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TITCHMARSH'S THEOREM FOR THE CHEREDNIK-OPDAM TRANSFORM IN THE SPACE $L^2_{\alpha,\beta}(\mathbb{R})$

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Abstract. In this paper, we prove the generalization of Titchmarshs theorem for the Cherednik-Opdam transform for functions satisfying the $(\psi,2)$-Cherednik-Opdam Lipschitz condition in the space $L^2_{\alpha,\beta}(\mathbb{R})$.

1. Introduction and Preliminaries

In [3], E. C. Titchmarsh's characterizes the set of functions in $L^2(\mathbb{R})$ satisfying the Cauchy-Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transform, namely we have

Theorem 1.1 [3] Let $\delta \in (0,1)$ and assume that $f \in L^2(\mathbb{R})$. Then the following are equivalents
$(i)\quad \|f(t+h)-f(t)\| = O(h^\delta), \quad \text{as} \quad h \to 0,$

$$(ii)\quad \int_{|\lambda|\geq r} |\widehat{f}(\lambda)|^2 d\lambda = O(r^{-2\delta}) \quad \text{as} \quad r \to \infty,$$

where $\widehat{f}$ stands for the Fourier transform of f.

In this paper, we prove the generalization of Theorem 1.1 for the Cherednik-Opdam transform for functions satisfying the $(\psi,2)$-Cherednik-Opdam Lipschitz condition in the space $L^2_{\alpha,\beta}(\mathbb{R})$. For this purpose, we use the generalized translation operator.

In this section, we develop some results from harmonic analysis related to the differential-difference operator $T^{(\alpha,\beta)}$. Further details can be found in [1] and [2]. In the following we fix parameters α, β subject to the constraints $\alpha \geq \beta \geq -\frac{1}{2}$ and $\alpha > \frac{-1}{2}$.
Let $\rho = \alpha+\beta+1$ and $\lambda \in \mathbb{C}$. The Opdam hypergeometric functions $G^{(\alpha,\beta)}_\lambda$ on $\mathbb{R}$ are eigenfunctions $T^{(\alpha,\beta)}G^{(\alpha,\beta)}_\lambda(x) = i\lambda G^{(\alpha,\beta)}_\lambda(x)$ of the differential-difference operator

$$T^{(\alpha,\beta)}f(x) = f'(x) + [(2\alpha+1)\coth x + (2\beta+1)\tanh x]\frac{f(x)-f(-x)}{2} - \rho f(-x),$$

that are normalized such that $G^{(\alpha,\beta)}_\lambda(0) = 1$. In the notation of Cherednik one would write $T^{(\alpha,\beta)}$ as

$$T(k_1+k_2)f(x) = f'(x) + \left\{\frac{2k_1}{1+e^{-2x}} + \frac{4k_2}{1-e^{-4x}}\right\}(f(x)-f(-x)) - (k_1+2k_2)f(x),$$

with $\alpha = k_1+k_2-\frac{1}{2}$ and $\beta = k_2-\frac{1}{2}$. Here k_1 is the multiplicity of a simply positive root and k_2 the (possibly vanishing) multiplicity of a multiple of this root. By [1] or [2], the eigenfunction

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$G_\lambda^{(\alpha,\beta)}$ is given by

$$G_\lambda^{(\alpha,\beta)}(x) = \varphi_\lambda^{\alpha,\beta}(x) - \frac{1}{\rho - i\lambda}\frac{\partial}{\partial x}\varphi_\lambda^{\alpha,\beta}(x) = \varphi_\lambda^{\alpha,\beta}(x) + \frac{\rho}{4(\alpha+1)}\sinh(2x)\varphi_\lambda^{\alpha+1,\beta+1}(x),$$

where $\varphi_\lambda^{\alpha,\beta}(x) ={}_2 F_1(\frac{\rho+i\lambda}{2}; \frac{\rho-i\lambda}{2}; \alpha+1; -\sinh^2 x)$ is the classical Jacobi function.

Lemma 1.2. [4] The following inequalities are valids for Jacobi functions $\varphi_\lambda^{\alpha,\beta}(x)$
(i) $|\varphi_\lambda^{\alpha,\beta}(x)| \leq 1$.
(ii) $1 - \varphi_\lambda^{\alpha,\beta}(x) \leq x^2(\lambda^2 + \rho^2)$.
(iii) there is a constant $c > 0$ such that

$$1 - \varphi_\lambda^{\alpha,\beta}(x) \geq c,$$

for $\lambda x \geq 1$.

Denote $L^2_{\alpha,\beta}(\mathbb{R})$, the space of measurable functions f on $\mathbb{R}$ such that

$$\|f\|_{2,\alpha,\beta} = \left(\int_{\mathbb{R}} |f(x)|^2 A_{\alpha,\beta}(x)dx\right)^{1/2} < +\infty,$$

where

$$A_{\alpha,\beta}(x) = (\sinh|x|)^{2\alpha+1}(\cosh|x|)^{2\beta+1}.$$

The Cherednik-Opdam transform of $f \in C_c(\mathbb{R})$ is defined by

$$\mathcal{H}f(\lambda) = \int_{\mathbb{R}} f(x)G_\lambda^{(\alpha,\beta)}(-x)A_{\alpha,\beta}(x)dx \quad \text{for all} \quad \lambda \in \mathbb{C}.$$

The inverse transform is given as

$$\mathcal{H}^{-1}g(x) = \int_{\mathbb{R}} g(\lambda)G_\lambda^{(\alpha,\beta)}(x)\left(1 - \frac{\rho}{i\lambda}\right)\frac{d\lambda}{8\pi|c_{\alpha,\beta}(\lambda)|^2},$$

here

$$c_{\alpha,\beta}(\lambda) = \frac{2^{\rho-i\lambda}\Gamma(\alpha+1)\Gamma(i\lambda)}{\Gamma(\frac{1}{2}(\rho+i\lambda))\Gamma(\frac{1}{2}(\alpha-\beta+1+i\lambda))}.$$

The corresponding Plancherel formula was established in [1], to the effect that

$$\int_{\mathbb{R}} |f(x)|^2 A_{\alpha,\beta}(x)dx = \int_0^{+\infty} \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2\right) d\sigma(\lambda),$$

where $\check{f}(x) := f(-x)$ and $d\sigma$ is the measure given by

$$d\sigma(\lambda) = \frac{d\lambda}{16\pi|c_{\alpha,\beta}(\lambda)|^2}.$$

According to [2] there exists a family of signed measures $\mu_{x,y}^{(\alpha,\beta)}$ such that the product formula

$$G_\lambda^{(\alpha,\beta)}(x)G_\lambda^{(\alpha,\beta)}(y) = \int_{\mathbb{R}} G_\lambda^{(\alpha,\beta)}(z)d\mu_{x,y}^{(\alpha,\beta)}(z),$$

holds for all $x, y \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, where

$$d\mu_{x,y}^{(\alpha,\beta)}(z) = \begin{cases} \mathcal{K}_{\alpha,\beta}(x,y,z)A_{\alpha,\beta}(z)dz & \text{if } xy \neq 0 \\ \\ d\delta_x(z) & \text{if } y = 0 \\ d\delta_y(z) & \text{if } x = 0 \end{cases}$$

and

$$\begin{aligned} \mathcal{K}_{\alpha,\beta}(x,y,z) &= M_{\alpha,\beta}|\sinh x.\sinh y.\sinh z|^{-2\alpha}\int_0^\pi g(x,y,z,\chi)_+^{\alpha-\beta-1} \\ &\times [1 - \sigma_{x,y,z}^\chi + \sigma_{x,z,y}^\chi + \sigma_{z,y,x}^\chi + \frac{\rho}{\beta+\frac{1}{2}}\coth x.\coth y.\coth z(\sin\chi)^2] \times (\sin\chi)^{2\beta}d\chi \end{aligned}$$

if $x, y, z \in \mathbb{R}\backslash\{0\}$ satisfy the triangular inequality $||x|-y|| < |z| < |x|+|y|$, and $\mathcal{K}_{\alpha,\beta}(x, y, z) = 0$ otherwise. Here

$$\forall x, y, z \in \mathbb{R}, \chi \in [0,1], \sigma^{\chi}_{x,y,z} = \begin{cases} \frac{\cosh x + \cosh y - \cosh z \cos\chi}{\sinh x \sinh y} & \text{if } xy \neq 0 \\ 0 & \text{if } xy = 0 \end{cases}$$

and $g(x, y, z, \chi) = 1 - \cosh^2 x - \cosh^2 y . \cosh^2 z + 2\cosh x . \cosh y . \cosh z . \cos\chi$.

Lemma 1.3. [2] For all $x, y \in \mathbb{R}$, we have
(i) $\mathcal{K}_{\alpha,\beta}(x, y, z) = \mathcal{K}_{\alpha,\beta}(y, x, z)$.
(ii) $\mathcal{K}_{\alpha,\beta}(x, y, z) = \mathcal{K}_{\alpha,\beta}(-x, z, y)$.
(iii) $\mathcal{K}_{\alpha,\beta}(x, y, z) = \mathcal{K}_{\alpha,\beta}(-z, y, -x)$.

The product formula is used to obtain explicit estimates for the generalized translation operators

$$\tau^{(\alpha,\beta)}_x f(y) = \int_{\mathbb{R}} f(z) d\mu^{(\alpha,\beta)}_{x,y}(z).$$

It is known from [2] that

$$\mathcal{H}\tau^{(\alpha,\beta)}_x f(\lambda) = G^{(\alpha,\beta)}_\lambda(x)\mathcal{H}f(\lambda), \tag{1.1}$$

for $f \in C_c(\mathbb{R})$.

2. Main Result

In this section we give the main result of this paper. We need first to define $(\psi, 2)$-Cherednik-Opdam Lipschitz class.
Denote N_h by

$$N_h = \tau^{(\alpha,\beta)}_h + \tau^{(\alpha,\beta)}_{-h} - 2I,$$

where I is the unit operator in the space $L^2_{\alpha,\beta}(\mathbb{R})$.

Definition 2.1. A function $f \in L^2_{\alpha,\beta}(\mathbb{R})$ is said to be in the $(\psi, 2)$-Cherednik-Opdam Lipschitz class, denoted by $Lip(\psi, 2)$, if

$$\|N_h f(x)\|_{2,\alpha,\beta} = O(\psi(h)) \quad \text{as} \quad h \to 0,$$

where ψ is a continuous increasing function on $[0, \infty), \psi(0) = 0$, $\psi(ts) = \psi(t)\psi(s)$ for all $t, s \in [0, \infty)$ and this function verify

$$\int_0^{1/h} s\psi(s^{-2}) ds = O(h^{-2}\psi(h^2)), \quad h \to 0.$$

Lemma 2.2. If $f \in C_c(\mathbb{R})$, then

$$\mathcal{H}\check{\tau}^{(\alpha,\beta)}_x f(\lambda) = G^{(\alpha,\beta)}_\lambda(-x)\mathcal{H}\check{f}(\lambda). \tag{2.1}$$

Proof. For $f \in C_c(\mathbb{R})$, we have

$$\begin{aligned} \mathcal{H}\check{\tau}^{(\alpha,\beta)}_x f(\lambda) &= \int_{\mathbb{R}} \tau^{(\alpha,\beta)}_x f(-y) G^{(\alpha,\beta)}_\lambda(-y) A_{\alpha,\beta}(y) dy \\ &= \int_{\mathbb{R}} \tau^{(\alpha,\beta)}_x f(y) G^{(\alpha,\beta)}_\lambda(y) A_{\alpha,\beta}(y) dy \\ &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(z) \mathcal{K}_{\alpha,\beta}(x, y, z) A_{\alpha,\beta}(z) dz \right] G^{(\alpha,\beta)}_\lambda(y) A_{\alpha,\beta}(y) dy \\ &= \int_{\mathbb{R}} f(z) \left[\int_{\mathbb{R}} G^{(\alpha,\beta)}_\lambda(y) \mathcal{K}_{\alpha,\beta}(x, y, z) A_{\alpha,\beta}(y) dy \right] A_{\alpha,\beta}(z) dz. \end{aligned}$$

Since $\mathcal{K}_{\alpha,\beta}(x,y,z)=\mathcal{K}_{\alpha,\beta}(-x,z,y)$, it follows from the product formula that

$$\begin{aligned}\mathcal{H}\check{\tau}_x^{(\alpha,\beta)}f(\lambda) &= G_\lambda^{(\alpha,\beta)}(-x)\int_{\mathbb{R}} f(z)G_\lambda^{(\alpha,\beta)}(z)A_{\alpha,\beta}(z)dz\\ &= G_\lambda^{(\alpha,\beta)}(-x)\int_{\mathbb{R}} f(-z)G_\lambda^{(\alpha,\beta)}(-z)A_{\alpha,\beta}(z)dz\\ &= G_\lambda^{(\alpha,\beta)}(-x)\mathcal{H}\check{f}(\lambda).\end{aligned}$$

Lemma 2.3. For $f\in L^2_{\alpha,\beta}(\mathbb{R})$, then

$$\|N_hf(x)\|^2_{2,\alpha,\beta}=4\int_0^{+\infty}|\varphi_\lambda^{\alpha,\beta}(h)-1|^2\left(|\mathcal{H}f(\lambda)|^2+|\mathcal{H}\check{f}(\lambda)|^2\right)d\sigma(\lambda).$$

Proof. From formulas (1) and (2), we have

$$\mathcal{H}(N_hf)(\lambda)=(G_\lambda^{(\alpha,\beta)}(h)+G_\lambda^{(\alpha,\beta)}(-h)-2)\mathcal{H}(f)(\lambda),$$

and

$$\mathcal{H}(\check{N}_hf)(\lambda)=(G_\lambda^{(\alpha,\beta)}(-h)+G_\lambda^{(\alpha,\beta)}(h)-2)\mathcal{H}(\check{f})(\lambda).$$

Since

$$G_\lambda^{(\alpha,\beta)}(h)=\varphi_\lambda^{\alpha,\beta}(h)+\frac{\rho}{4(\alpha+1)}\sinh(2h)\varphi_\lambda^{\alpha+1,\beta+1}(h),$$

and $\varphi_\lambda^{\alpha,\beta}$ is even, then

$$\mathcal{H}(N_hf)(\lambda)=2(\varphi_\lambda^{\alpha,\beta}(h)-1)\mathcal{H}(f)(\lambda)$$

and

$$\mathcal{H}(\check{N}_hf)(\lambda)=2(\varphi_\lambda^{\alpha,\beta}(h)-1)\mathcal{H}(\check{f})(\lambda).$$

Now by Plancherel Theorem, we have the result.

Theorem 2.4. Let $f\in L^2_{\alpha,\beta}(\mathbb{R})$. Then the following are equivalents

(a) $f\in Lip(\psi,2)$,

(b) $\displaystyle\int_r^{+\infty}\left(|\mathcal{H}f(\lambda)|^2+|\mathcal{H}\check{f}(\lambda)|^2\right)d\sigma(\lambda)=O(\psi(r^{-2})),\quad as\quad r\to\infty.$

Proof. $(a)\Rightarrow(b)$ Let $f\in Lip(\psi,2)$. Then we have

$$\|N_hf(x)\|_{2,\alpha,\beta}=O(\psi(h))\quad\text{as}\quad h\to0.$$

From Lemma 2.2, we have

$$\|N_hf(x)\|^2_{2,\alpha,\beta}=4\int_0^{+\infty}|1-\varphi_\lambda^{\alpha,\beta}(h)|^2\left(|\mathcal{H}f(\lambda)|^2+|\mathcal{H}\check{f}(\lambda)|^2\right)d\sigma\lambda.$$

If $\lambda\in[\frac{1}{h},\frac{2}{h}]$, then $\lambda h\geq1$ and (iii) of Lemma 1.2 implies that

$$1\leq\frac{1}{c^2}|1-\varphi_\lambda^{\alpha,\beta}(h)|^2.$$

Then

$$\begin{aligned}\int_{\frac{1}{h}}^{\frac{2}{h}}\left(|\mathcal{H}f(\lambda)|^2+|\mathcal{H}\check{f}(\lambda)|^2\right)d\sigma(\lambda) &\leq \frac{1}{c^2}\int_{\frac{1}{h}}^{\frac{2}{h}}|1-\varphi_\lambda^{\alpha,\beta}(h)|^2\left(|\mathcal{H}f(\lambda)|^2+|\mathcal{H}\check{f}(\lambda)|^2\right)d\sigma(\lambda)\\ &\leq \frac{1}{c^2}\int_0^{+\infty}|1-\varphi_\lambda^{\alpha,\beta}(h)|^2\left(|\mathcal{H}f(\lambda)|^2+|\mathcal{H}\check{f}(\lambda)|^2\right)d\sigma(\lambda)\\ &\leq \frac{1}{4c^2}\|N_hf(x)\|^2_{2,\alpha,\beta}\\ &= O(\psi(h^2)).\end{aligned}$$

We obtain

$$\int_r^{2r} \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2\right) d\sigma(\lambda) \leq C\psi(r^{-2}), \quad r \to \infty,$$

where C is a positive constant. Now,

$$\begin{aligned}\int_r^{+\infty} \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2\right) d\sigma(\lambda) &= \sum_{i=0}^{\infty} \int_{2^i r}^{2^{i+1} r} \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2\right) d\sigma(\lambda) \\ &\leq C\psi(r^{-2}) \sum_{i=0}^{\infty} (\psi(2^{-2}))^i \\ &\leq CC_\delta \psi(r^{-2}),\end{aligned}$$

where $C_\delta = (1 - \psi(2^{-2}))^{-1}$ since $\psi(2^{-2}) < 1$.
Consequently

$$\int_r^{+\infty} \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2\right) d\sigma(\lambda) = O(\psi(r^{-2})), \quad as \quad r \to \infty.$$

$(b) \Rightarrow (a)$. Suppose now that

$$\int_r^{+\infty} \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2\right) d\sigma(\lambda) = O(\psi(r^{-2})), \quad as \quad r \to \infty,$$

and write

$$\|N_h f(x)\|_{2,\alpha,\beta}^2 = 4(I_1 + I_2),$$

where

$$I_1 = \int_0^{\frac{1}{h}} |1 - \varphi_\lambda^{\alpha,\beta}(h)|^2 \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2\right) d\sigma\lambda,$$

and

$$I_2 = \int_{\frac{1}{h}}^{+\infty} |1 - \varphi_\lambda^{\alpha,\beta}(h)|^2 \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2\right) d\sigma\lambda.$$

Firstly, we use the formula $|\varphi_\lambda^{\alpha,\beta}(h)| \leq 1$ and

$$I_2 \leq 4 \int_{\frac{1}{h}}^{+\infty} \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2\right) d\sigma(\lambda) = O(\psi(h^2)), \quad as \quad h \to 0.$$

To estimate I_1, we use the inequalities (i) and (ii) of Lemma 1.2

$$\begin{aligned}I_1 &= \int_0^{\frac{1}{h}} |1 - \varphi_\lambda^{\alpha,\beta}(h)|^2 \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2\right) d\sigma\lambda \\ &\leq 2 \int_0^{\frac{1}{h}} |1 - \varphi_\lambda^{\alpha,\beta}(h)| \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2\right) d\sigma\lambda \\ &\leq 2h^2 \int_0^{\frac{1}{h}} (\lambda^2 + \rho^2) \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2\right) d\sigma\lambda = I_3 + I_4,\end{aligned}$$

where

$$I_3 = 2h^2\rho^2 \int_0^{\frac{1}{h}} \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2\right) d\sigma\lambda,$$

and

$$I_4 = 2h^2 \int_0^{\frac{1}{h}} \lambda^2 \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2\right) d\sigma\lambda.$$

Note that

$$\begin{aligned}I_3 &\leq 2h^2\rho^2 \int_0^{+\infty} \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2\right) d\sigma\lambda \\ &= 2h^2\rho^2 \|f\|_{2,\alpha,\beta}^2 = O(\psi(h^2)), \quad as \quad h \to 0.\end{aligned}$$

For a while, we put

$$\phi(s) = \int_s^{+\infty} \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2\right) d\sigma(\lambda).$$

Using integration by parts, we find that

$$\begin{aligned} h^2 \int_0^{1/h} \lambda^2 \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2\right) d\sigma\lambda &= h^2 \int_0^{1/h} -s^2\phi'(s)ds \\ &= h^2 \left(-\frac{1}{h^2}\phi(\frac{1}{h}) + 2\int_0^{1/h} s\phi(s)ds\right) \\ &= -\phi(\frac{1}{h}) + 2h^2 \int_0^{1/h} s\phi(s)ds. \end{aligned}$$

Since $\phi(s) = O(\psi(s^{-2}))$, we have $s\phi(s) = O(s\psi(s^{-2}))$ and

$$\int_0^{1/h} s\phi(s)ds = O\left(\int_0^{1/h} s\psi(s^{-2})ds\right) = O(h^{-2}\psi(h^2)).$$

Then

$$h^2 \int_0^{1/h} \lambda^2 \left(|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2\right) d\sigma\lambda \leq 2C_1 h^2 h^{-2}\psi(h^2),$$

where C_1 is a positive constant.

Finally

$$I_4 = O(\psi(h^2)),$$

which completes the proof of the theorem. □

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