International Journal of Analysis and Applications
ISSN 2291-8639
Volume 9, Number 1 (2015), 29-38
http://www.etamaths.com

REVERSE OF THE TRIANGLE INEQUALITY IN HILBERT C^*-MODULES

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Abstract. In this paper we prove the reverse of triangle inequality via Selberg's inequalities in the framework of Hilbert C^*-modules.

1. Introduction

In 1966, Diaz and Matcalf [4] proved the following reverse triangle inequality in setting of Hilbert spaces as follows .

Theorem 1.1. *Let $x_1, ..., x_n$ be vectors in a Hilbert space $\mathcal{H}$. If e is a unit vector of $\mathcal{H}$ such that $0 \leq r \leq \frac{Re\langle x_i, e\rangle}{||x_i||}$ for some $r \in \mathbb{R}$ and each $1 \leq i \leq n$, then*

$$r \sum_{j=1}^{n} \|x_i\| \leq \left\| \sum_{i=1}^{n} x_i \right\|$$

A number of mathematicians have represented several refinements of the reverse triangle inequality in Hilbert spaces and normed spaces, see[1, 2, 5, 8, 9, 12, 13]

Recently, M. Khosravi, H. Mahyar and M.S. Moslehian [12] obtained the following reverse of the triangle inequality in the framework of Hilbert C^*-modules.

Theorem 1.2. *Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module and $e_1, ..., e_m \in \mathcal{X}$ be a family of vectors with $\langle e_i, e_j\rangle = 0$ $(1 \leq i \neq j \leq m)$ and $||ei|| = 1$ $(1 \leq i \leq m)$. If the vectors $x_1, \ldots, x_n$ in $\mathcal{X}$ satisfy the conditions*

$$Re\, \langle e_k, x_j\rangle \geq \rho_k \, \|x_j\| \;\; , \;\; Im\, \langle e_k, x_j\rangle \geq \mu_k \, \|x_j\|$$

for $j \in \{1, \ldots, n\}$, $k \in \{1, \ldots, m\}$, where $\rho_k, \mu_k \in [0, \infty)\, k \in \{1, \ldots, m\}$, then

$$\left(\sum_{k=1}^{m} (\rho_k^2 + \mu_k^2)\right)^{\frac{1}{2}} \sum_{j=1}^{n} \|x_j\| \leq \left\| \sum_{j=1}^{n} x_j \right\| .$$

In [3] we obtained an extension of Selberg's inequality in the framework of Hilbert C^*-modules.

The goal of this paper is to show the reverse of triangle inequality via a extension of Selberg's inequality in the framework of Hilbert C^*-modules. Our results are extensions of theorem 2.1 and Corollary 2.3 obtained by Dragomir in [5] and theorem 9 obtained by Fujii and Nakamoto see [9] in the setting of Hilbert C^*-modules.

2010 *Mathematics Subject Classification.* Primary 46L08; Secondary 26D15, 46L05.
Key words and phrases. triangle inequality; reverse inequality; Hilbert C^*-module; C^*-algebra.

2. Preliminaries

In this section we briefly recall the definitions and examples of Hilbert C^*-modules. For information about Hilbert C^*-module, we refer to [10, 13]. Our reference for C^*-algebras as [15].

Let $\mathcal{A}$ be a C^*-algebra (not necessarily unitary) and $\mathcal{X}$ be a complex linear space.

Definition 2.1. A pre-Hilbert $\mathcal{A}$-module is a right $\mathcal{A}$-module $\mathcal{X}$ equipped with a sesquilinear map $\langle .,. \rangle : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ satisfying

(1) $\langle x, x\rangle \geq 0; \langle x, x\rangle = 0$ if and only if $x = 0$ for all x in $\mathcal{X}$,
(2) $\langle x, \alpha y + \beta z\rangle = \alpha \langle x, y\rangle + \beta(\langle x, z\rangle)$ for all x, y, z in $\mathcal{X}, \alpha, \beta$ in $\mathbb{C}$,
(3) $\langle x, y\rangle = \langle y, x\rangle^*$ for all x, y in $\mathcal{X}$,
(4) $\langle x, y.a\rangle = \langle x, y\rangle a$ for all x, y in $\mathcal{X}$, a in $\mathcal{A}$.

The map $\langle .,. \rangle$ is called an $\mathcal{A}$-valued inner product of $\mathcal{X}$ and for $x \in \mathcal{X}$, we define $||x|| = ||\langle x, x\rangle||^{\frac{1}{2}}$ as a norm on $\mathcal{X}$, where the latter is a norm in the C^*-algebra $\mathcal{A}$. This norm makes $\mathcal{X}$ into a right normed module over $\mathcal{A}$. A pre-Hilbert module $\mathcal{X}$ is called a Hilbert $\mathcal{A}$-module if it is complete with respect to its norm. Two typical examples of Hilbert C^*-modules are as follows:
(I) Every Hilbert space is a Hilbert C^*-module.
(II) Every C^*algebra $\mathcal{A}$ is a Hilbert $\mathcal{A}$ -module via $\langle a, b\rangle = a^*b \quad (a, b \in \mathcal{A})$.
Notice that the inner product structure of a C^*-algebra is essentially more complicated than complex numbers. One may define an $\mathcal{A}$ -valued norm $|.|$ by $|x| = \langle x, x\rangle^{\frac{1}{2}}$. Clearly, $\|x\| = \||x|\|$ for each $x \in \mathcal{X}$.

It is known that $|.|$ does not satisfy the triangle inequality in general (see [[13], p.4]). We also use the elementary C^*-algebra theory, we use the following property: if $a \leq b$ then $a^{\frac{1}{2}} \leq b^{\frac{1}{2}}$,where a, b are positive elements of C^*-algebra $\mathcal{A}$, and the relation $\frac{1}{2}(aa^* + a^*a) = Re(a)^2 + Im(a)^2$ where a is an arbitrary element of $\mathcal{A}$

3. Main result

Let $\mathcal{X}$ be a right Hilbert $\mathcal{A}$-module, which is an algebraic left $\mathcal{A}$-module satisfying:

$$\langle x, ay\rangle = a \langle x, y\rangle \ \ for\ all\ x, y \in \mathcal{X}\ and\ a \in \mathcal{A}.$$

For example if $\mathcal{A}$ is a unital C^*-algebra and $\mathcal{I}$ is a commutative right ideal of $\mathcal{A}$, then $\mathcal{I}$ is a right Hilbert module over $\mathcal{A}$ and

$$\langle x, ay\rangle = x^*(ay) = ax^*y \quad (x, y \in \mathcal{I}, a \in \mathcal{A}).$$

For a reverse of triangle inequality, we use the following lemma.

Lemma 3.1. *Let $\mathcal{X}$ be a right Hilbert $\mathcal{A}$- module which is an algebraic left $\mathcal{A}$-module and $y_1, \ldots, y_m$ be a non zero vectors in $\mathcal{X}$. If $x \in \mathcal{X}$ then*

$$\sum_{j=1}^{m} \frac{|\langle y_j, x\rangle|^2}{\sum_{k=1}^{m} \|\langle y_j, y_k\rangle\|} \leq |x|^2 \tag{3.1}$$

and

$$\sum_{j=1}^{m} \frac{|\langle x, y_j\rangle|^2}{\sum_{k=1}^{m} \|\langle y_j, y_k\rangle\|} \leq |x|^2 \tag{3.2}$$

Proof. Inequality (3.1) is proved in [[3], theorem 3.1].
Next we prove the inequality (3.2). Let $\alpha_j \in \mathcal{A}, j = 1, \dots, n$. We know that

$$\begin{aligned}
0 &\leq \left| x - \sum_{j=1}^{m} \alpha_j y_j \right|^2 = \left\langle x - \sum_{j=1}^{m} \alpha_j y_j, x - \sum_{j=1}^{m} \alpha_j y_j \right\rangle \\
&= \langle x, x\rangle - \left\langle x, \sum_{j=1}^{m} \alpha_j y_j \right\rangle - \left\langle \sum_{j=1}^{m} \alpha_j y_j, x \right\rangle + \left\langle \sum_{j=1}^{m} \alpha_j y_j, \sum_{j=1}^{m} \alpha_j y_j \right\rangle \\
&= |x|^2 - \sum_{j=1}^{m} \alpha_j \langle x, y_j\rangle - \sum_{j=1}^{m} \langle y_j, x\rangle \alpha_j^* + \sum_{j,k=1}^{m} \alpha_j \langle y_k, y_j\rangle \alpha_k^* \\
&= |x|^2 - \sum_{j=1}^{m} \alpha_j \langle x, y_j\rangle - \sum_{j=1}^{m} \langle y_j, x\rangle \alpha_j^* \\
&+ \frac{1}{2} \sum_{j,k=1}^{m} (\alpha_j \langle y_k, y_j\rangle \alpha_k^* + \alpha_k \langle y_j, y_k\rangle \alpha_j^*).
\end{aligned}$$

It follows from [[6], lemma 3.2] that

$$\alpha_j \langle y_j, y_k\rangle \alpha_k^* + \alpha_k \langle y_k, y_j\rangle \alpha_j^* \leq \left|\alpha_j^*\right|^2 \|\langle y_j, y_k\rangle\| + |\alpha_k^*|^2 \|\langle y_j, y_k\rangle\|,$$

so

$$\begin{aligned}
\left| x - \sum_{j=1}^{m} \alpha_j y_j \right|^2 &\leq |x|^2 - \sum_{j=1}^{m} \alpha_j \langle x, y_j\rangle - \sum_{j=1}^{m} \langle y_j, x\rangle \alpha_j^* \\
&+ \frac{1}{2} \sum_{j,k=1}^{m} (|\alpha_j^*|^2 | \langle y_j, y_k\rangle | + |\alpha_k^*|^2 \|\langle y_k, y_j\rangle\|),
\end{aligned}$$

Take

$$\alpha_j = \frac{\langle y_j, x\rangle}{\sum_{k=1}^{m} \|\langle y_j, y_k\rangle\|},$$

A simple calculation shows that

$$\begin{aligned}
\left| x - \sum_{j=1}^{m} \alpha_j y_j \right|^2 &\leq |x|^2 - \sum_{j=1}^{m} \frac{|\langle x, y_j\rangle|^2}{\sum_{k=1}^{m} \|\langle y_j, y_k\rangle\|} - \sum_{j=1}^{m} \frac{|\langle x, y_j\rangle|^2}{\sum_{k=1}^{m} \|\langle y_j, y_k\rangle\|} \\
&+ \frac{1}{2} \sum_{j,k=1}^{m} \frac{|\langle x, y_j\rangle|^2 \|\langle y_j, y_k\rangle\|}{(\sum_{k=1}^{m} \|\langle y_j, y_k\rangle\|)^2} + \frac{1}{2} \sum_{j,k=1}^{n} \frac{|\langle x, y_j\rangle|^2 \|\langle y_j, y_k\rangle\|}{(\sum_{k=1}^{m} \|\langle y_j, y_k\rangle\|)^2}.
\end{aligned}$$

Since

$$
\begin{aligned}
& |x|^2-\sum_{j=1}^{m}\frac{|\langle x,y_j\rangle|^2}{\sum_{k=1}^{m}\|\langle y_j,y_k\rangle\|}-\sum_{j=1}^{m}\frac{|\langle x,y_j\rangle|^2}{\sum_{k=1}^{m}\|\langle y_j,y_k\rangle\|}\\
+ & \frac{1}{2}\sum_{j,k=1}^{m}\frac{|\langle x,y_j\rangle|^2\|\langle y_j,y_k\rangle\|}{(\sum_{k=1}^{m}\|\langle y_j,y_k\rangle\|)^2}+\frac{1}{2}\sum_{j,k=1}^{m}\frac{|\langle x,y_j\rangle|^2\|\langle y_j,y_k\rangle\|}{(\sum_{k=1}^{m}\|\langle y_j,y_k\rangle\|)^2}\\
= & |x|^2-2\sum_{j=1}^{m}\frac{|\langle x,y_j\rangle|^2}{\sum_{k=1}^{m}\|\langle y_j,y_k\rangle\|}+\sum_{j=1}^{n}\frac{|\langle x,y_j\rangle|^2}{\sum_{k=1}^{m}\|\langle y_j,y_k\rangle\|}\\
= & |x|^2-\sum_{j=1}^{m}\frac{|\langle x,y_j\rangle|^2}{\sum_{k=1}^{m}\|\langle y_j,y_k\rangle\|}.
\end{aligned}
$$

It follows that

$$\left|x-\sum_{j=1}^{m}\alpha_j y_j\right|^2\leq |x|^2-\sum_{j=1}^{m}\frac{|\langle x,y_j\rangle|^2}{\sum_{k=1}^{m}\|\langle y_j,y_k\rangle\|},$$

hence

$$|x|^2-\sum_{j=1}^{m}\frac{|\langle x,y_j\rangle|^2}{\sum_{k=1}^{m}\|\langle y_j,y_k\rangle\|}\geq 0.$$

The proof is then completed. □

Corollary 3.2. *Let $\mathcal{X}$ be a right Hilbert $\mathcal{A}-$ module which is an algebraic left $\mathcal{A}$-module, then :*

$$|\langle y,x\rangle|^2\leq \|y\|^2|x|^2$$

and

$$|\langle x,y\rangle|^2\leq \|y\|^2|x|^2.$$

Corollary 3.3. *Let $\mathcal{X}$ be a right Hilbert $\mathcal{A}-$ module which is an algebraic left $\mathcal{A}$-module. If $y_1,\dots,y_n$ is a sequence of unit vectors in $\mathcal{X}$ such that $\langle y_j,y_k\rangle=0$ for $1\leq j\neq k\leq n$. Then*

$$\sum_{j=1}^{m}|\langle y_j,x\rangle|^2\leq |x|^2.$$

and

$$\sum_{j=1}^{m}|\langle x,y_j\rangle|^2\leq |x|^2.$$

Theorem 3.4. *Let $\mathcal{X}$ be a right Hilbert $\mathcal{A}$- module which is an algebraic left $\mathcal{A}$-module, $x_1,\dots,x_n$ and $y_1,\dots,y_m$ be a non zero vectors in $\mathcal{X}$ such that there exist the non-negative real numbers $\rho_j,\mu_j,j\in\{1,\dots,m\}$ with*

$$Re\langle y_j,x_i\rangle\geq\rho_j\|x_i\|\|y_j\|\ ,\ Im\langle y_j,x_i\rangle\geq\mu_j\|x_i\|\|y_j\| \tag{3.3}$$

for each $i\in\{1,\dots,n\}$, $j\in\{1,\dots,m\}$. Then

$$\Big(\sum_{j=1}^{m}\frac{(\rho_j^2+\mu_j^2)\|y_j\|^2}{\sum_{k=1}^{m}\|\langle y_j,y_k\rangle\|}\Big)^{\frac{1}{2}}\sum_{i=1}^{n}|x_i|\leq\left|\sum_{i=1}^{n}x_i\right|.$$

Proof. By (3.3), we have

$$\begin{aligned}(\sum_{i=1}^{n} Re\,\langle y_j, x_i\rangle)^2 + (\sum_{i=1}^{n} Im\,\langle y_j, x_i\rangle)^2 \quad &\geq \quad \rho_j^2 |y_j|^2 (\sum_{i=1}^{n} |x_i|)^2 + \mu_{kj}^2 \left\| y_j \right\|^2 (\sum_{i=1}^{n} \|x_i\|)^2 \\ &= \quad (\rho_j^2 + \mu_j^2)|y_j|^2 (\sum_{i=1}^{n} |x_i|)^2.\end{aligned}$$

By combining the above inequality and this equality

$\frac{1}{2}(|\langle \sum_{i=1}^{n} x_i, y_j\rangle|^2 + |\langle y_j, \sum_{i=1}^{n} x_i\rangle|^2) = (\sum_{i=1}^{n} Re\,\langle y_j, x_i\rangle)^2 + (\sum_{i=1}^{n} Im\,\langle y_j, x_i\rangle)^2,$

we deduce

$\frac{1}{2}\sum_{j=1}^{m} \frac{|\langle y_j, \sum_{i=1}^{n} x_i\rangle|^2}{\sum_{k=1}^{m}\|\langle y_j, y_k\rangle\|} + \frac{1}{2}\sum_{j=1}^{m} \frac{|\langle \sum_{i=1}^{n} x_i, y_j\rangle|^2}{\sum_{k=1}^{m}\|\langle y_j, y_k\rangle\|} \geq (\sum_{j=1}^{m} \frac{(\rho_j^2+\mu_j^2)\|y_j\|^2}{\sum_{k=1}^{m}\|\langle y_j, y_k\rangle\|})(\sum_{i=1}^{n} \|x_i\|)^2.$

Apply lemma 3.1, we get

$$\frac{1}{2}\sum_{j=1}^{m} \frac{|\langle y_j, \sum_{i=1}^{n} x_i\rangle|^2}{\sum_{k=1}^{m}\|\langle y_j, y_k\rangle\|} + \frac{1}{2}\sum_{j=1}^{m} \frac{|\langle \sum_{i=1}^{n} x_i, y_j\rangle|^2}{\sum_{k=1}^{m}\|\langle y_j, y_k\rangle\|} \leq |\sum_{i=1}^{n} x_i|^2.$$

Then

$$(\sum_{j=1}^{m} \frac{(\rho_j^2 + \mu_j^2)\left\| y_j \right\|^2}{\sum_{k=1}^{m}\|\langle y_j, y_k\rangle\|})(\sum_{i=1}^{n} \|x_i|\,|)^2 \leq \left\|\sum_{i=1}^{n} x_i\right\|^2.$$

And since $|x| \leq \|x\|$ and $|x|^2 \leq \|x\|^2$ for all $x \in \mathcal{X}$, then

$$(\sum_{j=1}^{m} \frac{(\rho_j^2 + \mu_j^2)\left\| y_j \right\|^2}{\sum_{k=1}^{m}\|\langle y_j, y_k\rangle\|})(\sum_{i=1}^{n} |x_i|)^2 \leq \left|\sum_{i=1}^{n} x_i\right|^2.$$

The desired result follows by taking the square roots. □

Remark 3.5. If the first condition of (3.3) is the only one available, then

$$(\sum_{j=1}^{m} \frac{\rho_j^2 \left\| y_j \right\|^2}{\sum_{k=1}^{m}\|\langle y_j, y_k\rangle\|})^{\frac{1}{2}} \sum_{i=1}^{n} |x_i| \leq \left|\sum_{i=1}^{n} x_i\right|.$$

Corollary 3.6. *Let $\mathcal{X}$ be a right Hilbert $\mathcal{A}-$module which is an algebraic left $\mathcal{A}$-module, $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ be a non zero vectors in $\mathcal{X}$ such that there exist the non-negative real numbers $\rho_j, \mu_j, j \in \{1, \ldots, m\}$ with*

$$Re\,\langle y_j, x_i\rangle \geq \rho_j \|x_i\| \|y_j\| \ , \ Im\,\langle y_j, x_i\rangle \geq \mu_j \|x_i\| \|y_j\| .$$

Then

$$(\sum_{j=1}^{m} \frac{(\rho_j^2 + \mu_j^2)|y_j|^2}{\max_{1\leq j\leq m} |y_j|^2 + (m-1)\max_{k\neq j} |\langle y_j, y_k\rangle|})^{\frac{1}{2}} \sum_{i=1}^{n} |x_i| \leq \left|\sum_{i=1}^{n} x_i\right|.$$

Proof. It is easy to show that

$$\sum_{k=1}^{m} \|\langle y_j, y_k\rangle\| \leq \max_{1\leq j\leq m} \left\| y_j \right\|^2 + (m-1)\max_{j\neq k} \|\langle y_j, y_k\rangle\| .$$

We thus have that

$$\frac{1}{\max_{1\leq j\leq m} \left\| y_j \right\|^2 + (m-1)\max_{j\neq k} \|\langle y_j, y_k\rangle\|} \leq \frac{1}{\sum_{k=1}^{m} \|\langle y_j, y_k\rangle\|},$$

and

$$\frac{(\rho_j^2+\mu_j^2)\,\|y_j\|^2}{\max_{1\le j\le m}\|y_j\|^2+(m-1)\max_{j\ne k}\|\langle y_j,y_k\rangle\|}\le\frac{(\rho_j^2+\mu_j^2)\,\|y_j\|^2}{\sum_{k=1}^m\|\langle y_j,y_k\rangle\|}.$$

Consequently

$$(\sum_{j=1}^n\frac{(\rho_j^2+\mu_j^2)\|y_j\|^2}{\max_{1\le j\le n}\|y_j\|^2+(m-1)\max_{j\ne k}\|\langle y_j,y_k\rangle\|})^{\frac12}\sum_{i=1}^n|x_i|\le(\sum_{j=1}^n\frac{(\rho_j^2+\mu_j^2)\|y_j\|^2}{\sum_{k=1}^n\|\langle y_j,y_k\rangle\|})^{\frac12}\sum_{i=1}^n|x_i|.$$

We apply the theorem 3.4 to get the result □

The next corollary is the theorem 2.5 in [12].

Corollary 3.7. *Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$-module, $x_1,\dots,x_n$ be a family of vectors in $\mathcal{X}$ and e_j be a unitary orthogonal vectors for $j\in\{1,\dots,m\}$ in $\mathcal{X}$ such that there exist the real numbers $\rho_j,\mu_j,j\in\{1,\dots,m\}$ with*

$$Re\,\langle\rho_j x_i,e_j\rangle\ge\rho_j^2\,\|x_i\|\;\;,\;Im\,\langle\mu_j x_i,e_j\rangle\ge\mu_j^2\,\|x_i\|$$

for each $i\in\{1,\dots,n\}$, $j\in\{1,\dots,m\}$. Then

$$(\sum_{j=1}^m(\rho_j^2+\mu_j^2))^{\frac12}\sum_{j=1}^n\|x_i\|\le\left\|\sum_{i=1}^n x_i\right\|.$$

Corollary 3.8. *Let $\mathcal{X}$ be a right Hilbert $\mathcal{A}-$module which is an algebraic left $\mathcal{A}$-module, $x_1,\dots,x_n$ and $y_1\dots,y_m$ be a non zero vectors in $\mathcal{X}$, such that there exist the non-negative real number in $[0;1]$ $p_j,q_j,j\in\{1,\dots,m\}$ with*

$$||x_i||^2-2Re\,\langle y_j,x_i\rangle+||y_j||^2\le p_j^2\le||y_j||^2 \tag{3.4}$$

and

$$||x_i||^2-2\;Im\,\langle y_j,x_i\rangle+||y_j||^2\le p_j^2\le||y_j||^2 \tag{3.5}$$

for each $i\in\{1,\dots,n\}$, $j\in\{1,\dots,m\}$. Then

$$(\sum_{j=1}^m\frac{(2\,\|y_j\|^2-p_j^2-qj^2)}{\sum_{k=1}^m\|\langle y_j,y_k\rangle\|})^{\frac12}\sum_{i=1}^n|x_i|\le\left|\sum_{i=1}^n x_i\right|.$$

Proof. By the inequality (3.4), we get

$$||x_i||^2+||y_j||^2-p_j^2\le 2Re\,\langle y_j,x_i\rangle\,.$$

Since $||y_j||^2-p_j^2\ge0$, then

$$2||x_i||\sqrt{||y_j||^2-p_j^2}\le||x_i||^2+||y_j||^2-p_j^2\le 2Re\,\langle y_j,x_i\rangle\,.$$

This implies that

$$Re\,\langle y_j,x_i\rangle\ge\frac{\sqrt{||y_j||^2-p_j^2}}{||y_j||}||y_j||||x_i||.$$

Even from (3.5), we get

$$Im\,\langle y_j,x_i\rangle\ge\frac{\sqrt{||y_j||^2-q_j^2}}{||y_j||}||y_j||||x_i||,$$

and if we let $\rho_j=\frac{\sqrt{||y_j||^2-p_j^2}}{||y_j}$ and $\mu_j=\frac{\sqrt{||y_j||^2-p_j^2}}{||y_j}$ in theorem 3.4,
then by simple computation, we get the desired result. □

The following lemma gives a refinement of Selberg's inequality in a right Hilbert $\mathcal{A}-$ module which is an algebraic left $\mathcal{A}$-module.

Lemma 3.9. *Let $\mathcal{X}$ be a right Hilbert $\mathcal{A}-$ module which is an algebraic left $\mathcal{A}$-module and $y_1, \ldots, y_m$ be a non zero vectors in $\mathcal{X}$. If $x \in \mathcal{X}$ then*

$$| \langle y, x \rangle |^2 + \sum_{j=1}^{m} \frac{|\langle y_j, x\rangle|^2}{\sum_{k=1}^{m} \|\langle y_j, y_k\rangle\|} ||y||^2 \leq |x|^2 \, ||y||^2, \tag{3.6}$$

$$| \langle x, y \rangle |^2 + \sum_{j=1}^{m} \frac{|\langle x, y_j\rangle|^2}{\sum_{k=1}^{m} \|\langle y_j, y_k\rangle\|} ||y||^2 \leq |x|^2 \, ||y||^2, \tag{3.7}$$

$$| \langle x, y \rangle |^2 + \sum_{j=1}^{m} \frac{|\langle y_j, x\rangle|^2}{\sum_{k=1}^{m} \|\langle y_j, y_k\rangle\|} ||y||^2 \leq |x|^2 \, ||y||^2, \tag{3.8}$$

and

$$| \langle y, x \rangle |^2 + \sum_{j=1}^{m} \frac{|\langle x, y_j\rangle|^2}{\sum_{k=1}^{m} \|\langle y_j, y_k\rangle\|} ||y||^2 \leq |x|^2 \, ||y||^2. \tag{3.9}$$

Proof. Inequality (3.6) is proved in [[3],theorem 3.3].
Now we prove the inequality (3.7), let

$$u = x - \sum_{j=1}^{m} \alpha_j y_j$$

where

$$\alpha_j = \frac{\langle y_j, x\rangle}{\sum_{k=1}^{n} \|\langle y_j, y_k\rangle\|}.$$

According to the proof of lemma 3.1, we have

$$\begin{aligned} |u|^2 &= |x - \sum_{j=1}^{m} \alpha_j y_j|^2 \\ &\leq |x|^2 - \sum_{j=1}^{m} \frac{| \langle y_j, x\rangle |^2}{\sum_{k=1}^{m} | \langle y_j, y_k\rangle |}. \end{aligned}$$

Hence it follows that

$$\|y\|^2 \left(|x|^2 - \sum_{j=1}^{m} \frac{|\langle x, y_j\rangle|^2}{\sum_{k=1}^{m} \|\langle y_j, y_k\rangle\|} \right) \geq \|y\|^2 \, |u|^2 .$$

Applying Cauchy Schwartz inequality, we get

$$\|y\|^2 \, |u|^2 \geq |\langle u, y\rangle|^2 .$$

and since $\langle y, y_j\rangle = 0$, so

$$|\langle u, y\rangle|^2 = \left| \left\langle x - \sum_{j=1}^{m} \alpha_j y_j , y \right\rangle \right|^2 = |\langle x, y\rangle|^2 .$$

It follows that

$$\|y\|^2\left(|x|^2-\sum_{j=1}^{m}\frac{|\langle x,y_j\rangle|^2}{\sum_{k=1}^{m}\|\langle y_j,y_k\rangle\|}\right)\geq|\langle x,y\rangle|^2\,,$$

which completes the proof of the inequality (3.7).

Similarly, we can get inequalities (3.8) and (3.9) . □

Lemma 3.10. *Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$ - module, $y_1,\cdots,y_m,y$ be non zero vectors in $\mathcal{X}$ and $x_1,\cdots,x_n\in\mathcal{X}$ such that such that there exist the real numbers $\rho_j,\mu_j,j\in\{1,\cdots,m\}$ with*

$$0\leq\rho_j\,\|x_i\|\,||y_j||\leq Re\,\langle y_j,x_i\rangle\,,\;0\leq\mu_j\,\|x_i\|\,||y_j||\leq Im\,\langle y_j,x_i\rangle \tag{3.10}$$

and $\langle y,y_j\rangle=0$, for each $i\in\{1,\ldots,n\}$, $j\in\{1,\ldots,m\}$. Then

$$|(y,\sum_{i=1}^{n}x_i)|^2+(\sum_{k=1}^{m}\frac{\rho_j^2+\mu_j^2}{c_j}||y_j^2)(\sum_{i=1}^{n}|x_i|)^2||y||^2\leq|\sum_{i=1}^{n}x_i|^2||y||^2, \tag{3.11}$$

and

$$|(\sum_{i=1}^{n}x_i,y)|^2+(\sum_{j=1}^{m}\frac{\rho_j^2+\mu_j^2}{c_j}||y_j||^2)(\sum_{i=1}^{n}|x_i|)^2||y||^2\leq|\sum_{i=1}^{n}x_i|^2||y||^2, \tag{3.12}$$

where $c_j=\sum_{k=1}^{m}\|\langle y_j,y_k\rangle\|$.

Proof. Let $x=\sum_{i=1}^{n}x_i$, from (3.10), we get

$$\begin{aligned}&||y||^2\{|x|^2-\textstyle\sum_{j=1}^{m}\frac{\rho_j^2+\mu_j^2}{c_j}(\sum_{i=1}^{n}||x_i||)^2\}\\&\geq||y||^2\{|x|^2-\textstyle\sum_{j=1}^{m}\frac{\mathrm{Re}\langle x,y_j\rangle^2+\mathrm{Im}\langle x,y_j\rangle^2}{c_j}\},\end{aligned}$$

Since

$||y||^2\{|x|^2-\sum_{j=1}^{m}\frac{\mathrm{Re}\langle x,y_j\rangle^2+\mathrm{Im}\langle x,y_j\rangle^2}{c_j}\}=\|y\|^2\{|x|^2-\frac{1}{2}\sum_{j=1}^{m}\frac{|\langle x,y_j\rangle|^2}{c_j}-\frac{1}{2}\sum_{j=1}^{m}\frac{\langle y_j,x\rangle^2}{c_j}\}.$

Then, from (3.6) and (3.9), we get

$$\|y\|^2\{|x|^2-\frac{1}{2}\sum_{j=1}^{m}\frac{|\langle x,y_j\rangle|^2}{c_j}-\frac{1}{2}\sum_{j=1}^{m}\frac{\langle y_j,x\rangle^2}{c_j}\}\geq|\langle y,x\rangle|^2,$$

it follows that

$$||y||^2\{|x|^2-\sum_{j=1}^{m}\frac{\rho_j^2+\mu_j^2}{c_j}(\|x_1\|+\cdots+\|x_n\|)^2\}\geq|\langle y,x\rangle|^2\,.$$

By using (3.7) and (3.8) and by similar argument, we get (3.12). □

Theorem 3.11. *Let $\mathcal{X}$ be a Hilbert $\mathcal{A}$ - module, $y_1,\cdots,y_m$ be non zero vectors in $\mathcal{X}$ and $x_1,\cdots,x_n\in\mathcal{X}$ such that such that there exist the real numbers $a,b,\rho_j,\mu_j,j\in\{1,\cdots,m\}$ with*

$$0\leq\rho_j\,\|x_i\|\,||y_j||\leq Re\,\langle y_j,x_i\rangle\,,\;0\leq\mu_j\,\|x_i\|\,||y_j||\leq Im\,\langle y_j,x_i\rangle\,, \tag{3.13}$$

$$0\leq a\,\|x_i\|\,||y||\leq Re\,\langle y,x_i\rangle\,,\;0\leq b\,\|x_i\|\,||y_j||\leq Im\,\langle y,x_i\rangle \tag{3.14}$$

and $\langle y, y_j\rangle = 0$, *for each* $i \in \{1,\dots,n\}$, $j \in \{1,\dots,m\}$. *Then*

$$(a^2+b^2+\sum_{j=1}^{m}\frac{\rho_j^2+\mu_j^2}{c_j})||y_j^2||^{\frac{1}{2}}(|x_1|+\cdots+|x_n|) \le |\sum_{i=1}^{n} x_i|.$$

where $c_j = \sum_{k=1}^{m} \|\langle y_j, y_k\rangle\|$.

Proof. From (3.11) and (3.12), we get

$$\begin{aligned}&\tfrac{1}{2}(|\langle \textstyle\sum_{i=1}^{n} x_i, y\rangle|^2 + |\langle y, \sum_{i=1}^{n} x_i\rangle|^2)\\ &+(\textstyle\sum_{j=1}^{m} \frac{\rho_j^2+\mu_j^2}{c_j}||y_|||^2)(\sum_{i=1}^{n}|x_i|)^2||y||^2\\ &\le |\textstyle\sum_{i=1}^{n} x_i|^2||y||^2,\end{aligned}$$

since

$$\tfrac{1}{2}(|\langle \textstyle\sum_{i=1}^{n} x_i, y\rangle|^2 + |\langle y, \sum_{i=1}^{n} x_i\rangle|^2) \ge (Re\,\langle y,x\rangle)^2 + (Im\,\langle y,x\rangle)^2$$

applying (3.13) and (3.14) and taking the square root, the desired result follows. □

Remark 3.12. If in Theorem 3.11 $y_1,\dots,y_m$ is a sequence orthonormal vectors, then

$$(a^2+b^2+\sum_{j=1}^{m}(\rho_j^2+\mu_j^2))^{\frac{1}{2}}(\sum_{i=1}^{n}|x_i|) \le \left|\sum_{i=1}^{n} x_i\right|.$$

This inequality is an extension of Diaz-Metcalf [4] inequality in C^*-module.

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