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HARMONIC ANALYSIS ASSOCIATED WITH THE GENERALIZED WEINSTEIN OPERATOR

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Abstract. In this paper we consider a generalized Weinstein operator $\Delta_{d,\alpha,n}$ on $\mathbb{R}^{d-1}\times]0,\infty[$, which generalizes the Weinstein operator $\Delta_{d,\alpha}$, we define the generalized Weinstein intertwining operator $\mathcal{R}_{\alpha,n}$ which turn out to be transmutation operator between $\Delta_{d,\alpha,n}$ and the Laplacian operator Δ_d. We build the dual of the generalized Weinstein intertwining operator ${}^t\mathcal{R}_{\alpha,n}$, another hand we prove the formula related $\mathcal{R}_{\alpha,n}$ and ${}^t\mathcal{R}_{\alpha,n}$. We exploit these transmutation operators to develop a new harmonic analysis corresponding to $\Delta_{d,\alpha,n}$.

1. Introduction

In this paper we consider a generalized Weinstein operator $\Delta_{d,\alpha,n}$ on $\mathbb{R}^{d-1}\times]0,\infty[$, defined by

$$\Delta_{d,\alpha,n}=\sum_{i=1}^{d}\frac{\partial^2}{\partial x_i^2}+\frac{2\alpha+1}{x_d}\frac{\partial}{\partial x_d}-\frac{4n(\alpha+n)}{x_d^2},\quad \alpha>-\frac{1}{2} \tag{1}$$

where $n=0,1,\dots$. For $n=0$, we regain the Weinstein operator

$$\Delta_{d,\alpha}=\sum_{i=1}^{d}\frac{\partial^2}{\partial x_i^2}+\frac{2\alpha+1}{x_d}\frac{\partial}{\partial x_d},\quad \alpha>-\frac{1}{2} \tag{2}$$

Through this paper, we provide a new harmonic analysis on $\mathbb{R}^{d-1}\times]0,\infty[$ corresponding to the generalized Weinstein operator $\Delta_{d,\alpha,n}$.
The outline of the content of this paper is as follows.
Section 2 is dedicated to some properties and results concerning the Weinstein transform.
In section 3, we construct a pair of transmutation operators $\mathcal{R}_{\alpha,n}$ and ${}^t\mathcal{R}_{\alpha,n}$, afterwards we exploit these transmutation operators to build a new harmonic analysis on $\mathbb{R}^{d-1}\times]0,\infty[$ corresponding to operator $\Delta_{d,\alpha,n}$.

2. Preliminaries

Throughout this paper, we denote by

- $\mathbb{R}_+^d=\mathbb{R}^{d-1}\times]0,\infty[$.
- $x=(x_1,...,x_d)=(x',x_d)\in\mathbb{R}^{d-1}\times]0,\infty[$.
- $\lambda=(\lambda_1,...,\lambda_d)=(\lambda',\lambda_d)\in\mathbb{C}^d$.
- $E(\mathbb{R}^d)$ (resp. $D(\mathbb{R}^d)$) the space of $\mathcal{C}^\infty$ functions on $\mathbb{R}^d$, even with respect to the last variable (resp. with compact support).

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- $S(\mathbb{R}^d)$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}^d$ which are even with respect to the last variable.

In this section, we recapitulate some facts about harmonic analysis related to the Weinstein operator $\Delta_{d,\alpha}$. We cite here, as briefly as possible, some properties. For more details we refer to [2, 3, 4].
The Weinstein operator $\Delta_{d,\alpha}$ defined on $\mathbb{R}^d_+$ by

$$\Delta_{d,\alpha} = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} + \frac{2\alpha+1}{x_d}\frac{\partial}{\partial x_d}, \quad \alpha > -\frac{1}{2} \tag{3}$$

Then

$$\Delta_{d,\alpha} = \Delta_d + \mathcal{B}_\alpha$$

where Δ_d is the Laplacian operator in $\mathbb{R}^{d-1}$ and $\mathcal{B}_\alpha$ the Bessel operator with respect to the variable x_d defined by

$$\mathcal{B}_\alpha = \frac{\partial^2}{\partial x_d^2} + \frac{2\alpha+1}{x_d}\frac{\partial}{\partial x_d}, \quad \alpha > -\frac{1}{2}. \tag{4}$$

The Weinstein kernel is given by

$$\Psi_{\lambda,\alpha}(x) = e^{-i<x',\lambda'>} j_\alpha(x_d\lambda_d), \quad \text{for all } (x,\lambda) \in \mathbb{R}^d \times \mathbb{C}^d. \tag{5}$$

Here $x' = (x_1, ..., x_{d-1}), \lambda' = (\lambda_1, ..., \lambda_{d-1})$ and j_α is the normalized Bessel function of index α defined by

$$j_\alpha(z) = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{2})^{2n}}{n!\, \Gamma(n+\alpha+1)} \quad (z \in \mathbb{C}). \tag{6}$$

Proposition 1. $\Psi_{\lambda,\alpha}$ *satisfies the differential equation*

$$\Delta_{d,\alpha}\Psi_{\lambda,\alpha} = -\|\lambda\|^2 \Psi_{\lambda,\alpha}.$$

Definition 1. *The Weinstein intertwining operator is the operator* $\mathcal{R}_\alpha$ *defined on* $\mathcal{C}(\mathbb{R}^d)$ *by*

$$\mathcal{R}_\alpha f(x) = a_\alpha x_d^{-2\alpha} \int_0^{x_d} (x_d^2 - t^2)^{\alpha-\frac{1}{2}} f(x',t)dt, \quad x_d > 0 \tag{7}$$

where

$$a_\alpha = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}. \tag{8}$$

Proposition 2. $\mathcal{R}_\alpha$ *is a topological isomorphism from* $E(\mathbb{R}^d)$ *onto itself satisfying the following transmutation relation*

$$\Delta_{d,\alpha}(\mathcal{R}_\alpha f) = \mathcal{R}_\alpha(\Delta_d f), \quad \textit{for all } f \in E(\mathbb{R}^d), \tag{9}$$

where Δ_d *is the Laplacian on* $\mathbb{R}^d$.

Proposition 3. $\Delta_{d,\alpha}$ *is self-adjoint, i.e*

$$\int_{\mathbb{R}^d_+} \Delta_{d,\alpha} f(x) g(x) d\mu_\alpha(x) = \int_{\mathbb{R}^d_+} f(x) \Delta_{d,\alpha} g(x) d\mu_\alpha(x)$$

for all $f \in E(\mathbb{R}^d)$ *and* $g \in D(\mathbb{R}^d)$.

Definition 2. *The dual of the Weinstein intertwining operator* $\mathcal{R}_\alpha$ *is the operator* ${}^t\mathcal{R}_\alpha$ *defined on* $D(\mathbb{R}^d)$ *by*

$${}^t\mathcal{R}_\alpha(f)(y) = a_\alpha \int_{y_d}^{\infty} (s^2 - y_d^2)^{\alpha-\frac{1}{2}} f(y',s)sds. \tag{10}$$

Proposition 4. *${}^t\mathcal{R}_\alpha$ is a topological isomorphism from $S(\mathbb{R}^d)$ onto itself satisfying the following transmutation relation*

$$ {}^t\mathcal{R}_\alpha(\Delta_{d,\alpha} f) = \Delta_d({}^t\mathcal{R}_\alpha f), \quad \textit{for all } f \in S(\mathbb{R}^d), \tag{11}$$

where Δ_d is the Laplacian on $\mathbb{R}^d$.
It satisfies for $f \in D(\mathbb{R}^d)$ and $g \in E(\mathbb{R}^d)$ the following relation

$$\int_{\mathbb{R}^d_+} {}^t\mathcal{R}_\alpha(f)(y) g(y) dy = \int_{\mathbb{R}^d_+} f(y) \mathcal{R}_\alpha(g)(y) d\mu_\alpha(y). \tag{12}$$

Definition 3. *The Weinstein transform $\mathcal{F}_{W,\alpha}$ is defined on $L^1_\alpha(\mathbb{R}^d_+)$ by*

$$\mathcal{F}_{W,\alpha}(f)(\lambda) = \int_{\mathbb{R}^d_+} f(x) \Psi_{\lambda,\alpha}(x) d\mu_{\alpha(x)}, \textit{ for all } \lambda \in \mathbb{R}^d. \tag{13}$$

Proposition 5. (i) *For all $f \in L^1(\mathbb{R}^d_+)$, the function $\mathcal{F}_{W,\alpha}(f)$ is continuous on $\mathbb{R}^d$ and we have*

$$\|\mathcal{F}_{W,\alpha}(f)\|_{\alpha,\infty} \leq \|f\|_{\alpha,1}. \tag{14}$$

(ii) *For all $f \in S(\mathbb{R}^d)$ we have*

$$\mathcal{F}_{W,\alpha}(f)(y) = \mathcal{F}_0 \circ^t \mathcal{R}_\alpha(f)(y), \quad \forall y \in \mathbb{R}^d_+, \tag{15}$$

where $\mathcal{F}_0$ is the transformation defined by, for all $y \in \mathbb{R}^d_+$

$$\mathcal{F}_0(f)(y) = \int_{\mathbb{R}^d_+} f(x) e^{-i<y',x'>} \cos(x_d y_d) dx, \quad \forall f \in D(\mathbb{R}^d). \tag{16}$$

(iii) *For all $f \in S(\mathbb{R}^d)$ and $m \in \mathbb{N}$, we have*

$$\mathcal{F}_{W,\alpha}(\Delta_{d,\alpha} f)(\lambda) = -\|\lambda\|^2 \mathcal{F}_{W,\alpha}(f)(\lambda). \tag{17}$$

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Theorem 1. (i) ***Plancherel formula:*** *For all $f \in S(\mathbb{R}^d)$ we have*

$$\int_{\mathbb{R}^d_+} |f(x)|^2 d\mu_\alpha(x) = C(\alpha) \int_{\mathbb{R}^d_+} |\mathcal{F}_{W,\alpha}(f)(\lambda)|^2 d\mu_\alpha(\lambda) \tag{18}$$

where

$$C(\alpha) = \frac{1}{(2\pi)^{d-1} 2^{2\alpha} (\Gamma(\alpha+1))^2}. \tag{19}$$

(ii) *For all $f \in L^1_\alpha(\mathbb{R}^d_+)$, if $\mathcal{F}_{W,\alpha}(f) \in L^1_\alpha(\mathbb{R}^d_+)$, then*

$$f(y) = C(\alpha) \int_{\mathbb{R}^d_+} \mathcal{F}_{W,\alpha}(f)(x) \Psi_{\lambda,\alpha}(-x) d\mu_\alpha(x) \tag{20}$$

where $C(\alpha)$ is given by (19).

Definition 4. *The translation operators τ^x_α, $x \in \mathbb{R}^d_+$, associated with the operator $\Delta_{d,\alpha}$ are defined by*

$$\tau^x_\alpha f(y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}) \int_0^\pi f(x'+y', \sqrt{x_d^2 + y_d^2 + 2x_d y_d \cos\theta}) (\sin\theta)^{2\alpha} d\theta \tag{21}$$

where $f \in C(\mathbb{R}^d_+)$.

Proposition 6. *For all $f \in L^p_{\alpha,n}(\mathbb{R}^d)$, $p \in [1,\infty]$, and for all $x \in \mathbb{R}^d_+$*

$$\tau^x_{\alpha,n}(\Phi_{\lambda,\alpha,n}(y)) = \Phi_{\lambda,\alpha,n}(x) \Phi_{\lambda,\alpha,n}(y).$$

Proposition 7. *The translation operator $\tau_\alpha^x,\ x \in \mathbb{R}_+^d$ satisfies the following properties:*

(i) $\forall x \in \mathbb{R}_+^d$, *we have*

$$\Delta_{d,\alpha} \circ \tau_\alpha^x = \tau_\alpha^x \circ \Delta_{d,\alpha}. \tag{22}$$

(ii) *For all f in $E(\mathbb{R}^d)$ and g in $S(\mathbb{R}^d)$ we have*

$$\int_{\mathbb{R}_+^d} \tau_\alpha^x f(y) g(y) d\mu_\alpha(y) = \int_{\mathbb{R}_+^d} f(y) \tau_\alpha^x g(y) d\mu_\alpha(y). \tag{23}$$

(iii) *For all f in $L_\alpha^p(\mathbb{R}_+^d)$, $p \in [1, \infty]$, and $x \in \mathbb{R}_+^d$ we have*

$$\|\tau_\alpha^x\|_{p,\alpha} \leq \|f\|_{p,\alpha}. \tag{24}$$

(iv) *For $f \in S(\mathbb{R}^d)$ and $y \in \mathbb{R}_+^d$ we have*

$$\mathcal{F}_{W,\alpha}\left(\tau_\alpha^y f\right)(x) = \Psi_{y,\alpha}(x) \mathcal{F}_{W,\alpha}(f)(x). \tag{25}$$

Definition 5. *The generalized convolution product $f *_{W,\alpha} g$ of functions $f, g \in L_\alpha^1(\mathbb{R}_+^d)$ is defined by*

$$f *_{W,\alpha} g(x) = \int_{\mathbb{R}_+^d} \tau_\alpha^x f(-y', y) g(y) d\mu_\alpha(y). \tag{26}$$

Proposition 8. *For all $f, g \in L_\alpha^1(\mathbb{R}_+^d)$, $f *_{W,\alpha} g$ belongs to $L_\alpha^1(\mathbb{R}_+^d)$ and*

$$\mathcal{F}_{W,\alpha}\left(f *_{W,\alpha} g\right) = \mathcal{F}_{W,\alpha}(f) \mathcal{F}_{W,\alpha}(g). \tag{27}$$

3. Harmonic analysis associated with the generalized Weinstein operator

Transmutation operators.

- $\mathcal{M}_n$ the map defined by $\mathcal{M}_n f(x', x_d) = x_d^{2n} f(x', x_d)$.
- $L_{\alpha,n}^p(\mathbb{R}_+^d)$ the class of measurable functions f on $\mathbb{R}_+^d$ for which

$$\|f\|_{\alpha,n,p} = \|\mathcal{M}_n^{-1} f\|_{\alpha+2n,p} < \infty.$$

- $E_n(\mathbb{R}^d)$ (resp. $D_n(\mathbb{R}^d)$ and $S_n(\mathbb{R}^d)$) stand for the subspace of $E(\mathbb{R}^d)$ (resp. $D(\mathbb{R}^d)$ and $S(\mathbb{R}^d)$) consisting of functions f such that

$$f(x', 0) = \left(\frac{d^k f}{dx_d^k}\right)(x', 0) = 0,\ \forall k \in \{1, ...2n-1\}.$$

Lemma 1. (i) *The map*

$$\mathcal{M}_n(f)(x) = x_d^{2n} f(x) \tag{28}$$

is an isomorphism

- *from $E(\mathbb{R}^d)$ onto $E_n(\mathbb{R}^d)$.*
- *from $S(\mathbb{R}^d)$ onto $S_n(\mathbb{R}^d)$.*

(ii) *For all $f \in E(\mathbb{R}^d)$ we have*

$$\mathcal{B}_{\alpha,n} \circ \mathcal{M}_n(f) = \mathcal{M}_n \circ \mathcal{B}_{\alpha+2n}(f), \tag{29}$$

where $\mathcal{B}_{\alpha,n}$ is the generalized Bessel operator given by (4).

(iii) *For all $f \in E(\mathbb{R}^d)$*

$$\Delta_{d,\alpha,n} \circ \mathcal{M}_n(f)(x) = \mathcal{M}_n \circ \Delta_{d,\alpha+2n} \tag{30}$$

where $\Delta_{d,\alpha+2n}$ is the Weinstein operator of order $\alpha + 2n$ given by (3).

(iv) $\Delta_{d,\alpha,n}$ *is self-adjoint, i.e*

$$\int_{\mathbb{R}^d_+} \Delta_{d,\alpha,n} f(x) g(x) d\mu_\alpha(x) = \int_{\mathbb{R}^d_+} f(x) \Delta_{d,\alpha,n} g(x) d\mu_\alpha(x) \tag{31}$$

for all $f \in E(\mathbb{R}^d)$ and $g \in D_n(\mathbb{R}^d)$.

Proof. Assertion (i) and (ii) (see [1]).
For assertion (iii) using (1) and (29) we obtain

$$\begin{aligned}
\Delta_{d,\alpha,n} \circ \mathcal{M}_n(f)(x^{'}, x_d) &= (\Delta_d + \mathcal{B}_{\alpha,n}) \circ \mathcal{M}_n(f)(x^{'}, x_d), \\
&= \Delta_d(\mathcal{M}_n f)(x^{'}, x_d) + \mathcal{B}_{\alpha,n}(\mathcal{M}_n f)(x^{'}, x_d), \\
&= \mathcal{M}_n(\Delta_d f)(x^{'}, x_d) + \mathcal{M}_n(\mathcal{B}_{\alpha+2n} f)(x^{'}, x_d), \\
&= \mathcal{M}_n \circ \Delta_{d,\alpha+2n} f(x^{'}, x_d).
\end{aligned}$$

which give (iii).
If $f \in E(\mathbb{R}^d)$ and $g \in D_n(\mathbb{R}^d)$, then by Proposition 3 we get

$$\begin{aligned}
\int_{\mathbb{R}^d_+} \Delta_{d,\alpha,n} f(x) g(x) d\mu_\alpha(x) &= \int_{\mathbb{R}^d_+} \left(\Delta_{d,\alpha} f(x) - \frac{4n(\alpha+n)}{x_d^2} f(x) \right) g(x) d\mu_\alpha(x), \\
&= \int_{\mathbb{R}^d_+} \Delta_{d,\alpha} f(x) g(x) d\mu_\alpha(x) - \int_{\mathbb{R}^d_+} \frac{4n(\alpha+n)}{x_d^2} f(x) g(x) d\mu_\alpha(x), \\
&= \int_{\mathbb{R}^d_+} f(x) \Delta_{d,\alpha} g(x) d\mu_\alpha(x) - \int_{\mathbb{R}^d_+} \frac{4n(\alpha+n)}{x_d^2} f(x) g(x) d\mu_\alpha(x), \\
&= \int_{\mathbb{R}^d_+} f(x) \left(\Delta_{d,\alpha} g(x) - \frac{4n(\alpha+n)}{x_d^2} g(x) \right) d\mu_\alpha(x), \\
&= \int_{\mathbb{R}^d_+} f(x) \Delta_{d,\alpha,n} g(x) d\mu_\alpha(x).
\end{aligned}$$

■

Definition 6. *The generalized Weirstein intertwining operator is the operator $\mathcal{R}_{\alpha,n}$ defined on $E(\mathbb{R}^{d+1})$ by*

$$\mathcal{R}_{\alpha,n} f(x) = a_{\alpha+2n} x_d^{-2(\alpha+n)} \int_0^{x_d} (x_d^2 - t^2)^{\alpha+2n-\frac{1}{2}} f(x', t) dt, \quad x_d > 0 \tag{32}$$

where $a_{\alpha+2n}$ is given by 8.

Remark 1. *by (7) and (32) we have*

$$\mathcal{R}_{\alpha,n} = \mathcal{M}_n \circ \mathcal{R}_{\alpha+2n}. \tag{33}$$

Proposition 9. *$\mathcal{R}_{\alpha,n}$ is a topological isomorphism from $E(\mathbb{R})$ onto $E_n(\mathbb{R})$ satisfying the following transmutation relation*

$$\Delta_{d,\alpha,n}(\mathcal{R}_{\alpha,n} f) = \mathcal{R}_{\alpha,n}(\Delta_d f), \quad forall f \in E(\mathbb{R}^{d+1}) \tag{34}$$

where Δ_d is the Laplacian on $\mathbb{R}^d$.

Proof. Using (9), (30) and (33) we obtain

$$\begin{aligned}
\Delta_{d,\alpha,n}(\mathcal{R}_{\alpha,n} f) &= \Delta_{d,\alpha,n} \left(\mathcal{M}_n \circ \mathcal{R}_{\alpha+2n} \right)(f), \\
&= \mathcal{M}_n \circ \Delta_{d,\alpha+2n}(\mathcal{R}_{\alpha+2n} f) \\
&= \mathcal{M}_n \left(\mathcal{R}_{\alpha+2n} \circ \Delta_d \right)(f) \\
&= \mathcal{R}_{\alpha,n}(\Delta_d f).
\end{aligned}$$

■

Definition 7. *The dual of the generalized Weinstein intertwining operator* $\mathcal{R}_{\alpha,n}$ *is the operator* ${}^t\mathcal{R}_{\alpha,n}$ *defined on* $D_n(\mathbb{R}^d)$ *by*

$$ {}^t\mathcal{R}_{\alpha,n}(f)(y) = a_{\alpha+2n}\int_{y_d}^{\infty}(s^2-y_d^2)^{\alpha+2n-\frac{1}{2}}f(y',s)s^{1-2n}ds. \tag{35}$$

Remark 2. *From (10) and (35) we have*

$$ {}^t\mathcal{R}_{\alpha,n} = {}^t\mathcal{R}_{\alpha+2n}\circ\mathcal{M}_n^{-1}. \tag{36}$$

Proposition 10. ${}^t\mathcal{R}_{\alpha,n}$ *is a topological isomorphism from* $S_n(\mathbb{R}^{d+1})$ *onto* $S(\mathbb{R}^{d+1})$ *satisfying the following transmutation relation*

$$ {}^t\mathcal{R}_{\alpha,n}(\Delta_{d,\alpha,n}f) = \Delta_d({}^t\mathcal{R}_{\alpha,n}f), \quad \text{for all } f\in S_n(\mathbb{R}^{d+1}) \tag{37}$$

where Δ_d *is the Laplacian on* $\mathbb{R}^d$.

Proof. An easily combination of (11), (30) and (37) shows that

$$\begin{aligned} {}^t\mathcal{R}_{\alpha,n}(\Delta_{d,\alpha,n}f) &= {}^t\mathcal{R}_{\alpha+2n}\circ\mathcal{M}_n^{-1}\left(\mathcal{M}_n\circ\Delta_{d,\alpha+2n}\circ\mathcal{M}_n^{-1}\right)(f),\\ &= {}^t\mathcal{R}_{\alpha+2n}\left(\Delta_{d,\alpha+2n}\circ\mathcal{M}_n^{-1}\right)(f),\\ &= \Delta_d\left(\mathcal{R}_{\alpha+2n}\circ\mathcal{M}_n^{-1}\right)(f),\\ &= \Delta_d({}^t\mathcal{R}_{\alpha,n}f). \end{aligned}$$

■

Proposition 11. *For all* $f\in D_n(\mathbb{R}^d)$ *and* $g\in E(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d_+}{}^t\mathcal{R}_{\alpha,n}(f)(y)g(y)dy = \int_{\mathbb{R}^d_+}f(y)\mathcal{R}_{\alpha,n}(g)(y)d\mu_\alpha(y). \tag{38}$$

Proof. Using (12), (33) and (37)

$$\begin{aligned} \int_{\mathbb{R}^d_+}{}^t\mathcal{R}_{\alpha,n}(f)(x)g(x)dx &= \int_{\mathbb{R}^d_+}{}^t\mathcal{R}_{\alpha+2n}\circ\mathcal{M}_n^{-1}f(x)g(x)dx\\ &= \int_{\mathbb{R}^d_+}\mathcal{M}_n^{-1}f(x){}^t\mathcal{R}_{\alpha+2n}(g)(x)d\mu_{\alpha+2n}(x)\\ &= \int_{\mathbb{R}^d_+}f(x)\mathcal{M}_n(\mathcal{R}_{\alpha+2n}(g))(x)d\mu_\alpha(x)\\ &= \int_{\mathbb{R}^d_+}f(y)\mathcal{R}_{\alpha,n}(g)(y)d\mu_\alpha(y). \end{aligned}$$

■

Generalized Weinstein transform.
Throughout this section assume $\alpha > -\frac{1}{2}$ and n a non-negative integer.
For all $\lambda = (\lambda_1, ..., \lambda_d)\in\mathbb{C}^d$ and $x = (x_1, ..., x_d)\in\mathbb{R}^d$, put

$$\Phi_{\lambda,\alpha,n}(x) = x_d^{2n}\Psi_{\lambda,\alpha+2n}(x) \tag{39}$$

where $\Psi_{\lambda,\alpha+2n}(x)$ is the Weinstein kernel of index $\alpha+2n$ is given by (5).

Proposition 12. $\Phi_{\lambda,\alpha,n}$ *satisfies the differential equation*

$$\Delta_{d,\alpha,n}\Phi_{\lambda,\alpha,n} = -\|\lambda\|^2\Phi_{\lambda,\alpha,n}. \tag{40}$$

Proof. From Proposition 1 and (39) we obtain

$$\begin{aligned}\Delta_{d,\alpha,n}\Phi_{\lambda,\alpha,n} &= \mathcal{M}_n\circ\Delta_{d,\alpha+2n}\mathcal{M}_n^{-1}\Phi_{\lambda,\alpha,n},\\ &= \mathcal{M}_n\circ\Delta_{d,\alpha+2n}\Psi_{\lambda,\alpha+2n},\\ &= -\|\lambda\|^2\mathcal{M}_n\Psi_{\lambda,\alpha+2n},\\ &= -\|\lambda\|^2\Phi_{\lambda,\alpha,n}.\end{aligned}$$

■

Definition 8. *The generalized Weinstein transform is defined on $L^1_{\alpha,n}(\mathbb{R}^d_+)$ by, for all $\lambda\in\mathbb{R}^d$*

(41)
$$\mathcal{F}_{W,\alpha,n}(f)(\lambda)=\int_{\mathbb{R}^d_+}f(x)\Phi_{\lambda,\alpha,n}(x)d\mu_\alpha(x).$$

Remark 3. *By (5), (13) and (41), we have*

(42)
$$\mathcal{F}_{W,\alpha,n}=\mathcal{F}_{W,\alpha+2n}\circ\mathcal{M}_n^{-1}.$$

Theorem 2. (i) *Inverse formula: Let $f\in L^1_{\alpha,n}(\mathbb{R}^d_+)$, if $\mathcal{F}_{W,\alpha,n}\in L^1_\alpha(\mathbb{R}^d_+)$ then*

(43)
$$f(x)=C(\alpha+2n)\int_{\mathbb{R}^d_+}\mathcal{F}_{W,\alpha,n}f(\lambda)\Phi_{\lambda,\alpha,n}(x)d\mu_{\alpha+2n}(\lambda).$$

(ii) *Plancherel formula:*

(44)
$$\int_{\mathbb{R}^d_+}|f(x)|^2d\mu_\alpha(x)=C(\alpha+2n)\int_{\mathbb{R}^d_+}|\mathcal{F}_{W,\alpha,n}f(\lambda)|^2d\mu_{\alpha+2n}(\lambda)$$

where $C(\alpha+2n)$ is given by (19).

Proof. By (20), (39) and (42) we obtain

$$\begin{aligned}C(\alpha+2n)\int_{\mathbb{R}^d_+}\mathcal{F}_{W,\alpha,n}f(\lambda)\Phi_{\lambda,\alpha,n}(x)d\mu_{\alpha+2n}(\lambda) &= C(\alpha+2n)\int_{\mathbb{R}^d_+}\mathcal{F}_{W,\alpha,n}f(\lambda)x_d^{2n}\Psi_{\lambda,\alpha+2n}(x)d\mu_{\alpha+2n}(\lambda),\\ &= x_d^{2n}C(\alpha+2n)\int_{\mathbb{R}^d_+}\mathcal{F}_{W,\alpha+2n}\left(\mathcal{M}_n^{-1}f\right)(\lambda)\Psi_{\lambda,\alpha+2n}(x)d\mu_{\alpha+2n}(\lambda),\\ &= x_d^{2n}\mathcal{M}_n^{-1}f(x),\\ &= f(x).\end{aligned}$$

which proves (i).
For (ii) an easily combination of (18), (39) and (42) shows that

$$\begin{aligned}\int_{\mathbb{R}^d_+}|f(x)|^2d\mu_\alpha(x) &= \int_{\mathbb{R}^d_+}|\mathcal{M}_n^{-1}f(x)|^2d\mu_{\alpha+2n}(x),\\ &= C(\alpha+2n)\int_{\mathbb{R}^d_+}\left|\mathcal{F}_{W,\alpha+2n}\left(\mathcal{M}_n^{-1}f(\lambda)\right)\right|^2d\mu_{\alpha+2n}(\lambda),\\ &= C(\alpha+2n)\int_{\mathbb{R}^d_+}|\mathcal{F}_{W,\alpha,n}f(\lambda)|^2d\mu_{\alpha+2n}(\lambda).\end{aligned}$$

■

Proposition 13. (i) *For all $f\in L^1_{\alpha,n}(\mathbb{R}^d_+)$, we have*

$$\|\mathcal{F}_{W,\alpha,n}(f)\|_{\alpha,\infty}\le\|f\|_{\alpha,n,1}.$$

(ii) *For all $f \in S_n(\mathbb{R}^d)$ we have*

$$\mathcal{F}_{W,\alpha,n}(f)(y) = \mathcal{F}_0 \circ^t \mathcal{R}_{\alpha,n}(f)(y), \quad \forall y \in \mathbb{R}^d_+,$$

where $\mathcal{F}_0$ is the transformation defined by ().

(iii) *For all $f \in S_n(\mathbb{R}^d)$ and $m \in \mathbb{N}$, we have*

$$\mathcal{F}_{W,\alpha,n}(\Delta_{d,\alpha} f)(\lambda) = -\|\lambda\|^2 \mathcal{F}_{W,\alpha,n}(f)(\lambda).$$

Proof. From (14) and (42) we have

$$\begin{aligned} \|\mathcal{F}_{W,\alpha,n}(f)\|_{\alpha,n,\infty} &= \|\mathcal{F}_{W,\alpha+2n} \circ \mathcal{M}_n^{-1}(f)\|_{\alpha,n,\infty} \\ &\leq \|\mathcal{M}_n^{-1} f\|_{\alpha+2n,1} \\ &\leq \|f\|_{\alpha,n,1} \end{aligned}$$

which proves assertion (i).
By (15), (36) and (42) we obtain

$$\begin{aligned} \mathcal{F}_{W,\alpha,n}(f) &= \mathcal{F}_{W,\alpha+2n} \circ \mathcal{M}_n^{-1}(f) \\ &= \mathcal{F}_0 \circ \ {}^t\mathcal{R}_{\alpha+2n} \circ \mathcal{M}_n^{-1}(f) \\ &= \mathcal{F}_0 \circ \ {}^t\mathcal{R}_{\alpha,n}(f), \end{aligned}$$

which proves assertion (ii).
Due to (16), (33) and (42) we have

$$\begin{aligned} \mathcal{F}_{W,\alpha,n}(\Delta_{d,\alpha,n} f)(\lambda) &= \mathcal{F}_{d,\alpha+2n} \circ \mathcal{M}_n^{-1}(\Delta_{d,\alpha,n} f)(\lambda) \\ &= \mathcal{F}_{W,\alpha+2n} \circ \mathcal{M}_n^{-1}(\Delta_{d,\alpha,n} f)(\lambda) \\ &= \mathcal{F}_{W,\alpha+2n}(\Delta_{d,\alpha+2n} \mathcal{M}_n^{-1} f)(\lambda) \\ &= -\|\lambda\|^2 \mathcal{F}_{W,\alpha+2n} \circ \mathcal{M}_n^{-1}(f)(\lambda) \\ &= -\|\lambda\|^2 \mathcal{F}_{W,\alpha,n}(f)(\lambda). \end{aligned}$$

■

Generalized convolution product.

Definition 9. *The generalized translation operators $\tau^x_{\alpha,n}$, $x \in \mathbb{R}^d$ associated with $\Delta_{d,\alpha,n}$ are defined on $\mathbb{R}^d_+$ by*

(45) $$\tau^x_{\alpha,n} f = x_d^{2n} \mathcal{M}_n \tau^x_{\alpha+2n} \mathcal{M}_n^{-1} f$$

where $\tau^x_{\alpha+2n}$ are the Weinstein translation operators of order $\alpha + 2n$ given by (21).

Definition 10. *The generalized convolution product of two functions $f \in E(\mathbb{R}^d)$ and $g \in D(\mathbb{R}^d)$ is defined by:*

(46) $$f *_{W,\alpha,n} g(x) = \int_{\mathbb{R}^d_+} \tau^x_{\alpha,n} f(y) g(y) d\mu_\alpha(y); \quad \forall x \in \mathbb{R}^d_+.$$

Proposition 14. *Let f and g in $D_n(\mathbb{R}^d)$, we have*

(47) $$f *_{W,\alpha,n} g = \mathcal{M}_n \left(\mathcal{M}_n^{-1} f *_{W,\alpha+2n} \mathcal{M}_n^{-1} g \right).$$

Proof. Using (23) and (45) we get

$$\begin{aligned} f *_{W,\alpha,n} g(x) &= \int_{\mathbb{R}^d_+} \tau^x_{\alpha,n} f(y) g(y) d\mu_\alpha(y) \\ &= \int_{\mathbb{R}^d_+} x_d^{2n} \mathcal{M}_n \tau^x_{\alpha+2n} \mathcal{M}_n^{-1} f(y) g(y) d\mu_\alpha(y) \\ &= x_d^{2n} \int_{\mathbb{R}^d_+} \tau^x_{\alpha+2n} \mathcal{M}_n^{-1} f(y) \mathcal{M}_n^{-1} g(y) d\mu_{\alpha+2n}(y) \\ &= \mathcal{M}_n \left(\mathcal{M}_n^{-1} f *_{W,\alpha+2n} \mathcal{M}_n^{-1} g \right)(x). \end{aligned}$$

■

Proposition 15. (i) *For all* $f \in L^p_{\alpha,n}(\mathbb{R}^d)$, $p \in [1,\infty]$, *and for all* $x \in \mathbb{R}^d_+$

(48) $$\|\tau^x_{\alpha,n} f\|_{p,\alpha,n} \leq x_d^{2n} \|f\|_{p,\alpha,n}.$$

(ii)

(49) $$\tau^x_{\alpha,n}(\Phi_{\lambda,\alpha,n}(y)) = \Phi_{\lambda,\alpha,n}(x)\Phi_{\lambda,\alpha,n}(y).$$

Proof. From (24) and (45) we have

$$\begin{aligned} \|\tau^x_{\alpha,n} f\|_{p,\alpha,n} &= x_d^2 \|\mathcal{M}_n \tau^x_{\alpha+2n} \mathcal{M}_n^{-1} f\|_{p,\alpha,n} \\ &= x_d^2 \|\tau^x_{\alpha+2n} \mathcal{M}_n^{-1} f\|_{p,\alpha+2n} \\ &\leq x_d^2 \|\tau^x_{\alpha+2n} \mathcal{M}_n^{-1} f\|_{p,\alpha+2n} \\ &\leq x_d^2 \|\mathcal{M}_n^{-1} f\|_{p,\alpha+2n} \\ &= x_d^{2n} \|f\|_{p,\alpha,n}. \end{aligned}$$

which give (i).
From (39), (45) and Proposition 6 we get

$$\begin{aligned} \tau^x_{\alpha,n} \Phi_{\lambda,\alpha,n}(y) &= x_d^{2n} \mathcal{M}_n \circ \tau^x_{\alpha+2n} \circ \mathcal{M}_n^{-1} \Phi_{\lambda,\alpha,n}(y) \\ &= x_d^{2n} \mathcal{M}_n \circ \tau^x_{\alpha+2n} \Psi_{\lambda,\alpha,n}(y) \\ &= x_d^{2n} y_d^{2n} \tau^x_{\alpha+2n} \Psi_{\lambda,\alpha,n}(y) \\ &= x_d^{2n} y_d^{2n} \Psi_{\lambda,\alpha,n}(x) \Psi_{\lambda,\alpha,n}(y) \\ &= \Phi_{\lambda,\alpha,n}(x)\Phi_{\lambda,\alpha,n}(y). \end{aligned}$$

which prove (ii). ■

Theorem 3. (i) *For* $f \in S(\mathbb{R}^d)$ *and* $y \in \mathbb{R}^d_+$

(50) $$\mathcal{F}_{W,\alpha,n}\left(\tau^x_{\alpha,n} f\right)(\lambda) = \Phi_{\lambda,\alpha,n}(x)\mathcal{F}_{W,\alpha,n}(f(\lambda)), \quad \lambda \in \mathbb{R}^d_+.$$

(ii) *For all* $f \in E(\mathbb{R}^d)$ *and* $g \in S(\mathbb{R}^d)$

(51) $$\int_{\mathbb{R}^d_+} \tau^x_{\alpha,n} f(y) g(y) d\mu_\alpha(y) = \int_{\mathbb{R}^d_+} f(y) \tau^x_{\alpha,n} g(y) d\mu_\alpha(y).$$

(iii) *For all* $f, g \in L^1_\alpha(\mathbb{R}^d_+)$, $f *_{W,\alpha,n} g \in L^1_\alpha(\mathbb{R}^d_+)$, *and*

(52) $$\mathcal{F}_{W,\alpha,n}\left(f *_{W,\alpha,n} g\right) = \mathcal{F}_{W,\alpha,n}(f)\mathcal{F}_{W,\alpha,n}(g).$$

Proof. An easily combination of (25), (39), (42) and (45) shows that

$$\begin{aligned}\mathcal{F}_{W,\alpha,n}\left(\tau^x_{\alpha,n}f\right)(\lambda) &= x_d^{2n}\mathcal{F}_{W,\alpha+2n}\left(\tau^x_{\alpha+2n}\mathcal{M}_n^{-1}f\right)(\lambda),\\ &= x_d^{2n}\Psi_{\lambda,\alpha+2n}(x)\mathcal{F}_{W,\alpha+2n}\mathcal{M}_n^{-1}(f)(\lambda),\\ &= \Phi_{\lambda,\alpha,n}(x)\mathcal{F}_{W,\alpha,n}(f(\lambda)).\end{aligned}$$

which prove (i).
For assertion (ii) using (23) and (45) we obtain

$$\begin{aligned}\int_{\mathbb{R}^d_+}\tau^x_{\alpha,n}f(y)g(y)d\mu_\alpha(y) &= x_d^2\int_{\mathbb{R}^d_+}\tau^x_{\alpha+2n}\left(\mathcal{M}_n^{-1}f(y)\right)\left(\mathcal{M}_n^{-1}g(y)\right)d\mu_{\alpha+2n}(y),\\ &= x_d^2\int_{\mathbb{R}^d_+}\left(\mathcal{M}_n^{-1}f(y)\right)\tau^x_{\alpha+2n}\left(\mathcal{M}_n^{-1}g(y)\right)d\mu_{\alpha+2n}(y),\\ &= \int_{\mathbb{R}^d_+}f(y)\tau^x_{\alpha,n}g(y)d\mu_\alpha(y).\end{aligned}$$

which prove (ii).
For the last assertion using (47) we get

$$f *_{W,\alpha,n} g = \mathcal{M}_n\left[(\mathcal{M}_n^{-1}f) *_{W,\alpha+2n}(\mathcal{M}_n^{-1}g)\right]$$

using (27) and (42) we get

$$\begin{aligned}\mathcal{F}_{W,\alpha,n}(f *_{W,\alpha,n} g) &= \mathcal{F}_{W,\alpha,n}\circ\mathcal{M}_n\left[(\mathcal{M}_n^{-1}f) *_{W,\alpha+2n}(\mathcal{M}_n^{-1}g)\right]\\ &= \mathcal{F}_{W,\alpha+2n}\circ\mathcal{M}_n^{-1}\circ\mathcal{M}_n\left[(\mathcal{M}_n^{-1}f) *_{W,\alpha+2n}(\mathcal{M}_n^{-1}g)\right]\\ &= \mathcal{F}_{W,\alpha+2n}\left[(\mathcal{M}_n^{-1}f) *_{W,\alpha+2n}(\mathcal{M}_n^{-1}g)\right]\\ &= \mathcal{F}_{W,\alpha+2n}(\mathcal{M}_n^{-1}f)\mathcal{F}_{W,\alpha+2n}(\mathcal{M}_n^{-1}g)\\ &= \mathcal{F}_{W,\alpha,n}(f)\mathcal{F}_{W,\alpha,n}(g).\end{aligned}$$

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