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FIXED POINT RESULTS OF ALTMAN INTEGRAL TYPE MAPPINGS IN S-METRIC SPACES

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Abstract. In this article, we introduce the concept of φ-weakly commuting self-mappings pairs in S-metric space. Using this idea we establish a common fixed point theorem of Altman integral type for four self-mappings in the context of S-metric space. Example is constructed to support our result.

1. Introduction and preliminaries

Fixed point theory is one of the most dynamic research subject in nonlinear analysis. In the field of metric fixed point theory the first important and significant result was proved by Banach in 1922 for contraction mapping in complete metric space. The well known Banach contraction theorem may be stated as follows: "Every contraction mapping of a complete metric space X into itself has a unique fixed point"(Bonsall 1962).

In [1] Altman proved a fixed point theorem for a single self-mapping in compact metric space satisfying the following contractive condition:

$$d(Tx,Ty) \leq Q(d(x,y)) \quad \forall\ x,y \in X$$

where $Q:[0,\infty) \to [0,\infty)$ is an increasing function satisfies the following conditions:

(1) $0 < Q(t) < t,\ t \in (0,\infty)$;
(2) $\rho(t) = \frac{t}{t-Q(t)}$ is a decreasing function;
(3) $\int_0^{t_1} \rho(t)dt < +\infty$ for some positive number t_1.

Remark 1.1. By (1) and that Q is increasing we have $Q(0)=0$ also $Q(t)=t \iff t=0$.

Common fixed point for Altman type mapping has been discussed by Garbone and Singh [2] and Li and Gu [3] in metric spaces. In 2006, Mustafa and Sims [4] introduced a new structure of generalized metric space called G-metric space. Gu and Ye [5] obtained a common fixed point theorem for Altman integral type mapping in complete G-metric space. Recently, Sedghi et al. [6] initiated the idea of S-metric space as a generalization of G-metric space. While in [7] Sedghi proved fixed point theorems for implicit relation in complete S-metric space. In this paper, we derive a common fixed point Altman integral type mapping for two pairs of φ-weakly commuting self-mappings in complete S-metric space.

We begin with the following definitions and results in the framework of S-metric space which can be found in [6, 7].

Definition 1.2. Let X be a non-empty set. An S-metric is a function $S: X \times X \times X \to [0,\infty)$ satisfying the following conditions for all $x,y,z,a \in X$

$S_1)$ $S(x,y,z)=0$ if and only if $x=y=z$;
$S_2)$ $S(x,y,z) \leq S(x,a,a)+S(y,a,a)+S(z,a,a)$.

The pair (X,S) is called S-metric space.

Example 1.3. Let $X=(-\infty,+\infty)$ the distance function $S: X \times X \times X \to [0,\infty)$ is defined by

$$S(x,y,z) = |x-z|+|y-z| \quad \text{for all } x,y,z \in X.$$

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Then (X, S) is a complete S-metric space.

Definition 1.4. Let (X, S) be an S-metric space. A sequence $\{x_n\}$ in X converges to $x \in X$ if

$$S(x_n, x_n, x) \to 0 \quad \text{as} \quad n \to \infty.$$

We write $x_n \to x$ for brevity.

Definition 1.5. Let (X, S) be an S-metric space. A sequence $\{x_n\}$ in X is called Cauchy sequence if for $\epsilon > 0$, there exists $n_0 \in N$ such that for all $n, m \geq n_0$ we have

$$S(x_n, x_n, x_m) < \epsilon.$$

Definition 1.6. An S-metric space (X, S) is said to be complete if every Cauchy sequence in X converges in X.

Lemma 1.7. *Limit of the convergent sequence in S-metric space is unique.*

Lemma 1.8. *S-metric is jointly continuous on all three variables.*

Lemma 1.9. *In an S-metric space, we have*

$$S(x, x, y) = S(y, y, x) \quad \textit{for all} \;\; x, y \in X.$$

Now we introduce the concept of φ-weakly commuting pairs of self-mappings in S-metric space as follows:

Definition 1.10. A pair of self-mappings (S, T) on S-metric space is called φ-weakly commuting. If there exist a continuous function $\varphi : [0, \infty) \to [0, \infty)$, $\varphi(0) = 0$ such that

$$S(STx, STx, TSx) \leq \varphi(S(Sx, Sx, Tx)) \;\; \forall \; x \in X.$$

Example 1.11. Let $X = [0, \infty)$, $S(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in X$. Let $S, T : X \to X$ are defined by $Sx = \frac{x}{8}$ and $Tx = \frac{x}{2}$ then

$$S(STx, STx, TSx) = S(\frac{x}{16}, \frac{x}{16}, \frac{x}{16}) \leq \frac{1}{2}\frac{3}{4}x = \frac{1}{2}S(Sx, Sx, Tx)$$
$$S(STx, STx, TSx) \leq \varphi(S(Sx, Sx, Tx)).$$

Lemma 1.12. [5]. *Let ρt ba a Lebesgue integrable function and $\rho(t) > 0$ for all $t > 0$. Let $F(x) = \int_0^x \rho(t)dt$, then $F(x)$ is an increasing function in $[0, +\infty)$.*

Definition 1.13. [8]. Let S and T be two self-mappings on a set X. Any point $x \in X$ is called coincidence point of S and T if $Sx = Tx$ for some $x \in X$ and we called $u = Sx = Tx$ is a point of coincidence of S and T.

Definition 1.14. A function $\phi : [0, \infty) \to [0, \infty)$ is called contractive modulus if it satisfy $\phi(t) \leq t$ for all $t \geq 0$.

2. Main results

Theorem 2.1. *Let (X, S) be a complete S-metric space and $P, T, f, g : X \to X$ be self-mappings. If there exists an increasing function $Q : [0, \infty) \to [0, \infty)$ satisfying conditions from (1)-(3) also the following conditions holds:*

(4) $P(X) \subseteq g(X)$ and $T(X) \subseteq f(X)$;

(5) $\int_0^{S(Px,Px,Ty)} \rho(t)dt \leq \phi(\int_0^{Q(S(fx,fx,gy))} \rho(t)dt)$ for all $x, y \in X$ and ϕ is contractive modulus where $\rho(t)$ is a Lebesgue integrable function which is summable nonnegative and such that

$$\int_0^{\delta} \rho(t)dt > 0 \;\; \forall \; \delta > 0.$$

(6) If (P, f) and (T, g) are two pairs of continuous φ-weakly commuting mappings.
Then P, T, f and g have a unique common fixed point in X.

Proof. Since $P(X) \subseteq g(X)$ and $T(X) \subseteq f(X)$ so we define two sequences $\{x_n\}$ and $\{y_n\}$ in X by the rule

$$y_{2n+1} = Px_{2n} = gx_{2n+1} \text{ and } y_{2n+2} = Tx_{2n+1} = fx_{2n+2} \; n = 0, 1, 2, ...$$

Now consider

$$\int_0^{S(y_{2n+1},y_{2n+1},y_{2n+2})} \rho(t)dt = \int_0^{S(Px_{2n},Px_{2n},Tx_{2n+1})} \rho(t)dt.$$

Using (5) we have

$$\leq \phi\Big(\int_0^{Q(S(fx_{2n},fx_{2n},gx_{2n+1}))} \rho(t)dt \Big) = \phi\Big(\int_0^{Q(S(y_{2n},y_{2n},y_{2n+1}))} \rho(t)dt \Big).$$

Using the property of ϕ we have

$$\leq \int_0^{Q(S(y_{2n},y_{2n},y_{2n+1}))} \rho(t)dt.$$

Let $t_n = S(y_n, y_{n+1})$ then the above inequality take the form

$$\int_0^{t_{2n+1}} \rho(t)dt \leq \int_0^{Q(t_{2n})} \rho(t)dt.$$

Now by the property of Q and Lemma 1.12 we have

$$t_{2n+1} \leq Q(t_{2n}) < t_{2n}.$$

Similarly we can show that

$$t_{2n} \leq Q(t_{2n-1}) < t_{2n-1}.$$

Hence $\{t_n\}$ is a nonnegative strictly decreasing sequence and hence convergent. Thus $t_{n+1} \leq Q(t_n) < t_n$ for all $n = 0, 1, 2, 3, ...$

Now to prove that $\{y_n\}$ is a Cauchy sequence consider for $m \geq n$ and by triangle inequality we have

$$S(y_n, y_n, y_m) \leq 2 \sum_{i=n}^{m-1} S(y_i, y_i, y_{i+1}) = 2 \sum_{i=n}^{m-1} t_i$$

$$= 2 \sum_{i=n}^{m-1} \frac{t_i(t_i - t_{i+1})}{(t_i - t_{i+1})} \leq 2 \sum_{i=n}^{m-1} \frac{t_i(t_i - t_{i+1})}{(t_i - Q(t_i))}$$

$$\leq 2 \sum_{i=n}^{m-1} \int_{t_{i+1}}^{t_i} \frac{t}{(t - Q(t))} dt = 2 \int_{t_m}^{t_n} \frac{t}{(t - Q(t))} dt = 2 \int_{t_m}^{t_n} P(t)dt.$$

It follows from the convergence of the sequence $\{t_n\}$ and by condition (3) we have

$$\lim_{n\to\infty} \int_{t_m}^{t_n} P(t)dt = 0.$$

Thus $\{y_n\}$ is a Cauchy sequence in X. Since X is complete so there must exists $u \in X$ such that

$$\lim_{n\to\infty} y_n = u.$$

Also the subsequences $\{y_{2n+1}\}$ and $\{y_{2n+2}\}$ converges to u. Therefore

$$\lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Px_{2n} = \lim_{n\to\infty} gx_{2n+1} = u$$

$$\lim_{n\to\infty} y_{2n+2} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} fx_{2n+2} = u.$$

Since (P, f) are continuous φ-weakly commuting pair so

$$S(Pfx_{2n}, Pfx_{2n}, fPx_{2n}) \leq \varphi(S(Px_{2n}, Px_{2n}, fx_{2n})).$$

Taking limit $n \to \infty$ and since (P, f) is continuous pair of mappings thus

$$S(Pu, Pu, fu) \leq \varphi(S(u, u, u)) = \varphi(0) = 0.$$

Which implies that $Pu = fu$. Similarly from continuous φ-weakly commuting pair (T, g) we can show that $Tu = gu$.

Now by condition (5) and using other given information we have

$$\int_0^{S(Pu,Pu,Tu)} \rho(t)dt \leq \phi\Bigg(\int_0^{Q(S(fu,fu,gu))} \rho(t)dt\Bigg) \leq \int_0^{Q(S(fu,fu,gu))} \rho(t)dt$$

$$\begin{aligned} S(Pu, Pu, Tu) &\leq Q(S(fu, fu, gu)) \leq S(fu, fu, gu) \\ &\leq S(fu, fu, Pu) + S(fu, fu, Pu) + S(gu, gu, Pu) \\ &\leq S(gu, gu, Tu) + S(gu, gu, Tu) + S(Pu, Pu, Tu) \\ &= S(Pu, Pu, Tu). \end{aligned}$$

Which implies that $fu = gu$. Thus $fu = gu = Pu = Tu$ and let $z = fu = gu = Pu = Tu$. Therefore u is the common coincidence point of mappings P, T, f and g.

Again since (P, f) are φ-weakly commuting pair so

$$S(Pz, Pz, fz) = S(Pfu, Pfu, fPu) \leq \varphi(S(Pu, Pu, fu)) = \varphi(0) = 0.$$

Implies that $Pz = fz$. Similarly we can show that $Tz = gz$. Thus

$$Pfu = fPu \ \text{ and } \ Tgu = gTu.$$

Again by condition (5) we have

$$\int_0^{S(Pz,Pz,z)} \rho(t)dt = \int_0^{S(PPu,PPu,Tu)} \rho(t)dt$$

$$\leq \phi\Bigg(\int_0^{Q(S(fPu,fPu,gu))} \rho(t)dt\Bigg) \leq \int_0^{Q(S(fPu,fPu,gu))} \rho(t)dt.$$

By Lemma 1.12 and using the property of Q we have

$$\begin{aligned} S(Pz, Pz, z) &\leq Q(S(fPu, fPu, gu)) \leq S(fPu, fPu, gu) \\ &= S(Pfu, Pfu, gu) = S(Pz, Pz, z). \end{aligned}$$

Which implies $Pz = z$ but $Pz = fz$ therefore $Pz = fz = z$. Similarly we can prove that $Tz = gz = z$. Hence $Pz = fz = gz = Tz = z$. Thus z is a common fixed point of mappings P, T, f and g.

Uniqueness. Assume that common fixed point of P, T, f and g is not unique i.e $z \neq w$ be two distinct fixed points of P, T, f and g. Then using condition (5) we have

$$\int_0^{S(z,z,w)} \rho(t)dt = \int_0^{S(Pz,Pz,Tw)} \rho(t)dt$$

$$\leq \phi\Bigg(\int_0^{Q(S(fz,fz,gw))} \rho(t)dt\Bigg) \leq \int_0^{Q(S(fz,fz,gw))} \rho(t)dt.$$

By Lemma 1.12 and using the property of Q we have

$$S(z, z, w) \leq Q(S(fz, fz, gw)) \leq S(z, z, w).$$

Which is a contradiction hence $z = w$. Therefore, fixed point of P, T, f and g is unique. □

Remark 2.2. If we take (1) $P = T$ (2) $f = g$ (3) $P = T$ and $f = g = I$ (4) $\phi = I$ in Theorem 2.1. Then we obtain several new results in the setting of S-metric space.

Corollary 2.3. *Let (X, S) be a complete S-metric space and $P, T, f, g : X \to X$ be self-mappings. If there exists an increasing function $Q : [0, \infty) \to [0, \infty)$ satisfying conditions from (1)-(3) also the following conditions holds:*

(4) $P(X) \subseteq g(X)$ and $T(X) \subseteq f(X)$;

(5) $S(Px, Px, Ty) \leq \phi(Q(S(fx, fx, gy)))$ for all $x, y \in X$ where ϕ is contractive modulus;

(6) If (P, f) and (T, g) are two pairs of continuous φ-weakly commuting mappings.

Then P, T, f and g have a unique common fixed point in X.

Proof. Putting $\rho(t) = I$ in Theorem 2.1, one can easily obtain the proof of Corollary 2.3 from Theorem 2.1. □

Now we construct an example to support Corollary 2.3.

Example 2.4. Let $X = [0, \infty))$ and $S(x, y, z) = |x - y| + |y - z|$ for all $x, y, z \in X$ with self-mappings defined on X is given by $Px = \frac{x}{8}, fx = x, Tx = \frac{x}{16}$ and $gx = \frac{x}{2}$. Clearly $P(X) \subseteq g(X)$ and $T(X) \subseteq f(X)$. Also we have

$$\begin{aligned} S(Px, Px, Ty) &= S\left(\frac{x}{8}, \frac{x}{8}, \frac{y}{16}\right) = |\frac{x}{8} - \frac{y}{16}| + |\frac{x}{8} - \frac{y}{16}| \\ &= \frac{1}{8}\left(|x - \frac{y}{2}| + |x - \frac{y}{2}|\right) = \frac{1}{8}S(fx, fx, gy) \\ S(Px, Px, Ty) &= \frac{1}{8}S(fx, fx, gy). \end{aligned}$$

Let $\phi(t) = \frac{3}{4}t$ and $Q(t) = \frac{1}{2}t$. Then $\phi(t) \leq t$ and $Q(t)$ satisfies conditions (1)-(3). Then we have

$$\begin{aligned} S(Px, Px, Ty) &= \frac{1}{8}S(fx, fx, gy) \leq \frac{3}{4} \cdot \frac{1}{2}S(fx, fx, gy) \\ &= \frac{3}{4}Q(S(fx, fx, gy)). \end{aligned}$$

On the other side if $\varphi(t) = \frac{1}{2}t$ for all $t \in [0, \infty)$. Then one can easily show that (P, f) and (T, g) are two pairs of continuous φ-weakly commuting mappings in X. So that all the conditions of Corollary 2.3 are satisfied. Therefore, 0 is the unique common fixed point of P, T, f and g.

Corollary 2.5. *Let (X, S) be a complete S-metric space and $P, T : X \to X$ be self-mappings. If there exists an increasing function $Q : [0, \infty) \to [0, \infty)$ satisfying conditions from (1)-(3) and ϕ is a contractive modulus also the following condition holds:*

$$S(Px, Px, Ty) \leq \phi(Q(S(x, x, y))) \quad \textit{for all} \;\; x, y \in X.$$

Then P and T have a unique common fixed point in X.

Corollary 2.6. *Let (X, S) be a complete S-metric space and $P : X \to X$ be self-mappings. If there exists an increasing function $Q : [0, \infty) \to [0, \infty)$ satisfying conditions from (1)-(3) and ϕ is a contractive modulus also the following condition holds:*

$$S(Px, Px, Py) \leq \phi(Q(S(x, x, y))) \quad \textit{for all} \;\; x, y \in X.$$

Then P has a unique fixed point in X.

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