International Journal of Analysis and Applications
ISSN 2291-8639
Volume 9, Number 1 (2015), 45-53
http://www.etamaths.com

APPLICATIONS OF SOME CLASSES OF SEQUENCES ON APPROXIMATION OF FUNCTIONS (SIGNALS) BY ALMOST GENERALIZED NÖRLUND MEANS OF THEIR FOURIER SERIES

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Abstract. In this paper, using rest bounded variation sequences and head bounded variation sequences, some new results on approximation of functions (signals) by almost generalized Nörlund means of their Fourier series are obtained. To our best knowledge this the first time to use such classes of sequences on approximations of the type treated in this paper. In addition, several corollaries are derived from our results as well as those obtained previously by others.

1. Introduction and preliminaries

Given two sequences $p := (p_n)$ and $q := (q_n)$ the convolution $(p * q)_n$ is defined by

$$R_n := (p * q)_n := \sum_{m=0}^{n} p_m q_{n-m},$$

and we also write $P_n := (p * 1)_n = \sum_{m=0}^{n} p_m$ and $Q_n := (1 * q)_n = \sum_{m=0}^{n} q_m = \sum_{m=0}^{n} q_{n-m}$.

Let (s_n) be a sequence. When $R_n \neq 0$ for all n, the generalized Nörlund transform of the sequence (s_n) is the sequence $\{t_n^{p,q}\}$ obtained by putting

$$t_n^{p,q} = \frac{1}{R_n} \sum_{m=0}^{n} p_{n-m} q_m s_m.$$

If $s_n \longrightarrow s(n \longrightarrow \infty)$ induces $t_n^{p,q} \longrightarrow s(n \longrightarrow \infty)$ then the method (N, p_n, q_n) is called to be regular. The necessary and sufficient condition for (N, p_n, q_n) method to be regular is $\sum_{m=0}^{n} |p_{n-m} q_m| = \mathcal{O}(|(p*q)_n|)$ and $p_{n-m} = o(|(p*q)_n|)$ as $n \longrightarrow \infty$ for every fixed $m \geq 0$ (see Borwein [1]).

The method (N, p_n, q_n) reduces to Nörlund method (N, p_n) if $q_n = 1$ for all n and to Riesz method $(\overline{N}, q_n)$ if $p_n = 1$ for all n. It is well-known that (N, p_n) mean or $(\overline{N}, q_n)$ mean includes as a special case Cesàro and harmonic means or logarithmic mean, respectively.

2010 *Mathematics Subject Classification.* 40C99, 40G99, 41A25, 42A16.

Key words and phrases. fourier series; almost generalized Nörlund means; degree of approximation.

Let f be a 2π periodic function (signal) and Lebesgue integrable i.e. $f \in L[0, 2\pi]$. Then the Fourier series of the function (signal) f at the point x is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx), \tag{1.1}$$

with its partial sums $s_n(f; x)$ being a trigonometric polynomial of order n with $n + 1$ terms.

A function (signal) $f \in \text{Lip } \alpha$ if $|f(x + t) - f(x)| = \mathcal{O}(|t|^\alpha)$ for $0 < \alpha \leq 1$.

A function (signal) $f \in \text{Lip } (\alpha, r)$ for $a \leq x \leq b$ if

$$w_r(t; f) = \left\{ \int_a^b |f(x + t) - f(x)|^r \right\}^{1/r} \leq M(|t|^\alpha) \tag{1.2}$$

for $r \geq 1$ and $0 < \alpha \leq 1$, where M is an absolute positive constant not necessarily the same at each occurrence (see McFadden [5]).

It should be noted that if $r \longrightarrow \infty$ in $\text{Lip}(p, r)$ class then this class reduces to $\text{Lip}\alpha$.

According to Lorentz [3] a bounded sequence (s_k) of k-th sums of the Fourier series (1.1) is said to be almost convergent to s, if

$$\lim_{n\to\infty} s_{n,r} = \lim_{n\to\infty} \frac{s_r + s_{r+1} + \cdots + s_{r+n}}{n + 1} = \lim_{n\to\infty} \frac{1}{n + 1} \sum_{k=r}^{n+r} s_k = s. \tag{1.3}$$

It is said (see [9]) that the the Fourier series (1.1) is said to be almost Riesz summable to the finite number s, if

$$\tau_{n,r} = \frac{1}{P_n} \sum_{m=0}^{n} p_m s_{m,r} \longrightarrow s \quad \text{as} \quad n \longrightarrow \infty$$

uniformly with respect to r, where

$$s_{m,r} = \frac{1}{m + 1} \sum_{j=r}^{m+r} s_j.$$

It is a well-known fact that a convergent sequence is almost convergent and the limits are the same. A bounded sequence (s_n) is said to be almost Riesz summable to s if the Riesz transform of (s_n) is almost convergent to s (see [2]).

The theory of approximation which is originated from a well-known theorem of Weierstrass has been an excitatory interdisciplinary field of study till nowadays. The approximations of the functions have a wide applications in signal analysis, digital communications, theory of machines in mechanical engineering and in particular in digital signal processing see [7] and [8] (also the interested reader could find several new results on these approximations and their applications into references given in [6]).

Very recently Mishra et al [6] determined the degree of approximation of a signal $f \in \text{Lip}(\alpha, r), (r \geq 1)$ by almost Riesz summability means of its Fourier series. Before we recall their results we need first some known definitions given below.

The L_r-norm of an function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by

$$\|f\|_r = \left(\int_0^{2\pi} |f(x)|^r dx \right)^{1/r}, \quad r \geq 1.$$

The L_∞-norm of an function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by

$$\|f\|_\infty = \sup\{|f(x)| : x \in \mathbb{R}\}.$$

A signal (function) f is approximated by trigonometric polynomial $\tau_n(f;x)$ of order n and the degree of approximation $E_n(f)$ of a function $f \in L_r$ is given by

$$E_n(f) = \min_n \|f(x) - \tau_n(f;x)\|_r,$$

in terms of n.

The degree of approximation of a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ by a trigonometric polynomial $\tau_n(f;x)$ of order n under sup norm $\|\cdot\|_\infty$ is defined by

$$\|f(x) - \tau_n(f;x)\|_\infty = \sup\{|f(x) - \tau_n(f;x)| : x \in \mathbb{R}\}.$$

Throughout this paper we will write

$$\psi(t) = f(x+t) - f(x-t) - 2f(x).$$

Now we are able to formulate the result obtained in [6]:

Theorem 1.1. *If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a 2π periodic function, Lebesgue integrable and belonging to the $Lip(\alpha, r), (r \geq 1)$ class, then the degree of approximation of the function f by almost Riesz means of its Fourier series is given by*

$$\|f(t) - \tau_n(f(t);x)\|_r = \mathcal{O}\left(P_n^{1/r-\alpha}\right), \forall n, \tag{1.4}$$

and $\psi(t)$ satisfies the following conditions

$$\left[\int_0^{\pi/P_n} \left(\frac{t|\psi(t)|}{t^\alpha}\right)^r dt\right]^{1/r} = \mathcal{O}\left(P_n^{-1}\right), \tag{1.5}$$

$$\left[\int_{\pi/P_n}^{\pi} \left(\frac{t^{-\delta}|\psi(t)|}{t^\alpha}\right)^r dt\right]^{1/r} = \mathcal{O}\left(P_n^{\delta}\right), \tag{1.6}$$

where δ is a finite quantity, Riesz means are regular and $r + s = rs$ such that $1 \leq r \leq \infty$.

Note that in this theorem is not mentioned explicitly that the sequence (p_n) is a non-decreasing one but in its proof it is used this property.

We say that the the Fourier series (1.1) is said to be almost generalized Nörlund summable to the finite number s ([1]), if

$$t_{n,r}^{p,q} = \frac{1}{R_n}\sum_{m=0}^{n} p_m q_{n-m} s_{m,r} \longrightarrow s \quad \text{as} \quad n \longrightarrow \infty$$

uniformly with respect to r, where

$$s_{m,r} = \frac{1}{m+1}\sum_{j=r}^{m+r} s_j.$$

Now we give definitions of two classes of sequences (see [4]).

A sequence $\mathbf{c} := \{c_n\}$ of nonnegative numbers tending to zero is called of Rest Bounded Variation, or briefly $\mathbf{c} \in RBVS$, if it has the property

$$\sum_{n=m}^{\infty} |c_n - c_{n+1}| \leq K(\mathbf{c})c_m \tag{1.7}$$

for all natural numbers m, where $K(\mathbf{c})$ is a constant depending only on $\mathbf{c}$.

A sequence $\mathbf{c} := \{c_n\}$ of nonnegative numbers will be called of Head Bounded Variation, or briefly $\mathbf{c} \in HBVS$, if it has the property

$$\sum_{n=0}^{m-1} |c_n - c_{n+1}| \leq K(\mathbf{c}) c_m \tag{1.8}$$

for all natural numbers m, or only for all $m \leq N$ if the sequence $\mathbf{c}$ has only finite nonzero terms, and the last nonzero term is c_N.

The purpose of this paper is to determine the degree of approximation of a function (signal) $f \in \text{Lip}(\alpha, r), (r \geq 1)$ by almost generalized Nörlund means of its Fourier series under conditions that $(p_n) \in HBVS$ and $(q_n) \in RBVS$. As is pointed out in Figure 2 constructed in [10] the class of sequences $RBVS$ is a wider one than that of monotone sequences. This fact shows that in some way our results are very extensive results.

2. Main Results

We prove the following main result.

Theorem 2.1. *Let $(p_n) \in HBVS$ and $(q_n) \in RBVS$. If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a 2π periodic function, Lebesgue integrable and belonging to the $Lip(\alpha, r), (r \geq 1)$ class, then the degree of approximation of the function f by almost generalized Nörlund means of its Fourier series $t^{p,q}_{n,r}(f(t); x)$ is given by*

$$\|f(t) - t^{p,q}_{n,r}(f(t); x)\|_r = \mathcal{O}\left(R_n^{1/r-\alpha}\right), \ \forall n, \tag{2.1}$$

and $\psi(t)$ satisfies the following conditions

$$\left[\int_0^{\pi/R_n} \left(\frac{t|\psi(t)|}{t^\alpha}\right)^r dt\right]^{1/r} = \mathcal{O}\left(R_n^{-1}\right), \tag{2.2}$$

$$\left[\int_{\pi/R_n}^{\pi} \left(\frac{t^{-\delta}|\psi(t)|}{t^\alpha}\right)^r dt\right]^{1/r} = \mathcal{O}\left(R_n^{\delta}\right), \tag{2.3}$$

where δ is a finite quantity, generalized Nörlund means are regular and $r + s = rs$ such that $1 \leq r \leq \infty$.

Proof. It is almost a routine that for partial sums $s_k(f(t); x)$ of the Fourier series (1.1) the equality

$$s_{k,r}(f(t); x) - f(t) = \frac{1}{2\pi(k+1)} \int_0^\pi \psi(t) \frac{\cos(rt) - \cos(k+r+1)t}{2\sin^2 \frac{t}{2}} \, dt$$

holds true.

Whence, for almost generalized Nörlund means of $s_{k,r}(f(t);x)$ we have

$$\begin{aligned}
& t_{n,r}^{p;q}(f(t);x)-f(t) \\
= & \frac{1}{R_n}\sum_{m=0}^{n} p_m q_{n-m}\{s_{m,r}(f(t);x)-f(t)\} \\
= & \frac{1}{2\pi R_n}\int_0^{\pi}\psi(t)\sum_{m=0}^{n}\frac{p_m q_{n-m}}{m+1}\cdot\frac{\cos(rt)-\cos(m+r+1)t}{2\sin^2\frac{t}{2}}\,dt \\
= & \frac{1}{2\pi R_n}\left(\int_0^{\pi/R_n}+\int_{\pi/R_n}^{\pi}\right)\psi(t)\sum_{m=0}^{n}\frac{p_m q_{n-m}}{m+1}\cdot\frac{\sin(m+2r+1)\frac{t}{2}\cdot\sin(m+1)\frac{t}{2}}{2\sin^2\frac{t}{2}}\,dt \\
= & L_1+L_2.
\end{aligned} \tag{2.4}$$

Applying Hölder's inequality, $f(t)\in \mathrm{Lip}(\alpha,s) \Longrightarrow \psi(t)\in \mathrm{Lip}(\alpha,s)$ on $[0,\pi]$ (see [5]), condition (2.2), the well-known inequalities

$$\sin u \geq \frac{2}{\pi}u,\ \text{ for } u\in(0,\pi/2], \tag{2.5}$$

$$|\sin(mu)|\leq m|\sin u| \text{ for all } u\in\mathbb{R}, m\in\mathbb{N}, \tag{2.6}$$

$r+s=rs$ such that $1\leq r\leq\infty$, we obtain

$$\begin{aligned}
|L_1| \leq & \frac{1}{2\pi R_n}\left[\int_0^{\pi/R_n}\left(\frac{t|\psi(t)|}{t^{\alpha}}\right)^r dt\right]^{1/r}\times \\
& \times\left[\int_0^{\pi/R_n}\left(\frac{1}{t^{1-\alpha}}\left|\sum_{m=0}^{n}\frac{p_m q_{n-m}}{m+1}\cdot\frac{\sin(m+2r+1)\frac{t}{2}\cdot\sin(m+1)\frac{t}{2}}{2\sin^2\frac{t}{2}}\right|\right)^s dt\right]^{1/s} \\
= & \mathcal{O}\left(R_n^{-2}\right)\left[\int_0^{\pi/R_n}\left(\frac{1}{t^{1-\alpha}}\left|\sum_{m=0}^{n}p_m q_{n-m}\cdot\frac{1}{t}\right|\right)^s dt\right]^{1/s} \\
= & \mathcal{O}\left(R_n^{-2}\right)\left[\int_0^{\pi/R_n}R_n^s\cdot t^{(\alpha-2)s}\,dt\right]^{1/s} \\
= & \mathcal{O}\left(R_n^{-1}\right)\cdot\mathcal{O}\left(\frac{1}{R_n^{\alpha-2+\frac{1}{s}}}\right)=\mathcal{O}\left(\frac{1}{R_n^{\alpha-\frac{1}{r}}}\right).
\end{aligned} \tag{2.7}$$

To estimate $|L_2|$ from above we again apply Hölder's inequality to obtain

$$\begin{aligned}
|L_2| \leq & \frac{1}{2\pi R_n}\left[\int_{\pi/R_n}^{\pi}\left(\frac{t^{-\delta}|\psi(t)|}{t^{\alpha}}\right)^r dt\right]^{1/r}\times \\
& \times\left[\int_{\pi/R_n}^{\pi}\left(t^{\delta+\alpha}\left|\sum_{m=0}^{n}\frac{p_m q_{n-m}}{m+1}\cdot\frac{\sin(m+2r+1)\frac{t}{2}\cdot\sin(m+1)\frac{t}{2}}{2\sin^2\frac{t}{2}}\right|\right)^s dt\right]^{1/s}.
\end{aligned}$$

Next, using again the fact that $f(t)\in \text{Lip}(\alpha,s)\Longrightarrow \psi(t)\in \text{Lip}(\alpha,s)$ on $[0,\pi]$ (see [5]), conditions (2.3), (2.5), (2.6), and $r+s=rs$ such that $1\le r\le\infty$, we get

$$\begin{aligned}|L_2| &= \mathcal{O}\left(R_n^{-1}\right)\times \\ &\times\left[\int_{\pi/R_n}^{\pi}\left(t^{\delta+\alpha}\sum_{m=0}^{n}\frac{\left|p_m q_{n-m}\sin(m+2r+1)\frac{t}{2}\right|\cdot(m+1)\left|\sin\frac{t}{2}\right|}{2(m+1)\sin^2\frac{t}{2}}\right)^s dt\right]^{1/s} \\ &= \mathcal{O}\left(R_n^{-1}\right)\mathcal{O}\left(R_n^{\delta}\right)\left[\int_{\pi/R_n}^{\pi}\left(\frac{t^{\delta+\alpha}}{\sin\frac{t}{2}}\sum_{m=0}^{n}\left|p_m q_{n-m}\sin(m+2r+1)\frac{t}{2}\right|\right)^s dt\right]^{1/s}.\end{aligned}$$

Since $(p_k)\in HBVS$, then by (1.8) we have

$$p_m-p_n\le|p_m-p_n|\le\sum_{k=m}^{n-1}|p_k-p_{k+1}|\le\sum_{k=0}^{n-1}|p_k-p_{k+1}|\le K(p)p_n$$

which implies

$$p_m\le(K(p)+1)p_n,\forall m\in[0,n]. \tag{2.8}$$

Also, since $(q_k)\in RBVS$, then by (1.7) we have

$$q_{n-m}\le\sum_{k=m}^{\infty}|q_{n-k}-q_{n-k-1}|\le\sum_{k=0}^{\infty}|q_{n-k}-q_{n-k-1}|\le K(q)q_n$$

which implies

$$q_{n-m}\le K(q)q_n,\forall m\in[0,n]. \tag{2.9}$$

Using the well-known fact

$$\sum_{\ell=j}^{d}e^{-i\ell t}=\mathcal{O}\left(t^{-1}\right),\ 0\le j\le d,$$

(2.8) and (2.9) we find that

$$\begin{aligned}&\sum_{m=0}^{n}\left|p_m q_{n-m}\sin(m+2r+1)\frac{t}{2}\right| \\ &\qquad\le(K(p)+1)K(q)p_nq_n\max_{0\le j\le n}\sum_{m=0}^{j}\sin(m+2r+1)\frac{t}{2}=\mathcal{O}\left(R_nt^{-1}\right).\end{aligned}$$

Subsequently, we obtain

$$|L_2| = \mathcal{O}\left(R_n^{\delta-1}\right)\left[\int_{\pi/R_n}^{\pi}\left(R_nt^{\delta+\alpha-2}\right)^s dt\right]^{1/s}=\mathcal{O}\left(\frac{1}{R_n^{\alpha-\frac{1}{r}}}\right). \tag{2.10}$$

Inserting (2.7) and (2.10) into (2.4) we immediately obtain

$$|f(t)-t_{n,r}^{p,q}(f(t);x)|=\mathcal{O}\left(R_n^{1/r-\alpha}\right).$$

Finaly, using L_r-norm and the lastest estimate we find that

$$\begin{aligned}\|f(t)-t_{n,r}^{p,q}(f(t);x)\|_r &= \left[\int_0^{2\pi}|f(t)-t_{n,r}^{p,q}(f(t);x)|^r\,dt\right]^{1/r}\\ &= \left[\int_0^{2\pi}\mathcal{O}\left(R_n^{1/r-\alpha}\right)^r dt\right]^{1/r} = \mathcal{O}\left(R_n^{1/r-\alpha}\right).\end{aligned}$$

The proof of the theorem is completed. □

If we take $q_n = 1$ for all $n \geq 0$ then we obtain:

Corollary 2.1. *Let $(p_n) \in HBVS$. If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a 2π periodic function, Lebesgue integrable and belonging to the $Lip(\alpha, r), (r \geq 1)$ class, then the degree of approximation of the function f by almost Riesz means $t_{n,r}^{p}(f(t);x)$ of its Fourier series is given by*

$$\|f(t)-t_{n,r}^{p}(f(t);x)\|_r = \mathcal{O}\left(P_n^{1/r-\alpha}\right),\ \forall n,$$

and $\psi(t)$ satisfies the following conditions

$$\left[\int_0^{\pi/P_n}\left(\frac{t|\psi(t)|}{t^\alpha}\right)^r dt\right]^{1/r} = \mathcal{O}\left(P_n^{-1}\right), \tag{2.11}$$

$$\left[\int_{\pi/P_n}^{\pi}\left(\frac{t^{-\delta}|\psi(t)|}{t^\alpha}\right)^r dt\right]^{1/r} = \mathcal{O}\left(P_n^{\delta}\right), \tag{2.12}$$

where δ is a finite quantity, Riesz means are regular and $r + s = rs$ such that $1 \leq r \leq \infty$.

If we take $p_n = 1$ for all $n \geq 0$ then we obtain:

Corollary 2.2. *Let $(q_n) \in RBVS$. If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a 2π periodic function, Lebesgue integrable and belonging to the $Lip(\alpha, r), (r \geq 1)$ class, then the degree of approximation of the function f by almost Nörlund means $t_{n,r}^{q}(f(t);x)$ of its Fourier series is given by*

$$\|f(t)-t_{n,r}^{q}(f(t);x)\|_r = \mathcal{O}\left(Q_n^{1/r-\alpha}\right),\ \forall n,$$

and $\psi(t)$ satisfies the following conditions

$$\left[\int_0^{\pi/Q_n}\left(\frac{t|\psi(t)|}{t^\alpha}\right)^r dt\right]^{1/r} = \mathcal{O}\left(Q_n^{-1}\right), \tag{2.13}$$

$$\left[\int_{\pi/Q_n}^{\pi}\left(\frac{t^{-\delta}|\psi(t)|}{t^\alpha}\right)^r dt\right]^{1/r} = \mathcal{O}\left(Q_n^{\delta}\right), \tag{2.14}$$

where δ is a finite quantity, Nörlund means are regular and $r + s = rs$ such that $1 \leq r \leq \infty$.

If we take $r \to \infty$ then $\mathrm{Lip}(\alpha, r) \equiv \mathrm{Lip}\alpha$ and we derive the following.

Corollary 2.3. *Let $(p_n) \in HBVS$ and $(q_n) \in RBVS$. If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a 2π periodic function, Lebesgue integrable and belonging to the Lipα class, then the degree of approximation of the function f by almost generalized Nörlund means of its Fourier series is given by*

$$|f(t) - t_{n,r}^{p;q}(f(t);x)| = \mathcal{O}\left(R_n^{-\alpha}\right), \ \forall n,$$

and $\psi(t)$ satisfies the following conditions (2.2) and (2.3), where δ is a finite quantity, generalized Nörlund means are regular and $r + s = rs$ such that $1 \leq r \leq \infty$.

Proof. For $r \to \infty$ and Theorem 2.1 we have

$$|f(t) - t_{n,r}^{p;q}(f(t);x)|_\infty = \sup_{0 \leq x \leq 2\pi} |f(t) - t_{n,r}^{p;q}(f(t);x)| = \mathcal{O}\left(R_n^{-\alpha}\right).$$

Subsequently, we find that

$$|f(t) - t_{n,r}^{p;q}(f(t);x)| \leq |f(t) - t_{n,r}^{p;q}(f(t);x)|_\infty = \mathcal{O}\left(R_n^{-\alpha}\right),$$

which completes the proof. □

Finally, if for all $n \geq 0$ we take $q_n = 1$ or $p_n = 1$ in Corollary 2.3 respectively, then we obtain the following two corollaries.

Corollary 2.4. *Let $(p_n) \in HBVS$. If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a 2π periodic function, Lebesgue integrable and belonging to the Lipα class, then the degree of approximation of the function f by almost Riesz means of its Fourier series is given by*

$$|f(t) - t_{n,r}^{p}(f(t);x)| = \mathcal{O}\left(P_n^{-\alpha}\right), \ \forall n,$$

and $\psi(t)$ satisfies the following conditions (2.11) and (2.12), where δ is a finite quantity, Riesz means are regular and $r + s = rs$ such that $1 \leq r \leq \infty$.

Corollary 2.5. *Let $(q_n) \in RBVS$. If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a 2π periodic function, Lebesgue integrable and belonging to the Lipα class, then the degree of approximation of the function f by almost Nörlund means of its Fourier series is given by*

$$|f(t) - t_{n,r}^{q}(f(t);x)| = \mathcal{O}\left(Q_n^{-\alpha}\right), \ \forall n,$$

and $\psi(t)$ satisfies the following conditions (2.13) and (2.14), where δ is a finite quantity, Nörlund means are regular and $r + s = rs$ such that $1 \leq r \leq \infty$.

Remark 2.1. *If we had assumed in Theorem 2.1 that (p_n) is a non-decreasing sequence and (q_n) is a non-increasing one, then it would also hold true. Thus, taking $q_n = 1$ for all $n \geq 0$, then all results obtained in* [6] *are immediate results of ours.*

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