be a biharmonic function in the unit disk $\mathbb{D}$, where G, K are harmonic. If K is univalent and $K(\mathbb{D})$ is a linearly connected domain with constant M, and if*

$$\frac{2|G| + |g_1'|(1 + |\omega_G|)}{|k_1'|(1 - |\omega_K|)} < \frac{1}{M},$$

then $F(z)$ is univalent. In the above ω_K, ω_G denotes the dilations $\omega_K = \dfrac{k_2'}{k_1'}$, $\omega_G = \dfrac{g_2'}{g_1'}$.

Proof. Let $H(z)=|z|^2G(z)$. We define

$$\varphi = H\circ K^{-1}.$$

Given $w\ \epsilon\ K(\mathbb{D})$, we claim $w+\varphi(w)$ is univalent.

Assume $w+\varphi(w)$ is not univalent, then there exists $w_1\neq w_2$ such that

$$\varphi(w_2)-\varphi(w_1)=w_1-w_2.$$

Let γ be a path in $K(\mathbb{D})$ joining w_1,w_2 such that $l(\gamma)\leq M|w_2-w_1|$.Then

$$|\varphi(w_2)-\varphi(w_1)|\leq\left|\int_\gamma \varphi_w dw+\varphi_{\overline{w}}d\overline{w}\right|\leq\int_\gamma\left(|\varphi_w|+|\varphi_{\overline{w}}|\right)|dw|.$$

But

$$\varphi_w = H_z(K^{-1})_w + H_{\overline{z}}(\overline{K^{-1}})_w$$

$$\varphi_{\overline{w}} = H_z(K^{-1})_{\overline{w}} + H_{\overline{z}}(\overline{K^{-1}})_{\overline{w}}.$$

Differentiating $K^{-1}(K(z))=z$, we show that

$$(K^{-1})_w=\frac{\overline{k_1'}}{|k_1'|^2-|k_2'|^2},\ \ (K^{-1})_{\overline{w}}=\frac{-\overline{k_2'}}{|k_1'|^2-|k_2'|^2}.$$

It follows

$$\begin{aligned}
\varphi_w &= H_z(K^{-1})_w + H_{\overline{z}}(\overline{K^{-1}})_w\\
&= (\overline{z}G+|z|^2g_1')\frac{\overline{k_1'}}{|k_1'|^2-|k_2'|^2}+(zG+|z|^2\overline{g_2'})\frac{-k_2'}{|k_1'|^2-|k_2'|^2},
\end{aligned}$$

$$\begin{aligned}
\varphi_{\overline{w}} &= H_z(K^{-1})_{\overline{w}} + H_{\overline{z}}(\overline{K^{-1}})_{\overline{w}}\\
&= (\overline{z}G+|z|^2g_1')\frac{-\overline{k_2'}}{|k_1'|^2-|k_2'|^2}+(zG+|z|^2g_2')\frac{k_1'}{|k_1'|^2-|k_2'|^2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|\varphi_w|+|\varphi_{\overline{w}}| &\leq \frac{2|z||G|(|k_1'|+|k_2'|)}{|k_1'|^2-|k_2'|^2}+\frac{|z|^2(|g_1'||k_2'|+|g_2'||k_1'|)}{|k_1'|^2-|k_2'|^2}\\
&= \frac{2|z||G|+|z|^2\left(|g_1'|+|g_2'|\right)}{|k_1'|-|k_2'|}\\
&= \frac{2|z||G|+|z|^2|g_1'|(1+|\omega_G|)}{|k_1'|(1-|\omega_K|)},
\end{aligned}$$

where $z=K^{-1}(w)\in\mathbb{D}$. Then we have

$$|\varphi(w_2)-\varphi(w_1)|\leq\int_\gamma\sup_{\mathbb{D}}\frac{2|G|+|g_1'|(1+|\omega_G|)}{|k_1'|(1-|\omega_K|)}|dw|<\frac{1}{M}l(\gamma)<|w_2-w_1|$$

which is a contradiction. Hence $F(z)$ is univalent. $\square$

Corollary 3. *Let $F(z)=|z|^2G(z)+K(z)$ be a biharmonic function in the unit disk $\mathbb{D}$, where G,K are harmonic. If K is univalent and convex with*

$$\frac{2|G|+|g_1'|(1+|\omega_G|)}{|k_1'|(1-|\omega_K|)}<1,$$

then $F(z)$ is univalent.

The following Corollary follows immediately from Theorem 2, for the case when G is analytic.

Corollary 4. *Let $F(z) = |z|^2G(z) + K(z)$ be a biharmonic function in the unit disk $\mathbb{D}$, where G is analytic, K is harmonic. If K is univalent and $K(\mathbb{D})$is a linearly connected domain with constant M, and if*

$$\frac{2|G| + |G'|}{|k_1'|(1 - |\omega_K|)} < \frac{1}{M},$$

then $F(z)$ is univalent. In the above ω_k denotes the dilation $\omega_K = \dfrac{k_2'}{k_1'}$.

In each of the previous results, it follows under the same conditions that $F(\mathbb{D})$ will also be linearly connected. We will prove it for the general case and the proof is along the same lines for the special cases.

Proposition 1. *Let $F(z) = |z|^2G(z) + K(z)$ be a biharmonic function in the unit disk $\mathbb{D}$, where G ,K are harmonic. If K is univalent and $K(\mathbb{D})$ is a linearly connected domain with constant M, and if*

$$\frac{2|G| + |g_1'|(1 + |\omega_g|)}{|k_1'|(1 - |\omega_k|)} \le C,$$

where $C < \frac{1}{M}$, then $F(\mathbb{D})$ is linearly connected.

Proof. Given $w \in \Omega = K(\mathbb{D})$, we let $\Psi(w) = w + \varphi(w)$, where $\varphi = H \circ K^{-1}$, and $H = |z|^2G$. Since K is univalent, we may look at $R = F(\mathbb{D})$, as the image of $\Omega = K(\mathbb{D})$ under the mapping Ψ, and we show $\Psi(\Omega)$ is linearly connected. Let $\varsigma_1 = \Psi(w_1), \varsigma_2 = \Psi(w_2)$, $w_1, w_2 \in \Omega$. Since $K(\mathbb{D})$ is a linearly connected domain, then there exists a curve $\gamma \subset \Omega$ satisfying $l(\gamma) \le M|w_2 - w_1|$.

Let $\Gamma = \Psi(\gamma)$. In the proof of Theorem 2, we have showed that

$$|\varphi_w| + |\varphi_{\overline{w}}| \le \frac{2|G| + |g_1'|(1 + |\omega_G|)}{|k_1'|(1 - |\omega_K|)} < C,$$

it follows

$$|\Psi_w| + |\Psi_{\overline{w}}| \le 1 + |\varphi_w| + |\varphi_{\overline{w}}| < 1 + C.$$

Hence we have,

$$l(\Gamma) = \int_\Gamma d\varsigma \le \int_\gamma (|\Psi_w| + |\Psi_{\overline{w}}|)\, dw < (1 + C)l(\gamma) \le (1 + C)M|w_2 - w_1|.$$

But,

$$\begin{aligned} |\varsigma_1 - \varsigma_2| &= |w_1 - w_2 + \varphi(w_1) - \varphi(w_2)| \ge |w_1 - w_2| - |\varphi(w_1) - \varphi(w_2)| \\ &\ge |w_1 - w_2| - \int_\gamma (|\varphi_w| + |\varphi_{\overline{w}}|)\, dw \\ &> |w_1 - w_2| - Cl(\gamma) \ge (1 - CM)|w_1 - w_2|. \end{aligned}$$

Then we get,

$$l(\Gamma) \le \frac{(1 + C)M}{1 - CM}|\varsigma_1 - \varsigma_2|$$

and so R is linearly connected with constant $\frac{(1+C)M}{1-CM}$. □

In our next result, we deduce the univalence of K from the univalence of F.

Theorem 3. *Let $F(z) = |z|^2 G(z) + K(z)$ be a biharmonic function in the unit disk $\mathbb{D}$. Suppose F is univalent and $F(\mathbb{D})$ is a linearly connected domain with constant M and satisfies*

$$\frac{2|G| + |G_z| + |G_{\overline{z}}|}{||F_{\overline{z}}| - |F_z||} < \frac{1}{M},$$

then $K(z)$ is univalent.

Proof. Let $H(z) = |z|^2 G(z)$. Aiming for a contradiction assume $K(z)$ is not univalent, then there exists $z_1 \neq z_2$, such that $K(z_1) = K(z_2)$. Hence we get

$$F(z_1) - F(z_2) = H(z_1) - H(z_2).$$

Given $w = F(z) \epsilon\, F(\mathbb{D})$, the above equation is equivalent to

$$w_1 - w_2 = \varphi(w_2) - \varphi(w_1),$$

where

$$\varphi = H \circ F^{-1}.$$

Let γ be a path in $F(\mathbb{D})$ joining w_1, w_2 such that $l(\gamma) \leq M|w_2 - w_1|$.
Then

$$|\varphi(w_2) - \varphi(w_1)| \leq \left| \int_\gamma \varphi_w dw + \varphi_{\overline{w}} d\overline{w} \right| \leq \int_\gamma (|\varphi_w| + |\varphi_{\overline{w}}|)\, |dw|.$$

But

$$\varphi_w = H_z (F^{-1})_w + H_{\overline{z}} (\overline{F^{-1}})_w$$

$$\varphi_{\overline{w}} = H_z (F^{-1})_{\overline{w}} + H_{\overline{z}} (\overline{F^{-1}})_{\overline{w}}.$$

Differentiating $F^{-1}(F(z)) = z$, we get the following two equations

$$(F^{-1})_w F_z + (F^{-1})_{\overline{w}} \overline{F}_z = 1$$

$$(F^{-1})_w F_{\overline{z}} + (F^{-1})_{\overline{w}} \overline{F}_{\overline{z}} = 0.$$

Solving the above system we get

$$(F^{-1})_w = \frac{\overline{F_z}}{J_F}, \quad (K^{-1})_{\overline{w}} = \frac{-F_{\overline{z}}}{J_F},$$

where J_F denotes the jacobian $J_f = |F_z|^2 - |F_{\overline{z}}|^2$.
It follows,

$$\varphi_w = H_z (F^{-1})_w + H_{\overline{z}} (\overline{F^{-1}})_w = H_z \frac{\overline{F_z}}{J_F} - H_{\overline{z}} \frac{\overline{F_{\overline{z}}}}{J_F}$$

$$\varphi_{\overline{w}} = H_z (F^{-1})_{\overline{w}} + H_{\overline{z}} (\overline{F^{-1}})_{\overline{w}} = -H_z \frac{F_{\overline{z}}}{J_F} + H_{\overline{z}} \frac{F_z}{J_F}.$$

Therefore,

$$\begin{aligned} |\varphi_w| + |\varphi_{\overline{w}}| &\leq \frac{|H_z||F_z| + |H_{\overline{z}}||F_{\overline{z}}| + |H_z||F_{\overline{z}}| + |H_{\overline{z}}||F_z|}{|J_F|} \\ &= \frac{(|H_z| + |H_{\overline{z}}|)(|F_z| + |F_{\overline{z}}|)}{|J_F|} \\ &= \frac{|H_z| + |H_{\overline{z}}|}{||F_z| - |F_{\overline{z}}||} \end{aligned}$$

$$\leq \quad \frac{2|G| + |G_z| + |G_{\overline{z}}|}{||F_{\overline{z}}| - |F_z||}.$$

Hence

$$|\varphi(w_2) - \varphi(w_1)| \leq \int_\gamma \sup_{\mathbb{D}} \frac{2|G| + |G_z| + |G_{\overline{z}}|}{||F_{\overline{z}}| - |F_z||} |dw| < \frac{1}{M} l(\gamma) < |w_2 - w_1|,$$

which is a contradiction. Thus $K(z)$ is univalent. □

Corollary 5. *Let $F(z) = |z|^2 G(z) + K(z)$ be a biharmonic function in the unit disk $\mathbb{D}$, where G , K are harmonic. If K is univalent and convex with*

$$\frac{2|G| + |g_1'|(1 + |\omega_G|)}{|k_1'|(1 - |\omega_K|)} < 1,$$

then $K(z)$ is univalent.

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