International Journal of Analysis and Applications
ISSN 2291-8639
Volume 10, Number 1 (2016), 1-8
http://www.etamaths.com

APPLICATIONS OF EXTREMAL THEOREM TO A CLASS OF P-VALENT ANALYTIC FUNCTIONS

LIANGPENG XIONG* AND XIAOLI LIU

Abstract. A subclass $\mathscr{J}_{p,\lambda}^{m,l}(\xi,\alpha)$ of p-valent analytic functions with a generalized multiplier transformation operator is introduced. We discuss the compactness as well as the extreme points of $\mathscr{J}_{p,\lambda}^{m,l}(\xi,\alpha)$ under the topology of uniform convergence. Finally, as one of the applications of extremal theorem, we solve the sharp distortion inequalities problem as

$$\max_{f\in\mathscr{J}_{p,\lambda}^{m,l}(\xi,\alpha)} |f^{(\chi)}(z)|, \chi=0,1,2,...$$

Several related basic results and remarks about the old or new classes are also presented.

1. Introduction

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z)=z^p+\sum_{k=p+1}^{\infty}a_kz^k, p\in\mathbb{N}=\{1,2,3,...\}, \tag{1}$$

which are analytic in $\triangle=\{z: z\in\mathbb{C}, |z|<1\}$.

In fact, $\mathcal{A}_p$ is a vector space over $\mathbb{C}$ with the usual definitions of addition and scalar multiplication for functions. Let the topology on $\mathcal{A}_p$ be given by a metric ρ which is equivalent to the topological of uniform convergence on compact subsets, where the ρ is determined as

$$\rho(f,g)=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{\|f-g\|_n}{1+\|f-g\|_n}$$

whenever f and g belong to $\mathcal{A}_p$, and $\|f\|_n=\max\{|f(z)|:|z|=r_n, 0<r_n<1, \lim_{n\to\infty}r_n=1\}$. It follows from theorems of Weierstrass and Montel that this topology space is complete(see [11], P_{38}).

If $\mathcal{F}\subset\mathcal{A}_p$ then $\mathcal{F}$ is called locally uniformly bounded if there is a constant M such that $|f(z)|\leqslant M$ whenever $f\in\mathcal{F}$. Moreover, Montel's theorem implies that $\mathcal{F}\subset\mathcal{A}_p$ is compact if and only if $\mathcal{F}$ is closed and locally uniformly bounded(see [11], P_{39}).

We use the notation $H\mathcal{F}$ for the closed convex hull of $\mathcal{F}$, where

$$H\mathcal{F}=\Big\{\sum_{k=1}^{\infty}t_kf_k, f_k\in\mathcal{F}, t_k\geqslant 0, \sum_{k=1}^{\infty}t_k=1\Big\}.$$

Let $\mathcal{V}$ be a subclass of a linear topological space. If $v\in\mathcal{V}$ and if $v=tf_1+(1-t)f_2, 0<t<1, f_1\in\mathcal{V}, f_2\in\mathcal{V}$ can make sure that $f_1=f_2$, then we say $v\in E\mathcal{V}$, where $E\mathcal{V}$ denotes the set of extreme points of $\mathcal{V}$. Again, suppose that $\mathcal{M}$ is a convex subset of $\mathcal{A}_p$ and $J:\mathcal{M}\to\mathbb{R}$, if

$$J(tf+(1-t)g)\leqslant tJ(f)+(1-t)J(g)$$

whenever $f,g\in\mathcal{M}$ and $0\leqslant t\leqslant 1$, then linear functional J is called convex on $\mathcal{M}$.

2010 *Mathematics Subject Classification.* 30C35, 30C45, 35Q30.
Key words and phrases. P-valent functions; extreme points; linear topological space; distortion inequalities; multiplier transformation operator.

Recently, Prajapat [18] and Sharma et al. [20] studied a generalized multiplier transformation operator $J_p^m(\lambda,l):\mathcal{A}_p\to\mathcal{A}_p$ as

$$J_p^m(\lambda,l)f(z)=\begin{cases}\dfrac{p+l}{\lambda}z^{p-\frac{p+l}{\lambda}}\displaystyle\int_0^z t^{\frac{p+l}{\lambda}-p-1}J_p^{m+1}(\lambda,l)f(t)dt, & m\in\mathbb{Z}^-,\\ \dfrac{\lambda}{p+l}z^{1+p-\frac{p+l}{\lambda}}\Big(z^{\frac{p+l}{\lambda}-p}J_p^{m-1}(\lambda,l)f(z)\Big)', & m\in\mathbb{Z}^+,\\ f(z), & m=0.\end{cases}\tag{2}$$

where $m\in\mathbb{Z}=\{..,-2,-1,0,1,2,..\},\lambda>0,l>-p$. It easily follows from the above definition of the operator that the series expansion of $J_p^m(\lambda,l)f(z)$ for $f(z)$ of the form (1.1) is given by

$$J_p^m(\lambda,l)f(z)=z^p+\sum_{k=p+1}^{\infty}\Big(1+\frac{\lambda(k-p)}{p+l}\Big)^m a_kz^k.\tag{3}$$

We note that operator $J_p^m(\lambda,l)$ contains kinds of operators introduced and studied by different mathematicians (for details, see [3, 4, 9, 13]). Now, we reproduce here briefly some of these special cases as follows.

Remark 1.1. (i) For $m\in\mathbb{Z}^+\cup\{0\}$, $J_p^m(\lambda,l)\equiv I_p^m(\lambda,l)$, (see Cătaş[5]).
(ii) For $m\in\mathbb{Z}^+\cup\{0\}$, $J_p^m(1,l)\equiv I_p(m,l)$, (see Kumar et al. [12]).
(iii) For $m\in\mathbb{Z}^+\cup\{0\}$, $J_p^m(1,0)\equiv D_p^m$, (see Orhan et al. [16]).
(iv) For $m\in\mathbb{Z}^-\cup\{0\}$, $J_p^m(\lambda,l)$ operator, (see El-Ashwah and Aouf [10]).
(v) For $m\in\mathbb{Z}^-\cup\{0\}$, $J_p^m(1,1)$ operator, (see Patel and Sahoo [17]).

For various other special cases studied earlier of the operator $J_p^m(\lambda,l)$, one can refer to [8, 14, 15, 18]. Let V_p be the subclass of $\mathcal{A}_p$ consisting functions of the form

$$f_\nabla(z)=z^p-\sum_{k=p+1}^{\infty}a_kz^k,a_k\geqslant 0,p\in\mathbb{N}=\{1,2,3,...\},\tag{4}$$

It is easy to see that

$$J_p^m(\lambda,l)f_\nabla(z)=z^p-\sum_{k=p+1}^{\infty}\Big(1+\frac{\lambda(k-p)}{p+l}\Big)^m a_kz^k.\tag{5}$$

We introduce the class $\mathscr{I}_{p,\lambda}^{m,l}(\xi,\alpha)$ as a subclass of V_p consisting of functions f which satisfy

$$\Re\left\{\frac{z\big(J_p^m(\lambda,l)f_\nabla(z)\big)'+\xi z^2\big(J_p^m(\lambda,l)f_\nabla(z)\big)''}{\xi z\big(J_p^m(\lambda,l)f_\nabla(z)\big)'+(1-\xi)J_p^m(\lambda,l)f_\nabla(z)}\right\}>\alpha,z\in\triangle,\tag{6}$$

where $0\leqslant\alpha<p$, $0\leqslant\xi\leqslant 1,m\in\mathbb{Z}^+\cup\{0\},\lambda>0,l>-p$.

Here, for reader's convenience, we depict some of subclasses related the above $\mathscr{I}_{p,\lambda}^{m,l}(\xi,\alpha)$.

Remark 1.2. (i) $\mathscr{I}_{p,\lambda}^{0,l}(\xi,\alpha)$ was studied by Altintaş et al.[1] and Xiong [21].
(ii) $\mathscr{I}_{1,\lambda}^{0,l}(\xi,\alpha)$ was studied by Altintaş [2].
(iii) $\mathscr{I}_{p,\lambda}^{0,l}(0,\alpha)\equiv S_p^*(\alpha)$ was studied by Darwish and Aouf [6].
(iv) $\mathscr{I}_{p,\lambda}^{0,l}(1,\alpha)\equiv K_p(\alpha)$ was studied by Darwish and Aouf [6].
(v) $\mathscr{I}_{1,\lambda}^{0,l}(0,\alpha)\equiv S^*(\alpha)$ and $\mathscr{I}_{1,\lambda}^{0,l}(1,\alpha)\equiv K(\alpha)$ were studied by Srivastava et al. [19] and Domokos [7], respectively.

In this paper we obtain the extreme points for class $\mathscr{I}_{p,\lambda}^{m,l}(\xi,\alpha)$. Furthermore, the sharp distortion inequalities are given by using the extreme points theory.

2. Preliminary results

In this section, we give a sufficient and necessary condition for the functions $f_{\nabla}(z) \in V_p$ to be in $\mathscr{I}_{p,\lambda}^{m,l}(\xi,\alpha)$. As a Lemma, we find that the class $\mathscr{I}_{p,\lambda}^{m,l}(\xi,\alpha)$ is the closed convex hull of a set $\mathscr{M}$.

Lemma 2.1. *A function* $f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k$ *belongs to* $\mathscr{I}_{p,\lambda}^{m,l}(\xi,\alpha)$ *if and only if*

$$\sum_{k=p+1}^{\infty} \psi(k) a_k \leqslant (p-\alpha)[\xi(p-1)+1] \quad (p \in \mathbb{N} = \{1,2,..\}). \tag{7}$$

where

$$\psi(k) = (k-\alpha)[\xi(k-1)+1]\Big(1+\frac{\lambda(k-p)}{p+l}\Big)^m. \tag{8}$$

Proof. If we taking $n=1$ and

$$b_k = \Big(1+\frac{\lambda(k-p)}{p+l}\Big)^m a_k,$$

then this Lemma is an immediate consequence of Altintas et al. [1] (Also see Xiong [21], Lemma 1]) in function $f_{\nabla}(z) = z^p - \sum_{k=p+1}^{\infty} b_k z^k$. □

Lemma 2.2. *If* $\mathscr{M} = \{f_k(z) : k = p, p+1, p+2, ...\}$, *then* $\mathscr{I}_{p,\lambda}^{m,l}(\xi,\alpha) = H\mathscr{M}$, *where* $f_k(z)$ *are determined by*

$$\begin{cases} f_p(z) = z^p, & k = p, \\ f_k(z) = z^p - \dfrac{(p-\alpha)[\xi(p-1)+1]}{\psi(k)} z^k, & k \geqslant p+1. \end{cases}$$

Proof. Suppose that $f(z) \in H\mathscr{M}$ and

$$f(z) = \lambda_p f_p(z) + \sum_{k=p+1}^{\infty} \lambda_k f_k(z), \lambda_k \geqslant 0, \lambda_p + \sum_{k=p+1}^{\infty} \lambda_k = 1.$$

Making use of the elements in $\mathscr{M}$, we can express

$$\begin{aligned} f(z) &= \lambda_p z^p + \sum_{k=p+1}^{\infty} \lambda_k \Big[z^p - \frac{(p-\alpha)[\xi(p-1)+1]}{\psi(k)} z^k\Big] \\ &= \lambda_p z^p + \sum_{k=p+1}^{\infty} \lambda_k z^p - \sum_{k=p+1}^{\infty} \lambda_k \frac{(p-\alpha)[\xi(p-1)+1]}{\psi(k)} z^k \\ &= \Big(\lambda_p + \sum_{k=p+1}^{\infty} \lambda_k\Big) z^p - \sum_{k=p+1}^{\infty} \lambda_k \frac{(p-\alpha)[\xi(p-1)+1]}{\psi(k)} z^k \\ &= z^p - \sum_{k=p+1}^{\infty} \lambda_k \frac{(p-\alpha)[\xi(p-1)+1]}{\psi(k)} z^k = z^p - \sum_{k=p+1}^{\infty} b_k z^k, \end{aligned}$$

where

$$b_k = \lambda_k \frac{(p-\alpha)[\xi(p-1)+1]}{\psi(k)}. \tag{9}$$

Consequently, by using (2.3) and the Lemma 2.1, we are lead to

$$\sum_{k=p+1}^{\infty} \frac{\psi(k)}{(p-\alpha)[\xi(p-1)+1]} b_k = \sum_{k=p+1}^{\infty} \lambda_k = 1 - \lambda_p \leqslant 1,$$

thus, it follows that $f(z) \in \mathscr{I}_{p,\lambda}^{m,l}(\xi,\alpha)$. This implies $H\mathscr{M} \subset \mathscr{I}_{p,\lambda}^{m,l}(\xi,\alpha)$.

We next consider $\mathscr{J}_{p,\lambda}^{m,l}(\xi,\alpha)\subset H\mathscr{M}$. If a function $f(z)\in\mathscr{J}_{p,\lambda}^{m,l}(\xi,\alpha)$, recalling Lemma 2.1 we see that

$$a_k\leqslant\frac{(p-\alpha)[\xi(p-1)+1]}{\psi(k)}\quad(k\geqslant p+1).$$

Here, taking

$$\lambda_k=\frac{\psi(k)}{(p-\alpha)[\xi(p-1)+1]}a_k,$$

where $k\geqslant p+1$ and $\lambda_p=1-\sum\limits_{k=p+1}^{\infty}\lambda_k$, then we can know $0\leqslant\lambda_k\leqslant 1, k\geqslant p$. Hence, we conclude that

$$\begin{aligned}
f(z) &= z^p-\sum_{k=p+1}^{\infty}a_kz^k=z^p-\sum_{k=p+1}^{\infty}\frac{(p-\alpha)[\xi(p-1)+1]}{\psi(k)}\lambda_kz^k\\
&= z^p-\sum_{k=p+1}^{\infty}\lambda_k\left[z^p-\Big(z^p-\frac{(p-\alpha)[\xi(p-1)+1]}{\psi(k)}z^k\Big)\right]\\
&= z^p-\sum_{k=p+1}^{\infty}\lambda_kz^p+\sum_{k=p+1}^{\infty}\lambda_k\Big(z^p-\frac{(p-\alpha)[\xi(p-1)+1]}{\psi(k)}z^k\Big)\\
&= \Big(1-\sum_{k=p+1}^{\infty}\lambda_k\Big)z^p+\sum_{k=p+1}^{\infty}\lambda_kf_k(z)=\sum_{k=p}^{\infty}\lambda_kf_k(z).
\end{aligned}$$

This proves that $\mathscr{J}_{p,\lambda}^{m,l}(\xi,\alpha)\subset H\mathscr{M}$ and completes the proof of Lemma 2.2. □

Lemma 2.3. *(*[11],P_{44}*) Let X be a locally convex linear topological space and let U be a compact subset of X. If HU be a compact then $EHU\subset U$.*

3. Compactness and Extreme points

In this section, we prove that the class $\mathscr{J}_{p,\lambda}^{m,l}(\xi,\alpha)$ is compact subset of $\mathcal{A}_p$, which can help us to obtain the extreme points of $\mathscr{J}_{p,\lambda}^{m,l}(\xi,\alpha)$ and to complete the works in section 4.

Theorem 3.1. *The class $\mathscr{J}_{p,\lambda}^{m,l}(\xi,\alpha)$ is a compact subset of $\mathcal{A}_p$.*

Proof. We need to prove $\mathscr{J}_{p,\lambda}^{m,l}(\xi,\alpha)$ is closed and locally uniformly bounded. Suppose that

$$f(z)=z^p-\sum_{k=p+1}^{\infty}a_kz^k\in\mathscr{J}_{p,\lambda}^{m,l}(\xi,\alpha)$$

and $|z|\leqslant r$, $0<r<1$, then

$$|f(z)|\leqslant r^p+\sum_{k=p+1}^{\infty}\psi(k)|a_k|\frac{r^k}{\psi(k)}\leqslant r^p+(p-\alpha)[\xi(p-1)+1]\sum_{k=p+1}^{\infty}\frac{r^k}{\psi(k)}.$$

Moreover, it is easy to see that

$$\lim_{k\longrightarrow\infty}\sup\Big(\frac{r^k}{\psi(k)}\Big)^{\frac{1}{k}}=r\lim_{k\longrightarrow\infty}\sup(\psi(k))^{-\frac{1}{k}}=r<1,$$

this shows that the series $\sum\limits_{k=p+1}^{\infty}\frac{r^k}{\psi(k)}$ is convergent, so the results above assert that $\mathscr{J}_{p,\lambda}^{m,l}(\xi,\alpha)$ is locally uniformly bounded.

We next prove that the $\mathscr{J}_{p,\lambda}^{m,l}(\xi,\alpha)$ is sequentially closed and suppose that

$$\{f^{\jmath}(z)\}\subset\mathscr{J}_{p,\lambda}^{m,l}(\xi,\alpha)\subset V_p\subset\mathcal{A}_p$$

and $\{f^{\mathcal{J}}(z)\} \to f(z)$ as $\mathcal{J} \to \infty$, where $f^{\mathcal{J}}(z) = z^p - \sum_{k=p+1}^{\infty} a_k^{\mathcal{J}} z^k$. In fact, Weierstrass' theorem asserts $f(z) \in V_p$([11], P_{38}), so we can set $f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k$, thus, it is easy to know that $a_k^{\mathcal{J}} \to a_k$ as $\mathcal{J} \to \infty$. Because of Lemma 2.1 and $f^{\mathcal{J}}(z) \in \mathscr{J}_{p,\lambda}^{m,l}(\xi, \alpha)$, for a positive integer $\mathcal{H}$, then we have

$$\sum_{k=p+1}^{\mathcal{H}} \psi(k) a_k^{\mathcal{J}} \leqslant \sum_{k=p+1}^{\infty} \psi(k) a_k^{\mathcal{J}} \leqslant (p-\alpha)[\xi(p-1)+1].$$

Choosing $\mathcal{J} \to \infty$, it follows that

$$\sum_{k=p+1}^{\mathcal{H}} \psi(k) a_k \leqslant (p-\alpha)[\xi(p-1)+1].$$

Furthermore, taking $\mathcal{H} \to +\infty$, this leads to

$$\sum_{k=p+1}^{\infty} \psi(k) a_k \leqslant (p-\alpha)[\xi(p-1)+1],$$

which implies that $f(z) \in \mathscr{J}_{p,\lambda}^{m,l}(\xi, \alpha)$. This completes the proof of Theorem 3.1. □

Theorem 3.2. *$E\mathscr{J}_{p,\lambda}^{m,l}(\xi, \alpha) = \mathscr{M}$, where $\mathscr{M}$ is defined as Lemma 2.2.*

Proof. Suppose that

$$z^p - \frac{(p-\alpha)[\xi(p-1)+1]}{\psi(k)} z^k = t g_1(z) + (1-t) g_2(z),$$

where

$$g_i(z) = z^p - \sum_{k=p+1}^{\infty} a_{k,\,i} z^k \in \mathscr{J}_{p,\lambda}^{m,l}(\xi, \alpha),\ 0 < t < 1,\ i = 1, 2,$$

then we have

$$\frac{(p-\alpha)[\xi(p-1)+1]}{\psi(k)} = t a_{k,1} + (1-t) a_{k,2}. \tag{10}$$

Since $g_i(z) \in \mathscr{J}_{p,\lambda}^{m,l}(\xi, \alpha)$, Lemma 2.1 gives

$$a_{k,i} \leqslant \frac{(p-\alpha)[\xi(p-1)+1]}{\psi(k)},\ i = 1, 2. \tag{11}$$

The (10) and (11) imply that

$$a_{k,1} = a_{k,2} = \frac{(p-\alpha)[\xi(p-1)+1]}{\psi(k)}.$$

Thus, we can know that $g_1(z) = g_2(z)$, which gives us

$$f_k(z) = z^p - \frac{(p-\alpha)[\xi(p-1)+1]}{\psi(k)} z^k \in E\mathscr{J}_{p,\lambda}^{m,l}(\xi, \alpha),\ k \geqslant p+1.$$

Likewise, we also can obtain $f_p(z) = z^p \in E\mathscr{J}_{p,\lambda}^{m,l}(\xi, \alpha)$. In fact, we get $\mathscr{M} \subset E\mathscr{J}_{p,\lambda}^{m,l}(\xi, \alpha)$. Again, using the Lemma 2.2, we know $\mathscr{J}_{p,\lambda}^{m,l}(\xi, \alpha) = H\mathscr{M}$. Moreover, Theorem 3.1 shows that $\mathscr{M} \subset \mathscr{J}_{p,\lambda}^{m,l}(\xi, \alpha)$ is a compact set, thus, using the Lemma 2.3, it suffices to verify that $E\mathscr{J}_{p,\lambda}^{m,l}(\xi, \alpha) = EH\mathscr{M} \subset \mathscr{M}$. This completes the proof of Theorem 3.2. □

4. Applications of extreme points theorem

We want to maximize the $|f^{(\mathcal{X})}(z)|(\mathcal{X}=0,1,2,...)$ over $\mathscr{J}^{m,l}_{p,\lambda}(\xi,\alpha)$ by making critical use of the information about extreme points. For this point, let linear functional $J:\mathscr{J}^{m,l}_{p,\lambda}(\xi,\alpha)\to\mathbb{R}$ be defined as:

$$J(f)=|f^{(\mathcal{X})}(z)|, f(z)\in\mathscr{J}^{m,l}_{p,\lambda}(\xi,\alpha), \mathcal{X}=\{0,1,2,...\}.$$

It is easy to see that J is a continuous and convex functional. By using the known Krein-Milman theorem, ([11], P_{45}, Theorem 4.6) gives a important result: let $\mathscr{F}$ be a compact subset of $\mathcal{A}$ and J is a real-valued, continuous, convex functional on $H\mathscr{F}$, then

$$\max\{J(f):f\in H\mathscr{F}\}=\max\{J(f):f\in\mathscr{F}\}=\max\{J(f):f\in EH\mathscr{F}\}.$$

Thus, following the Lemma 2.2, Theorem 3.1 and Theorem 3.2, we can know that

$$\begin{aligned}\max\{|f^{(\mathcal{X})}(z)|:f\in\mathscr{J}^{m,l}_{p,\lambda}(\xi,\alpha)\} &= \max\{|f^{(\mathcal{X})}(z)|:f\in H\mathscr{M}\}\\ &= \max\{|f^{(\mathcal{X})}(z)|:f\in\mathscr{M}\}.\end{aligned}$$

Hence, in order to solve the extremal problems on $|f^{(\mathcal{X})}(z)|$ over $\mathscr{J}^{m,l}_{p,\lambda}(\xi,\alpha)$, it needs only to solve them over the smaller class $\mathscr{M}$. We next turn to consider the $|f^{(\mathcal{X})}(z)|$ over $\mathscr{M}$. In Lemma 2.2, taking the function

$$f(z)=z^p-\frac{(p-\alpha)[\xi(p-1)+1]}{\psi(k)}z^k,\ k\geqslant p+1,$$

and after some simplifications, then we have

$$\begin{cases}f^{(\chi)}(z)=\dfrac{p!}{(p-\chi)!}z^{p-\chi}-\dfrac{(p-\alpha)[\xi(p-1)+1]k!}{(k-\chi)!\psi(k)}z^{k-\chi}, & \chi=0,1,2,...,p\\ f^{(\chi)}(z)=-\dfrac{(p-\alpha)[\xi(p-1)+1]k!}{(k-\chi)!\psi(k)}z^{k-\chi}, & \chi=p+1,p+2,...,k\\ f^{(\chi)}(z)=0, & \chi>k.\end{cases}$$

Setting $\chi\in N_0=\{0,1,2,...\}$, $0<r<1$, we define the sequence $\{\mathscr{J}^{(\chi)}_k\}$ as follows:
Case I: if $\chi=0$, then

$$\mathscr{J}^{(\chi)}_k=\begin{cases}0, & k<p+1,\\ \frac{(p-\alpha)[\xi(p-1)+1]}{\psi(k)}r^k, & k\geqslant p+1.\end{cases}$$

Case II: if $\chi\in N=\{1,2,3,...\}$, then

$$\mathscr{J}^{(\chi)}_k=\begin{cases}0, & k<\max\{\chi,p+1\},\\ \dfrac{(p-\alpha)[\xi(p-1)+1]k!}{(k-\chi)!\psi(k)}r^{k-\chi}, & k\geqslant\max\{\chi,p+1\}.\end{cases}$$

We can easily prove that $\mathscr{J}^{(\chi)}_k\to 0$ as $k\to\infty$, this implies that there is a $k_\chi\in\{p+1,p+2,...\}(\chi\in N_0)$, such that

$$\mathscr{J}^{(\chi)}_{k_\chi}=\max\{\mathscr{J}^{(\chi)}_k:k=p+1,p+2,...\}. \tag{12}$$

Finally, we now present the deserved results according to the analysis above.

Theorem 4.1. *Suppose that* $f(z)=z^p-\sum\limits_{k=p+1}^{\infty}a_kz^k\in\mathscr{J}^{m,l}_{p,\lambda}(\xi,\alpha)$, *where* $|z|=r<1$, *then*

(1) If $\chi=0,1,2,...,p$, *we have* $\mathscr{N}_1\leqslant|f^{(\chi)}(z)|\leqslant\mathscr{N}_2$, *where*

$$\mathscr{N}_1=\frac{p!}{(p-\chi)!}r^{p-\chi}-\frac{(p-\alpha)[\xi(p-1)+1]k_\chi!}{(k_\chi-\chi)!\psi(k_\chi)}r^{k_\chi-\chi}$$

and

$$\mathscr{N}_2=\frac{p!}{(p-\chi)!}r^{p-\chi}+\frac{(p-\alpha)[\xi(p-1)+1]k_\chi!}{(k_\chi-\chi)!\psi(k_\chi)}r^{k_\chi-\chi}.$$

(2)If $\chi = p+1, p+2, ...,$ we have

$$-\frac{(p-\alpha)[\xi(p-1)+1]k_\chi!}{(k_\chi-\chi)!\psi(k_\chi)}r^{k_\chi-\chi} \leqslant |f^{(\chi)}(z)| \leqslant \frac{(p-\alpha)[\xi(p-1)+1]k_\chi!}{(k_\chi-\chi)!\psi(k_\chi)}r^{k_\chi-\chi}.$$

All the above k_χ is defined as (12), and $\psi(k_\chi)$ are the values of $\psi(k)$ in (2.2) whenever $k = k_\chi$. The results are sharp and the extremal functions are given by the $\mathscr{M}$ of Lemma 2.2.

Two special cases of Theorem 4.1 when $m = 0, \xi = 0$ and $m = 0, \xi = 1$ yield, respectively,

Corollary 4.1. *Suppose that $f(z) = z^p - \sum\limits_{k=p+1}^{\infty} a_k z^k \in S_p^*(\alpha)$, where $|z| = r < 1$, then*
(1) If $\chi = 0, 1, 2, ..., p$, we have $\mathscr{N}_1 \leqslant |f^{(\chi)}(z)| \leqslant \mathscr{N}_2$, where

$$\mathscr{N}_1 = \frac{p!}{(p-\chi)!}r^{p-\chi} - \frac{(p-\alpha)k_\chi!}{(k_\chi-\chi)!(k_\chi-\alpha)}r^{k_\chi-\chi}$$

and

$$\mathscr{N}_2 = \frac{p!}{(p-\chi)!}r^{p-\chi} + \frac{(p-\alpha)k_\chi!}{(k_\chi-\chi)!(k_\chi-\alpha)}r^{k_\chi-\chi}$$

(2)If $\chi = p+1, p+2, ...,$ we have

$$-\frac{(p-\alpha)k_\chi!}{(k_\chi-\chi)!(k_\chi-\alpha)}r^{k_\chi-\chi} \leqslant |f^{(\chi)}(z)| \leqslant \frac{(p-\alpha)k_\chi!}{(k_\chi-\chi)!(k_\chi-\alpha)}r^{k_\chi-\chi},$$

All the above k_χ is defined as (12). The results are sharp and the extremal functions are given by the $\mathscr{M}$ of Lemma 2.2.

Corollary 4.2. *Suppose that $f(z) = z^p - \sum\limits_{k=p+1}^{\infty} a_k z^k \in K_p(\alpha)$, where $|z| = r < 1$, then*
(1) If $\chi = 0, 1, 2, ..., p$, we have $\mathscr{N}_1 \leqslant |f^{(\chi)}(z)| \leqslant \mathscr{N}_2$, where

$$\mathscr{N}_1 = \frac{p!}{(p-\chi)!}r^{p-\chi} - \frac{(p-\alpha)p\cdot k_\chi!}{(k_\chi-\chi)!(k_\chi-\alpha)k_\chi}r^{k_\chi-\chi}$$

and

$$\mathscr{N}_2 = \frac{p!}{(p-\chi)!}r^{p-\chi} - \frac{(p-\alpha)p\cdot k_\chi!}{(k_\chi-\chi)!(k_\chi-\alpha)k_\chi}r^{k_\chi-\chi}$$

(2)If $\chi = p+1, p+2, ...,$ we have

$$-\frac{(p-\alpha)p\cdot k_\chi!}{(k_\chi-\chi)!(k_\chi-\alpha)k_\chi}r^{k_\chi-\chi} \leqslant |f^{(\chi)}(z)| \leqslant \frac{(p-\alpha)p\cdot k_\chi!}{(k_\chi-\chi)!(k_\chi-\alpha)k_\chi}r^{k_\chi-\chi}.$$

All the above k_χ is defined as (12). The results are sharp and the extremal functions are given by the $\mathscr{M}$ of Lemma 2.2.

Remark 4.1. In fact, various other interesting consequences of our main results (which are asserted by Theorems 3.1, 3.2 and 4.1 and Corollaries 4.1 to 4.2 above) can be derived by appropriately choosing special parameters as the Remark 1.1 and Remark 1.2. The details may be left as an exercise for the interested reader.

Acknowledgement: The authors are thankful to the referees for reading this paper and this work was supported by Scientific Research Fund of SiChuan Provincial Education Department of China, Grant No.14ZB0364.

References

[1] O.Altintaş, H.Irmak and H.M.Srivastava, Fractional calculus and certain starlike func- tions with negative coefficients, Comput. Math. Appl., 30(2)(1995), 9-16.

[2] O.Altintaş, On a subclass of certain starlike functions with negative coefficients, Math. Japonica, 36(3)(1991), 1-7.

[3] Y.Abu Muhanna, L.L.Li and S.Ponnusamy, Extremal problems on the class of convex functions of order $-1/2$, Archiv der mathematik, 103(6)(2014), 461-471.

[4] R.M.Ali, S.R.Mondal and V.Ravichandran, On the Janowski convexity and starlikeness of the confluent hypergeometric function, Bulletin of the belgian mathematical society-simon stevin, 22(2)(2015), 227-250.

[5] A.Cătaş, On certain classes of p-valent functions defined by multiplier transformations, in Proceedings of the international Symposium on Geometric Function Theory and Applications: GFTA 2007 (Eds. S. Owa, Y. Polatoglu), TC Istanbul University Publications, Turkey, 2008, pp. 241-250.

[6] H.E.Darwish and M.K.Aouf, Generalizations of modified-Hadamard products of p-valent functions with negative coefficients, Math. Comput. Modelling 49(1-2)(2009), 38-45.

[7] T.Domokos, On a subclass of certain starlike functions with negative coefficients, Stud. Univ. Babeş-Bolyai Math., 36(1991), 29-36.

[8] J.Dziok, Applications of extreme points to distortion estimates, Appl.Math.Comput. 215(2009), 71-77.

[9] E.Deniz and H.Orhan, Loewner chains and univalence criteria related with ruscheweyh and salagean derivatives, Journal of applied analysis and computation, 5(3)(2015), 465-478.

[10] R.M.El-Ashwah and M.K.Aouf, Some properties of new integral operator, Acta Univ.Apulensis Math.Inform., 24(2010), 51-61.

[11] D.Hallenbeck and T.H.MacGregor, Linear Problems and Convexity Techniques in Geometric Function Theory, 39-46. Pitman Advanced Publishing Program, Boston, Pitman, (1984)

[12] S.S.Kumar, H.C.Taneja and V.Ravichandran, Classes of multivalent functions defined by Dzoik-Srivastava linear operator and multiplier transformations, Kyungpook Math. J., 46(2006), 97-109.

[13] S.Kanas and D.Raducanu, Some class of analytic functions related to conic domains, Mathematica slovaca, 64(5)(2014), 1183-1196.

[14] J.L.Liu, Certain sufficient conditions for strongly starlike functions associated with an integral operator, Bull.Malays.Math.Sci.Soc., 34(1)(2011), 21-30.

[15] G.Murugusundaramoorthy and K.Vijaya, Second hankel determinant for bi-univalent analytic functions associated with hohlov operator, International journal of analysis and applications, 8(1)(2015), 22-29.

[16] H.Orhan and H.Kiziltunc, A generalization on subfamily of p-valent functions with negative coefficients, Appl. Math.Comput., 155(2004), 521-530.

[17] J.Patel, P.Sahoo, Certain subclasses of multivalent analytic functions, Indian J. Pure Appl. Math., 34(3)(2003), 487-500.

[18] J.K.Prajapat, Subordination and superordination preserving properties for generalized multiplier transformation operator, Math. Comput. Modelling, 55(2012), 1456-1465.

[19] H.M.Srivastava, S.Owa and S.K.Chatterjea, A note on certain classes of starlike functions, Rend.Sem.Mat.Univ.Padova, 77(1987), 115-124.

[20] Poonam Sharma, J.K.Prajapat and R.K.Raina, Certain subordination results involving a generalized multiplier transformation operator, Journal of Classical Analysis, 2(1)(2013), 85-106.

[21] L.P.Xiong, Some general results and extreme points of p-valent functions with negative coefficients, Demonstratio mathematica, 44(2)(2011), 261-272.

Department of Mathematics, Engineering and Technical College of Chengdu University of Technology, Leshan, Sichuan, 614007, P.R.China

*Corresponding author: xlpwxf@163.com