

DUNKL LIPSCHITZ FUNCTIONS FOR THE GENERALIZED FOURIER-DUNKL TRANSFORM IN THE SPACE $L^2_{\alpha,n}$

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ABSTRACT. In this paper, using a generalized translation operator, we prove the estimates for the generalized Fourier-Dunkl transform in the space $L^2_{\alpha,n}$ on certain classes of functions.

1. Introduction and Preliminaries

In [5], E. C. Titchmarsh's characterizes the set of functions in $L^2(\mathbb{R})$ satisfying the Cauchy-Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transform, namely we have

Theorem 1.1 Let $\delta \in (0, 1)$ and assume that $f \in L^2(\mathbb{R})$. Then the following are equivalents

$$(i) \quad \|f(t+h) - f(t)\| = O(h^\delta), \quad \text{as } h \rightarrow 0,$$

$$(ii) \quad \int_{|\lambda| \geq r} |\hat{f}(\lambda)|^2 d\lambda = O(r^{-2\delta}) \quad \text{as } r \rightarrow \infty,$$

where \hat{f} stands for the Fourier transform of f .

In this paper, we consider a first-order singular differential-difference operator Λ on \mathbb{R} which generalizes the Dunkl operator Λ_α . We prove an analog of Theorem 1.1 in the generalized Fourier-Dunkl transform associated to Λ in $L^2_{\alpha,n}$. For this purpose, we use a generalized translation operator. We point out that similar results have been established in the context of non compact rank one Riemannian symmetric spaces [4].

In this section, we develop some results from harmonic analysis related to the differential-difference operator Λ . Further details can be found in [1] and [6]. In all what follows assume where $\alpha > -1/2$ and n a non-negative integer.

Consider the first-order singular differential-difference operator on \mathbb{R}

$$\Lambda f(x) = f'(x) + \left(\alpha + \frac{1}{2}\right) \frac{f(x) - f(-x)}{x} - 2n \frac{f(-x)}{x}.$$

For $n = 0$, we regain the differential-difference operator

$$\Lambda_\alpha f(x) = f'(x) + \left(\alpha + \frac{1}{2}\right) \frac{f(x) - f(-x)}{x},$$

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which is referred to as the Dunkl operator of index $\alpha + 1/2$ associated with the reflection group \mathbb{Z}_2 on \mathbb{R} . Such operators have been introduced by Dunkl (see [3], [7]) in connection with a generalization of the classical theory of spherical harmonics. Let M be the map defined by

$$Mf(x) = x^{2n} f(x), \quad n = 0, 1, \dots$$

Let $L_{\alpha,n}^p$, $1 \leq p < \infty$, be the class of measurable functions f on \mathbb{R} for which

$$\|f\|_{p,\alpha,n} = \|M^{-1}f\|_{p,\alpha+2n} < \infty,$$

where

$$\|f\|_{p,\alpha} = \left(\int_{\mathbb{R}} |f(x)|^p |x|^{2\alpha+1} dx \right)^{1/p}.$$

If $p = 2$, then we have $L_{\alpha,n}^2 = L^2(\mathbb{R}, |x|^{2\alpha+1})$.

The one-dimensional Dunkl kernel is defined by

$$(1) \quad e_{\alpha}(z) = j_{\alpha}(iz) + \frac{z}{2(\alpha+1)} j_{\alpha+1}(iz), \quad z \in \mathbb{C},$$

where

$$(2) \quad j_{\alpha}(z) = \Gamma(\alpha+1) \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m}}{m! \Gamma(m+\alpha+1)}, \quad z \in \mathbb{C},$$

is the normalized spherical Bessel function of index α . It is well-known that the functions $e_{\alpha}(\lambda)$, $\lambda \in \mathbb{C}$, are solutions of the differential-difference equation

$$\Lambda_{\alpha} u = \lambda u, \quad u(0) = 1.$$

Lemma 1.2 For $x \in \mathbb{R}$ the following inequalities are fulfilled

- i) $|j_{\alpha}(x)| \leq 1$,
- ii) $|1 - j_{\alpha}(x)| \leq |x|$,
- iii) $|1 - j_{\alpha}(x)| \geq c$ with $|x| \geq 1$, where $c > 0$ is a certain constant which depends only on α .

Proof. Similarly as the proof of Lemma 2.9 in [2].

For $\lambda \in \mathbb{C}$, and $x \in \mathbb{R}$, put

$$\varphi_{\lambda}(x) = x^{2n} e_{\alpha+2n}(i\lambda x),$$

where $e_{\alpha+2n}$ is the Dunkl kernel of index $\alpha + 2n$ given by (1).

Proposition 1.3

i) φ_{λ} satisfies the differential equation

$$\Lambda \varphi_{\lambda} = i\lambda \varphi_{\lambda}.$$

ii) For all $\lambda \in \mathbb{C}$, and $x \in \mathbb{R}$

$$|\varphi_{\lambda}(x)| \leq |x|^{2n} e^{|Im\lambda||x|}.$$

The generalized Fourier-Dunkl transform we call the integral transform

$$\mathcal{F}_{\Lambda} f(\lambda) = \int_{\mathbb{R}} f(x) \varphi_{-\lambda}(x) |x|^{2\alpha+1} dx, \quad \lambda \in \mathbb{R}, \quad f \in L_{\alpha,n}^1.$$

Let $f \in L_{\alpha,n}^1$ such that $\mathcal{F}_\Lambda(f) \in L_{\alpha+2n}^1 = L^1(\mathbb{R}, |x|^{2\alpha+4n+1}dx)$. Then the inverse generalized Fourier-Dunkl transform is given by the formula

$$f(x) = \int_{\mathbb{R}} \mathcal{F}_\Lambda f(\lambda) \varphi_\lambda(x) d\mu_{\alpha+2n}(\lambda),$$

where

$$d\mu_{\alpha+2n}(\lambda) = a_{\alpha+2n} |\lambda|^{2\alpha+4n+1} d\lambda, \quad a_\alpha = \frac{1}{2^{2\alpha+2} (\Gamma(\alpha+1))^2}.$$

Proposition 1.4

i) For every $f \in L_{\alpha,n}^2$,

$$\mathcal{F}_\Lambda(\Lambda f)(\lambda) = i\lambda \mathcal{F}_\Lambda(f)(\lambda).$$

ii) For every $f \in L_{\alpha,n}^1 \cap L_{\alpha,n}^2$ we have the Plancherel formula

$$\int_{\mathbb{R}} |f(x)|^2 |x|^{2\alpha+1} dx = \int_{\mathbb{R}} |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

iii) The generalized Fourier-Dunkl transform \mathcal{F}_Λ extends uniquely to an isometric isomorphism from $L_{\alpha,n}^2$ onto $L^2(\mathbb{R}, \mu_{\alpha+2n})$.

The generalized translation operators τ^x , $x \in \mathbb{R}$, tied to Λ are defined by

$$\begin{aligned} \tau^x f(y) &= \frac{(xy)^{2n}}{2} \int_{-1}^1 \frac{f(\sqrt{x^2+y^2-2xyt})}{(x^2+y^2-2xyt)^n} \left(1 + \frac{x-y}{\sqrt{x^2+y^2-2xyt}}\right) A(t) dt \\ &+ \frac{(xy)^{2n}}{2} \int_{-1}^1 \frac{f(-\sqrt{x^2+y^2-2xyt})}{(x^2+y^2-2xyt)^n} \left(1 - \frac{x-y}{\sqrt{x^2+y^2-2xyt}}\right) A(t) dt, \end{aligned}$$

where

$$A(t) = \frac{\Gamma(\alpha+2n+1)}{\sqrt{\pi}\Gamma(\alpha+2n+1/2)} (1+t)(1-t^2)^{\alpha+2n-1/2}.$$

Proposition 1.5 Let $x \in \mathbb{R}$ and $f \in L_{\alpha,n}^2$. Then $\tau^x f \in L_{\alpha,n}^2$ and

$$\|\tau^x f\|_{2,\alpha,n} \leq 2x^{2n} \|f\|_{2,\alpha,n}.$$

Furthermore,

$$(3) \quad \mathcal{F}_\Lambda(\tau^x f)(\lambda) = x^{2n} e_{\alpha+2n}(i\lambda x) \mathcal{F}_\Lambda(f)(\lambda).$$

2. Main Results

In this section we give the main result of this paper. We need first to define (ψ, δ, β) -generalized Dunkl Lipschitz class.

Definition 2.1. Let $\delta > 1$ and $\beta > 0$. A function $f \in L_{\alpha,n}^2$ is said to be in the (ψ, δ, β) -generalized Dunkl Lipschitz class, denoted by $DLip(\psi, \delta, \beta)$, if

$$\|\tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x)\|_{2,\alpha,n} = O(h^{\delta+2n} \psi(h^\beta)) \quad \text{as } h \rightarrow 0,$$

where

(a) ψ is a continuous increasing function on $[0, \infty)$,

- (b) $\psi(0) = 0$, $\psi(ts) = \psi(t)\psi(s)$ for all $t, s \in [0, \infty)$,
(c) and

$$\int_0^{1/h} s^{1-2\delta} \psi(s^{-2\beta}) ds = O(h^{2\delta-2} \psi(h^{2\beta})), \quad h \rightarrow 0.$$

Theorem 2.2. Let $f \in L^2_{\alpha, n}$. Then the following are equivalents

- (a) $f \in DLip(\psi, \delta, \beta)$
(b) $\int_{|\lambda| \geq r} |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = O(r^{-2\delta} \psi(r^{-2\beta})), \quad as \quad r \rightarrow \infty.$

Proof. (a) \Rightarrow (b). Let $f \in DLip(\psi, \delta, \beta)$. Then we have

$$\|\tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x)\|_{2, \alpha, n} = O(h^{\delta+2n} \psi(h^\beta)) \quad as \quad h \rightarrow 0.$$

From formulas (1), (2) and (3), we have the generalized Fourier-Dunkl transform of $\tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x)$ is $2h^{2n}(j_{\alpha+2n}(\lambda h) - 1)\mathcal{F}_\Lambda f(\lambda)$.

By Plancherel equality, we obtain

$$\|\tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x)\|_{2, \alpha, n}^2 = 4h^{4n} \int_{-\infty}^{+\infty} |j_{\alpha+2n}(\lambda h) - 1|^2 |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

If $|\lambda| \in [\frac{1}{h}, \frac{2}{h}]$, then $|\lambda h| \geq 1$ and (iii) of Lemma 1.2 implies that

$$1 \leq \frac{1}{c^2} |j_{\alpha+2n}(\lambda h) - 1|^2.$$

Then

$$\begin{aligned} \int_{\frac{1}{h} \leq |\lambda| \leq \frac{2}{h}} |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) &\leq \frac{1}{c^2} \int_{\frac{1}{h} \leq |\lambda| \leq \frac{2}{h}} |j_{\alpha+2n}(\lambda h) - 1|^2 |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) \\ &\leq \frac{1}{c^2} \int_{-\infty}^{+\infty} |j_{\alpha+2n}(\lambda h) - 1|^2 |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) \\ &\leq \frac{h^{-4n}}{4c^2} \|\tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x)\|_{2, \alpha, n}^2 \\ &= O(h^{2\delta} \psi(h^{2\beta})). \end{aligned}$$

We obtain

$$\int_{r \leq |\lambda| \leq 2r} |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) \leq Cr^{-2\delta} \psi(r^{-2\beta}), \quad r \rightarrow \infty,$$

where C is a positive constant. Now,

$$\begin{aligned} \int_{|\lambda| \geq r} |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) &= \sum_{i=0}^{\infty} \int_{2^i r \leq |\lambda| \leq 2^{i+1} r} |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) \\ &\leq Cr^{-2\delta} \psi(r^{-2\beta}) \sum_{i=0}^{\infty} (2^{-2\delta} \psi(2^{-2\beta}))^i \\ &\leq CC_{\delta, \beta} r^{-2\delta} \psi(r^{-2\beta}), \end{aligned}$$

where $C_{\delta, \beta} = (1 - 2^{-2\delta} \psi(2^{-2\beta}))^{-1}$ since $2^{-2\delta} \psi(2^{-2\beta}) < 1$.

Consequently

$$\int_{|\lambda| \geq r} |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = O(r^{-2\delta} \psi(r^{-2\beta})), \quad as \quad r \rightarrow \infty.$$

(b) \Rightarrow (a). Suppose now that

$$\int_{|\lambda| \geq r} |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = O(r^{-2\delta} \psi(r^{-2\beta})), \quad \text{as } r \rightarrow \infty.$$

and write

$$\|\tau^h f(x) + \tau^{-h} f(x) - 2h^{2n} f(x)\|_{2,\alpha,n}^2 = 4h^{4n} (I_1 + I_2),$$

where

$$I_1 = \int_{|\lambda| < \frac{1}{h}} |j_{\alpha+2n}(\lambda h) - 1|^2 |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda),$$

and

$$I_2 = \int_{|\lambda| \geq \frac{1}{h}} |j_{\alpha+2n}(\lambda h) - 1|^2 |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

Firstly, we use the formulas $|j_{\alpha+2n}(\lambda h)| \leq 1$ and

$$I_2 \leq 4 \int_{|\lambda| \geq \frac{1}{h}} |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) = O(h^{2\delta} \psi(h^{2\beta})), \quad \text{as } h \rightarrow 0.$$

Set

$$\phi(x) = \int_x^{+\infty} |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda).$$

Integrating by parts we obtain

$$\begin{aligned} \int_0^x \lambda^2 |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) &= \int_0^x -\lambda^2 \phi'(\lambda) d\lambda = -x^2 \phi(x) + 2 \int_0^x \lambda \phi(\lambda) d\lambda \\ &\leq C_1 \int_0^x \lambda^{1-2\delta} \psi(\lambda^{-2\beta}) d\lambda = O(x^{2-2\delta} \psi(x^{-2\beta})), \end{aligned}$$

where C_1 is a positive constant.

We use the formula (ii) of Lemma 1.2

$$\begin{aligned} \int_{-\infty}^{+\infty} |j_{\alpha+2n}(\lambda h) - 1|^2 |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda) &= O\left(h^2 \int_{|\lambda| < \frac{1}{h}} \lambda^2 |\mathcal{F}_\Lambda f(\lambda)|^2 d\mu_{\alpha+2n}(\lambda)\right) \\ &+ O(h^{2\delta} \psi(h^{2\beta})) \\ &= O(h^2 h^{-2+2\delta} \psi(h^{2\beta})) + O(h^{2\delta} \psi(h^{2\beta})) \\ &= O(h^{2\delta} \psi(h^{2\beta})), \end{aligned}$$

and this ends the proof. \square

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