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QUASI-COMPACT PERTURBATIONS OF THE WEYL ESSENTIAL SPECTRUM AND APPLICATION TO SINGULAR TRANSPORT OPERATORS

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Abstract. This paper is devoted to the investigation of the stability of the Weyl essential spectrum of closed densely defined linear operator A subjected to additive perturbation K such that $(\lambda - A - K)^{-1}K$ or $K(\lambda - A - K)^{-1}$ is a quasi-compact operator. The obtained results are used to describe the Weyl essential spectrum of singular neutron transport operator.

1. Introduction and preliminaries

Let X and Y be complex infinite dimensional Banach spaces, and let $\mathcal{C}(X, Y)$ (resp. $\mathcal{L}(X, Y)$) denote the set of all closed, densely defined linear operators from X into Y (resp. the Banach algebra of all bounded operators), abbreviate $\mathcal{C}(X, X)$ (resp. $\mathcal{L}(X, X)$) to $\mathcal{C}(X)$ (resp. $\mathcal{L}(X)$). For $A \in \mathcal{C}(X)$, write $\mathcal{D}(A) \subset X, N(A)$, $R(A)$, $\sigma(A)$ and $\rho(A)$ respectively, the domain, the null space, the range, the spectrum and the resolvent set of A. The subset of all compact operators of $\mathcal{L}(X, Y)$ is noted by $\mathcal{K}(X, Y)$ and if $X = Y$ we write $\mathcal{K}(X)$. Let I denote the identity operator in X. Let $A \in \mathcal{C}(X)$, we know that $\mathcal{D}(A)$ provided with the graph norm $\|x\|_A = \|x\| + \|Ax\|$ is a Banach space denoted by X_A. Recall that an operator B is relatively bounded with respect to A or simply A-bounded if $\mathcal{D}(A) \subseteq \mathcal{D}(B)$ and B is bounded on X_A.

Definition 1.1. An operator $V \in \mathcal{L}(X)$ is said to be quasi-compact operator if there exists a compact operator K and an integer m such that $\|V^m - K\| < 1$.

We denote by $\mathcal{QK}(X)$ the set of all quasi-compact operators. If $r = r(V)$ is the spectral radius of a bounded linear operator V on X, another equivalent definition is given in [2], to quasi-compactness, is that if there exists M and N closed V-invariant subspaces of X such that $X = M \oplus N$ with $r(V/M) < r$ and $dim N < \infty$. We refer the reader to [2] for a detailed presentation of the quasi-compactness. For $A \in \mathcal{C}(X, Y)$, the nullity, $\alpha(A)$, of A is defined as the dimension of $N(A)$ and the deficiency, $\beta(A)$, of A is defined as the codimension of $R(A)$ in Y. The set of Fredholm operators from X into Y is defined by

$$\Phi(X, Y) = \{A \in \mathcal{C}(X, Y) : R(A) \text{ is closed in } Y, \beta(A) < \infty \text{ and } \alpha(A) < \infty\},$$

If $A \in \Phi(X, Y)$, the number $\text{ind}(A) = \alpha(A) - \beta(A)$ is called the index of A. The operator A is Weyl if it is Fredholm of index zero. The Fredholm essential spectrum $\sigma_{ef}(A)$ and the Weyl essential spectrum $\sigma_{ew}(A)$ are defined by:

$$\sigma_{ef}(A) := \{\lambda \in \mathbb{C} : (\lambda - A) \text{ is not Fredholm }\}$$

and

$$\sigma_{ew}(A) := \{\lambda \in \mathbb{C} : (\lambda - A) \text{ is not Weyl }\}.$$

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For $A \in \mathcal{L}(X)$, the Fredholm essential spectrum of A is also equal to the spectrum of $A+\mathcal{K}(X)$ in the Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$. Accordingly, the essential spectral radius of A, denoted by $r_e(A)$, is given by the formula

$$r_e(A) = \lim_{n\to+\infty} dist(A^n, \mathcal{K}(X))^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} \operatorname{dist}(A^n, \mathcal{K}(X))^{\frac{1}{n}}$$

where $\operatorname{dist}(A, \mathcal{K}(X)) = \inf_{K\in\mathcal{K}(X)} \|A - K\|$. Thus A is quasi-compact if and only if $r_e(A) < 1$. We say that A is a Riesz operator if $\lambda - A$ is Fredholm for all non-zero complex numbers λ. Thus A is Riesz if and only if $r_e(A) = 0$. A useful property of a bounded linear operator A on a Banach space is that each spectral element of A which lies in the unbounded component of the complement of the essential spectrum of A is an eigenvalue of finite multiplicity. Further, if there are infinitely many of them, then they cluster only on the essential spectrum. Certainly we have the implications: compact $\Rightarrow$ Riesz $\Rightarrow$ quasi-compact. The following theorem gives an important and useful characterization of quasi-compact operators:

Theorem 1.2. [1, Theorem I.6] *If $V \in \mathcal{QK}(X)$ then for all for all complex number λ such that $|\lambda| \geq 1$ then $(\lambda - A)$ is a Weyl operator.*

As a consequence of this theorem, V is quasi-compact if (and only if) the peripheral spectrum of V contains only many poles of the resolvent $R(., V)$ of V, and if V is power-bounded operator the residue of $R(., V)$ at each peripheral pole is of finite rank. However, the concept of quasi-compact operator plays a crucial role and seems to be more appropriate because it evokes not only the special configuration of the spectrum of V, but also the fact that the spectral values of greatest modulus are eigenvalues associated with finite dimensional generalized eigenspaces.

The purpose of this paper is to point out how, by means of the concept of the quasi-compactness it is possible to improve the definition of the Weyl essential spectrum. More precisely, we establish that $\sigma_{ew}(A) = \sigma_{ew}(A+K)$ for all closed densely defined linear operator A, with $K \in \mathcal{C}(X)$ such that K is A-bounded and $(\lambda - A - K)^{-1}K$ or $K(\lambda - A - K)^{-1}$ is a quasi-compact operator for all $\lambda \in \rho(A+K)$. Our results generalize many known ones in the literature.

The next Section is concerned with the definition of the Weyl essential spectrum of closed densely defined linear operator A subjected to additive perturbation K, A-bounded, such that $(\lambda - A - K)^{-1}K$ or $K(\lambda - A - K)^{-1}$ is a quasi-compact operator for some $\lambda \in \rho(A+K)$ and we establish some properties of quasi-compact operators. In the last Section we apply the results obtained in the second section to investigate the Weyl essential spectrum of the following singular neutron transport operator

$$\begin{aligned} A\psi(x,\xi) &= -\xi\nabla_x\psi(x,\xi) - \sigma(\xi)\psi(x,\xi) + \int_{\mathbb{R}^n} \kappa(x,\xi')\psi(x,\xi')\,d\xi' \qquad (x,\xi) \in \Omega\times V \\ &= T\psi(x,\xi) + K\psi(x,\xi), \end{aligned}$$

with the vacum boundary conditions

$$\psi_{|\Gamma_-}(x,\xi) = 0 \qquad\qquad (x,\xi) \in \Gamma_- = \{(x,\xi)\ \in \partial\Omega\times V\ ;\ \xi\cdot n(x) < 0\}.$$

where Ω is a smooth open subset of $\mathbb{R}^n$ $(n \geq 1)$, V is the support of a positive Radon measure $d\mu$ on $\mathbb{R}^n$ and $n(x)$ stands for the outward normal unit at $x \in \partial\Omega$. The operator A describes the transport of particle (neutrons, photons, molecules of gas, etc.) in the domain Ω. The function $\psi \in L^p(\Omega\times V, dxd\mu(\xi))$ $(1 \leq p < \infty)$ represents the number (or probability) density of particles having the position x and the velocity ξ. The functions $\sigma(.)$ and $\kappa(.,.)$ are called, respectively, the collision frequency and the scattering kernel and will be assumed to be unbounded. More precisely, we will assume that there exist a closed subset $E \subset \mathbb{R}^n$ with zero $d\mu$ measure and a constant $\sigma_0 > 0$ such that

$$\sigma(.) \in L^\infty_{loc}(\mathbb{R}^n \setminus E), \quad \sigma(\xi) > \sigma_0; \tag{1.1a}$$

$$\left[\int_{\mathbb{R}^n}(\frac{\kappa(.,\xi')}{\sigma(\xi')^{\frac{1}{p}}})^q d\mu(\xi')\right]^{\frac{1}{q}} \in L^p(\mathbb{R}^n). \tag{1.1b}$$

where q denotes the conjugate exponent of p. We describe here the Weyl essential spectrum of the operator A subjected to assumptions (1.1a), (1.1b) and the scattering kernel taking the form $\kappa(.,.) = \kappa_1(.,.) + \kappa_2(.,.)$ with $\kappa_i(.,.)$ are non-negative $i = 1, 2$. Theorem 3.3 assert that if conditions (1.1a) and (1.1b) are satisfied, the hyperplanes of $\mathbb{R}^n$ have zero $d\mu$-measure, (i.e. for each $e \in S^{n-1}, d\mu\{\xi \in \mathbb{R}^n, \xi.e = 0\} = 0$, where S^{n-1} denotes the unit ball of $\mathbb{R}^n$), the collision operator $K_1 : \psi(x,\xi) \longrightarrow \int_{\mathbb{R}^n} \kappa_1(x,\xi')\psi(x,\xi')d\xi'$ is compact from $L^p(\mathbb{R}^n, \sigma(\xi)d\mu(\xi))$ into $L^p(\mathbb{R}^n, d\mu(\xi))$, and if further

$$\sup_{\xi\in\mathbb{R}^n} \frac{\sigma(\xi)}{(Re\lambda + \sigma(\xi))^p} \left\| \left[\int_{\mathbb{R}^n}(\frac{\kappa_2(.,\xi')}{\sigma(\xi')^{\frac{1}{p}}})^q d\mu(\xi')\right]^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} < 1 \quad \text{for } Re\lambda > -\sigma_0.$$

Then $K(\lambda-A-K)^{-1}$ is a sum of a compact operator $K_1(\mu-T)^{-1}[I+(\mu-\lambda+K)(\lambda-T-K)^{-1}]+K_2(\mu-T)^{-1}K_1(\lambda-T-K)^{-1}$ and a pure contraction $K_2(\mu-T)^{-1}[I+(\mu-\lambda+K_2)(\lambda-T-K)^{-1}]$ on X_p. Hence $K(\lambda - A - K)^{-1}$ is a quasi-compact operator on X_p. Now, by the knowledge of the Weyl essential spectrum of the streaming operator T and Theorem 2.3 we assert that $\sigma_{ew}(A) = \{\lambda \in \mathbb{C} \ : \ Re\lambda \le -\sigma_0\}$.

2. Invariance of the Weyl essential spectrum

If $A \in \mathcal{C}(X)$ we define the sets

$$\mathcal{R}_A(X) = \{K \in \mathcal{L}(X) \text{ such that } (\lambda - A - K)^{-1}K \in \mathcal{QK}(X) \text{ for some } \lambda \in \rho(A+K)\},$$

$$\mathcal{L}_A(X) = \{K \in \mathcal{L}(X) \text{ such that } K(\lambda - A - K)^{-1} \in \mathcal{QK}(X) \text{ for some } \lambda \in \rho(A+K)\},$$

Theorem 2.1. *Let $A \in \mathcal{C}(X)$ with nonempty resolvent set. Then*

$$\sigma_{ew}(A) = \bigcap_{K\in\mathcal{R}_A(X)} \sigma(A+K) = \bigcap_{K\in\mathcal{L}_A(X)} \sigma(A+K). \tag{2.1}$$

Proof. Set $\Sigma_1 = \bigcap_{K\in\mathcal{R}_A(X)} \sigma(A+K)$ (resp. $\Sigma_2 = \bigcap_{K\in\mathcal{L}_A(X)} \sigma(A+K)$). We first claim that $\sigma_{ew}(A) \subseteq \Sigma_1$ (resp. $\sigma_{ew}(A) \subseteq \Sigma_2$). Indeed, if $\lambda \notin \Sigma_1$ (resp. $\lambda \notin \Sigma_2$), then there exists $K \in \mathcal{R}_A(X)$ (resp. $K \in \mathcal{L}_A(X)$) such that $\lambda \in \rho(A+K)$ and $(\lambda - A - K)^{-1}K \in \mathcal{QK}(X)$ (resp. $K(\lambda - A - K)^{-1} \in \mathcal{QK}(X)$). Therefore Theorem 1.2 proves that $I + (\lambda - A - K)^{-1}K$ (resp. $I + K(\lambda - A - K)^{-1}$) is a Weyl operator. Next, using the relation

$$\lambda - A = (\lambda - A - K)(I + (\lambda - A - K)^{-1}K)$$

$$(\text{ resp. } \lambda - A = (I + K(\lambda - A - K)^{-1})(\lambda - A - K))$$

together with Atkinson's theorem we get $(\lambda-A)$ is a Weyl operator. This shows that $\lambda \notin \sigma_{ew}(A)$. The opposite inclusion follows from $\mathcal{K}(X) \subseteq \mathcal{R}_A(X)$ (resp. $\mathcal{K}(X) \subseteq \mathcal{L}_A(X)$). □

It follows, immediately, from Theorem 2.1 that

Corollary 2.2. *let $U(X)$ a subset of $\mathcal{L}(X)$ (not necessarily an ideal). If $\mathcal{K}(X) \subseteq U(X) \subseteq \mathcal{R}_A(X)$ or $\mathcal{K}(X) \subseteq U(X) \subseteq \mathcal{L}_A(X)$. Then $\sigma_{ew}(A) = \sigma_{ew}(A+K)$ for all $K \in U(X)$.*

Proof. We have

$$\bigcap_{K\in\mathcal{R}_A(X)} \sigma(A+K) \subseteq \bigcap_{K\in U(X)} \sigma(A+K) \subseteq \sigma_{ew}(A). \tag{2.2}$$

and

$$\bigcap_{K\in\mathcal{L}_A(X)} \sigma(A+K) \subseteq \bigcap_{K\in U(X)} \sigma(A+K) \subseteq \sigma_{ew}(A). \tag{2.3}$$

Now, the result follows from Theorem 2.1. $\square$

Let $A \in \mathcal{C}(X)$ and let J be an arbitrary A-bounded operator on X. Hence we can regard A and J as operators from X_A into X. They will be denoted by $\hat{A}$ and $\hat{J}$, respectively. These belong to $\mathcal{L}(X_A, X)$. Furthermore, we have the obvious relations

$$\left\{\begin{array}{l}\alpha(\hat{A}) = \alpha(A),\ \beta(\hat{A}) = \beta(A),\ R(\hat{A}) = R(A)\\ \alpha(\hat{A}+\hat{J}) = \alpha(A+J)\\ \beta(\hat{A}+\hat{J}) = \beta(A+J),\ \text{and } R(\hat{A}+\hat{J}) = R(A+J)\end{array}\right. \tag{2.4}$$

Theorem 2.3. *Let $A \in \mathcal{C}(X)$ with nonempty resolvent set. Then*

$$\sigma_{ew}(A) = \bigcap_{K \in \Delta_A(X)} \sigma(A+K). \tag{2.5}$$

where
$\Delta_A(X) = \{K \in \mathcal{C}(X), K$ *is A-bounded and* $K(\lambda - A - K)^{-1} \in \mathcal{QK}(X)$ *for all* $\lambda \in \rho(A+K)\}$.

Proof. Since $\mathcal{K}(X) \subset \Delta_A(X)$, we refer that $\bigcap_{K \in \Delta_A(X)} \sigma(A+K) \subset \sigma_{ew}(A)$. Conversely, suppose that there exists $K \in \Delta_A(X)$, hence by using (2.4) and Theorem 2.1, we infer that $I + K(\lambda - A - K)^{-1}$ is a Weyl operator for all $\lambda \in \rho(A+K)$. The fact that $\lambda - A = (I + K(\lambda - A - K)^{-1})(\lambda - A - K)$ and by using the Atkinson′s theorem we get $(\lambda - A)$ is a Weyl operator. This shows that $\lambda \notin \sigma_{ew}(A)$. $\square$

Remark 2.4. Since $\mathcal{K}(X) \subset \Delta_A(X)$, $\mathcal{K}(X)$ is the minimal subset of $\mathcal{C}(X)$ (in the sense of inclusion) which characterize the essential Weyl spectrum . Hence Theorem 2.3 provides an improvement of the definition of $\sigma_{ew}(A)$ and is valid for a somewhat large variety of subsets of $\mathcal{C}(X)$. Furthermore,

$$\sigma_{ew}(A+K) = \sigma_{ew}(A), \qquad \text{for all } K \in \Delta_A(X).$$

It follows, immediately, from Theorem 2.3 that

Corollary 2.5. *Let $A \in \mathcal{C}(X)$ and let $M(X)$ be any subset of $\mathcal{QK}(X)$ (not necessarily an ideal) satisfying the condition*

$$\mathcal{K}(X) \subseteq M(X) \subseteq \mathcal{QK}(X). \tag{2.6}$$

Then

$$\sigma_{ew}(A) = \sigma_{ew}(A+K) \qquad \text{for all } K \in H_A(X).$$

$H_A(X) = \{K \in \mathcal{C}(X), K$ *is A-bounded and* $K(\lambda - A - K)^{-1} \in M(X)$ *for all* $\lambda \in \rho(A+K)\}$.

Proof. We have from (2.6) that $\mathcal{K}(X) \subseteq H_A(X) \subseteq \Delta_A(X)$. From this we infer that

$$\bigcap_{K \in \Delta_A(X)} \sigma(A+K) \subseteq \bigcap_{K \in H_A(X)} \sigma(A+K) \subseteq \sigma_{ew}(A). \tag{2.7}$$

Now, the result follows from Theorem 2.3. $\square$

Note that in applications (transport operators, operators arising in dynamic populations, etc.), we deal with operators A and B such that $B = A + K$ where $A \in \mathcal{C}(X)$ and K is, in general, a closed (or closable) A-bounded linear operator. The operator K does not necessarily satisfy the hypotheses of the previous results. For some physical conditions on K, we have information about the operator $(\lambda - A)^{-1} - (\lambda - B)^{-1}$ ($\lambda \in \rho(A) \cap \rho(B)$). So we have the following useful stability result.

Theorem 2.6. *Let $A, B \in \mathcal{C}(X)$ such that $\rho(A) \cap \rho(B) \neq \emptyset$. If for some $\lambda \in \rho(A) \cap \rho(B)$ the operator $(\lambda - A)^{-1} - (\lambda - B)^{-1} \in \Delta_A(X)$, then*

$$\sigma_{ef}(A) = \sigma_{ef}(B) \quad \text{and} \quad \sigma_{ew}(A) = \sigma_{ew}(B).$$

Proof. Without loss of generality, we suppose that $\lambda = 0$. Hence $0 \in \rho(A) \cap \rho(B)$. Therefore, we can write for $\mu \neq 0$

$$\mu - A = -\mu(\mu^{-1} - A^{-1})A.$$

Since, A is one to one and onto, then

$$\alpha(\mu - A) = \alpha(\mu^{-1} - A^{-1}) \text{ and } \beta(\mu - A) = \beta(\mu^{-1} - A^{-1}).$$

This shows that $(\mu - A)$ is a Fredholm operator if and only if $(\mu^{-1} - A^{-1})$ is a Fredholm operator and $\mathrm{ind}(\mu - A) = \mathrm{ind}(\mu^{-1} - A^{-1})$. Now, assume that $A^{-1} - B^{-1} \in \Delta_A(X)$. Hence using Theorem 2.3 we conclude that $(\mu - A)$ is a Fredholm operator if and only if $(\mu - B)$ is a Fredholm operator and $\mathrm{ind}(\mu - A) = \mathrm{ind}(\mu - B)$ for each $\mu \notin \sigma_{ef}(A)$. This proves $\sigma_{ef}(A) = \sigma_{ef}(B)$ and $\sigma_{ew}(A) = \sigma_{ew}(B)$. □

3. Application to transport operator

In this section we are concerned with the Weyl essential spectrum of singular transport operators

$$A\psi(x,\xi) = -\xi \cdot \nabla_x \psi(x,\xi) - \sigma(\xi)\psi(x,\xi) + \int_{\mathbb{R}^n} \kappa(x,\xi')\psi(x,\xi')\, d\xi' \qquad (x,\xi) \in \Omega \times V, \tag{3.1a}$$

$$\psi_{|\Gamma_-}(x,\xi) = 0 \qquad (x,\xi) \in \Gamma_-. \tag{3.1b}$$

where Ω is a smooth open subset of $\mathbb{R}^n$ $(n \geq 1)$, V is the support of a positive Radon measure $d\mu$ on $\mathbb{R}^n$ and $\psi \in L^p(\Omega \times V, dxd\mu(\xi))$ $(1 \leq p < \infty)$. In (3.1b) Γ_- denotes the incoming part of the boundary of the phase space $\Omega \times V$

$$\Gamma_- = \{(x,\xi) \in \partial\Omega \times V \; ; \; \xi \cdot n(x) < 0\},$$

where $n(x)$ stands for the outward normal unit at $x \in \partial\Omega$. The operator A describes the transport of particle (neutrons, photons, molecules of gas, etc.) in the domain Ω. The function ψ represents the number (or probability) density of particles having the position x and the velocity ξ. The functions $\sigma(.)$ and $\kappa(.,.)$ are called, respectively, the collision frequency and the scattering kernel. Let us first introduce the functional setting we shall use in the sequel. Let

$$\begin{aligned} X_p &= L^p(\Omega \times V, dxd\mu(\xi)), \\ X_p^\sigma &= L^p(\Omega \times V, \sigma(\xi)dxd\mu(\xi)) \qquad\qquad 1 \leq p < \infty, \\ L^p_\sigma(\mathbb{R}^n) &= L^p(\mathbb{R}^n, \sigma(\xi)d\mu(\xi)). \end{aligned}$$

We define the partial Sobolev space

$$W_p = \{\psi \in X_p \, ; \, \xi \cdot \nabla_x \psi \in X_p\}.$$

For any $\psi \in W_p$, one can define the space traces $\psi_{|\Gamma_-}$ on Γ_-,

$$\widetilde{W}_p = \{\psi \in W_p \, ; \, \psi_{|\Gamma_-} = 0\}.$$

The streaming operator T associated with the boundary condition (3.1b) is

$$\begin{cases} T \quad : \quad D(T) \subset \quad X_p \to X_p \\ \qquad\qquad\qquad\qquad \psi \mapsto T\psi(x,\xi) := -\xi \cdot \nabla_x \psi(x,\xi) - \sigma(\xi)\psi(x,\xi), \end{cases}$$

with domain

$$D(T) := \widetilde{W}_p \cap X_p^\sigma.$$

The transport operator (3.1) can be formulated as follows $A = T + K$, where K denotes the following collision operator

$$\begin{aligned} K : X_p &\to X_p \\ \psi &\mapsto \int_{\mathbb{R}^n} \kappa(x,\xi')\psi(x,\xi')\, d\xi' \end{aligned}.$$

We will assume that the scattering kernel $\kappa(.,.) = \kappa_1(.,.) + \kappa_2(.,.)$, $\kappa_i(.,.)$ are non-negative $i = 1, 2$ and there exist a closed subset $E \subset \mathbb{R}^n$ with zero $d\mu$ measure and a constant $\sigma_0 > 0$ such that

$$\sigma(.) \in L^{\infty}_{loc}(\mathbb{R}^n \setminus E), \quad \sigma(\xi) > \sigma_0; \tag{3.2a}$$

$$\left[\int_{\mathbb{R}^n} (\frac{\kappa_i(.,\xi')}{\sigma(\xi'^{\frac{1}{p}})})^q d\mu(\xi')\right]^{\frac{1}{q}} \in L^p(\mathbb{R}^n), \quad i = 1, 2. \tag{3.2b}$$

where q denotes the conjugate exponent of p. Denote by $K_i\psi(x,\xi) = \displaystyle\int_{\mathbb{R}^n} \kappa_i(x,\xi')\psi(x,\xi')\,d\xi'$ $i = 1, 2$. Using boundedness of Ω and the assumption (3.2b) we can fined that $K_i \in \mathcal{L}(X_p^\sigma, X_p)$ with

$$\|K_i\|_{\mathcal{L}(X_p^\sigma, X_p)} \leq \left\| \left[\int_{\mathbb{R}^n} (\frac{\kappa_i(.,\xi')}{\sigma(\xi'^{\frac{1}{p}})})^q d\mu(\xi')\right]^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} \qquad i = 1, 2. \tag{3.3}$$

Note that a simple calculation using the assumption (3.2a) shows that X_p^σ is a subset of X_p and the the embedding $X_p^\sigma \hookrightarrow X_p$ is continuous.

Let us now consider the resolvent equation

$$(\lambda - T)\psi = \varphi, \tag{3.4}$$

where φ is a given element of X_p and the unknown ψ must be founded in $\mathcal{D}(T)$. For Re$\lambda > -\sigma_0$, the solution of (3.4) reads as follows

$$\psi(x,\xi) = \int_0^{t(x,\xi)} e^{-(\lambda+\sigma(\xi))s}\varphi(x - s\xi, \xi)ds, \tag{3.5}$$

where $t(x,\xi) = \sup\{\, t > 0\,; x - s\xi \in \Omega,\ \forall\, 0 < s < t\,\} = \inf\{\, s > 0\,;\ x - s\xi \notin \Omega\}$. An immediate consequence of these facts is that $\sigma(T) \subseteq \{\lambda \in \mathbb{C}\ :\ Re\lambda \leq -\sigma_0\}$, and in [8, 7] shows that $\sigma(T)$ is reduced to the continuous spectrum $\sigma_c(T)$ of T, that is

$$\sigma(T) = \sigma_c(T) = \{\lambda \in \mathbb{C}\ :\ Re\lambda \leq -\sigma_0\}, \tag{3.6}$$

Since all essential spectra are enlargement of the continuous spectrum we infer that

$$\sigma_{ew}(T) = \sigma_{ef}(T) = \{\lambda \in \mathbb{C}\ :\ Re\lambda \leq -\sigma_0\}. \tag{3.7}$$

Lemma 3.1. *The collision operator K is T-bounded.*

Proof. Let $\lambda \in \mathbb{C}$ be such that $Re\lambda > -\sigma_0$ and consider $\psi \in X_p$. It follows from (3.5) that

$$\int_\Omega \left|(\lambda - T)^{-1}\psi(x,\xi)\right|^p dx \leq \frac{1}{(Re\lambda + \sigma(\xi))^p}\int_\Omega |\psi(x,\xi)|^p\,dx$$

Therefore,

$$\left\|(\lambda - T)^{-1}\psi\right\|_{X_p^\sigma} \leq \sup_{\xi\in\mathbb{R}^n} \frac{\sigma(\xi)}{(Re\lambda + \sigma(\xi))^p}\,\|\psi\|_{X_p} \tag{3.8}$$

Hence, $(\lambda - T)^{-1} \in \mathcal{L}(X_p, X_p^\sigma)$. Using now the equation (3.3) to deduce that the operator K is T-bounded. □

Proposition 3.2 ([7, Proposition 4.1])**.** *Let Ω be a bounded subset of $\mathbb{R}^n$ and $1 < p < \infty$. If the hypotheses* (3.2a) *and* (3.2b) *are satisfied for $\kappa_1(.,.)$, the measure $d\mu$ satisfies*

$$\begin{cases} \textit{the hyperplanes have zero } d\mu\textit{-measure, i.e.,} \\ \textit{for each, } e \in S^{n-1}, d\mu\{\xi \in \mathbb{R}^n, \xi.e = 0\} = 0, \end{cases} \tag{3.9}$$

where S^{n-1} denotes the unit ball of $\mathbb{R}^n$ and the collision operator $K_1 : L^p_\sigma(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n)$ is compact. Then for any $\lambda \in \mathbb{C}$ such that $Re\lambda > -\sigma_0$, the operator $K_1(\lambda - T)^{-1}$ is compact on X_p.

Now we are in position to state the main result of this section.

Theorem 3.3. *Assume that the hypotheses of Proposition 3.2 are satisfied and*

$$\sup_{\xi\in\mathbb{R}^n}\frac{\sigma(\xi)}{(Re\lambda+\sigma(\xi))^p}\left\|\left[\int_{\mathbb{R}^n}\Big(\frac{\kappa_2(.,\xi')}{\sigma(\xi'^{\frac{1}{p}})}\Big)^q d\mu(\xi')\right]^{\frac{1}{q}}\right\|_{L^p(\mathbb{R}^n)}<1 \quad \textit{for } Re\lambda>-\sigma_0. \tag{3.10}$$

Then

(1) *For all $\lambda\in\rho(A+K)$, we have $K(\lambda-A-K)^{-1}$ is quasi-compact on X_p,*
(2) $\sigma_{ew}(A)=\sigma_{ew}(T)=\{\lambda\in\mathbb{C}\ :\ Re\lambda\le-\sigma_0\}$.

Proof. (1) Let $\lambda\in\rho(A+K)$ and $\mu\in\rho(T)$ such that

$$\left\|I+(\mu-\lambda+K_2)(\lambda-T-K)^{-1}\right\|<1.$$

By this estimate and (3.10) we can deduce that $K_2(\mu-T)^{-1}[I+(\mu-\lambda+K_2)(\lambda-T-K)^{-1}]$ is a pure contraction. On other hand, we have

$$\begin{aligned}K(\lambda-A-K)^{-1}&=K(\mu-T)^{-1}[I+(\mu-\lambda+K)(\lambda-T-K)^{-1}]\\&=K_1(\mu-T)^{-1}[I+(\mu-\lambda+K)(\lambda-T-K)^{-1}]\\&+K_2(\mu-T)^{-1}K_1(\lambda-T-K)^{-1}\\&+K_2(\mu-T)^{-1}[I+(\mu-\lambda+K_2)(\lambda-T-K)^{-1}].\end{aligned}$$

By this equation and Proposition 3.2 we have $K(\lambda-A-K)^{-1}$ is a sum of a compact operator and a pure contraction on X_p. Hence $K(\lambda-A-K)^{-1}$ is a quasi-compact operator on X_p.

(2) The first item together with hypotheses on K implies that $K\in\Delta_A(X_p)$. Now the second item follows from Theorem 2.3, Remark 2.4 and the relation (3.7).

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