International Journal of Analysis and Applications
ISSN 2291-8639
Volume 9, Number 1 (2015), 9-18
http://www.etamaths.com

FIXED POINTS OF α-ADMISSIBLE MAPPINGS IN CONE METRIC SPACES WITH BANACH ALGEBRA

S.K. MALHOTRA[1] J.B. SHARMA[2] AND SATISH SHUKLA[3,*]

Abstract. In this paper, we introduce the α-admissible mappings in the setting of cone metric spaces equipped with Banach algebra and solid cones. Our results generalize and extend several known results of metric and cone metric spaces. An example is presented which illustrates and shows the significance of results proved herein.

Keywords. Cone metric space; α-admissible mapping; Solid cone; Banach algebra; Fixed point.
AMC 2000. 47H10; 54H25.

1. Introduction

Huang and Zhang [7] introduced the notion of cone metric spaces as a generalization of metric spaces. They defined the distance of two points of space in terms of a vector lying in a particular subset of a Banach space called cone. They also defined the Cauchy sequence and convergence of a sequence in such spaces in terms of interior points of the underlying cone. Moreover, they proved the Banach contraction principle in the setting of cone metric spaces with the assumption that the cone is normal. Later, the assumption of normality of cone was removed by Rezapour and Hamlbarani [10]. Huang and Zhang [7] also gave an example and showed the dependency of contractive nature of mappings on the cone metric spaces. Although, some authors (see, e.g., [3, 16, 13, 14]) showed that the fixed point results proved on cone metric spaces are the simple consequences of corresponding results of usual metric spaces.

Liu and Xu [4] used the cones over a Banach algebra and proved some fixed point theorems on cone metric spaces. They improved the contractive condition on self-maps of cone metric spaces by replacing the contractive constant by a vector of cone. They also gave an example which shows that their fixed point results cannot be obtained by the corresponding results on usual metric spaces with an approach used, e.g., in [3, 16, 13, 14]. The results proved by Liu and Xu [4] demands the normality of the underlying cone. Later on, Xu and Radenović [9] showed that the condition of normality of cone can be removed, and so, the results of Liu and Xu [4] are also true in case of a non-normal cone.

On the other hand, Samet et al. [2] introduced the study of fixed points for the α-admissible mappings and generalized several known results of metric spaces. In this paper, we use the concept of α-admissibility of mappings defined on cone

2010 *Mathematics Subject Classification.* 47H10.
Key words and phrases. Fixed points; univalent; α-admissible mappings.

metric spaces with Banach algebra and define the generalized Lipschitz contractions on such spaces. Our results extend and generalize several known results of metric and cone metric spaces. An example is also provided which verifies the usability and significance of our results.

2. Preliminaries

First, we recall some definitions and results about the Banach algebras and cone metric spaces.

Let A be a real Banach algebra, i.e., A is a real Banach space in which an operation of multiplication is defined, subject to the following properties: for all $x, y, z \in A, a \in \mathbb{R}$

(1) $x(yz) = (xy)z$;
(2) $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$;
(3) $a(xy) = (ax)y = x(ay)$;
(4) $\|xy\| \leq \|x\|\|y\|$.

In this paper, we shall assume that the Banach algebra A has a unit, i.e., a multiplicative identity e such that $ex = xe = x$ for all $x \in A$. An element $x \in A$ is said to be invertible if there is an inverse element $y \in A$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1}. For more details we refer to [11].

The following proposition is well known [11].

Proposition 2.1. Let A be a real Banach algebra with a unit e and $x \in A$. If the spectral radius $\rho(x)$ of x is less than *one*.,

$$\rho(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf_{n\geq 1} \|x^n\|^{\frac{1}{n}} < 1$$

then $e - x$ is invertible. Actually,

$$(e-x)^{-1} = \sum_{i=0}^{\infty} x^i.$$

A subset P of A is called a cone if

(1) P is non-empty, closed and $\{\theta, e\} \subset P$, where θ is the zero vector of A;
(2) $a_1P + a_2P \subset P$ for all non-negative real numbers a_1, a_2;
(3) $P^2 = PP \subset P$
(4) $P \bigcap (-P) = \{\theta\}$.

For a given cone $P \subset A$, we can define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. The notation $x \ll y$ will stand for $y - x \in P^\circ$, where P° denotes the interior of P.

The cone P is called normal if there exists a number $K > 0$ such that for all $a, b \in A$,

$$a \preceq b \quad \text{implies} \quad \|a\| \leq K\|b\|.$$

The least positive value of K satisfying the above inequality is called the normal constant (see [7]). Note that, for any normal cone P we have $K \geq 1$ (see [10]). In the following we always assume that P is a cone in a real Banach algebra A with $P^\circ \neq \phi$ (i.e., the cone P is a solid cone) and $\preceq$ is the partial ordering with respect to P.

The following lemmas and remark will be useful in the sequel.

Lemma 2.2 (See [15]). *If E is a real Banach space with a cone P and if $a \preceq \lambda a$ with $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.*

Lemma 2.3 (See [8]). *If E is a real Banach space with a solid cone P and if $\theta \preceq u \ll c$ for each $\theta \ll c$, then $u = \theta$.*

Lemma 2.4 (See [8]). *If E is a real Banach space with a solid cone P and if $\|x_n\| \to 0$ as $n \to \infty$, then for any $\theta \ll c$, there exists $n_0 \in \mathbb{N}$ such that, $x_n \ll c$ for all $n < n_0$.*

Remark 2.5 (See [9]). If $\rho(x) < 1$ then $\|x^n\| \to 0$ as $n \to \infty$.

Definition 2.6 (See [4, 5, 7]). Let X be a non-empty set. Suppose that the mapping $d\colon X \times X \to A$ satisfies:

(1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$.
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$.
(3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space over the Banach algebra A.

Definition 2.7 (See [7]). Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then:

(1) The sequence $\{x_n\}$ converges to x whenever for each $c \in A$ with $\theta \ll c$, there is $n_0 \in \mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n > n_0$. We denote this by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
(2) The sequence $\{x_n\}$ is a Cauchy sequence whenever for each $c \in A$ with $\theta \ll c$, there is $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \ll c$ for all $n, m > n_0$.
(3) (X, d) is a complete cone metric space if every Cauchy sequence is convergent in X.

It is obvious that the limit of a convergent sequence in a cone metric space is unique. A mapping $T\colon X \to X$ is called continuous at $x \in X$, if for every sequence $\{x_n\}$ in X such that $x_n \to x$ as $n \to \infty$, we have $Tx_n \to Tx$ as $n \to \infty$.

Definition 2.8 (See [2]). Let X be a nonempty set and $\alpha\colon X \times X \to [0, \infty)$ be a function. We say that T is α-admissible if $(x, y) \in X$, $\alpha(x, y) \geq 1 \implies \alpha(Tx, Ty) \geq 1$.

Now, we define the generalized Lipschitz contractions on the cone metric spaces with a Banach algebra (see also, [4]).

Definition 2.9. Let (X, d) be a complete cone metric space over a Banach algebra A and P be the underlying solid cone. Then the mapping $T\colon X \to X$ is said to be generalized Lipschitz contraction if there exists $k \in P$ such that $\rho(k) < 1$ and,

$$d(Tx, Ty) \preceq kd(x, y)$$

for all $x, y \in X$ with $\alpha(x, y) \geq 1$. Here, the vector k is called the Lipschitz vector of T.

Now we can state our main results.

3. Main results

First, we prove an existence theorem for a generalized Lipschitz contraction on cone metric space over Banach algebras.

Theorem 3.1. *Let (X,d) be a complete cone metric space over a Banach algebra A and P be the underlying solid cone. Suppose, $T\colon X\to X$ be a generalized Lipschitz contraction with Lipschitz vector k and the following conditions are satisfied:*

(i) T is α-admissible;
(ii) there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq 1$;
(iii) T is continuous.

Then T has a fixed point $x^*\in X$.

Proof. Let $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq 1$ and define a sequence $\{x_n\}$ in X such that $x_n=Tx_{n-1}$ for all $n\in\mathbb{N}$. If $x_n=x_{n+1}$ for some $n\in\mathbb{N}$, then $x^*=x_n$ is a fixed point for T. Assume that $x_n\neq x_{n+1}$ for all $n\in\mathbb{N}$. Since T is α-admissible we have

$$\alpha(x_0,x_1)=\alpha(x_0,Tx_0)\geq 1 \implies \alpha(Tx_0,T^2x_0)=\alpha(x_1,x_2)\geq 1.$$

By induction, we get

$$\alpha(x_n,x_{n+1})\geq 1 \quad \text{for all} \quad n\in\mathbb{N}. \tag{1}$$

Since T is generalized Lipscitz contraction, then

$$\begin{aligned} d(x_n,x_{n+1}) &= d(Tx_{n-1},Tx_n)\\ &\preceq kd(x_{n-1},x_n)\\ &\vdots\\ &\preceq k^n d(x_0,x_1). \end{aligned}$$

Thus, for $n<m$ we have

$$\begin{aligned} d(x_n,x_m) &\preceq d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+\cdots+d(x_{m-1},x_m)\\ &\preceq k^n d(x_0,x_1)+k^{n+1}d(x_0,x_1)+\cdots+k^{m-1}d(x_0,x_1)\\ &= (e+k+...+k^{m-n-1})k^n d(x_0,x_1)\\ &\preceq \left(\sum_{i=0}^{\infty}k^i\right)k^n d(x_0,x_1)\\ &= (e-k)^{-1}k^n d(x_0,x_1). \end{aligned}$$

Since $\rho(k)<1$, by Remark 2.5 we have $\|k^n\|\to 0$ as $n\to\infty$. Therefore, by Lemma 2.4 it follows that: for every $c\in A$ with $\theta\ll c$ there exists $n_0\in\mathbb{N}$ such that

$$d(x_n,x_m)\preceq (e-k)^{-1}k^n d(x_0,x_1)\ll c$$

for all $n>n_0$. It implies that $\{x_n\}$ is a Cauchy sequence. By completeness of X, there exists $x^*\in X$ such that $x_n\to x^*$ as $n\to\infty$. Since T is continuous, it follows that $x_{n+1}=Tx_n\to Tx^*$ as $n\to\infty$. By the uniqueness of limit we get $x^*=Tx^*$, that is x^* is a fixed point of T. □

In the above theorem, we use the continuity of the mapping T. Now, we show that the assumption of continuity can be replaced by another condition.

Theorem 3.2. *Let (X,d) be a complete cone metric space over a Banach algebra A and P be the underlying solid cone. Suppose, $T\colon X\to X$ be a generalized Lipschitz contraction with Lipschitz vector k and the following conditions are satisfied:*

(i) T is α-admissible;
(ii) there exists $x_0\in X$ such that $\alpha(x_0,Tx_0)\geq 1$;
(iii) if x_n is a sequence in X such that $\alpha(x_n,x_{n+1})\geq 1$ for all n and $x_n\to x\in X$ as $n\to\infty$, then $\alpha(x_n,x)\geq 1$ for all $n\in\mathbb{N}$.

Then T has a fixed point $x^*\in X$.

Proof. By proof of theorem 3.1, we know that the sequence $\{x_n\}$ is a Cauchy sequence in complete cone metric space (X,d). Then, there exists $x^*\in X$ such that $x_n\to x^*$ as $n\to\infty$. On the other hand, from (1) and hypothesis (iii), we have

$$\alpha(x_n,x^*)\geq 1,\quad \text{for all}\quad n\in\mathbb{N}. \tag{2}$$

Since T is a generalized Lipschitz contraction, using (2) we obtain

$$\begin{aligned} d(x^*,Tx^*) &\preceq d(x^*,x_{n+1})+d(x_{n+1},Tx^*)\\ &= d(x^*,x_{n+1})+d(Tx_n,Tx^*)\\ &\preceq d(x^*,x_{n+1})+kd(x_n,x^*). \end{aligned}$$

As $x_n\to x^*$ as $n\to\infty$, for every $c\in P$ with $\theta\ll c$ and for every $m\in\mathbb{N}$ there exists $n(m)$ such that $d(x_n,x^*)\ll\frac{c}{2m}$ for all $n>n(m)$. Therefore, $kd(x_n,x^*)\ll\frac{kc}{2m}$ and it follows from the above inequality that

$$d(x^*,Tx^*)\preceq\frac{c}{2m}+\frac{kc}{2m}=\frac{c}{2m}(e+k)\text{ for all }n>n(m),m\in\mathbb{N}.$$

It implies that $\frac{c}{2m}(e+k)-d(x^*,Tx^*)\in P$ for all $m\in\mathbb{N}$. Since P is closed, letting $m\to\infty$ we obtain $\theta-d(x^*,Tx^*)\in P$. By definition, we must have $d(x^*,Tx^*)=\theta$, i.e., $Tx^*=x^*$. Thus, x^* is a fixed point of T. □

Next, we give an example which illustrate the above result.

Example 3.3. Let $A=C^1_{\mathbb{R}}[0,1]\times C^1_{\mathbb{R}}[0,1]$ with the norm

$$\|(x_1,x_2)\|=\|x_1\|_\infty+\|x_2\|_\infty+\|x_1'\|_\infty+\|x_2'\|_\infty.$$

Define the multiplication on X by

$$xy=(x_1y_1,x_1y_2+x_2y_1)\quad\text{for all}\quad x=(x_1,x_2),y=(y_1,y_2)\in X.$$

Then, A is a Banach algebra with usual sum of functions and scalar product on cartesian product $C^1_{\mathbb{R}}[0,1]\times C^1_{\mathbb{R}}[0,1]$ and with unit $e=(0,1)$. Let $P=\{(x_1(t),x_2(t))\in A\colon x_1(t),x_2(t)\geq 0,t\in[0,1]\}$. Then P is a cone which is not normal.

Let $X=\mathbb{R}^2$ and define the cone metric $d\colon X\times X\to P$ by

$$d((x_1,x_2),(y_1,y_2))=(|x_1-y_1|,|x_2-y_2|)\,e^t\in P.$$

Then, (X,d) is a complete cone metric space. For a constant $a\in\mathbb{Q}$, define the mappings $T\colon X\to X$ and $\alpha\colon X\times X\to[0,\infty)$ by:

$$T(x_1,x_2)=\begin{cases}\left(\dfrac{x_1}{2},\dfrac{x_2}{3}+ax_1\right), & \text{if }(x_1,x_2)\in\mathbb{Q}\times\mathbb{Q};\\ (x_1,x_2), & \text{otherwise}\end{cases}$$

and

$$\alpha((x_1,x_2),(y_1,y_2)) = \begin{cases} 1, & \text{if } (x_1,x_2),(y_1,y_2) \in \mathbb{Q}\times\mathbb{Q}; \\ 0, & \text{otherwise.} \end{cases}$$

Then, T is a generalized Lipschitz contraction with Lipschitz vector $k = \left(\frac{1}{2}, a\right)$, where $\rho(k) = \frac{1}{2} < 1$. Indeed, $\alpha(x_1,x_2) \geq 1$ implies that $(x_1,x_2),(y_1,y_2) \in \mathbb{Q}\times\mathbb{Q}$. Therefore,

$$\begin{aligned} d(T(x_1,x_2),T(y_1,y_2)) &= \left(\frac{1}{2}|x_1-y_1|, \frac{1}{3}|x_2-y_2| + a|x_1-y_1|\right)e^t \\ &\preceq \left(\frac{1}{2}|x_1-y_1|, \frac{1}{2}|x_2-y_2| + a|x_1-y_1|\right)e^t \\ &= \left(\frac{1}{2}, a\right) d((x_1,x_2),(y_1,y_2)). \end{aligned}$$

Since $a \in \mathbb{Q}$, the mapping T is an α-admissible mapping, and for every $(x_1,x_2) \in \mathbb{Q}\times\mathbb{Q}$ we have $\alpha((x_1,x_2),T(x_1,x_2)) = 1$. Therefore, the conditions (i) and (ii) of Theorem 3.2 are satisfied. Finally, since $\mathbb{Q}$ is complete, the condition (iii) of Theorem 3.2 is satisfied. Thus, all the conditions of Theorem 3.2 are satisfied and we conclude the existence of at least one fixed point of T. Indeed, $(0,0)$ and all the points of $(X\times X)\setminus(\mathbb{Q}\times\mathbb{Q})$ are the fixed points of T.

Remark 3.4. In the above example, the mapping T is not a continuous mapping on the space X. Also, $\left(\frac{1}{2}, a\right) \not\preceq (1,0) = e$ and $\left\|\left(\frac{1}{2}, a\right)\right\| = \frac{1+2a}{2} > 1$ (for $a > 1$). For large enough a one can see that the mapping is not a contraction in the sense of Samet et al. [2] with respect to Euclidian metric on X. Again, it is easy to see that the mapping T is not a contraction in the sense of Liu and Xu [4], and so, we can not apply these known results on the mapping T. Moreover, following similar arguments to those in the Remark 2.3 of the paper [4] we can say that our results are actual generalization of the known results.

In the Example 3.3 we can see that the mapping T may have more than one fixed points. Let us denote the set of all fixed points of T by $\mathrm{Fix}(T)$.

Next, to assure the uniqueness of fixed point of a generalized Lipschitz mapping we use the following property (see [2]):

(H) $$\forall\, x,y \in \mathrm{Fix}(T)\ \exists\, z \in X\colon\ \ \alpha(x,z) \geq 1, \alpha(y,z) \geq 1.$$

Theorem 3.5. *Adding condition (H) to the hypothesis of Theorem 3.1 (resp. Theorem 3.2) we obtain the uniqueness of the fixed point of T.*

Proof. Following similar arguments to those in the proof of Theorem 3.1 (resp. Theorem 3.2) we obtain the existence of fixed point. Let the condition (H) is satisfied and $x^*, y^* \in \mathrm{Fix}(T)$ and $x^* \neq y^*$. By (H) there exists $z \in X$ such that

(3) $$\alpha(x^*, z) \geq 1 \quad \text{and} \quad \alpha(y^*, z) \geq 1.$$

Since T is α-admissible and $x^*, y^* \in \mathrm{Fix}(T)$, therefore from (3) we obtain

(4) $$\alpha(x^*, T^n z) \geq 1 \quad \text{and} \quad \alpha(y^*, T^n z) \geq 1. \quad \text{for all} \quad n \in \mathbb{N}.$$

Since T is generalized Lipschitz contraction, using (4), we have

$$\begin{aligned} d(x^*, T^n z) &= d(Tx^*, T(T^{n-1}z)) \\ &\preceq kd(x^*, T^{n-1}z) \\ &\vdots \\ &\preceq k^n d(x^*, z) \text{ for all } n \in \mathbb{N}. \end{aligned}$$

Since $\rho(k) < 1$, by Remark 2.5 we have $\|k^n\| \to 0$ as $n \to \infty$, and so,

$$\|k^n d(x^*, z)\| \leq \|k^n\| \|d(x^*, z)\| \to 0 \text{ as } n \to \infty.$$

Therefore, by Lemma 2.4 it follows that: for every $c \in A$ with $\theta \ll c$ there exists $n_0 \in \mathbb{N}$ such that

$$d(x^*, T^n z) \preceq k^n d(x^*, z) \ll c.$$

it implies that

$$T^n z \to x^* \text{ as } n \to \infty.$$

Similarly we get

$$T^n z \to y^* \text{ as } n \to \infty.$$

Therefore, by uniqueness of the limit we obtain $x^* = y^*$. This finishes the proof. □

4. Some consequences

In this section, we give some consequences of the results of previous section. The following result is an improved version of Theorem 2.1 of Liu and Xu [4].

Theorem 4.1 (Xu and Radenović [9]). *Let (X, d) be a complete cone metric space over a Banach algebra A and P be the underlying solid cone with $k \in P$ where $\rho(k) < 1$. Suppose the mapping $T\colon X \to X$ satisfies generalized Lipschitz condition :*

$$d(Tx, Ty) \preceq kd(x, y) \quad \text{for all} \quad x, y \in X.$$

Then T has a unique fixed point in X. Moreover, for any $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point of X.

Proof. Define the function $\alpha\colon X \times X \to [0, \infty)$ by $\alpha(x, y) = 1$ for all $x, y \in X$. Then, all the conditions of Theorem 3.5 are satisfied, and so, the mapping T has a unique fixed point in X. □

Next, we derive the ordered and cyclic versions of Banach contraction principle. In the next theorems, we generalize and unify the results of Ran and Reurings [1], Liu and Xu [4] and Nieto, Rodríguez-López [6] and Kirk et al. [12].

The following theorem is the cone metric version of Ran and Reurings [1] when the cone metric is endowed with a Banach algebra.

Theorem 4.2. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) be a complete cone metric space (X, d) over a Banach algebra A with P the underlying solid cone. Let $T\colon X \to X$ be a continuous nondecreasing mapping with respect to $\sqsubseteq$. Suppose that the following two assumptions hold:*

(i) there exists $k \in P$ such that $\rho(k) < 1$ and $d(Tx, Ty) \preceq kd(x, y)$ for all $x, y \in X$ with $x \sqsubseteq y$;

(ii) there exists $x_0 \in X$ such that $x_0 \sqsubseteq Tx_0$.

Then, T has a fixed point in X.

Proof. Define the mapping $\alpha_r : X \times X \to [0, \infty)$ by

$$\alpha_r(x, y) = \begin{cases} 1, & \text{if } x \sqsubseteq y; \\ 0, & \text{otherwise.} \end{cases}$$

Note that, the condition (i) implies that the mapping T a generalized Lipschitz contraction with Lipschitz vector k, where $\rho(k) < 1$. Since T is nondecreasing it is an α_r-admissible mapping. The condition (ii) implies that, there exists $x_0 \in X$ such that $\alpha_r(x_0, Tx_0) = 1$. Therefore, all the conditions of Theorem 3.1 are satisfied, and so, the mapping T has a fixed point in X. □

The following theorem is the cone metric version of Nieto, Rodríguez-López [6] when the cone metric is endowed with a Banach algebra.

Theorem 4.3. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) be a complete cone metric space (X, d) over a Banach algebra A with P the underlying solid cone. Let $T : X \to X$ be a nondecreasing mapping with respect to $\sqsubseteq$. Suppose that the following three assumptions hold:*

(i) there exists $k \in P$ such that $\rho(k) < 1$ and $d(Tx, Ty) \preceq kd(x, y)$ for all $x, y \in X$ with $x \sqsubseteq y$;
(ii) there exists $x_0 \in X$ such that $x_0 \sqsubseteq Tx_0$;
(iii) if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x \in X$ as $n \to \infty$, then $x_n \sqsubseteq x$ for all $n \in \mathbb{N}$.

Then, T has a fixed point in X.

Proof. Define the mapping $\alpha_r : X \times X \to [0, \infty)$ similar to that as in the proof of Theorem 4.2. Now, the proof follows from the Theorem 3.2. □

Next, we define the cyclic contractions (see [12]) in cone metric spaces.

Let X be a nonempty set, $T : X \to X$ a mapping and $A_1, A_2, \dots, A_m$ be subsets of X. Then $X = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to T if

(1) A_i, $i = 1, 2, \dots, m$ are nonempty sets;
(2) $T(A_1) \subset A_2, \dots, T(A_{m-1}) \subset T(A_m), T(A_m) \subset T(A_1)$.

Remark 4.4. (See [12]) If $X = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to T, then $\mathrm{Fix}(T) \subset \bigcap_{i=1}^{m} A_i$.

A cyclic contraction on a cone metric space is defined as follows.

Definition 4.5. Let (X, d) be a complete cone metric space over a Banach algebra A and P be the underlying solid cone. Suppose, $A_1, A_2, \dots, A_m$ be subsets of X and $Y = \bigcup_{i=1}^{m} A_i$. A mapping $T : Y \to Y$ is called a generalized cyclic Lipschitz contraction with Lipschitz vector k if following conditions hold:

(1) $Y = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to T;
(2) there exists $k \in P$ such that $\rho(k) < 1$ and

$$d(Tx, Ty) \preceq kd(x, y) \tag{5}$$

for any $x \in A_i, y \in A_{i+1}$ ($i = 1, 2, \dots, m$ where $A_{m+1} = A_1$).

The following theorem is the cone metric version of Kirk et al. [12] when the cone metric is endowed with a Banach algebra.

Theorem 4.6. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) be a complete cone metric space (X, d) over a Banach algebra A with P the underlying solid cone. Suppose, $A_1, A_2, \ldots, A_m$ be closed subsets of X and $Y = \bigcup_{i=1}^{m} A_i$ and $T\colon Y \to Y$ be a a generalized cyclic Lipschitz contraction with Lipschitz vector k. Then, T has a unique fixed point in X.*

Proof. Define the mapping $\alpha_c\colon X \times X \to [0, \infty)$ by:

$$\alpha_c(x, y) = \begin{cases} 1, & \text{if } (x, y) \in A_i \times A_{i+1} \ (i = 1, 2, \ldots, m \text{ where } A_{m+1} = A_1); \\ 0, & \text{otherwise.} \end{cases}$$

First, by definition of the function α and the cyclic representation, T is α_c-admissible. Again, by definition of the function α_c, T is a generalized cyclic Lipschitz contraction with Lipschitz vector k. Suppose, for a sequence $\{x_n\}$ we have $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$. Then, as $Y = \bigcup_{i=1}^{m} A_i$ is a cyclic representation with respect to T, we must have $x \in \bigcap_{i=1}^{m} A_i$. Therefore, $\alpha_c(x_n, x) \geq 1$ for all $n \in \mathbb{N}$. Now, the proof of existence of fixed point of T follows from Theorem 3.2. For uniqueness, if $x^*, y^* \in \mathrm{Fix}(T)$, then by Remark 4.4 we have $x^*, y^* \in \bigcap_{i=1}^{m} A_i$. Since each A_i, $i \in \{1, 2, \ldots, m\}$ is nonempty, there exists $z \in Y$ such that $x^*, y^* \in A_i, z \in A_{i+1}$ for some $i \in \{1, 2, \ldots, m\}$, and so $\alpha_c(x^*, z) = \alpha_c(y^*, z) = 1$. Thus, the condition (H) is satisfied and the uniqueness of fixed point follows from Theorem 3.5. □

References

[1] A.C.M. Ran, M.C.B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc., 132 (2003) 1435-1443.

[2] B. Samet, C. Vetro, and P. Vetro, Fixed point theorems for α-ψ-contractive type mappings, Nonlinear Analysis, 75 (2012) 2154-2165.

[3] H. Çakallı, A. Sönmez, Ç. Genç, On an equivalence of topological vector space valued cone metric spaces and metric spaces, Appl. Math. Lett., 25, (2012) 429-433.

[4] H. Liu and S.-Y. Xu, Cone metric spaces with Banach algebras and fixed point theorems of generalized Lipschitz mappings, Fixed Point Theory Appl., 2013, 2013:320.

[5] H. Liu and S.-Y. Xu, Fixed point theorems of quasi-contractions on cone metric spaces with Banach algebras, Abstarct and Applied Analysis, Volume 2013, Article ID 187348, 5 pages.

[6] J.J. Nieto, R. Rodríguez-López, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, Order 22 (2005) 223-239.

[7] L.-G. Huang and X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, J. Math. Anal. Appl., 332 (2007) 1468-1476.

[8] S. Radenović, B.E. Rhoades, Fixed point theorem for two non-self mappings in cone metric spaces, Comput. Math. Appl., 57, 1701-1707 (2009)

[9] S. Xu, S. Radenović, Fixed point theorems of generalized Lipschitz mappings on cone metric spaces over Banach algebras without assumption of normality, Fixed Point Theory Appl., 2014, 2014:102.

[10] Sh. Rezapour and R. Hamlbarani, Some notes on the paper Cone metric spaces and fixed point theorems of contractive mappings, Math. Anal. Appl., 345 (2008), 719-724.

[11] W. Rudin, Functional Analysis, 2nd ed., McGraw-Hill, 1991.

[12] W.A. Kirk, P.S. Srinivasan, P. Veeramani, Fixed points for mappings satisfying cyclical contractive conditions, Fixed Point Theory 4(1)(2003), 79-89.

[13] W.S. Du, A note on cone metric fixed point theory and its equivalence, Nonlinear Anal., 72(5), (2010) 2259-2261.
[14] Y. Feng, W. Mao, The equivalence of cone metric spaces and metric spaces, Fixed Point Theory, 11(2), (2010) 259-264.
[15] Z. Kadelburg, M. Pavlović, S. Radenović, Common fixed point theorems for ordered contractions and quasi-contractions in ordered cone metric spaces, Comput. Math. Appl. 59, 3148-3159 (2010)
[16] Z. Kadelburg, S. Radenović, V. Rakočević, A note on the equivalence of some metric and cone metric fixed point results, Appl. Math. Lett., 24, (2011) 370-374.

[1]Department of Mathematics, Govt. S.G.S.P.G. College Ganj Basoda, Distt Vidisha M.P., India

[2]Department of Mathematics, Choithram College of Professional Studies, Dhar Road, Indore (M.P.) 453001, India

[3]Department of Applied Mathematics, Shri Vaishnav Institute of Technology & Science, Gram Baroli, Sanwer Road, Indore (M.P.) 453331, India

*Corresponding author