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CONVERGENCE THEOREM FOR FINITE FAMILY OF TOTAL ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract. In this paper we introduce an explicit iteration process and prove strong convergence of the scheme in a real Hilbert space H to the common fixed point of finite family of total asymptotically nonexpansive mappings which is nearest to the point $u \in H$. Our results improve previously known ones obtained for the class of asymptotically nonexpansive mappings. As application, iterative method for: approximation of solution of variational Inequality problem, finite family of continuous pseudocontractive mappings, approximation of solutions of classical equilibrium problems and approximation of solutions of convex minimization problems are proposed. Our theorems unify and complement many recently announced results.

1. Introduction

Let K be a nonempty subset of a real Hilbert space H. A mapping $T : K \longrightarrow K$ is called *nonexpansive* if and only if for all $x, y \in K$, we have that

$$\|Tx - Ty\| \leq \|x - y\|. \tag{1}$$

The mapping T is called *asymptotically nonexpansive mapping* if and only if there exists a sequence $\{\mu_n\}_{n\geq 1} \subset [0, +\infty)$, with $\lim_{n\to\infty} \mu_n = 0$ such that for all $x.y \in K$,

$$\|T^n x - T^n y\| \leq (1 + \mu_n)\|x - y\| \quad \forall\, n \in \mathbb{N}. \tag{2}$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [11] as a generalisation of nonexpansive mappings. As further generalisation of class of nonexpansive mappings, Alber, Chidume and Zegeye [2] introduced the class of total asymptotically nonexpansive mappings, where a mapping $T : K \longrightarrow K$ is called *total asymptotically nonexpansive* if and only if there exist two sequences $\{\mu_n\}_{n\geq 1}, \{\eta_n\}_{n\geq 1} \subset [0, +\infty)$, with $\lim_{n\to\infty} \mu_n = 0 = \lim_{n\to\infty} \eta_n$ and nondecreasing continous function $\phi : [0, +\infty) \longrightarrow [0, +\infty)$ with $\phi(0) = 0$ such that for all $x, y \in K$,

$$\|T^n x - T^n y\| \leq \|x - y\| + \mu_n \phi(\|x - y\|) + \eta_n \quad n \geq 1 \tag{3}$$

Observe that if $\phi(t) = 0\ \forall\, t \in [0, +\infty)$, then equation (3) becomes

$$\|T^n x - T^n y\| \leq \|x - y\| + \eta_n \quad n \geq 1, \tag{4}$$

so that if K is bounded and T^N is continuous for some integer $N \geq 1$, then the mapping T is of asymptotically nonexpansive type. The class of asymptotically

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nonexpansive type mappings includes the class of mappings which are asymptotically nonexpansive in the intermediate sense and the class of nearly asymptotically nonexpansive mappings. These classes of mappings had been studied extensively by several authors (see e.g.[11], [15], [31]). If $\phi(t) = t \; \forall \; t \in [0, +\infty)$, then equation (3) becomes

$$\|T^n x - T^n y\| \leq (1 + \mu_n)\|x - y\| + \eta_n \quad n \geq 1 \tag{5}$$

In addition, if $\eta_n = 0$ for all $n \in \mathbb{N}$, then we easily see that every asymptotically nonexpansive mapping is total asymptotically nonexpansive. If $\mu_n = 0$ and $\eta_n = 0 \; \forall \; n \geq 1$ we obtain from equation (3) the class of mappings which includes the class of nonexpansive mappings. The class of total asymptotically nonexpansive mappings properly includes the class of asymptotically nonexpansive mappings (See Example 2 of [20]). A point $x_0 \in K$ is called a fixed point of a mapping $T : K \longrightarrow K$ if and only if $Tx_0 = x_0$. We denote the set of fixed points of T by $F(T)$, that is, $F(T) = \{x \in K : Tx = x\}$. A point $x^* \in K$ is called a minimum norm fixed point of T if and only if $x^* \in F(T)$ and $\|x^*\| = min\{\|x\| : x \in F(T)\}$.

Let D_1 and D_2 be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. The split feasibility problem is formulated as finding a point x satisfying

$$x \in D_1 \text{ such that } Ax \in D_2, \tag{6}$$

where A is bounded linear operator from H_1 into H_2. A split feasibility problem in finite dimensional Hilbert spaces was first studied by Censor and Elfving [8] for modeling inverse problems which arise in medical image reconstruction, image restoration and radiation therapy treatment planning (see e.g [6], [7], [8]). It is clear that $x \in D_1$ is a solution of the split feasibility problem (6) if and only if $Ax - P_{D_2}Ax = 0$, where P_{D_2} is the metric projection from H_2 onto D_2. Consider the minimization problem:

$$\text{find } \; x^* \in D_1 \text{ such that } \frac{1}{2}\|Ax^* - P_{D_2}Ax\|^2 = \min_{x \in D_1} \frac{1}{2}\|Ax - P_{D_2}Ax\|^2, \tag{7}$$

then x^* is a solution of (6) if and only if x^* solves the minimization problem (7) with the minimum equal to zero. Suppose that problem (6) has solution and let Ω denote the (closed convex) set of solutions of (6) (or equivalently, solution of (7), then Ω is a singleton if and only if it is a set of solutions of the following variational inequality problem:

$$\text{find } \; x \in D_1 \text{ such that } \langle A^*(I - P_{D_2})Ax, y - x\rangle \geq 0 \quad \forall \;\; y \in D_2, \tag{8}$$

where A^* is the adjoint of the linear operator A . Moreover, problem (8) can be rewritten as

$$\text{find } \; x \in D_1 \text{ such that } \langle x - rA^*(I - P_{D_2})Ax - x, y - x\rangle \leq 0 \quad \forall \, y \in D_2, \tag{9}$$

where $r > 0$ is any positive scalar. Using the nature of projection, (9) is equivalent to the fixed point equation

$$x = P_{D_1}(x - rA^*(I - P_{D_2})Ax). \tag{10}$$

Thus, finding a solution of split feasibility problem (7) is equivalent to finding the minimum-norm fixed point of the mapping $x \longrightarrow P_{D_1}(x - rA^*(I - P_{D_2})Ax)$.
Approximation of solutions of equations involving nonexpansive mappings and their

generalization by iterative methods has been of increasing research interest for numerous mathematicians in recent years. One of the first results of this nature was obtained by Browder [5] for nonexpansive self mappings in Hilbert spaces. Suppose K is a closed convex nonempty subset of a real Hilbert space H. Browder [5] studied the path $u \in K,\ x_t = tu + (1-t)Tx_t,\ t \in (0,1)$, where $T : K \longrightarrow K$ is a nonexpansive mapping. In [5], Browder proved that $\lim_{t\to 0} x_t$ exists and $\lim_{t\to 0} x_t \in F(T)$. The result was extended by Reich [24] to uniformly smooth real Banach spaces. Reich [24] proved, in fact, that $\lim_{t\to 0} x_t$ is a sunny nonexpansive retraction of K onto $F(T)$. In [12], Halpern studied the convergence of the explicit iteration method defined from $x_1 \in K$ by

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n;\ n \geq 1 \tag{11}$$

in the frame work of real Hilbert spaces. Under appropriate conditions on the iterative parameter α_n, it had been shown by Halpern [12], Lions [16], Wittmann [26] and Banschke [3] that the sequence $\{x_n\}$ generated by (11) converges strongly to a fixed point of T nearest to u, that is, $P_{F(T)}u$, Browder and Halpern iterative methods had motivated different iterative methods for approximation of fixed points of asymptotically nonexpansive mappings. In this regard, Lim and Xu [15] introduced and studied the following implicit iterative method for asymptotically nonexpansive mapping T,

$$z_n = \alpha_n u + (1-\alpha_n)T^n z_n;\ n \geq 1. \tag{12}$$

They showed that the sequence $\{z_n\}_{n\geq 1}$ generated by (12) converges strongly to a fixed point of T in the frame work of uniformly smooth real Banach spaces under suitable conditions on the iterative parameters. In [10], Chidume, Li and Udomene proved the strong convergence of the explicit iterative method generated from $x_1, u \in K$ by

$$x_{n+1} = \alpha_n u + (1-\alpha_n)T^n x_n;\ n \geq 1, \tag{13}$$

where $\lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=0}^{\infty} \alpha_n = +\infty$ and T is asymptotically nonexpansive.

Yao, Zhou and Lion [28], studied a modified Mann iteration algorithm $\{x_n\}$ generated from $x_1, \in H$ by

$$\begin{aligned} \nu_n &= (1-t_n)x_n, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T\nu_n,\ n \geq 1, \end{aligned} \tag{14}$$

where $\{t_n\}_{n\geq 1}, \{\alpha_n\}_{n\geq 1}$ are sequences in $(0,1)$ satisfying appropriate conditions. They proved the strong convergence of the modified algorithm to the fixed point of a nonexpansive mapping $T : H \longrightarrow H$ when $F(T) \neq \emptyset$. Osilike etal [23] modified the algorithm (14) with $\{x_n\}$ generated from $x_1, \in K$ by

$$\begin{aligned} \nu_n &= P_K[(1-t_n)x_n], \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n\nu_n;\ n \geq 1, \end{aligned} \tag{15}$$

where $\{t_n\}_{n\geq 1}, \{\alpha_n\}_{n\geq 1}$ are sequences in $(0,1)$ satisfying appropriate conditions. They proved the strong convergence of the modified algorithm to the fixed point of assymptotically nonexpansive mapping $T : K \longrightarrow K$ when $F(T) \neq \emptyset$.

Recently, Alber, Espinola and Lorenzo [2] obtained strong convergence of (13) for a total asymptotically nonexpansive self map T on K in the setting of smooth reflexive real Banach space with weakly sequentially continuous duality mapping.

In connection with the iterative approximation of minimum norm fixed point of the mapping T, Yang, Lion and Yao [27] introduced an explicit iterative method generated from $x_1 \in K$ by

$$x_{n+1} = \beta_n T x_n + (1-\beta_n) P_K[(1-\alpha_n)x_n];\ n \geq 1, \tag{16}$$

They proved under appropriate conditions on $\{\alpha_n\}_{n\geq 1}$ and$\{\beta_n\}_{n\geq 1}$ that the sequence $\{x_n\}_{n\geq 1}$ converges strongly to the minimum norm fixed point of T in Hilbert spaces. Yang et al [27] proved that the explicit iterative method generated from $x_1 \in K$ defined by

$$x_{n+1} = P_K[(1-\alpha_n)Tx_n];\ n \geq 1, \tag{17}$$

converges strongly to the minimum norm fixed point of nonexpansive mapping $T : K \longrightarrow K$ provided that $\{\alpha_n\}_{n\geq 1}$ satisfies appropriate condition. Recently, Zegeye and Shahzad [31] proved that the iterative method generated from arbitrary $x_1 \in K$ by

$$\begin{aligned} y_n &= P_K[(1-\alpha_n)x_n], \\ x_{n+1} &= \beta_n x_n + (1-\beta_n)T^n y_n;\ n \geq 1, \end{aligned} \tag{18}$$

converges strongly to minimum norm fixed point of asymptotically nonexpansive self map T on K.

Motivated by the results of these authors, it is our aim in this paper to prove strong convergence theorem to the common fixed point of finite family of total asymptotically nonexpansive mappings which is nearest to the point $u \in H$. Our theorems generalize and unify the corresponding results of Osilike etal [23], Yao, Zhou and Lion [28], Yang, Lion and Yao [27], Zegeye and Shahzad [31]. Our method of proof is of independent interest.

2. Preliminaries

We shall make use of the following lemmas and propositions.

Lemma 2.1. *Let H be a real Hilbert space. Then for all $x, y \in H$ the following inequality holds.*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$$

Lemma 2.2. *For any x, y, z in a real Hilbert space H and a real number $\lambda \in [0,1]$,*

$$\|\lambda x + (1-\lambda)y - z\|^2 = \lambda\|x-z\|^2 + (1-\lambda)\|y-z\|^2 - \lambda(1-\lambda)\|x-y\|^2.$$

Lemma 2.3. [25] *Let K be a closed convex nonempty subset of a real Hilbert space H. Let $x \in H$, then $x_0 = P_K x$ if and only if*

$$\langle z - x_0, x - x_0\rangle \leq 0\ \forall\ z \in K$$

Let $T : K \longrightarrow K$ be a mapping and I be the identity mapping of K, we say that $(I-T)$ is demiclose at zero if and only if for any sequence $\{x_n\}_{n\geq 1}$ in K such that x_n converges weakly to x and $x_n - Tx_n \to 0$, as $n \to \infty$, we have that $x = Tx$.

Lemma 2.4. *(*see Corollary 2.6 of [1]*) Let E be a reflexive Banach space with weakly continuous normalized duality mapping. Let K be a closed convex subset of E and let T be a uniformly continuous total asymptotically nonexpansive mapping from K into itself with bounded orbit, then $(I - T)$ is* demiclose *at zero.*

Lemma 2.5. [1] *Let $\{a_n\}$ be a sequence of nonegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n;\ \ n \geq 1.$$

Suppose that for $n \geq 1, \frac{\delta_n}{\alpha_n} \leq c_1$ and $\alpha_n \leq \alpha$ (for some $\alpha, c_1 > 0$), then $a_n \leq \max\{a_1, (1+\alpha)c_1\}$. Moreover, if $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\delta_n, = o(\alpha_n)$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6. *(*see [17]*) Let $\{\Gamma_n\}$ be sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_j}\}$of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_j} < \Gamma_{n_j+1}\ \forall\ j \in \mathbb{N}$. Define the sequence $\{\tau(n)\}_{n\geq n_0}$ of integers as follows*

$$\tau(n) = max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in \mathbb{N}$ and that the set $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\}$ is not empty, then the following hold (i) $\tau(n_0) \leq \tau(n_0 + 1)$ and $\tau(n) \to \infty$ as $n \to \infty$ (ii) $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ and $\Gamma_n \leq \Gamma_{\tau(n)+1}\ \forall\ \ n \in \mathbb{N}$.

Proposition 2.1. *(*see Proposition 8 of [21]*) Let H be a real Hilbert space, let K be a nonempty closed convex subset of H and let $T_i : K \longrightarrow K$,*where *$i \in I = \{1, 2, ..., m\}$, be m uniformly continuous total asymptotically nonexpansive mappings from K into itself with sequences $\{\mu_{n,i}\}_{n\geq 1}, \{\eta_{n,i}\}_{n\geq 1} \subset [0, +\infty)$ such that $\lim_{n\to\infty} \mu_{n,i} = 0 = \lim_{n\to\infty} \eta_{n,i}$ and with function $\phi_i : [0, +\infty) \longrightarrow [0, +\infty)$ satisfying $\phi_i(t) \leq M_0 t\ \ \forall\ \ t > M_1$ for some constants $M_0, M_1 > 0$. Let $\mu_n = \max_{i\in I}\{\mu_{n,i}\}$ and $\eta_n = \max_{i\in I}\{\eta_{n,i}\}$ and, $\phi(t) = \max_{i\in I}\{\phi_i(t)\}\forall\ t \in [0, \infty)$. Suppose that $F(T) = \bigcap_{i=1}^m F(T_i)$, then $F(T)$ is closed and convex.*

Proposition 2.2. [20] *Let K be a nonempty subset of a real normed space E and $T_i : K \longrightarrow K$,*where *$i \in I = \{1, 2, ..., m\}$, be m total asymptotically nonexpansive mappings, then there exist sequences $\{\mu_n\}_{n\geq 1}, \{\eta_n\}_{n\geq 1} \subset [0, +\infty)$, with $\lim_{n\to\infty} \mu_n = 0 = \lim_{n\to\infty} \eta_n$ and nondecreasing continous function $\phi : [0, +\infty) \longrightarrow [0, +\infty)$ with $\phi(0) = 0$ such that for all $x, y \in K$,*

$$\|T_i^n x - T_i^n y\| \leq \|x - y\| + \mu_n \phi(\|x - y\|) + \eta_n;\quad n \geq 1, \forall\ i \in I. \tag{19}$$

3. Main results

Let K be a nonempty closed and convex subset of a real Hilbert space H. Let $T_i : K \longrightarrow K$, where $i \in I = \{1, 2, ..., m\}$, be m total asymptotically nonexpansive mappings and $\{\alpha_n\}_{n\geq 1}, \{\beta_n\}_{n\geq 1}$ be sequences in $(0, 1)$, we define the explicit iteration process $\{x_n\}_{n\geq 1}$ from $x_1 \in K, u \in H$ by

$$
\begin{aligned}
y_1 &= P_K[\alpha_1 u + (1-\alpha_1)x_1],\\
x_2 &= (1-\beta_1)x_1 + \beta_1 T_1 y_1,\\
y_2 &= P_K[\alpha_2 u + (1-\alpha_2)x_2],\\
x_3 &= (1-\beta_2)x_2 + \beta_2 T_2 y_2,\\
&\vdots\\
y_{m-1} &= P_K[\alpha_{m-1} u + (1-\alpha_{m-1})x_{m-1}],\\
x_m &= (1-\beta_{m-1})x_m + \beta_{m-1} T_{m-1} y_{m-1},\\
y_m &= P_K[\alpha_m u + (1-\alpha_{)}x_m],\\
x_{m+1} &= (1-\beta_m)x_m + \beta_m T_m^1 y_m,\\
y_{m+1} &= P_K[\alpha_{m+1} u + (1-\alpha_{m+1})x_{m+1}],\\
x_{m+2} &= (1-\beta_{m+1})x_{m+1} + \beta_{m+1} T_1^2 y_{m+1}\\
y_{m+2} &= P_K[\alpha_{m+2} u + (1-\alpha_{m+2})x_{m+2}],\\
x_{m+3} &= (1-\beta_{m+2})x_{m+2} + \beta_{m+2} T_2^2 y_{m+2}\\
&\vdots\\
y_{2m-1} &= P_K[\alpha_{2m-1} u + (1-\alpha_{2m-1})x_{2m-1}],\\
x_{2m} &= (1-\beta_{2m-1})x + \beta_{2m-1} T_{m-1}^2 y_{2m-1}\\
y_{2m} &= P_K[\alpha_{2m} u + (1-\alpha_{2m})x_{2m}],\\
x_{2m+1} &= (1-\beta_{2m})x_{2m} + \beta_{2m} T_m^2 y_{2m}\\
y_{2m+1} &= P_K[\alpha_{2m+1} u + (1-\alpha_{2m+1})x_{2m+1}],\\
x_{2m+2} &= (1-\beta_{2m+1})x_{2m+1} + \beta_{2m+1} T_1^3 y_{2m+1}\\
&\vdots
\end{aligned} \tag{20}
$$

(21)

Since $\forall z \in \mathbb{Z}$ (where Z is the set of integers), there exists $j(z) \in I$ such that $z - j(z)$ is divisible by m (that is $j(z) = z\ mod(m)$), then there exists $q(z) \in \mathbb{Z}$ with $\lim_{z\to\infty} q_(z) = +\infty$ such that

$$z = \big(q(z)-1\big)m + j(z) \tag{22}$$

so we may write (20) in a more compact form as

$$\begin{aligned} x_1 \in K, u \in H, y_n &= P_K[\alpha_n u + (1-\alpha_n)x_n], \\ x_{n+1} &= (1-\beta_n)x_n + \beta_n T_{j(n)}^{q(n)} y_n. \end{aligned} \tag{23}$$

Remark 3.1. *Since $n - m \in \mathbb{Z}\, \forall\, n \in \mathbb{N}$, we obtain from (22) for $z = n - m$ that*

$$n - m = \big(q(n-m) - 1\big)m + j(n-m). \tag{24}$$

Also, substituting $n \in \mathbb{N}$ for z in (22) and subtracting m from both sides of the resulting equation gives

$$n - m = \big((q(n) - 1) - 1\big)m + j(n) \tag{25}$$

Comparing (24) and (25) we obtain (by unique representation theorem) that

$$q(n-m) = q(n) - 1 \text{ and } j(n-m) = j(n)\ \forall\ n \in \mathbb{N}. \tag{26}$$

Theorem 3.1. *Let H be a real Hilbert space, let K be a closed convex nonempty subset of H and let $T_i : K \longrightarrow K$, where $i \in I = \{1, 2, ..., m\}$, be m uniformly continuous total asymptotically nonexpansive mapping from K into itself with sequences $\{\mu_{in}\}_{n\geq 1}, \{\eta_{in}\}_{n\geq 1} \subset [0, +\infty)$ such that $\lim_{n\to\infty} \mu_{in} = 0 = \lim_{n\to\infty} \eta_{in}$ and with function $\phi_i : [0, +\infty) \longrightarrow [0, +\infty)$ satisfying $\phi_i(t) \leq M_0 t \quad \forall \quad t > M_1$ for some constants $M_0, M_1 > 0$, Let $\mu_n = \max_{i\in I}\{\mu_{in}\}$ and $\eta_n = \max_{i\in I}\{\eta_{in}\}$ and, $\phi(t) = \max_{i\in I}\{\phi_i(t)\} \forall\, t \in [0, \infty)$. Suppose that $F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$ and let $\{x_n\}_{n\geq 1}$ be a sequence generated iteratively by (23), where $\{\alpha_n\}_{n\geq 1}, \{\beta_n\}_{n\geq 1}$ are sequences in $(0, 1)$ satisfying the following conditions:*
$\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \alpha_n^{-1}\mu_n = \lim_{n\to\infty} \alpha_n^{-1}\eta_n = 0$ and $0 < \zeta < \beta_n < \epsilon < 1\ \forall\ n \geq 1$, then $\{x_n\}_{n\geq 1}$ converges strongly to $P_F(u)$.

Proof. Let $x^* \in F$, then from (23) and hypothesis on T_i we have that

$$\begin{aligned} \|y_n - x^*\| &= \|P_K[\alpha_n u + (1-\alpha_n)x_n] - P_K x^*\| \\ &\leq \|\alpha_n u + (1-\alpha_n)x_n - x^*\| \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|u - x^*\| \end{aligned} \tag{27}$$

and

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|(1-\beta_n)x_n + \beta_n T_{j(n)}^{q(n)} y_n - x^*\| \\ &\leq (1-\beta_n)\|x_n - x^*\| + \beta_n\|T_{j(n)}^{q(n)} y_n - x^*\| \\ &\leq (1-\beta_n)\|x_n - x^*\| + \beta_n\Big[\|y_n - x^*\| + \mu_{q(n)}\phi(\|y_n - x^*\|) + \eta_{q(n)}\Big]. \end{aligned} \tag{28}$$

Since ϕ is continuous, it follows that ϕ attains its maximum (say M) on the interval $[0, M_1]$, moreover, $\phi(t) \leq M_0 t$ whenever $t > M_1$. Thus,

$$\phi(t) \leq M + M_0 t \ \ \forall\, t \in [0, +\infty). \tag{29}$$

Using (27) and (29) we obtain from (28) that

$$\begin{aligned}\|x_{n+1}-x^*\| &\leq (1-\beta_n)\|x_n-x^*\| \\ &\quad +\beta_n\Big[\|y_n-x^*\|+\mu_{q(n)}\Big(M+M_0\|y_n-x^*\|\Big)+\eta_{q(n)}\Big] \\ &\leq (1-\beta_n)\|x_n-x^*\|+\beta_n\Big[(1+\mu_{q(n)}M_0)\|y_n-x^*\|+\mu_{q(n)}M+\eta_{q(n)}\Big] \\ &\leq (1-\beta_n)\|x_n-x^*\|+\beta_n\Big[(1+\mu_{q(n)}M_0)\|y_n-x^*\|\Big] \\ &\quad +\beta_n\mu_{q(n)}M+\beta_n\eta_{q(n)} \\ &\leq (1-\beta_n)\|x_n-x^*\|+\beta_n\mu_{q(n)}M+\beta_n\eta_{q(n)} \\ &\quad +\beta_n\Big[(1+\mu_{q(n)}M_0)(1-\alpha_n)]\|x_n-x^*\|+\alpha_n\|u-x^*\|\Big] \\ &= \Big[1-\alpha_n\beta_n+(1-\alpha_n)\beta_n\mu_{q(n)}M_0\Big]\|x_n-x^*\| \\ &\quad +\alpha_n\beta_n(1+\mu_{q(n)}M_0)\|u-x^*\|+\beta_n\mu_{q(n)}M+\beta_n\eta_{q(n)} \\ &= \Big[1-\alpha_n\beta_n+(1-\alpha_n)\beta_n\mu_{q(n)}M_0\Big]\|x_n-x^*\|+\delta_n, \qquad (30)\end{aligned}$$

where $\delta_n=\alpha_n\beta_n(1+\mu_{q(n)}M_0)\|u-x^*\|+\beta_n\mu_{q(n)}M+\beta_n\eta_{q(n)}$. Since $\lim_{n\to\infty}\alpha_n^{-1}\mu_{q(n)}=0=\lim_{n\to\infty}\alpha_n^{-1}\eta_{q(n)}$, we may assume without loss of generality that there exists $k_0\in(0,1)$ and $M_2>0$ such that $\alpha_n^{-1}\mu_{q(n)}<\frac{(1-k_0)}{(1-\alpha_n)M_0}$ and $\frac{\delta_n}{\alpha_n\beta_n}<M_2$. Thus, we obtain from (30) that

$$\|x_{n+1}-x^*\|\leq\|x_n-x^*\|-k_0\alpha_n\beta_n\|x_n-x^*\|+\delta_n. \qquad (31)$$

So, Lemma 2.5 gives $\|x_n-x^*\|\leq\max\{\|x_1-x^*\|,(1+k_1)M_2\}$. Therefore, $\{x_n\}_{n\geq1}$ is bounded and by (27) we obtain that $\{y_n\}_{n\geq1}$ is bounded. Moreover, using Lemma 2.1, we obtain that

$$\begin{aligned}\|y_n-x^*\|^2 &= \|P_K[\alpha_nu+(1-\alpha_n)x_n]-P_Kx^*\|^2 \\ &\leq \|\alpha_nu+(1-\alpha_n)x_n-x^*\|^2 \\ &\leq (1-\alpha_n)^2\|x_n-x^*\|^2 \\ &\quad +2\alpha_n(1-\alpha_n)\langle u-x^*,x_n-x^*\rangle+2\alpha_n^2\|u-x^*\|^2. \qquad (32)\end{aligned}$$

Furthermore, using Lemma 2.2, we obtain that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|(1-\beta_n)x_n+\beta_nT_{j(z)}^{q(n)}y_n-x^*\|^2 \\ &= (1-\beta_n)\|x_n-x^*\|^2+\beta_n\|T_{j(n)}^{q(n)}y_n-x^*\|^2-\beta_n(1-\beta_n)\|x_n-T_{j(n)}^{q(n)}y_n\|^2 \qquad (33)\end{aligned}$$

But

$$\begin{aligned}\|T_{j(n)}^{q(n)}y_n-x^*\|^2 &\leq \Big[(1+\mu_{q(n)}M_0)\|y_n-x^*\|+\mu_{q(n)}M+\eta_{q(n)}\Big]^2 \\ &= (1+\mu_{q(n)}M_0)^2\|y_n-x^*\|^2 \\ &\quad +(\mu_{q(n)}M+\eta_{q(n)})\Big[2(1+\mu_{q(n)}M_0)\|y_n-x^*\|+\mu_{q(n)}M+\eta_{q(n)}\Big]^2 \qquad (34)\end{aligned}$$

so that putting (32) in (34), we have

$$\|T_{j(n)}^{q(n)}y_n - x^*\|^2 \leq (1+\mu_{q(n)}M_0)^2\Big((1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n(1-\alpha_n)\langle u - x^*, x_n - x^*\rangle + 2\alpha_n^2\|u - x^*\|^2\Big) + (\mu_{q(n)}M + \eta_{q(n)})\Big[2(1+\mu_{q(n)}M_0)\|y_n - x^*\| + \mu_{q(n)}M + \eta_{q(n)}\Big]^2 \tag{35}$$

and

$$\beta_n\|T_{j(n)}^{q(n)}y_n - x^*\|^2 \leq \beta_n(1+\mu_{q(n)}M_0)^2(1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n\beta_n(1+\mu_{q(n)}M_0)^2(1-\alpha_n)\langle u - x^*, x_n - x^*\rangle + 2\alpha_n^2\beta_n(1+\mu_{q(n)}M_0)^2\|u - x^*\|^2 + (\mu_{q(n)}M + \eta_{q(n)})\Big[2(1+\mu_{q(n)}M_0)\|y_n - x^*\| + \mu_{q(n)}M + \eta_{q(n)}\Big]. \tag{36}$$

Now, substituting (36) in (33), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|(1-\beta_n)x_n + \beta_n T_{j(z)}^{q(n)}y_n - x^*\|^2\\ &\leq (1-\beta_n)\|x_n - x^*\|^2 + \beta_n(1+\mu_{q(n)}M_0)^2(1-\alpha_n)^2\|x_n - x^*\|^2\\ &\quad +2\alpha_n\beta_n(1+\mu_{q(n)}M_0)^2(1-\alpha_n)\langle u - x^*, x_n - x^*\rangle\\ &\quad +2\alpha_n^2\beta_n(1+\mu_{q(n)}M_0)^2\|u - x^*\|^2 - \beta_n(1-\beta_n)\|x_n - T_{j(n)}^{q(n)}y_n\|^2\\ &\quad +(\mu_{q(n)}M + \eta_{q(n)})\Big[2(1+\mu_{q(n)}M_0)\|y_n - x^*\| + \mu_{q(n)}M + \eta_{q(n)}\Big]\\ &\leq (1-\gamma_n)\|x_n - x^*\|^2 + 2\gamma_n(1-\alpha_n)\langle u - x^*, x_n - x^*\rangle\\ &\quad +\theta_n - \beta_n(1-\beta_n)\|x_n - T_{j(n)}^{q(n)}y_n\|^2,\end{aligned} \tag{37}$$

where $\gamma_n = \beta_n\alpha_n(1+\mu_n M_0)^2$ and $\theta_n = 2\alpha_n^2\beta_n(1+\mu_n M_0)^2\|u2 - x^*\|^2 + \beta_n\mu_{q(n)}M_0(2+\mu_{q(n)}M_0\sup_{n\geq 1}\|x_n - x^*\|^2 + \beta_n(\mu_{q(n)}M + \eta_{q(n)})\Big[2(1+\mu_{q(n)}M_0)\sup_{n\geq 1}\|y_n - x^*\| + \mu_{q(n)}M + \eta_{q(n)}\Big]$.

Two cases arise Case 1: Suppose$\{\|x_n - x^*\|\}_{n\geq 1}$ is nonincreasing for $n \geq n_0$, for some $n_0 \in \mathbb{N}$, this implies that $\|x_{n+1} - x^*\| \leq \|x_n - x^*\| \quad \forall\ n \geq n_0$. Thus, $\lim_{n\to\infty}\|x_n - x^*\|$ exist and $\lim_{n\to\infty}\Big(\|x_{n+1} - x^*\| - \|x_n - x^*\|\Big) = 0$. Moreover, using the fact that $0 < \xi_0 < \beta_n < \zeta_0 < 1$, we obtain that

$$\lim_{n\to\infty}\|x_n - T_{j(n)}^{q(n)}y_n\| = 0. \tag{38}$$

Next, we observe that

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|(1-\beta_n)x_n + \beta_n T_{j(n)}^{q(n)}y_n - x_n\|\\ &= \beta_n\|T_{j(n)}^{q(n)}y_n - x_n\|.\end{aligned} \tag{39}$$

Thus, by (38)

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{40}$$

Observe that by(40)

$$\lim_{n\to\infty}\|x_n - x_{n-i}\| = 0 = \lim_{n\to\infty}\|x_n - x_{n+i}\|, i \in I. \tag{41}$$

Moreso,

$$\begin{aligned}\|y_n - x_n\| &= \|P_K[\alpha_n u + (1-\alpha_n)x_n] - P_K x_n\| \\ &\leq \|\alpha_n u + (1-\alpha_n)x_n] - x_n\| \\ &= \alpha_n\|u - x_n\| \end{aligned} \tag{42}$$

so that by our hypothesis

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{43}$$

Furthermore,

$$\|y_{n+1} - y_n\| \leq \|y_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - y_n\| \tag{44}$$

which by (40) and (43) gives

$$\lim_{n\to\infty}\|y_{n+1} - y_n\| = 0. \tag{45}$$

Observe that by (45) we have

$$\lim_{n\to\infty}\|y_{n+i} - y_n\| = 0 = \lim_{n\to\infty}\|y_{n-i} - y_n\| \ \forall\, i \in I \tag{46}$$

Now,

$$\|y_n - T_{j(n)}^{q(n)}y_n\| \leq \|y_n - x_n\| + \|x_n - T_{j(n)}^{q(n)}y_n\|. \tag{47}$$

Using (38) and(43) in (47) gives

$$\lim_{n\to\infty}\|y_n - T_{j(n)}^{q(n)}y_n\| = 0. \tag{48}$$

By uniform continuity of $T_i; i \in I$ there exists a continuous increasing function $\Pi_i : R \longrightarrow R$ with $\Pi_i(0) = 0$ such that

$$\|T_i x - T_i y\| \leq \Pi_i(\|x - y\|)\forall x, y \in K. \tag{49}$$

Thus, defining $\Pi_0 : \mathbb{R} \longrightarrow \mathbb{R}$ by $\Pi_0(t) = \max_{i\in I}\{\Pi_i(t)\}\ \forall\ t \in \mathbb{R}$, we have that Π_0 is a continuous increasing function with $\Pi_0(0) = 0$ and

$$\begin{aligned}\|y_n - T_{j(n)}y_n\| &\leq \|y_n - T_{j(n)}^{q(n)}y_n\| + \|T_{j(n)}^{q(n)}y_n - T_j(n)y_n\| \\ &\leq \|y_n - T_{j(n)}^{q(n)}y_n\| + \Pi_0(\|T_{j(n)}^{q(n)-1}y_n - y_n\|). \end{aligned} \tag{50}$$

Consider the argument of Π_0 in (50),

$$\begin{aligned}\|T_{j(n)}^{q(n)-1}y_n - y_n\| &\leq \|T_{j(n)}^{q(n)-1}y_n - T_{j(n-m)}^{q(n)-1}y_{n-m}\| + \|T_{j(n-m)}^{q(n)-1}y_{n-m} - y_{n-m}\| \\ &\quad +\|y_{n-m} - y_n\|\ . \end{aligned} \tag{51}$$

By (26) we have that

$$\begin{aligned}\|T_{j(n)}^{q(n)-1}y_n - T_{j(n-m)}^{q(n)-1}y_{n-m}\| &\leq \|T_{j(n-m)}^{q(n)-1}y_n - T_{j(n-m)}^{q(n)-1}y_{n-m}\| \\ &\leq \|y_{n-m} - y_n\| + \mu_{q(n)-1} + \phi(\|y_{n-m} - y_n\|) \\ &\quad +\eta_{q(n)-1}\ . \end{aligned} \tag{52}$$

Using (46) in (52) and by hypothesis we have that

$$\lim_{n\to\infty}\|T_{j(n)}^{q(n)-1}y_n - T_{j(n-m)}^{q(n)-1}y_{n-m}\| = 0. \tag{53}$$

Moreso, by (26) we have that

$$\|T_{j(n-m)}^{q(n)-1}y_{n-m} - y_{n-m}\| = \|T_{j(n-m)}^{q(n-m)}y_{n-m} - y_{n-m}\|. \tag{54}$$

Thus,

$$\lim_{n\to\infty} \|T_{j(n-m)}^{q(n)-1}y_{n-m} - y_{n-m}\| = 0. \tag{55}$$

Now, using (53) and (54) in (50) we obtain that

$$\lim_{n\to\infty} \|T_{j(n)}^{q(n)-1}y_n - y_n\| = 0. \tag{56}$$

Consequently, we obtain from (48) and (50) that

$$\lim_{n\to\infty} \|y_n - T_{j(n)}y_n\| \quad = \quad 0. \tag{57}$$

Furthermore, we obtain for $i \in I$ that

$$\begin{aligned} \|y_n - T_{j(n)+i}y_n\| &\leq \|y_n - y_{n+i}\| + \|y_{n+i} - T_{j(n)+i}y_{n+i}\| + \|T_{j(n)+i}y_{n+i} - T_{j(n)+i}y_n\| \\ &\leq \|y_n - y_{n+i}\| + \|y_{n+i} - T_{j(n)+i}y_{n+i}\| + \Pi_0(\|y_{n+i} - y_n\|). \end{aligned} \tag{58}$$

So,using (46), (57) and (58) we have

$$\lim_{n\to\infty} \|y_n - T_{j(n)+i}y_n\| = 0 \; \forall i \in I. \tag{59}$$

But $\forall i \in I$ there exists $\vartheta_i \in I$ such that $j(n) + \vartheta_i = i \; mod(m)$ so that from (59), we have that

$$\lim_{n\to\infty} \|y_n - T_i y_n\| = \lim_{n\to\infty} \|y_n - T_{j(n)+i}y_n\| = 0 \; \forall i \in I. \tag{60}$$

But

$$\|x_n - T_i x_n\| = \|x_n - y_n\| + \|y_n - T_i y_n\| + \|T_i y_n - T_i x_n\| \quad \forall \, n \in \mathbb{N}. \tag{61}$$

Hence, using (43), uniform continuity of the mapping T and (60) we obtain from (61) that

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = 0, \; \forall i \in I. \tag{62}$$

Now, let $\{x_{n_k}\}_{k\geq 1}$ be a subsequence of $\{x_n\}_{n\geq 1}$ such that

$$\limsup_{n\to\infty} \langle u - x^*, x_n - x^* \rangle = \lim_{k\to\infty} \langle u - x^*, x_{n_k} - x^* \rangle, \tag{63}$$

then, there exist a subsequence$\{x_{n_{k_j}}\}_{j\geq 1}$ of $\{x_{n_k}\}_{k\geq 1}$ that converges weakly to some $z \in H$. Thus, (63) gives

$$\limsup_{n\to\infty} \langle u - x^*, x_n - x^* \rangle = \lim_{k\to\infty} \langle u - x^*, x_{n_k} - x^* \rangle = \lim_{j\to\infty} \langle u - x^*, x_{n_{k_j}} - x^* \rangle. \tag{64}$$

Furthermore, by (62) $\lim_{j\to\infty} \|x_{n_{k_j}} - T_i x_{n_{k_j}}\| = 0$ and by Lemma 2.4, $I - T_i$ is demiclose at 0, we obtain that $z \in F$. So using(64) and the fact that $x^* = P_K u$, we obtain from Lemma 2.3 that

$$\begin{aligned} \limsup_{n\to\infty} \langle u - x^*, x_n - x^* \rangle &= \lim_{k\to\infty} \langle u - x^*, x_{n_k} - x^* \rangle \\ &= \lim_{j\to\infty} \langle u - x^*, x_{n_{k_j}} - x^* \rangle \\ &= \langle u - x^*, z - x^* \rangle \leq 0. \quad . \end{aligned} \tag{65}$$

Therefore, defining

$$\nu_n = max\{0, \langle u - x^*, x_n - x^*\rangle\}, \tag{66}$$

Then it is easy to see that $\lim_{n\to\infty} \nu_n = 0$. Moreover we obtain from (37)(using (66)) that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\le \|x_n - x^*\|^2 - \gamma_n\|x_n - x^*\|^2 + 2\gamma_n(1-\alpha_n)\langle u - x^*, x_n - x^*\rangle + \theta_n \\ &\le (1-\gamma_n)\|x_n - x^*\|^2 + 2\gamma_n(1-\alpha_n)\nu_n + \theta_n \\ &= (1-\gamma_n)\|x_n - x^*\|^2 + \sigma_n \end{aligned} \tag{67}$$

where $\sigma_n = 2\gamma_n(1-\alpha_n)\nu_n + \theta_n$. Conditions on our iterative parameter easily give that $\sigma_n = o(\gamma_n)$. Hence, we obtain from (67) using Lemma 2.5 that $\{x_n\}_{n\ge0}$ converges strongly to $x^* = P_K u$

CASE 2: Suppose there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\|x_{n_i} - x^*\| \le \|x_{n_i+1} - x^*\| \;\forall\; i \in \mathbb{N}$, then by lemma 2.6 there exist a nondecreasing sequence $\{\tau(n)\} \subset \mathbb{N}$ such that (i) $\lim_{n\to\infty} \tau(n) = \infty$ (ii) $\|x_{\tau(n)} - x^*\| \le \|x_{\tau(n)+1} - x^*\| \;\forall\; n \in \mathbb{N}$. So, from (37), we have that

$$\begin{aligned} \gamma_n\|x_{\tau(n)} - x^*\|^2 &\le \|x_{\tau(n)} - x^*\|^2 - \|x_{\tau(n)+1} - x^*\|^2 \\ &\quad +2\gamma_{\tau(n)}\langle u - x^*, x_{\tau(n)} - x^*\rangle + \theta_{\tau(n)} \;\;\forall\; n \in \mathbb{N} \end{aligned} \tag{68}$$

Thus, using the fact that $\gamma_{\tau(n)} > 0$, we have that

$$\|x_{\tau(n)} - x^*\|^2 \le 2\langle u - x^*, x_{\tau(n)} - x^*\rangle + \frac{\theta_{\tau(n)}}{\gamma_{\tau(n)}} \quad \forall\; n \in \mathbb{N}. \tag{69}$$

Observe that following the argument of case 1 we have that $\lim_{n\to\infty} \|x_{\tau(n+1)} - x_{\tau(n)}\| = \lim_{n\to\infty} \|y_{\tau(n)} - T_i y_{\tau(n)}\| = 0 \;\forall\; i \in I$ and $\limsup_{n\to\infty}\langle u - x^*, x_n - x^*\rangle \le 0$. Thus, setting $\nu_{\tau(n)} = max\{0, \langle u - x^*, x_{\tau(n)} - x^*\rangle\}$, we obtain that $\nu_{\tau(n)} \to 0$ as $n \to \infty$. Furthermore, from conditions on our iterative parameters, we obtain that $\frac{\theta_{\tau(n)}}{\gamma_{\tau(n)}} \to 0$. So we obtain from (69) that $\|x_{\tau(n)} - x^*\|^2 \le 2\nu_{\tau(n)} + \frac{\theta_{\tau(n)}}{\gamma_{\tau(n)}} \quad \forall\; n \in \mathbb{N}$. Thus, $\lim_{n\to\infty} \|x_{\tau(n)} - x^*\| = 0$. Also from Lemma 2.6 we have that $\|x_n - x^*\| \le \|x_{\tau(n+1)} - x^*\|^2 \;\forall\; n \in \mathbb{N}$. Thus we obtain using sandwich theorem that $\lim_{n\to\infty} \|x_n - x^*\| = 0$. Hence, x_n converges strongly to $x^* = P_K u$.

Remark 3.2. *Observe that if $T_i, i \in I$ in Theorem 3.1 were asymptotically nonexpansive mappings, the condition there exist $M_0 > 0$ and $M_1 > 0$ such that $\phi(t) \le M_0 t \;\forall\; t > M_1$ is not needed. Moreso, every asymptotically nonexpansive mapping $T_i : K \longrightarrow K$ is uniformly L-Lipschitzian thus uniformly continuous.*

Hence we have the folowing theorems as an easy corollaries of Theorem 3.1 above:

Theorem 3.2. *Let K be a closed convex nonempty subset of a real Hilbert space H and let $T_i : K \longrightarrow K$, $i \in I$, be asymptotically nonexpansive mappings such that $F = \bigcap_{i=1}^m F(T_i) \ne \emptyset$ and let $\{x_n\}_{n\ge1}$ be a sequence generated iteratively by (23), where $\{\alpha_n\}_{n\ge1}, \{\beta_n\}_{n\ge1}$ are sequences in $(0,1)$ satisfying the following conditions:*

$\sum_{n=1}^{\infty}\alpha_n = \infty$, $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\alpha_n^{-1}\mu_n = 0$ *and* $0 < \zeta < \beta_n < \epsilon < 1 \ \forall\ n \geq 1$, *then* $\{x_n\}_{n\geq 1}$ *converges strongly to* $P_F(u)$.

Theorem 3.3. *Let* K *be a closed convex nonempty subset of a real Hilbert space* H *and let* $T_i : K \longrightarrow K$, $i \in I$, *be finite family of nonexpansive mappings from* K *into itself. Suppose that* $F = \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$ *and let* $\{x_n\}_{n\geq 1}$ *be a sequence generated iteratively by (23), where* $\{\alpha_n\}_{n\geq 1}, \{\beta_n\}_{n\geq 1}$ *are sequences in* $(0,1)$ *satisfying the following conditions:*
$\sum_{n=1}^{\infty}\alpha_n = \infty$, $\lim_{n\to\infty}\alpha_n = 0$ *and* $0 < \zeta < \beta_n < \epsilon < 1 \ \forall\ n \geq 1$, *then* $\{x_n\}_{n\geq 1}$ *converges strongly to* $P_F u$.

.

Corollary 3.1. *Suppose in our Theorems the finite family is a singleton (that is if* $m = 1$*), our results hold.*

Remark 3.3. *If* $u = 0$ *in the recursion formulas of our theorems, we obtain what authors now call the Minimum norm iteration process. We observe that all our theorems in this paper carry over trivially to the so called minimum norm iteration process.*

Remark 3.4. *If* $f : K \longrightarrow K$ *is a contraction map and we replace* u *by* $f(x_n)$ *in the recursion formulas of our theorems, we obtain what some authors now call viscosity iteration process. We observe that all our theorems in this paper carry over trivially to the so-called viscosity process.*

4. Application to approximation of fixed points of continuous pseudocontractive mappings

The most important generalization of the class of nonexpansive mappings is, perhaps, the class of pseudocontractive mappings. These mappings are intimately connected with the important class of nonlinear monotone operators. For the importance of monotone operators and their connections with evolution equations, the reader may consult [9], [19].
Due to the above connection, fixed point theory of pseudocontractive mappings became a flourishing area of intensive research for several authors. Recently, H. Zegeye [29] established the following Lemmas:

Lemma 4.1. [29] *Let* K *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $T : K \longrightarrow H$ *be a continuous pseudocontractive mapping, then for all* $r > 0$ *and* $x \in H$, *there exists* $z \in K$ *such that*

$$\langle y - z, Tz\rangle - \frac{1}{r}\langle y - z, (1+r)z - x\rangle \leq 0; \quad \forall \ y \in K \tag{70}$$

Lemma 4.2. [29] *Let* K *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $T : K \longrightarrow K$ *be a continuous pseudocontractive mapping, then for all* $r > 0$ *and* $x \in H$, *there exists* $z \in K$, *define a mapping* $F_r : H \longrightarrow K$ *by*

$$F_r(x) = \{z \in K : \langle y - z, Tz\rangle - \frac{1}{r}\langle y - z, (1+r)z - x\rangle \leq 0 \quad \forall \ y \in K\} \tag{71}$$

then the following hold:

(1) F_r *is single-valued*
(2) F_r *is firmly nonexpansive type mapping i.e for all* $x, y, z \in H$

$$\|F_r(x) - F_r(y)\|^2 \leq \langle F_r(x) - F_r(y), x - y\rangle \tag{72}$$

(3) $Fix(F_r)$ *is closed and convex; and* $Fix(F_r) = Fix(T)$*; for all* $r > 0$.

Remark 4.1. *We observe that Lemmas 4.1 and 4.2 hold in particular for r = 1. Thus, if* T_i, $i \in I = \{1, 2, ..., m\}$ *is finite family of continuous pseudocontractive mapping and we define* $F_{1(i)} : H \longrightarrow K$ *by*

$$F_{1(i)}(x) = \{z \in K : \langle y - z, T_i z\rangle - \langle y - z, 2z - x\rangle \leq 0 \quad \forall \ y \in K\} \tag{73}$$

then $F_{1(i)}$ *satisfies the conditions of Lemma 4.2* $\forall\, i \in I$. *Hence, we easily see that* $F_{1(i)}$ *is nonexpansive and* $Fix(F_{1(i)}) = Fix(T_i) \forall\, i \in I$.

Thus, we have the following theorem.

Theorem 4.1. *Let K be a closed convex nonempty subset of a real Hilbert space H and let* $T_i : K \longrightarrow K\, i \in I$ *be finite family of continous pseudocontractive mappings from K into itself. Suppose that* $F' = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$ *and let* $\{x_n\}_{n\geq 1}$ *be a sequence generated iteratively by*

$$\begin{aligned} x_1 \in K, u \in H, y_n &= P_K[\alpha_n u + (1 - \alpha_n)x_n], \\ x_{n+1} &= (1 - \beta_n)x_n + \beta_n F_{1j(n)}^{q(n)} y_n, \end{aligned} \tag{74}$$

where $\{\alpha_n\}_{n\geq 1}, \{\beta_n\}_{n\geq 1}$ *are sequences in* $(0, 1)$ *satisfying the following conditions:* $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$ *and* $0 < \zeta < \beta_n < \epsilon < 1\ \forall\ n \geq 1$, *then* $\{x_n\}_{n\geq 1}$ *converges strongly to* $P_{F'}u$.

Furthermore, if $u = 0, \{x_n\}_{n\geq 1}$ converges strongly to a minimum norm fixed point of the finite family.

5. Application to approximation of solutions of classical equilibrium problems

Let K be a closed convex nonempty subset of a real Hilbert space H. Let $f : KXK \longrightarrow \mathbb{R}$ be a bifunction. The classical equilibrium problem (abbreviated EP) for f is to find $u^* \in K$ such that

$$f(u^*, y) \geq 0 \quad \forall \ y \in K \tag{75}$$

The set of solutions of classical equilibrium problem is denoted by $EP(f)$, where $EP(f) = \{u \in K : f(u, y) \geq 0\, \forall \ y \in K\}$. The classical equilibrium problem (EP) includes as special cases the monotone inclusion problems, saddle point problems, variational inequality problems, mini- mization problems, optimization problems, vector equilibrium problems, Nash equilibria in noncooperative games. Furthermore, there are several other problems, for example, the complementarity problems and fixed point problems, which can also be written in the form of the classical equilibrium problem. In other words, the classical equilibrium problem is a unifying model for several problems arising from engineering, physics, statistics, computer

science, optimization theory, operations research, economics and countless other fields. For the past 20 years or so, many existence results have been published for various equilibrium problems (see e.g.[4], [14],[30]).
In the sequel, we shall require that the bifunction $f : KxK \longrightarrow R$ satisfies the following conditions: (A1) $f(x, x) = 0\ \forall\ x\ \in K$; (A2)$f$ is monotone, in the sense that $f(x, y)+f(y, x) \leq 0\ \ \forall\ x, y\ \in K$; (A3)$\limsup\limits_{t\to 0^+} f(tz+(1-t)x, y) \leq f(x, y)\ \forall\ x, y, z\ \in K$;. (A4) the function $y \mapsto f(x, y)$ is convex and lower semicontinuous for all $x \in K$

Lemma 5.1. *[(compare with lemma 2.4 of* [14]*)] Let K be a nonempty closed convex subset of a real Hilbert space H. Let$f_i : KxK \longrightarrow R$ be finite family of bifunction satisfying conditions (A1) - (A4) for each $i \in I = \{1, 2, ..., m\}$ then for all $r > 0$ and $x \in H$, there exists $u \in K$ such that*

$$f_i(u, y) + \frac{1}{r}\langle y - u, u - x\rangle \geq 0\ \ \forall\ y\ \in K\ i \in I. \tag{76}$$

moreover' if for all $x \in H$ we define $G_{ir} : H \longrightarrow 2^K$ by

$$G_{ir}(x) = \{u \in K : f_i(u, y) + \frac{1}{r}\langle y - u, u - x\rangle \geq 0\ \ \forall\ y\ \in K.\} \tag{77}$$

then the following hold:

(1) *G_{ir} is single-valued for all $r \geq 0\ i \in I$*
(2) *$Fix(G_{ir}) = EP(f_i)$ for all $r > 0$*
(3) *$EP(f_i)$ is closed and convex*

Remark 5.1. *We observe that Lemmas 5.1 holds in particular for r = 1. Thus, if we define $G_{i1} : H \longrightarrow 2^K$ by*

$$G_{i1}(x) = \{u \in K : f_i(u, y) + \langle y - u, u - x\rangle \geq 0\ \ \forall\ y\ \in K.\} \tag{78}$$

then G_{i1} satisfies the conditions of Lemma 5.1 $\forall\, i \in I$. Hence, we easily see that G_{i1} is nonexpansive and $Fix(G_{i1}) = EP(f_i)\ \forall\, i \in I$.

Thus, we have the following theorem:

Theorem 5.1. *Let K be a closed convex nonempty subset of a real Hilbert space H and let Let$f_i : KxK \longrightarrow R$ be finite family of bifunction satisfying conditions (A1) - (A4) for each $i \in I = \{1, 2, ..., m\}$. Suppose that $F'' = \bigcap_{i=1}^{m} EP(f_i) \neq \emptyset$ and let $\{x_n\}_{n\geq 1}$ be a sequence generated iteratively by*

$$\begin{aligned} x_1 \in K, u \in H, y_n &= P_K[\alpha_n u + (1-\alpha_n)x_n], \\ x_{n+1} &= (1-\beta_n)x_n + \beta_n G_{1j(n)}^{q(n)} y_n\ n \geq 0 \end{aligned} \tag{79}$$

where $\{\alpha_n\}_{n\geq 1}, \{\beta_n\}_{n\geq 1}$ are sequences in $(0, 1)$ satisfying the following conditions: $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$, $\lim\limits_{n\to\infty} \alpha_n = 0$ and $0 < \zeta < \beta_n < \epsilon < 1\ \forall\ n \geq 1$, then $\{x_n\}_{n\geq 1}$ converges strongly to $P_{F''}u$.

. Furthermore, if $u = 0, \{x_n\}_{n\geq 1}$ converges strongly to a minimum norm fixed point of the finite family.

Remark 5.2. *Several authors (see e.g.*[14], [18] *and references therein) have studied the following problem:Let K be a closed convex nonempty subset of a real Hilbert space H. Let $f : KxK \longrightarrow \mathbb{R}$ be a bifunction and $\Phi : K \longrightarrow \mathbb{R}$ be a proper extended real valued function, where $\mathbb{R}$ denotes the real numbers. let $\Theta : K \longrightarrow H$ be a nonlinear monotone mapping. The* generalised mixed equilibrium problem *(abbreviated GMEP) for $f, \Phi and \Theta$ is to find $u^* \in K$ such that*

$$f(u^*, y) + \Phi(y) - \Phi(u^*) + \langle \Theta u^*, y - u^* \rangle \geq 0 \quad \forall \ y \ \in K \tag{80}$$

Observe that if we define$\Gamma : KxK \longrightarrow \mathbb{R}$

$$\Gamma(x, y) = f(x, y) + \Phi(y) - \Phi(x) + \langle \Theta x, y - x \rangle \tag{81}$$

then it could be easily checked that Γ is a bi-function and satisfies properties (A1)to(A4). Thus, the so called generalized mixed equilibrium problem reduces to the classical equilibrium problem for the bifunction Γ. Thus, consideration of the so called generalized mixed equilibrium problem in place of the classical equilibrium problem studied in this section leads to no further generalization.

6. Applications to Convex optimization)

Let us look at the problem of minimizing a continuously Frechet-differentiable convex functional with minimum norm in Hilbert spaces.
Let K be a closed convex subset of a real Hilbert space H, Consider the minimization problem given by

$$\min_{x \in K} \phi(x) \tag{82}$$

where ϕ is a Frechet-differentiable convex functional. Let Ω the solution set of (82) be nonempty. It is known that a point $z \in K$ is a solution of (82) if and only if the following optimality condition holds:

$$z \in K, \ \langle \nabla \phi(z), x - z \rangle \geq 0, x \in K, \tag{83}$$

where ∇ is the gradient of ϕ at $x \in K$. It is also known that the optimality condition (83) is equivalent to the following fixed point problem:

$$z = T_\gamma(z), \text{where} T_\gamma := P_K(I - \gamma \nabla \phi), \tag{84}$$

for all $\gamma > 0$. So, we have the following corollary deduced from theorem 3.1

Theorem 6.1. *Let H be a real Hilbert space, let K be a closed convex nonempty subset of H. Let ψ be a continuously Frechet-differentiable convex functional on K such that $T_{\gamma_{(i)}} := P_K(I - \gamma_{(i)} \nabla \psi)$ be finite family of uniformly continuous total asymptotically nonexpansive mapping from K into itself with sequences $\{\mu_{in}\}_{n \geq 1}, \{\eta_{in}\}_{n \geq 1} \subset [0, +\infty)$ such that $\lim_{n \to \infty} \mu_{in} = 0 = \lim_{n \to \infty} \eta_{in}$ and with function $\phi_i : [0, +\infty) \longrightarrow [0, +\infty)$ satisfying $\phi_i(t) \leq M_0 t \quad \forall \quad t > M_1$ for some constants $M_0, M_1 > 0$, Let $\mu_n = \max_{i \in I} \{\mu_{in}\}$ and $\eta_n = \max_{i \in I} \{\eta_{in}\}$ and $\phi(t) = \max_{i \in I} \{\phi_i(t)\} \forall \ t \in [0, \infty)$. Suppose that $F = \bigcap_{i=1}^N F(T_{\gamma_{(i)}}) \neq \emptyset$ and $\{x_n\}_{n \geq 1}$ is a sequence generated iteratively by*

$$\begin{aligned} x_1 \in K, \ y_n &= P_K[(1 - \alpha_n) x_n], \\ x_{n+1} &= (1 - \beta_n) x_n + \beta_n [P_K(I - \gamma_{(i)} \nabla \psi)]_{j(n)}^{q(n)} y_n; \ n \geq 1 \end{aligned} \tag{85}$$

where $\{\alpha_n\}_{n\geq 1}, \{\beta_n\}_{n\geq 1}$ *are sequences in* $(0,1)$ *satisfying the following conditions:* $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \alpha_n^{-1}\mu_n = 0$ *and* $0 < \zeta < \beta_n < \epsilon < 1 \; \forall \; n \geq 1$, *then* $\{x_n\}_{n\geq 1}$ *converges strongly to the minimum norm solution of the minimization problem (82).*

A prototype of $\phi_i : [0,+\infty) \longrightarrow [0,+\infty)$ in Theorem 3.1 is $\Phi(\lambda) = \lambda^s$, where $0 < s \leq 1$. Moreso, prototype of the sequences used in the same Theorem 3.1 are: Take $\alpha_n = \frac{1}{n}$, $\mu_n = \frac{1}{n^{1+\epsilon}}$, for $\epsilon > 0, \eta_n = \frac{1}{n \log n}$.

Remark 6.1. *Our results extends and unify most of the results that have been proved for the class of assymptotically nonexpansive mappings of which the results obtained in* [10], [15],[27], [31] *are examples.*

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