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ADDITIVE UNITS OF PRODUCT SYSTEM OF HILBERT MODULES

BILJANA VUJOŠEVIĆ*

Abstract. In this paper we consider the notion of additive units and roots of a central unital unit in a spatial product system of two-sided Hilbert C^*-modules. This is a generalization of the notion of additive units and roots of a unit in a spatial product system of Hilbert spaces introduced in [B. V. R. Bhat, M. Lindsay, M. Mukherjee, *Additive units of product system,* arXiv:1501.07675v1 [math.FA] 30 Jan 2015]. We introduce the notion of continuous additive unit and continuous root of a central unital unit ω in a spatial product system over C^*-algebra $\mathcal{B}$ and prove that the set of all continuous additive units of ω can be endowed with a structure of two-sided Hilbert $\mathcal{B}-\mathcal{B}$ module wherein the set of all continuous roots of ω is a Hilbert $\mathcal{B}-\mathcal{B}$ submodule.

1. Introduction

The notion of additive units and roots of a unit in a spatial product system of Hilbert spaces is introduced and studied in [1, Section 3]. In more details, an additive unit of a unit $u = (u_t)_{t>0}$ in a spatial product system E is a measurable section $a = (a_t)_{t>0},\ a_t \in E_t$, that satisfies

$$a_{s+t} = a_s u_t + u_s a_t$$

for all $s, t > 0$, i.e. a is "additive with respect to the given unit u". An additive unit $a = (a_t)_{t>0}$ of a unit $u = (u_t)_{t>0}$ is a root if for all $t > 0$

$$\langle a_t, u_t \rangle = 0.$$

In the same paper it is, also, proved that the set of all additive units of a unit u is a Hilbert space wherein the set of all roots of u is a Hilbert subspace.

The goal of this paper is to generalize the notion of additive units and roots of a unit in a spatial product system of Hilbert spaces (from [1, Section 3]) and to obtain some similar results as therein but in a more general context. To this purpose, we observe a spatial product system of two-sided Hilbert modules over unital C^*-algebra $\mathcal{B}$ (it presents a product system that contains a central unital unit). We introduce the notion of continuous additive unit and continuous root of a central unital unit. Also, we show that the set of all continuous additive units of a central unital unit is continuous in a certain sense. Finally, we prove that the set of all continuous additive units of a central unital unit ω can be provided with a structure of two-sided Hilbert $\mathcal{B}-\mathcal{B}$ module wherein the set of all continuous roots of ω is a Hilbert $\mathcal{B}-\mathcal{B}$ submodule.

Throughout the whole paper, $\mathcal{B}$ denotes a unital C^*-algebra and 1 denotes its unit. Also, we use $\otimes$ for tensor product, although $\odot$ is in common use.

The rest of this section is devoted to basic definitions.

Definition 1.1. *a) A Hilbert $\mathcal{B}$-module F is a right $\mathcal{B}$-module with a map $\langle\ ,\ \rangle : F \times F \to \mathcal{B}$ which satisfies the following properties:*

- $\langle x, \lambda y + \mu z \rangle = \lambda \langle x, y \rangle + \mu \langle x, z \rangle$ *for* $x, y, z \in F$ *and* $\lambda, \mu \in C$;
- $\langle x, y\beta \rangle = \langle x, y \rangle \beta$ *for* $x, y \in F$ *and* $\beta \in \mathcal{B}$;
- $\langle x, y \rangle = \langle y, x \rangle^*$ *for* $x, y \in F$;
- $\langle x, x \rangle \geq 0$ *and* $\langle x, x \rangle = 0 \Leftrightarrow x = 0$ *for* $x \in F$;

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and F is complete with respect to the norm $\|\cdot\| = \|\langle\cdot,\cdot\rangle\|^{\frac{1}{2}}$.

b) A Hilbert $\mathcal{B}-\mathcal{B}$ module is a Hilbert $\mathcal{B}$-module with a non-degenerate $*-$representation of $\mathcal{B}$ by elements in the C^*-algebra $B^a(F)$ of adjointable (and, therefore, bounded and right linear) mappings on F. The homomorphism $j:\mathcal{B}\to B^a(F)$ is contractive. In particular, since C^*-algebra $\mathcal{B}$ is unital, the unit of $\mathcal{B}$ acts as the unit of $B^a(F)$. Also, for $x,y\in F$ and $\beta\in\mathcal{B}$ there holds $\langle x,\beta y\rangle = \langle\beta^* x, y\rangle$ where $\beta y = j(\beta)(y)$.

For basic facts about Hilbert C^*-modules we refer the reader to [5] and [6].

Definition 1.2. *a) A product system over C^*-algebra $\mathcal{B}$ is a family $(E_t)_{t\geq 0}$ of Hilbert $\mathcal{B}-\mathcal{B}$ modules, with $E_0\cong\mathcal{B}$, and a family of (unitary) isomorphisms*

$$\varphi_{t,s}: E_t\otimes E_s\to E_{t+s},$$

where $\otimes$ stands for the so-called inner tensor product obtained by identifications $ub\otimes v\sim u\otimes bv$, $u\otimes vb\sim(u\otimes v)b$, $bu\otimes v\sim b(u\otimes v)$, ($u\in E_t$, $v\in E_s$, $b\in\mathcal{B}$) and then completing in the inner product $\langle u\otimes v, u_1\otimes v_1\rangle = \langle v, \langle u,u_1\rangle v_1\rangle$;

b) Unit on E is a family $u=(u_t)_{t\geq 0}$, $u_t\in E_t$, so that $u_0=1$ and $\varphi_{t,s}(u_t\otimes u_s)=u_{t+s}$, which we shall abbreviate to $u_t\otimes u_s = u_{t+s}$. A unit $u=(u_t)$ is unital if $\langle u_t,u_t\rangle = 1$. It is central if for all $\beta\in\mathcal{B}$ and all $t\geq 0$ there holds $\beta u_t = u_t\beta$.

Definition 1.3. *The spatial product system is a product system that contains a central unital unit.*

For a more detailed approach to this topic, we refer the reader to [2], [8], [9], [4].

2. Additive units

In this section we define all notions and prove auxiliary statements that are necessary for the proof of main result that we present in Section 3.

Throughout the whole paper, $\omega = (\omega_t)_{t\geq 0}$ is a central unital unit in a spatial product system $E=(E_t)_{t\geq 0}$ over unital C^*-algebra $\mathcal{B}$.

Definition 2.1. *A family $a=(a_t)$, $a_t\in E_t$, is said to be an additive unit of ω if $a_0=0$ and*

$$a_{s+t} = a_s\otimes\omega_t + \omega_s\otimes a_t,\quad s,t\geq 0.$$

Definition 2.2. *An additive unit $a=(a_t)$ of a unit $\omega=(\omega_t)$ is said to be a root if $\langle a_t,\omega_t\rangle = 0$ for all $t\geq 0$.*

The previous definitions do not include any technical condition, such as measurability or continuity. It occurs that it is sometimes more convenient to pose the continuity condition directly on units.

Definition 2.3. *For $\beta\in\mathcal{B}$, let $F^{a,b}_\beta:[0,\infty)\to\mathcal{B}$ be the map defined by*

$$F^{a,b}_\beta(s) = \langle a_s, \beta b_s\rangle,\ s\geq 0, \tag{1}$$

where a,b are additive units of ω in E.

We say that the set of additive units of ω $\mathcal{S}$ is continuous if the map $F^{a,b}_\beta$ is continuous for all $a,b\in\mathcal{S}$, $\beta\in\mathcal{B}$. We say that a is a continuous additive unit of ω if the set $\{a\}$ is continuous, i.e. if the map $F^{a,a}_\beta$ is continuous for each $\beta\in\mathcal{B}$. Denote the set of all continuous additive units of ω by $\mathcal{A}_\omega$ and the set of all continuous roots of ω by $\mathcal{R}_\omega$.

Remark 2.4. *We should tell the difference between the continuous set of additive units of ω and the set of continuous additive units of ω. In the second case only $F^{a,a}_\beta$ should be continuous for all $a\in\mathcal{S}$, $\beta\in\mathcal{B}$, whereas in the first case all $F^{a,b}_\beta$ should be continuous.*

The following example assures us that the set of all continuous additive units of a central unital unit ω in a spatial product system is not empty.

Example 2.5. *For $\gamma \in \mathcal{B}$, the family $(a_s)_{s\geq 0}$, where $a_s = s\gamma\omega_s = s\omega_s\gamma$, is an additive unit of ω since for $s, t \geq 0$ there holds*

$$a_{s+t} = (s+t)\gamma\omega_s \otimes \omega_t = s\gamma\omega_s \otimes \omega_t + t\omega_s\gamma \otimes \omega_t = s\gamma\omega_s \otimes \omega_t + t\omega_s \otimes \gamma\omega_t =$$
$$= s\gamma\omega_s \otimes \omega_t + \omega_s \otimes t\gamma\omega_t = a_s \otimes \omega_t + \omega_s \otimes a_t$$

and $a_0 = 0$. Since $F_\beta^{a,a} : s \mapsto \langle s\omega_s\gamma, \beta(s\omega_s\gamma)\rangle = s^2\gamma^*\beta\gamma$ is a continuous mapping for all $\beta \in \mathcal{B}$, the additive unit a belongs to $\mathcal{A}_\omega$.

The properties of additive units of ω are given in the following lemma:

Lemma 2.6. 1. *If a is a continuous additive unit of ω, then*

$$\langle \omega_s, a_s\rangle = s\langle \omega_1, a_1\rangle,\ \ s \geq 0. \tag{2}$$

2. *If a, b are continuous roots of ω and $\beta \in \mathcal{B}$, then*

$$F_\beta^{a,b}(s) = sF_\beta^{a,b}(1),\ \ s \geq 0. \tag{3}$$

3. *If a is a continuous additive unit of ω, then a family $(a'_s)_{s\geq 0}$, where*

$$a'_s = a_s - \langle \omega_s, a_s\rangle\omega_s, \tag{4}$$

is a continuous root of ω.

Proof. 1. Let $G^a : [0, \infty) \to \mathcal{B}$ be the map defined by $G^a(s) = \langle \omega_s, a_s\rangle,\ s \geq 0$. For $s, t \geq 0$ we obtain

$$G^a(s+t) = \langle \omega_{s+t}, a_s \otimes \omega_t + \omega_s \otimes a_t\rangle = \langle \omega_s \otimes \omega_t, a_s \otimes \omega_t\rangle + \langle \omega_s \otimes \omega_t, \omega_s \otimes a_t\rangle =$$
$$= \langle \omega_t, \langle \omega_s, a_s\rangle\omega_t\rangle + \langle \omega_t, \langle \omega_s, \omega_s\rangle a_t\rangle = \langle \omega_s, a_s\rangle + \langle \omega_t, a_t\rangle = G^a(s) + G^a(t)$$

and

$$\|G^a(s) - G^a(0)\|^2 = \|\langle \omega_s, a_s\rangle\|^2 \leq \|\omega_s\|^2\|a_s\|^2 =$$
$$= \|\langle a_s, a_s\rangle\| = \|F_1^{a,a}(s)\| \to \|F_1^{a,a}(0)\| = 0,\ \ s \to 0.$$

Hence, the map G^a is continuous. Therefore, $G^a(s) = sG^a(1)$, i.e.

$$\langle \omega_s, a_s\rangle = s\langle \omega_1, a_1\rangle.$$

2. Let $s, t \geq 0$. Since $a, b \in \mathcal{R}_\omega$, we see that

$$F_\beta^{a,b}(s+t) = \langle a_s \otimes \omega_t + \omega_s \otimes a_t, \beta(b_s \otimes \omega_t + \omega_s \otimes b_t)\rangle =$$
$$= \langle \omega_t, \langle a_s, \beta b_s\rangle\omega_t\rangle + \langle a_t, \langle \omega_s, \beta\omega_s\rangle b_t\rangle = \langle a_s, \beta b_s\rangle + \langle a_t, \beta b_t\rangle = F_\beta^{a,b}(s) + F_\beta^{a,b}(t)$$

and

$$\|F_\beta^{a,b}(s) - F_\beta^{a,b}(0)\|^2 = \|\langle a_s, \beta b_s\rangle\|^2 \leq \|\langle a_s, a_s\rangle\|\|\beta\|^2\|\langle b_s, b_s\rangle\| \to 0,\ \ s \to 0.$$

Hence, the map $F_\beta^{a,b}$ is continuous and, therefore, $F_\beta^{a,b}(s) = sF_\beta^{a,b}(1)$.

3. For $s, t \geq 0$, we obtain that

$$a'_{s+t} = a_s \otimes \omega_t + \omega_s \otimes a_t - \langle \omega_s \otimes \omega_t, a_s \otimes \omega_t + \omega_s \otimes a_t\rangle\omega_s \otimes \omega_t =$$
$$= a_s \otimes \omega_t + \omega_s \otimes a_t - (\langle \omega_t, \langle \omega_s, a_s\rangle\omega_t\rangle + \langle \omega_t, \langle \omega_s, \omega_s\rangle a_t\rangle)\omega_s \otimes \omega_t =$$
$$= a_s \otimes \omega_t + \omega_s \otimes a_t - \langle \omega_s, a_s\rangle\omega_s \otimes \omega_t - \langle \omega_t, a_t\rangle\omega_s \otimes \omega_t =$$
$$= (a_s - \langle \omega_s, a_s\rangle\omega_s) \otimes \omega_t + \omega_s \otimes (a_t - \langle \omega_t, a_t\rangle\omega_t) = a'_s \otimes \omega_t + \omega_s \otimes a'_t$$

and

$$\langle a'_s, \omega_s\rangle = 0.$$

Therefore, a' is a root of ω.
Let $\beta \in \mathcal{B}$. By (4) and (2), it follows that

$$F_\beta^{a',a'}(s) = F_\beta^{a,a}(s) - s^2\langle a_1, \omega_1\rangle\beta\langle \omega_1, a_1\rangle,\ \ s \geq 0.$$

Hence, the map $F_\beta^{a',a'}$ is continuous which implies that $a' \in \mathcal{R}_\omega$. □

Remark 2.7. *Let a be a continuous additive unit of ω. By (2) and (4), it can be decomposed as $a_s = s\langle \omega_1, a_1\rangle\omega_s + a'_s,\ s \geq 0$, where a' is a continuous root of ω.*

Let a, b be two continuous additive units of ω. By Remark 2.7, we can decompose them as

$$a_s = s\langle\omega_1, a_1\rangle\omega_s + a'_s,\ \ b_s = s\langle\omega_1, b_1\rangle\omega_s + b'_s,\ \ s \geq 0, \tag{5}$$

where $a', b' \in \mathcal{R}_\omega$. Therefore,

$$F_\beta^{a',b'}(1) = \langle a_1 - \langle\omega_1, a_1\rangle\omega_1, \beta(b_1 - \langle\omega_1, b_1\rangle\omega_1)\rangle =$$
$$= F_\beta^{a,b}(1) - \langle a_1, \omega_1\rangle\beta\langle\omega_1, b_1\rangle,\ \ \beta \in \mathcal{B}.$$

Let $s \geq 0$ and $\beta \in \mathcal{B}$. Since, by (3), there holds

$$F_\beta^{a',b'}(s) = sF_\beta^{a',b'}(1),$$

it follows that

$$F_\beta^{a',b'}(s) = sF_\beta^{a,b}(1) - s\langle a_1, \omega_1\rangle\beta\langle\omega_1, b_1\rangle. \tag{6}$$

Now, by (5) and (6), we obtain that

$$F_\beta^{a,b}(s) = sF_\beta^{a,b}(1) + (s^2 - s)\langle a_1, \omega_1\rangle\beta\langle\omega_1, b_1\rangle. \tag{7}$$

It follows that the map $F_\beta^{a,b}$ is continuous.

Therefore, we conclude that the set of all continuous additive units of ω is continuous in the sense of Definition 2.3.

3. The result

In this section we prove the main result.

Throughout the whole section, $\omega = (\omega_t)_{t\geq 0}$ is a central unital unit in a spatial product system $E = (E_t)_{t\geq 0}$ over unital C^*-algebra $\mathcal{B}$.

Theorem 3.1. *The set $\mathcal{A}_\omega$ (the set of all continuous additive units of ω) is a $\mathcal{B} - \mathcal{B}$ module under the point-wise addition and point-wise scalar multiplication. The set $\mathcal{R}_\omega$ (the set of all continuous roots of ω) is a $\mathcal{B} - \mathcal{B}$ submodule in $\mathcal{A}_\omega$.*

Proof. Let $a = (a_s),\ b = (b_s) \in \mathcal{A}_\omega$ and $\beta \in \mathcal{B}$. For $s \geq 0$, $(a + b)_s = a_s + b_s$, $(a\beta)_s = a_s\beta$ and $(\beta a)_s = \beta a_s$.

Let $s, t \geq 0$. Since $(a+b)_{s+t} = (a+b)_s \otimes \omega_t + \omega_s \otimes (a+b)_t$ and $F_\beta^{a+b,a+b} = F_\beta^{a,a} + F_\beta^{b,a} + F_\beta^{a,b} + F_\beta^{b,b}$, it follows that $a + b \in \mathcal{A}_\omega$.

Let $\gamma \in \mathcal{B}$. Since the unit ω is central, we obtain that $(a\gamma)_{s+t} = (a\gamma)_s \otimes \omega_t + \omega_s \otimes (a\gamma)_t$. Also, $F_\beta^{a\gamma,a\gamma}(s) = \gamma^* F_\beta^{a,a}(s)\gamma$ which implies that the map $F_\beta^{a\gamma,a\gamma}$ is continuous. Therefore, $a\gamma \in \mathcal{A}_\omega$.

Similarly, $(\gamma a)_{s+t} = (\gamma a)_s \otimes \omega_t + \omega_s \otimes (\gamma a)_t$. By Remark 2.7, $a_s = s\langle\omega_1, a_1\rangle\omega_s + a'_s,\ a' \in \mathcal{R}_\omega$, and we obtain that $F_\beta^{\gamma a,\gamma a}(s) = s^2\langle a_1, \omega_1\rangle\gamma^*\beta\gamma\langle\omega_1, a_1\rangle + F_{\gamma^*\beta\gamma}^{a',a'}(s)$. By (3), the map $F_\beta^{\gamma a,\gamma a}$ is continuous. Therefore, $\gamma a \in \mathcal{A}_\omega$.

The associativity and the commutativity follow directly. The neutral element is $0 = (0_s)$ and the inverse of a is $-a = (-a_s)$. The other axioms of two-sided $\mathcal{B}-\mathcal{B}$ module $(a\beta)\gamma = a(\beta\gamma)$, $\beta(\gamma a) = (\beta\gamma)a$, $\beta(a + b) = \beta a + \beta b$, $(a + b)\beta = a\beta + b\beta$, $(\beta + \gamma)a = \beta a + \gamma a$, $a(\beta + \gamma) = a\beta + a\gamma$, $1a = a1 = a$ follow directly.

If $a, b \in \mathcal{R}_\omega$, then $\langle a_s + b_s, \omega_s\rangle = 0$, $\langle a_s\beta, \omega_s\rangle = \beta^*\langle a_s, \omega_s\rangle = 0$ and $\langle\beta a_s, \omega_s\rangle = \langle a_s, \beta^*\omega_s\rangle = \langle a_s, \omega_s\beta^*\rangle = \langle a_s, \omega_s\rangle\beta^* = 0$. Hence, $a + b, a\beta, \beta a \in \mathcal{R}_\omega$. Since also $0 = (0_s)$ and $-a = (-a_s) \in \mathcal{R}_\omega$, we see that $\mathcal{R}_\omega$ is a $\mathcal{B} - \mathcal{B}$ submodule in $\mathcal{A}_\omega$. □

For every $\mathcal{B} \ni \beta \geq 0$ there is a map $\langle\ ,\ \rangle_\beta : \mathcal{A}_\omega \times \mathcal{A}_\omega \to \mathcal{B}$ given by

$$\langle a, b\rangle_\beta = \langle a_1, \beta b_1\rangle. \tag{8}$$

Proposition 3.2. *The pairing (8) satisfies the following properties:*

1. $\langle a, \lambda b + \mu c\rangle_\beta = \lambda\langle a, b\rangle_\beta + \mu\langle a, c\rangle_\beta$ *for all* $a, b, c \in \mathcal{A}_\omega$ *and* $\lambda, \mu \in \mathbb{C}$;
2. $\langle a, b\gamma\rangle_\beta = \langle a, b\rangle_\beta\gamma$ *for all* $a, b \in \mathcal{A}_\omega$ *and* $\gamma \in \mathcal{B}$;
3. $\langle a, b\rangle_\beta = \langle b, a\rangle_\beta^*$ *for all* $a, b \in \mathcal{A}_\omega$;
4. $\langle a, a\rangle_\beta \geq 0$ *for all* $a \in \mathcal{A}_\omega$;

5. $\langle a, a\rangle_1 = 0 \Leftrightarrow a = 0$ *for all* $a \in \mathcal{A}_\omega$*;*
6. $\langle a, \gamma b\rangle_1 = \langle \gamma^* a, b\rangle_1$ *for all* $a, b \in \mathcal{A}_\omega$ *and* $\gamma \in \mathcal{B}$*.*

Proof. 1, 2, 3 - Straightforward calculation.

4 - Since $\beta \geq 0$, it follows that $\beta = \gamma^*\gamma$ for some $\gamma \in \mathcal{B}$. Thus, $\langle a, a\rangle_\beta = \langle a_1, \gamma^*\gamma a_1\rangle = \langle \gamma a_1, \gamma a_1\rangle \geq 0$.

5 - If $\langle a, a\rangle_1 = 0$, then $a_1 = 0$ by (8). By Remark 2.7, $a_s = s\langle\omega_1, a_1\rangle\omega_s + a'_s$, $a' \in \mathcal{R}_\omega$ and $s \geq 0$, implying that $a_s = a'_s$. Therefore, $\langle a_s, a_s\rangle = s\langle a'_1, a'_1\rangle$ by (3). Now, it follows that $\langle a_s, a_s\rangle = 0$, i.e. $a_s = 0$ for all $s \geq 0$.

6 - Straightforward calculation. □

Theorem 3.3. *The set* $\mathcal{A}_\omega$ *(the set of all continuous additive units of* ω*) is a Hilbert* $\mathcal{B} - \mathcal{B}$ *module under the inner product* $\langle\ ,\ \rangle : \mathcal{A}_\omega \times \mathcal{A}_\omega \to \mathcal{B}$ *defined by*

$$\langle a, b\rangle = \langle a_1, b_1\rangle, \quad a, b \in \mathcal{A}_\omega. \tag{9}$$

The set $\mathcal{R}_\omega$ *(the set of all continuous roots of* ω*) is a Hilbert* $\mathcal{B} - \mathcal{B}$ *submodule in* $\mathcal{A}_\omega$*.*

Proof. We notice that the mapping $\langle\ ,\ \rangle$ in (9) is equal to the mapping $\langle\ ,\ \rangle_1$ in (8). Therefore, by Theorem 3.1 and Proposition 3.2, we obtain that $\langle\ ,\ \rangle$ is a $\mathcal{B}$-valued inner product on $\mathcal{B} - \mathcal{B}$ module $\mathcal{A}_\omega$. Therefore, $\mathcal{A}_\omega$ is a pre-Hilbert $\mathcal{B} - \mathcal{B}$ module. Now, we need to prove that $\mathcal{A}_\omega$ is complete with respect to the inner product (9).

Let (a^n) be a Cauchy sequence in $\mathcal{A}_\omega$ and $s \geq 0$. If $\beta = 1$ and $a = b = a^m - a^n$ in (7), it follows that

$$\|a_s^m - a_s^n\|^2 \leq (s^2 + 2s)\|a_1^m - a_1^n\|^2 = (s^2 + 2s)\|a^m - a^n\|^2.$$

(The last equality follows by (9).) Thus, (a_s^n) is a Cauchy sequence in E_s and denote

$$a_s = \lim_{n\to\infty} a_s^n. \tag{10}$$

Let $\varepsilon > 0$ and $s, t \geq 0$. There is $n_0 \in \mathbb{N}$ so that $\|a_s^n - a_s\| \leq \frac{\varepsilon}{3}$, $\|a_t^n - a_t\| \leq \frac{\varepsilon}{3}$ and $\|a_{s+t}^n - a_{s+t}\| \leq \frac{\varepsilon}{3}$ for $n > n_0$. Then,

$$\begin{aligned}\|a_{s+t} - a_s \otimes \omega_t - \omega_s \otimes a_t\| &\leq \|a_{s+t} - a_{s+t}^n\| + \|a_{s+t}^n - a_s \otimes \omega_t - \omega_s \otimes a_t\| \leq \\ &\leq \|a_{s+t} - a_{s+t}^n\| + \|(a_s^n - a_s) \otimes \omega_t\| + \|\omega_s \otimes (a_t^n - a_t)\| \leq \varepsilon.\end{aligned}$$

Hence, a is an additive unit of ω. Let $\beta \in \mathcal{B}$. By (1), (10) and (7),

$$\begin{aligned}F_\beta^{a,a}(s) = \lim_{n\to\infty} F_\beta^{a^n,a^n}(s) &= \lim_{n\to\infty}[sF_\beta^{a^n,a^n}(1) + (s^2 - s)\langle a_1^n, \omega_1\rangle\beta\langle\omega_1, a_1^n\rangle] = \\ &= sF_\beta^{a,a}(1) + (s^2 - s)\langle a_1, \omega_1\rangle\beta\langle\omega_1, a_1\rangle.\end{aligned}$$

Hence, the map $F_\beta^{a,a}$ is continuous, i.e. $a \in \mathcal{A}_\omega$. By (9) and (10), $\|a^n - a\| = \|a_1^n - a_1\| \to 0,\ n \to \infty$. Therefore, $\mathcal{A}_\omega$ is complete with respect to the inner product (9).

Let (a^n) be a sequence in $\mathcal{R}_\omega$ satisfying $\lim\limits_{n\to\infty} a^n = a$. The only question is whether the continuous additive unit a belongs to $\mathcal{R}_\omega$. However, this immediately follows from (10) since $\langle a_s, \omega_s\rangle = \lim\limits_{n\to\infty}\langle a_s^n, \omega_s\rangle = 0$ for all $s \geq 0$. □

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Faculty of Mathematics, University of Belgrade, Studentski trg 16-18, 11000 Beograd, Serbia

*Corresponding author: bvujosevic@matf.bg.ac.rs