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NEW FIXED POINT RESULTS FOR RATIONAL TYPE CONTRACTIONS IN PARTIALLY ORDERED $b-$METRIC SPACES

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ABSTRACT. The purpose of this paper is to establish some fixed point theorems for a mapping having a monotone property satisfying a contractive condition of rational type in the partially ordered $b-$metric spaces. The results presented in the paper generalize and extend several well-known results in the literature. An example is given to support the usability of our results.

1. INTRODUCTION

In [11, 12], S. Czerwik introduced the notion of a b-metric space which is a generalization of usual metric space and generalized the Banach contraction principle in the context of complete b-metric spaces. After that, many authors have carried out further studies on b-metric space and their topological properties (see e.g., [2, 3, 4, 5, 8, 17]) and the references therein.
The following definitions and results will be needed in what follows.

Definition 1.1. [11] *Let X be a (nonempty) set and $s \geq 1$ be given a real number. A function $d : X \times X \longrightarrow \mathbb{R}^+$ is said to be a b-metric space if and only if for all $x, y, z \in X$, the following conditions are satisfied:*
(i) $d(x, y) = 0$ *if and only if* $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \leq s[d(x, z) + d(z, y)]$.

Then the triplet (X, d, s) is called a $b-$metric space with the parameter s. Clearly, a (standard)metric space is also a $b-$metric space, but the converse is not always true.

Example 1.1. *Let $X = [0, 1]$ and $d : X \times X \longrightarrow \mathbb{R}^+$ be defined by $b(x, y) = |x - y|^2$ for all $x, y \in X$. Clearly, $(X, d, s = 2)$ is a $b-$metric space that is not a metric space.*

Also, the following example of a $b-$metric space is given in [6].

Example 1.2. *Let $p \in (0, 1)$. Then the space $L^p([0, 1])$ of all real functions $f : [0, 1] \longrightarrow \mathbb{R}$ such that $\int_0^1 |f(x)|^p dx < \infty$ endowed with the functional $d : L^p([0, 1]) \times L^p([0, 1]) \longrightarrow \mathbb{R}$ given by*

$$d(f, g) = (\int_0^1 |f(x) - g(x)|^p dx)^{\frac{1}{p}},$$

for all $f, g \in L^p([0, 1])$ is a $b-$metric space with $s = 2^{\frac{1}{p}}$.

Since in general a b-metric is not continuous, we need the following simple lemma about the b-convergent sequences in the proof of our main result.

Lemma 1.1. [1] *Let (X, d) be a $b-$metric space with $s \geq 1$ and suppose that $\{x_n\}$ and $\{y_n\}$ are $b-$convergent to x, y, respectively. Then we have,*

$$\frac{1}{s^2} d(x, y) \leq \liminf d(x_n, y_n) \leq \limsup d(x_n, y_n) \leq s^2 d(x, y).$$

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In particular, if $x = y$, then we have $\lim d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have,

$$\frac{1}{s}d(x,z) \leq \liminf d(x_n,z) \leq \limsup d(x_n,z) \leq sd(x,z).$$

In [13], Dass and Gupta presented the following fixed point theorem.

Theorem 1.2. *Let (X,d) be a complete metric space and $T : X \longrightarrow X$ a mapping such that there exist $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$ satisfying*

$$d(Tx,Ty) \leq \alpha \frac{d(y,Ty)(1 + d(x,Tx)}{1 + d(x,y)} + \beta d(x,y),$$

for all $x, y \in X$. Then T has a unique fixed point.

In [9], Cabrera, Harjani and Sadarangani proved the above theorem in the context of partially ordered metric spaces.

Theorem 1.3. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X,d) is a complete metric space. Let $T : X \longrightarrow X$ be a continuous and non-decreasing mapping such that there exist $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$ satisfying*

$$d(Tx,Ty) \leq \alpha \frac{d(y,Ty)(1 + d(x,Tx)}{1 + d(x,y)} + \beta d(x,y),$$

for all $x, y \in X$ with $x \leq y$. If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$, then T has a fixed point.

Notice that Theorem 1.3 is Theorem 1.2 in the context of ordered metric spaces. Harjani et al. [14] extended the result of Jaggi [13] and established a fixed point result in partially ordered metric spaces. Recently, Chandok et al. [10] proved the following Theorem.

Theorem 1.4. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Suppose that T is a continuous self-mapping on X, T is monotone nondecreasing mapping and*

$$d(Tx,Ty) \leq \alpha \frac{d(y,Ty)d(x,Tx)}{d(x,y)} + \beta d(x,y) + \gamma(d(x,Tx) + d(y,Ty)) + \delta(d(y,Tx) + d(x,Ty)),$$

for all $x, y \in X$ with $x \geq y, x \neq y$ and for some $\alpha, \beta, \gamma, \delta \in [0,1)$ with $\alpha + \beta + 2\gamma + 2\delta < 1$. If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$, then T has a fixed point.

The purpose of this paper is to establish some fixed point results satisfying a generalized contraction mapping of rational type in $b-$metric spaces endowed with partial order. Also, we establish a result for existence and uniqueness of fixed point for such class of mappings.

2. MAIN RESULTS

In this section, we will present some fixed point theorems for contractive mappings in the setting of b-metric spaces. Furthermore, we will give examples to support our main results. The first result in this paper is the following fixed point theorem.

Theorem 2.1. *Suppose that $(X, d, \leq)$ is a partially ordered complete b-metric space. Let $T : X \longrightarrow X$ be a continuous and nondecreasing mapping. Suppose there exist mappings $a_i : X \times X \longrightarrow [0,1)$ such that for all $x, y \in X$ and $i = 1, 2, \cdots, 7$*

$$a_i(Tx,Ty) \leq a_i(x,y).$$

Also, for all $x, y \in X$ with $x \leq y$,

$$\begin{aligned} d(Tx,Ty) \leq & a_1(x,y)d(x,y) + a_2(x,y)[d(x,Tx) + d(y,Ty)] + a_3(x,y)\frac{d(y,Tx) + d(x,Ty)}{s} \\ & + a_4(x,y)d(y,Ty)\varphi(d(x,y),d(x,Tx)) + a_5(x,y)d(y,Tx)\varphi(d(x,y),d(x,Ty)) \\ & + a_6(x,y)d(x,y)\varphi(d(x,y),d(x,Tx) + d(y,Tx)) + a_7(x,y)d(y,Tx), \end{aligned} \tag{2.1}$$

where $\varphi : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a function such that $\varphi(t,t) = 1$ for all $t \in \mathbb{R}^+$ and

$$\sup_{x,y \in X}\{a_1(x,y) + a_2(x,y) + a_3(x,y) + a_4(x,y) + a_6(x,y)\} \leq \frac{1}{s+1}.$$

If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$, then T has a fixed point.

Proof 1. *If $x_0 = Tx_0$, then we have the result. Suppose that $x_0 < Tx_0$. Then we construct a sequence $\{x_n\}$ in X such that*

$$x_{n+1} = Tx_n \ for\ every\ n = 0, 1, 2, \cdots . \tag{2.2}$$

Since T is a nondecreasing mapping, we obtain by induction that

$$x_0 \leq Tx_0 = x_1 \leq Tx_1 = x_2 \leq \cdots \leq Tx_{n-1} = x_n \leq Tx_n = x_{n+1} \leq \cdots . \tag{2.3}$$

If there exists some $k \in \mathbb{N}$ such that $x_{k+1} = x_k$, then from (2.2), $x_{k+1} = Tx_k = x_k$, that is, x_k is a fixed point of T and the proof is finished. So, we suppose that $x_{n+1} = x_n$, for all n in$\mathbb{N}$. Since $x_n < x_{n+1}$, for all $n \in \mathbb{N}$, we set $x = x_n$ and $y = x_{n+1}$ in (2.1), we have

$$\begin{aligned} d(Tx_n, Tx_{n+1}) \leq & a_1(x_n, x_{n+1})d(x_n, x_{n+1}) + a_2(x_n, x_{n+1})[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] \\ & + a_3(x_n, x_{n+1})\frac{d(x_{n+1}, x_{n+1}) + d(x_n, x_{n+2})}{s} \\ & + a_4(x_n, x_{n+1})d(x_{n+1}, x_{n+2})\varphi(d(x_n, x_{n+1}), d(x_n, x_{n+1})) \\ & + a_5(x_n, x_{n+1})d(x_{n+1}, x_{n+1})\varphi(d(x_n, x_{n+1}), d(x_n, x_{n+2}))) \\ & + a_6(x_n, x_{n+1})d(x_n, x_{n+1})\varphi(d(x_n, x_{n+1}), d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+1}))) \\ & + a_7(x_n, x_{n+1})d(x_{n+1}, x_{n+1}). \end{aligned} \tag{2.4}$$

that is,

$$\begin{aligned} d(x_{n+1}, x_{n+2}) \leq & a_1(x_n, x_{n+1})d(x_n, x_{n+1}) + a_2(x_n, x_{n+1})[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] \\ & + a_3(x_n, x_{n+1})[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})] + a_4(x_n, x_{n+1})d(x_{n+1}, x_{n+2}) \\ & + a_6(x_n, x_{n+1})d(x_n, x_{n+1}) \\ = & [a_1(x_n, x_{n+1}) + a_2(x_n, x_{n+1}) + a_3(x_n, x_{n+1}) + a_6(x_n, x_{n+1})]d(x_n, x_{n+1}) \\ & + [a_2(x_n, x_{n+1}) + a_3(x_n, x_{n+1}) + a_4(x_n, x_{n+1})]d(x_{n+1}, x_{n+2}) \\ = & [a_1 + a_2 + a_3 + a_6](Tx_{n-1}, Tx_n)d(x_n, x_{n+1}) + [a_2 + a_3 + a_4](Tx_{n-1}, Tx_n)d(x_{n+1}, x_{n+2}) \\ \leq & [a_1 + a_2 + a_3 + a_6](x_{n-1}, x_n)d(x_n, x_{n+1}) + [a_2 + a_3 + a_4](x_{n-1}, x_n)d(x_{n+1}, x_{n+2}) \\ & \vdots \\ \leq & [a_1 + a_2 + a_3 + a_6](x_0, x_1)d(x_n, x_{n+1}) + [a_2 + a_3 + a_4](x_0, x_1)d(x_{n+1}, x_{n+2}), \end{aligned}$$

which implies that

$$d(x_{n+1}, x_{n+2}) \leq \frac{(a_1 + a_2 + a_3 + a_6)(x_0, x_1)}{1 - (a_2 + a_3 + a_4)(x_0, x_1)}\ d(x_n, x_{n+1}).$$

Now,

$$\begin{aligned} & (a_1 + 2a_2 + 2a_3 + a_4 + a_6)(x_0, x_1) \\ & \leq \{(s+1)\,a_1 + (s+1)\,a_2 + (s+1)\,a_3 + (s+1)\,a_4 + (s+1)\,a_6\}(x_0, x_1) \\ & \leq \sup_{x,y \in X}\{(s+1)\,a_1(x,y) + (s+1)\,a_2(x,y) + (s+1)\,a_3(x,y) + (s+1)\,a_4(x,y) + (s+1)\,a_6(x,y)\} \\ & < 1. \end{aligned}$$

Thus we get $d(x_{n+1}, x_{n+2}) \leq \lambda\ d(x_n, x_{n+1})$, where $\lambda = \dfrac{(a_1 + a_2 + a_3 + a_6)(x_0, x_1)}{1 - (a_2 + a_3 + a_4)(x_0, x_1)} < 1$. Obviously, $0 \leq \lambda < \dfrac{1}{s}$. Then by repeated application (2.4), we have

$$d(x_{n+1}, x_{n+2}) \leq \lambda\ d(x_n, x_{n+1}) \leq \lambda^2\ d(x_{n-1}, x_n) \leq \cdots \leq \lambda^{n+1}\, d(x_0, x_1). \tag{2.5}$$

Thus, setting any positive integers m and n ($m > n$), we have

$$\begin{aligned}d(x_n, x_m) \leq & sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \cdots + s^{m-n} d(x_{m-1}, x_m)\\ \leq & [s\lambda^n + s^2\lambda^{n+1} + \cdots + s^{m-n}\lambda^{m-1}]d(x_0, x_1)\\ = & s\lambda^n[1 + s\lambda + (s\lambda)^2 + \cdots + (s\lambda)^{m-n-1}]d(x_0, x_1)\\ \leq & s\lambda^n[1 + s\lambda + (s\lambda)^2 + \cdots]d(x_0, x_1)\\ \leq & \frac{s\lambda^n}{1 - s\lambda} d(x_0, x_1).\end{aligned}$$

Since $0 \leq \lambda < \frac{1}{s}$, we notice that $\frac{s\lambda^n}{1-s\lambda} \longrightarrow 0$ as $n \longrightarrow \infty$ for any $m \in \mathbb{N}$. So $\{x_n\}$ is Cauchy in a complete b-metric space X, there exist $x \in X$ such that

$$\lim_{n\longrightarrow\infty} x_{n+1} = x. \tag{2.6}$$

Letting $n \longrightarrow \infty$ in (2.2) and from the continuity of T, we get

$$x = \lim_{n\longrightarrow\infty} x_{n+1} = \lim_{n\longrightarrow\infty} T(x_n) = T(\lim_{n\longrightarrow\infty} x_n) = T(x).$$

This implies that x is a fixed point of T.

Example 2.1. *Let $X = [0, 1]$ with the usual order $\leq$. Define $d(x, y) = |x - y|^2$. Then d is a $b-$metric with $s = 2$. Also define $a_1(x, y) = \frac{x+y+1}{32}$ and $Tx = \frac{1}{16}x^2$. We observe that*

$$a_1(Tx, Ty) = \frac{1}{32}(\frac{1}{16}x^2 + \frac{1}{16}y^2 + 1) = \frac{1}{32}(\frac{1}{16}x.x + \frac{1}{16}y.y + 1) \leq \frac{x + y + 1}{32} = a_1(x, y),$$

and for all comparable $x, y \in X$, we get

$$\begin{aligned}d(Tx, Ty)) = & |Tx - Ty|^2 = |\frac{1}{16}x^2 - \frac{1}{16}y^2|^2 = \frac{1}{16^2}|x + y||x + y||x - y|^2\\ & \leq \frac{1}{16 \times 8}|x + y||x - y|^2 \leq \frac{1}{8}\frac{x + y + 1}{16}|x - y|^2 = \frac{1}{8}a_1(x, y)d(x, y)\\ & \leq a_1(x, y)d(x, y)\end{aligned}$$

Moreover, T is a nondecreasing continuous mapping with respect to the usual order $\leq$. Hence, all conditions of Theorem 2.1 are satisfied. Therefore, T has a fixed point $x = 0$.

Corollary 2.2. *Suppose that $(X, d, \leq)$ is a partially ordered complete b-metric space. Let $T : X \longrightarrow X$ be a continuous and nondecreasing mapping such that the following conditions hold:*

$$\begin{aligned}d(Tx, Ty) \leq & a_1 d(x, y) + a_2[d(x, Tx) + d(y, Ty)] + a_3\frac{d(y, Tx) + d(x, Ty)}{s}\\ & + a_4 d(y, Ty)\varphi(d(x, y), d(x, Tx)) + a_5 d(y, Tx)\varphi(d(x, y), d(x, Ty))\\ & + a_6 d(x, y)\varphi(d(x, y), d(x, Tx) + d(y, Tx)) + a_7 d(y, Tx),\end{aligned}$$

for all $x, y \in X$ with $x \leq y$, where a_i are nonnegative coefficients for $i = 1, 2, \cdots, 7$ with

$$a_1 + a_2 + a_3 + a_4 + a_6 \leq \frac{1}{s + 1},$$

and $\varphi : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a function such that $\varphi(t, t) = 1$ for all $t \in \mathbb{R}^+$. If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$, then T has a fixed point.

Example 2.2. *Let $X=[0,1]$ with the usual order $\leq$. Define $d(x,y)=|x-y|^2$. Then d is a $b-$metric with $s=2$. Also define $Tx=\frac{1}{2}x-\frac{1}{4}x^2$. For all comparable $x,y\in X$, we get*

$$\begin{aligned}d(Tx,Ty)) =&|Tx-Ty|^2=|\frac{1}{2}x-\frac{1}{4}x^2-\frac{1}{2}y+\frac{1}{4}y^2|^2=|\frac{1}{2}(x-y)-\frac{1}{4}(x-y)(x+y)|^2\\ =&|x-y||^2.|\frac{1}{2}-\frac{1}{4}(x+y)|^2\\ \leq&\frac{1}{4}|x-y||^2\\ =&a_1d(x,y).\end{aligned}$$

Moreover, T is a nondecreasing continuous mapping with respect to the usual order $\leq$ and $a_1=\frac{1}{4}<\frac{1}{s+1}$. Hence, all conditions of Corollary 2.2 are satisfied. Therefore, T has a fixed point $x=0$.

If we take $\varphi(t,s)=\frac{1+s}{1+t}$ for all $t,s\in\mathbb{R}^+$ in Theorem 2.1 and Corollary 2.2, we have the following Theorem and Corollary.

Theorem 2.3. *Suppose that $(X,d,\leq)$ is a partially ordered complete b-metric space. Let $T:X\longrightarrow X$ be a continuous and nondecreasing mapping. Suppose there exist mappings $a_i:X\times X\longrightarrow[0,1)$ such that for all $x,y\in X$ and $i=1,2,\cdots,7$*

$$a_i(Tx,Ty)\leq a_i(x,y).$$

Also, for all $x,y\in X$ with $x\leq y$,

$$\begin{aligned}d(Tx,Ty)\leq&a_1(x,y)d(x,y)+a_2(x,y)[d(x,Tx)+d(y,Ty)]+a_3(x,y)\frac{d(y,Tx)+d(x,Ty)}{s}\\ &+a_4(x,y)\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}+a_5(x,y)\frac{d(y,Tx)[1+d(x,Ty)]}{1+d(x,y)}\\ &+a_6(x,y)\frac{d(x,y)[1+d(x,Tx)+d(y,Tx)]}{1+d(x,y)}+a_7(x,y)d(y,Tx),\end{aligned}$$

and

$$\sup_{x,y\in X}\{a_1(x,y)+a_2(x,y)+a_3(x,y)+a_4(x,y)+a_6(x,y)\}\leq\frac{1}{s+1}.$$

If there exists $x_0\in X$ such that $x_0\leq Tx_0$, then T has a fixed point.

Corollary 2.4. *Suppose that $(X,d,\leq)$ is a partially ordered complete b-metric space. Let $T:X\longrightarrow X$ be a continuous and nondecreasing mapping such that the following conditions hold:*

$$\begin{aligned}d(Tx,Ty)\leq&a_1d(x,y)+a_2[d(x,Tx)+d(y,Ty)]+a_3\frac{d(y,Tx)+d(x,Ty)}{s}\\ &+a_4\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}+a_5\frac{d(y,Tx)[1+d(x,Ty)]}{1+d(x,y)}\\ &+a_6\frac{d(x,y)[1+d(x,Tx)+d(y,Tx)]}{1+d(x,y)}+a_7d(y,Tx),\end{aligned}$$

for all $x,y\in X$ with $x\leq y$, where a_i are nonnegative coefficients for $i=1,2,\cdots,7$ with

$$a_1+a_2+a_3+a_4+a_6\leq\frac{1}{s+1}.$$

If there exists $x_0\in X$ such that $x_0\leq Tx_0$, then T has a fixed point.

In our next theorem we relax the continuity assumption of the mapping T in Theorem 2.1 by imposing the following order condition of the metric space X:
If $\{x_n\}$ is a non-decreasing sequence in X such that $x_n\longrightarrow x$, then $x_n\leq x$ for all $n\in\mathbb{N}$.

Theorem 2.5. *Suppose that $(X, d, \leq)$ is a partially ordered complete b-metric space. Assume that if $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \longrightarrow x$, then $x_n \leq x$ for all $n \in \mathbb{N}$. Let $T : X \longrightarrow X$ be a nondecreasing mapping. Suppose there exist continuous mappings $a_i : X \times X \longrightarrow [0, 1)$ such that for all $x, y \in X$ and $i = 1, 2, \cdots, 7$*

$$a_i(Tx, Ty) \leq a_i(x, y).$$

Also, for all $x, y \in X$ with $x \leq y$,

$$\begin{aligned} d(Tx, Ty) \leq & a_1(x, y)d(x, y) + a_2(x, y)[d(x, Tx) + d(y, Ty)] + a_3(x, y)\frac{d(y, Tx) + d(x, Ty)}{s} \\ & + a_4(x, y)d(y, Ty)\varphi(d(x, y), d(x, Tx)) + a_5(x, y)d(y, Tx)\varphi(d(x, y), d(x, Ty)) \\ & + a_6(x, y)d(x, y)\varphi(d(x, y), d(x, Tx) + d(y, Tx)) + a_7(x, y)d(y, Tx), \end{aligned} \tag{2.7}$$

where $\varphi : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a continuous function such that $\varphi(t, t) = 1$ for all $t \in \mathbb{R}^+$ and

$$\sup_{x,y \in X} \{a_1(x, y) + a_2(x, y) + a_3(x, y) + a_4(x, y) + a_6(x, y)\} \leq \frac{1}{s + 1}.$$

If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$, then T has a fixed point.

Proof 2. *We take the same sequence $\{x_n\}$ as in the proof of Theorem 2.1. Then we have $x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} \leq \cdots$. that is, $\{x_n\}$ is a nondecreasing sequence. Also, this sequence converges to x. Then $x_n \leq x$, for alln $\in \mathbb{N}$. Suppose that $Tx \neq x$, that is, $d(x, Tx) > 0$. Since $x_n \leq x$ for all n, applying (2.7) and using Lemma 1.1, we have*

$$\begin{aligned} & \frac{1}{s}d(x, Tx) \leq \limsup_{n\to\infty} d(x_{n+1}, Tx) = \limsup_{n\to\infty} d(Tx_n, Tx) \\ \leq & \limsup_{n\to\infty} \Big\{ a_1(x_n, x)d(x_n, x) + a_2(x_n, x)[d(x_n, x_{n+1}) + d(x, Tx)] + a_3(x_n, x)\frac{d(x, x_{n+1}) + d(x_n, Tx)}{s} \\ & + a_4(x_n, x)d(x, Tx)\varphi(d(x_n, x), d(x_n, x_{n+1})) + a_5(x_n, x)d(x, x_{n+1})\varphi(d(x_n, x), d(x_n, Tx))) \\ & + a_6(x_n, x)d(x_n, x)\varphi(d(x_n, x), d(x_n, x_{n+1}) + d(x, x_{n+1}))) + a_7(x_n, x)d(x, x_{n+1}) \Big\} \\ \leq & [a_2(x, x) + a_3(x, x) + a_4(x, x)]d(x, Tx) \\ \leq & \frac{1}{s + 1}d(x, Tx) < \frac{1}{s}d(x, Tx), \end{aligned}$$

which is a contradiction. Hence, $Tx = x$, that is, x is a fixed point of T.

Corollary 2.6. *Suppose that $(X, d, \leq)$ is a partially ordered complete b-metric space. Assume that if $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \longrightarrow x$, then $x_n \leq x$ for all $n \in \mathbb{N}$. Let $T : X \longrightarrow X$ be a nondecreasing mapping such that the following conditions hold:*

$$\begin{aligned} d(Tx, Ty) \leq & a_1 d(x, y) + a_2[d(x, Tx) + d(y, Ty)] + a_3\frac{d(y, Tx) + d(x, Ty)}{s} \\ & + a_4 d(y, Ty)\varphi(d(x, y), d(x, Tx)) + a_5 d(y, Tx)\varphi(d(x, y), d(x, Ty)) \\ & + a_6 d(x, y)\varphi(d(x, y), d(x, Tx) + d(y, Tx)) + a_7 d(y, Tx), \end{aligned}$$

for all $x, y \in X$ with $x \leq y$, where a_i are nonnegative coefficients for $i = 1, 2, \cdots, 7$ with

$$a_1 + a_2 + a_3 + a_4 + a_6 \leq \frac{1}{s + 1},$$

and $\varphi : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a function such that $\varphi(t, t) = 1$ for all $t \in \mathbb{R}^+$. If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$, then T has a fixed point.

Remark 2.1. *Since a $b-$metric space is a metric space when $s = 1$, so our results can be viewed as the generalization and the extension of several comparable results.*

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