International Journal of Analysis and Applications
ISSN 2291-8639
Volume 10, Number 1 (2016), 48-57
http://www.etamaths.com

CONVERGENCE THEOREM FOR GENERALIZED MIXED EQUILIBRIUM PROBLEM AND COMMON FIXED POINT PROBLEM FOR A FAMILY OF MULTIVALUED MAPPINGS

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Abstract. In this paper, a new hybrid iterative algorithm is constructed using the shrinking projection method introduced by Takahashi. The sequence of the algorithm is proved to converge strongly to a common element of the set of solutions of generalized mixed equilibrium problem and the set of common fixed points of a finite family of multivalued strictly pseudocontractive mappings in real Hilbert spaces. Furthermore, we apply our main result to convex minimization problem.

1. Introduction

Let H be a real Hilbert space with inner product $\langle ., . \rangle$ and norm $||\cdot||$, and let K be a nonempty closed convex subset of H. Let $B : K \to H$ be a nonlinear mapping, $\varphi : K \to \mathbb{R} \cup \{+\infty\}$ be a function and $F : K \times K \to \mathbb{R}$, be a bifunction where $\mathbb{R}$ is the set of real numbers. The *generalized mixed equilibrium problem* is defined as follows:

$$\text{find } x \in K : F(x, y) + \varphi(y) - \varphi(x) + \langle Bx, y - x\rangle \geq 0 \ \forall \ y \in K. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $GMEP(F, \varphi, B)$.
If $B = 0$, problem (1.1) reduces to the following *mixed equilibrium* problem:

$$\text{find } x \in K : F(x, y) + \varphi(y) - \varphi(x) \geq 0 \ \forall \ y \in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by $MEP(F, \varphi)$.
If $\varphi = 0$, problem (1.1) becomes the following *generalized equilibrium* problem:

$$\text{find } x \in K : F(x, y) + \langle Bx, y - x\rangle \geq 0 \ \forall \ y \in C. \tag{1.3}$$

The set of solutions of (1.3) is denoted by $GEP(F, B)$.
If $\varphi = 0$ and $B = 0$, problem (1.1) becomes the following *equilibrium* problem:

$$\text{Find } x \in K : F(x, y) \geq 0 \ \forall \ y \in K. \tag{1.4}$$

The set of solutions of (1.4) is denoted by $EP(F)$.
If $F(x, y) = 0$ for all $x, y \in K$, problem (1.1) becomes the following *generalized variational inequality* problem:

$$\text{Find } x \in K : \varphi(y) - \varphi(x) + \langle Bx, y - x\rangle \geq 0 \ \forall \ y \in K. \tag{1.5}$$

If $\varphi = 0$ and $F(x, y) = 0$ for all $x, y \in K$, problem (1.1) becomes the following *variational inequality* problem:

$$\text{Find } x \in K : \langle Bx, y - x\rangle \geq 0 \ \forall \ y \in K. \tag{1.6}$$

If $B = 0$ and $F(x, y) = 0$ for all $x, y \in K$, problem (1.1) becomes the following *convex minimization* problem:

$$\text{Find } x \in C : \varphi(y) \geq \varphi(x) \ \forall \ y \in K. \tag{1.7}$$

2010 *Mathematics Subject Classification.* 47H04, 47H06, 47H15, 47H17, 47J25.
Key words and phrases. equilibrium problem; strict pseudo-contractive mappings; multivalued mappings; Hausdorff metric.

Let X be a normed space. A subset K of X is called *proximinal* (see [36]) if for each $x \in X$, there exists an element $k \in K$ such that

$$d(x, k) = d(x, K),$$

where $d(x, K) = \inf\{||x - y|| \; \forall \; y \in K\}$ is the distance from the point x to the set K.
Let K be a nonempty closed convex subset of X. We denote by $CB(K)$, the family of nonempty closed, bounded subsets of K, $P(K)$ the family of nonempty proximinal bounded subsets of K. The Hausdorff metric (see [29]) on $CB(X)$ is defined by

$$D(A, B) = max\Big\{ \sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\Big\} \; \forall A, B \in CB(X).$$

Let $T : D(T) \subset X \to CB(X)$. An element $x \in D(T)$ is called a fixed point of T if $x \in Tx$. The set of fixed points of T is denoted by $F(T)$.
For multivalued mappings $T : K \to P(K)$, the best approximation operator, P_T (see [21]) is defined by $P_T(x) := \{y \in T(x) : ||x - y|| = d(x, T(x))\} \; \forall x \in K$.

A multi-valued mapping $T : D(T) \subseteq X \to CB(X)$ is called L-Lipschitzian if there exists $L > 0$ such that

$$D(Tx, Ty) \le L||x - y|| \; \forall \; x, y \in D(T). \tag{1.8}$$

When $L \in (0, 1)$ in (1.8), we say that T is a contraction, and T is called nonexpansive if $L = 1$.

A multi-valued map $T : D(T) \subset H \to CB(H)$ is called k-strictly pseudo-contractive (see [10]) if there exists $k \in (0, 1)$ such that for all $x, y \in D(T)$,

$$(D(Tx, Ty))^2 \le ||x - y||^2 + k||x - y - (u - v)||^2 \; \forall \; u \in Tx, v \in Ty. \tag{1.9}$$

It is well known that the generalized mixed equilibrium problem and indeed equilibrium problem include variational inequality problem, optimization problem, problems of Nash equilibria, saddle point problems, fixed point problems and complementarity problems as special cases.(see [5, 19, 20, 28, 33] and the references therein).
Different iterative algorithms for solving generalized mixed equilibrium problems, mixed equilibrium problems and equilibrium problems have been developed and studied by many authors. See for instance, [6, 7, 12, 14, 17, 26, 34, 37, 39] and the references therein.
For several years, the study of fixed point theory for multi-valued nonlinear mappings has attracted the interest of several well known mathematicians (see, for example, Brouwer [4], Chidume et al. [10], Denavari [13], Kakutani [22], Nash [31, 32] Geanakoplos [18], Nadler [29], Downing and Kirk[15]).
Interest in such studies stems, perhaps, mainly from the usefulness of such fixed point theory in real-world applications, such as in Game Theory, Market Economy, Non-Smooth Differential Equations and so on (see e.g., [10, 16]). Game theory is perhaps the most successful area of application of fixed point theory for multi-valued mappings. However, it has been remarked that the applications of this theory to equilibrium problems in game theory are mostly static in the sense that while they enhance the understanding of conditions under which equilibrium may be achieved, they do not indicate how to construct a process starting from a non-equilibrium point that will converge to an equilibrium solution. Iterative methods for fixed points of multivalued mappings are designed to address this problem. For more detail, see [8, 10, 16].

The classical *Mann iteration process* has been employed successfully to approximate fixed points of nonlinear mappings (single valued or multi valued).
However, it is known to yield only weak convergence even in Hilbert spaces. To overcome this weakness, Takahashi [40], introduced a method known as the *shrinking projection method*, and obtained strong convergence results of the method.

The shrinking projection method has been studied extensively in the literature. See for instance, Tada and Takahashi [39], Aoyama et al. [2], Yao et al. [41], Kang *et. al.* [23], Kimura *et. al.* [25], Cholamjiak and Suantai [42] and the references contained therein.

Motivated by the result of Takahashi [40], Bunyawat and Suantai [6] used the shrinking projection method and defined a hybrid method for Mixed equilibrium problem and fixed point problem for a family of nonexpansive multivalued mappings in real Hilbert spaces. Precisely, they proved the following result:

Theorem S (Bunyawat and Suantai [6]).

Let D be a nonempty closed and convex subset of a real Hilbert space H. Let F be a bifunction from $D\times D$ to $\mathbb{R}$ satisfying $(A1)-(A4)$, and let φ be a proper, lower semi continuous and convex function from D to $\mathbb{R}\cup\{+\infty\}$ such that $D\cap dom\varphi\neq\emptyset$. Let $T_i: D\to P(D)$ be multivalued nonexpansive mappings for all $i\in\mathbb{N}$ with $\Omega:=\bigcap_{i=1}^{\infty}F(T_i)\cap MEP(F,\varphi)\neq\emptyset$ such that all P_{T_i} are nonexpansive. Assume that either $(B1)$ or $(B2)$ holds and $\{\alpha_{n,i}\}\subset(0,1)$ satisfies the condition $\liminf_{n\to\infty}\alpha_{n,i}\alpha_{n,0}>0$ for all $i\in\mathbb{N}$. Define $\{x_n\}$ as follows: $x_1\in D=C_1$,

$$\begin{cases} F(u_n,y)+\varphi(y)-\varphi(u_n)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0\ \forall\ y\in D,\\ y_n=\alpha_{n,0}u_n+\sum_{i=1}^{n}\alpha_{n,i}x_{n,i},\ x_{n,i}\in P_{T_i}(u_n),\\ C_{n+1}=\{z\in C_n:||y_n-z||\leq||x_n-z||\},\\ x_{n+1}=P_{C_{n+1}}x_0,\ n\geq 0.\end{cases} \tag{1.10}$$

where P_{C_n} is the metric projection of H onto C_n. They proved that $\{x_n\}$ converges strongly to $P_\Omega x_0,\ \Omega=\cap_{i=1}^{\infty}F(T_i)\cap MEP(F,\varphi)$.

The class of strictly pseudocontractive mappings was introduced in 1967 by Browder and Petryshyn [3] as a generalization of the class of nonexpansive mappings. This class of operators have been studied by several authors under different assumptions. See for instance, [1, 9, 27, 35, 38, 43] and the references therein.

In 2013, Chidume *et. al.* [10] introduced and studied the class of multivalued strictly pseudocontractive mappings as a generalization of the class of multi valued nonexpansive mappings in real Hilbert spaces.

Recently, Chidume and Ezeora [11] introduced a Krasnoselskii-type sequence and proved that the sequence converges strongly to a common fixed point of a finite family of multivalued strictly pseudo contractive mappings in real Hilbert spaces under some compactness assumption on the operators.

Motivated by the results of Takahashi [40], Bunyawat and Suantai [6], Chidume and Ezeora [11], it is our purpose in this paper to introduce a new hybrid iterative algorithm based on the shrinking projection method and prove that the sequence of the scheme converges strongly to a common element of the set of solution of generalized mixed equilibrium problem and the set of common fixed points of a finite family of multivalued strictly pseudocontractive mappings in real Hilbert spaces. Our result extends that of Bunyawat and Suantai [6] from multivalued nonexpansive mappings to the more general class of multivalued strictly pseudocontractive mappings and many other important results. In proving our result, compactness assumption imposed on the operators by Chidume and Ezeora [11] was dispensed with.

2. PRELIMINARIES

Lemma 2.1. [30] *Let K be a nonempty closed convex subset of a real Hilbert space H and $P_K: H\to K$ be the metric projection from H onto K. Then the following inequality holds:*
$\|y-P_Kx\|^2+\|x-P_Kx\|^2\leq\|x-y\|^2,\forall\ x\in H,y\in K.$

Lemma 2.2. *(see [11]) Let H be a real Hilbert space and $\{x_i\}_{i=1}^m\subset H$. For $\alpha_i\in(0,1),\ i=1,\dots,m$ such that $\sum_{i=1}^m\alpha_i=1$, the following identity holds:*

$$\left|\left|\sum_{i=1}^m\alpha_ix_i\right|\right|^2=\sum_{i=1}^m\alpha_i\,||x_i||^2-\sum_{i,j=1,i\neq j}^m\alpha_i\alpha_j\,||x_i-x_j||^2 \tag{2.1}$$

Lemma 2.3. *(see [10]) Let X be a reflexive real Banach space and let $A,B\in CB(X)$. Assume that B is weakly closed. Then, for every $a\in A$, there exists $b\in B$ such that*

$$||a-b||\leq D(A,B) \tag{2.1}$$

Lemma 2.4. *(see [10]) Let K be a nonempty subset of a real Hilbert space H and let $T : K \to CB(K)$ be a multivalued k-strictly pseudocontractive mapping. Assume that for every $x \in K$, the set Tx is weakly closed. Then, T is Lipschitzian. That is*

$$D(Tx, Ty) \leq L||x - y|| \; \forall \; x, y \in K. \tag{2.2}$$

Lemma 2.5. [24] *Let D be a nonempty closed and convex subset of a real Hilbert space H. Given $x, y, z \in H$ and also given $a \in R$, the set*
$\{v \in D : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v\rangle + a\}$
is convex and closed.

For solving the generalized mixed equilibrium problem, we assume the bifunction F, φ and the set K satisfy the following conditions:
(A1) $F(x, x) = 0 \; for \; all \; x \in K$;
(A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0 \; \forall \; x, y \in K$;
(A3) for each $y \in K, x \mapsto F(x, y)$ is weakly upper semicontinuous
(A4) for each $x \in K, y \mapsto F(x, y)$ is convex and lower semicontinuous;
(B1) for each $x \in H$ and $r > 0$, there exist a bounded subset $K_x \subseteq K$ and $y_x \in K \cap dom\varphi$ such that for any $z \in K \setminus K_x$,
$F(z, y_x) + \varphi(y_x) + \langle Bz, y_x - z\rangle + \dfrac{1}{r}\langle y_x - z, z - x\rangle < \varphi(z)$;
(B2) K is a bounded set.

Lemma 2.6. [26] *Let K be a nonempty closed and convex subset of a real Hilbert space H. Let $F : K \times K \to R$ be a bifunction satisfying conditions (A1)-(A4) and $\varphi : K \to R \cup \{+\infty\}$ be a paper lower semicontinuous and convex function such that $K \cap dom\varphi \neq \emptyset$. For $r > 0$ and $x \in K$, define a mapping $T_r : H \to K$ as follows:*
$T_r(x) = \{z \in K : F(z, y) + \varphi(y) + \langle Bz, y_x - z\rangle + \dfrac{1}{r}\langle y - z, z - x\rangle \geq \varphi(z), \; \forall y \in K\}$
for all $x \in H$. Assume that either (B1)or (B2) holds. Then the following conditions hold:
(1) for all $x \in H, T_r(x) \neq \emptyset$;
(2) T_r is single- valued;
(3) T_r is firmly nonexpansive, that is, for any $x, y \in H$,
$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle$;
(4) $F(T_r(I - rB)) = GMEP(F, \varphi, B)$;
(5) $GMEP(F, \varphi, B)$ is closed and convex.

3. Main Result

Let K be a nonempty closed convex subset of a real Hilbert space H. In this section, we denote by $CB(K)$, the family of nonempty, closed,convex and bounded subsets of K.

Theorem 3.1. *Let K be a nonempty closed convex subset of a real Hilbert space H, F be a bifunction from $K \times K$ to $\mathbb{R}$ satisfying $(A1) - (A4)$, and let φ be a proper lower semicontinuous and convex function from K to $\mathbb{R} \cup \{+\infty\}$ such that $K \cap dom\varphi \neq \emptyset$ and B an α-inverse strongly monotone mapping from K into H. Let $T_i : K \to CB(K)$ be multivalued k_i-strictly pseudo-contractive mappings, $k_i \in (0, 1)$, $i = 1, \dots, m$ with $\Omega := \cap_{i=1}^m F(T_i) \cap GMEP(F, \varphi, B) \neq \emptyset$. Assume that for $p \in \bigcap_{i=1}^m F(T_i)$, $T_i p = \{p\}$ and that either $(B1)$ or $(B2)$ holds with $\{\alpha_{n,i}\} \subset (k, 1)$, $i = 0, 1, \cdots, m$. Define the sequence $\{x_n\}$ as follows: $x_1 \in K = C_1$,*

$$\begin{cases} F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Bx_n, y - u_n\rangle \\ +\dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall \; y \in K, \\ y_n = \alpha_{n,0}u_n + \sum_{i=1}^m \alpha_{n,i}x_n^i, \quad x_n^i \in T_i u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \; n \geq 0, \end{cases} \tag{3.1}$$

where the sequence $r_n \in (0, \infty)$ *with* $\liminf_{n\to\infty} r_n > 0$ *and* $\sum_{i=0}^{m} \alpha_{n,i} = 1$. *Then, the sequence* $\{x_n\}$ *converges strongly to* $P_\Omega x_0$.

Proof. We split the proof into steps.

Step 1. We show that $P_{C_{n+1}}x_0$ is well defined for every $x_0 \in K$.
By Lemma 2.6 and the condition on $CB(K)$, we obtain that GMEP (F, φ, B) and $\cap_{i=1}^{m} F(T_i)$ are closed and convex subsets of K. Hence Ω is a closed and convex subset of K. From Lemma 2.5, we have that C_{n+1} is closed and convex for each $n \geq 0$. Let $p \in \Omega$, then $T_i(p) = \{p\},\ i = 1, 2, \cdots, m$. Since $u_n = T_{r_n}(x_n - r_n x_n)$, we have using Lemma 2.6 that

$$\|u_n - p\| = \|T_{r_n}(x_n - r_n Bx_n) - T_{r_n}(p - r_n Bp)\| \leq \|x_n - p\|,\ \forall\ n \geq 0.$$

Using Lemma 2.2 and Lemma 2.3, we obtain the following estimates:

$$\begin{aligned}
||y_n - p||^2 &= \alpha_{n,0}||u_n - p||^2 + \sum_{i=1}^{m} \alpha_{n,i}||x_n^i - p||^2 \\
&- \sum_{i=1}^{m} \alpha_{n,i}\alpha_{n,0}||u_n - x_n^i||^2 - \sum_{i,j=1, i\neq j}^{m} \alpha_{n,i}\alpha_{n,j}||x_n^i - x_n^j||^2 \\
&\leq \alpha_{n,0}||u_n - p||^2 + \sum_{i=1}^{m} \alpha_{n,i}||x_n^i - p||^2 - \sum_{i=1}^{m} \alpha_{n,i}\alpha_{n,0}||u_n - x_n^i||^2 \\
&\leq \alpha_{n,0}||u_n - p||^2 + \sum_{i=1}^{m} \alpha_{n,i}(D(T_i u_n, T_i p))^2 - \sum_{i=1}^{m} \alpha_{n,i}\alpha_{n,0}||u_n - x_n^i||^2 \\
&\leq \alpha_{n,0}||u_n - p||^2 + \sum_{i=1}^{m} \alpha_{n,i}||u_n - p||^2 + \sum_{i=1}^{m} \alpha_{n,i}k||u_n - x_n^i||^2 \\
&- \sum_{i=1}^{m} \alpha_{n,i}\alpha_{n,0}||u_n - x_n^i||^2 \\
&= ||u_n - p||^2 - \sum_{i=1}^{m} \alpha_{n,i}(\alpha_{n,0} - k)||u_n - x_n^i||^2. \qquad (3.2)
\end{aligned}$$

Since $\alpha_{n,i} \in (k, 1)$, we obtain

$$\begin{aligned}
||y_n - p||^2 &\leq ||u_n - p||^2 \text{ so that} \\
||y_n - p|| &\leq ||u_n - p|| \leq ||x_n - p||.
\end{aligned}$$

This implies that

$$||y_n - p|| \leq ||x_n - p||. \qquad (3.3)$$

Hence $p \in C_{n+1}$, and so $\Omega \subset C_{n+1}$. Therefore, $P_{C_{n+1}}x_0$ is well defined.

Step 2. We show that $\lim_{n\to\infty} \|x_n - x_0\|$ exists.

Since Ω is a nonempty closed convex subset of H, there exists a unique $v \in \Omega$ such that $v = P_\Omega x_0$. Since $x_n = P_{C_n}x_0$ and $x_{n+1} \in C_{n+1} \subset C_n,\ \forall\ n \geq 0$, we have $\|x_n - x_0\| \leq \|x_{n+1} - x_0\|,\ \forall\ n \geq 0$.
On the other hand, since $v \in \Omega \subset C_n$, we obtain

$$\|x_n - x_0\| \leq \|v - x_0\|,\ \forall\ n \geq 0.$$

It follows that the sequence $\{x_n\}$ is bounded and $\{||x_n - x_0||\}$ is non decreasing and bounded. Therefore,

$$\lim_{n\to\infty} \|x_n - x_0\| \quad \text{exists.}$$

Step 3. We show that $\lim_{n\to\infty} x_n$ exists in K.

For $m > n$, by the definition of C_n, we get $x_m = P_{C_m}x_0 \in C_m \subset C_n$. By applying Lemma 2.1, we have

$$\|x_m - x_n\|^2 \le \|x_m - x_0\|^2 - \|x_n - x_0\|^2.$$

Since $\lim_{n\to\infty} \|x_n - x_0\|$ exists, it follows that $\{x_n\}$ is a Cauchy sequence. Hence, there exists $x^* \in K$ such that $\lim_{n\to\infty} x_n = x^*$.

Step 4. We show that $\|x_n^i - x_n\| \to 0$ *as* $n \to \infty$, $i = 1, 2, \cdots, m$.

From $x_{n+1} \in C_{n+1}$, we have

$$\|x_n - y_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \le 2\|x_n - x_{n+1}\| \to 0 \ as\ n \to \infty. \tag{3.4}$$

For $p \in \Omega$, using inequality (3.2), we get

$$\begin{aligned} \|y_n - p\|^2 &\le \alpha_{n,0}||u_n - p||^2 + \sum_{i=1}^m \alpha_{n,i}||u_n - p||^2 + \sum_{i=1}^m \alpha_{n,i}k||u_n - x_n^i||^2 \\ &- \sum_{i=1}^m \alpha_{n,i}\alpha_{n,0}||u_n - x_n^i||^2 \\ &= ||u_n - p||^2 - \sum_{i=1}^m \alpha_{n,i}(\alpha_{n,0} - k)||u_n - x_n^i||^2 \\ &\le ||x_n - p||^2 - \sum_{i=1}^m \alpha_{n,i}(\alpha_{n,0} - k)||u_n - x_n^i||^2. \end{aligned}$$

Thus,

$$\begin{aligned} \alpha_{n,i}\alpha_{n,0}\|x_n^i - u_n\|^2 &\le \sum_{i=1}^m \alpha_{n,i}\alpha_{n,0}\|x_n^i - u_n\|^2 \\ &\le \|x_n - p\|^2 - \|y_n - p\|^2 \\ &\le M\|x_n - y_n\|, \end{aligned} \tag{3.5}$$

where $M = \sup_{n\ge 0}\{\|x_n - p\| + \|y_n - p\|\}$. By the given condition on $\{\alpha_{n,i}\}$ and (3.5), we get

$$\lim_{n\to\infty} \|x_n^i - u_n\| = 0,\ i = 1, 2, \cdots, m.$$

By Lemma 2.6, we have

$$\begin{aligned} \|u_n - p\|^2 &= \|T_{r_n}(x_n - r_nBx_n) - T_{r_n}(p - r_nBp)\|^2 \le \langle T_{r_n}(x_n - r_nBx_n) - T_{r_n}(p - r_nBp), x_n - p\rangle \\ &= \langle u_n - p, x_n - p\rangle \\ &= \frac{1}{2}\{||u_n - p||^2 + ||x_n - p||^2 - ||x_n - u_n||^2\}. \text{ Hence,} \\ \|u_n - p\|^2 &\le \|x_n - p\|^2 - \|x_n - u_n\|^2. \end{aligned}$$

From inequality (3.2), we get

$$\begin{aligned} \|y_n - p\|^2 &\le \|x_n - p\|^2 - \|x_n - u_n\|^2. \\ \Rightarrow \|x_n - u_n\|^2 &\le \|x_n - p\|^2 - \|y_n - p\|^2 \\ &\le M\|x_n - y_n\|, \text{ where } M = \sup_{n\ge 0}\{\|x_n - p\| + \|y_n - p\|\}. \end{aligned}$$

Applying (3.4), we have

$$\|x_n - u_n\| \to 0, n \to \infty.$$

Hence,

$$\|x_n^i - x_n\| \le \|x_n^i - u_n\| + \|u_n - x_n\| \to 0 \text{ as } n \to \infty,\ i = 1, 2, \cdots, m.$$

Step 5. We show that $x^* \in \Omega$.

Using the assumption that $\liminf_{n\to\infty} r_n > 0$, we have

$$||\frac{x_n - u_n}{r_n}|| = \frac{1}{r_n}\|x_n - u_n\| \to 0, n \to \infty. \tag{3.6}$$

So, since $\lim_{n\to\infty} x_n = x^*$, we obtain

$$\lim_{n\to\infty} u_n = x^*.$$

First, we show that $x^* \in GMEP(F, \varphi, B)$. This proof follows as in the proof of Theorem 3.1 of [26], we omit the proof. Hence, $x^* \in GMEP(F, \varphi, B)$.

Next, we have to show that $x^* \in \cap_{i=1}^m F(T_i)$. For each $i = 1, 2, ..., m$, using Lemma 2.4, we have

$$\begin{aligned} d(x^*, T_i x^*) &\le d(x^*, x_n) + d(x_n, x_n^i) + d(x_n^i, T_i x^*) \\ &\le d(x^*, x_n) + d(x_n, x_n^i) + D(T_i u_n, T_i x^*) \\ &\le d(x^*, x_n) + d(x_n, x_n^i) + L_i||u_n - x^*|| \\ &\le d(x^*, x_n) + d(x_n, x_n^i) + L||u_n - x^*||, \end{aligned}$$

where $L = \max_{1\le i\le m}\{L_i\}$.

Applying Step 3-4, we have $d(x^*, T_i x^*) = 0$. Hence $x^* \in T_i x^*$ for all $i = 1, 2, ..., m$. That is, $x^* \in \cap_{i=1}^m F(T_i)$.

Step 6. We show that $x^* = P_\Omega x_0$.

Since $x_n = P_{C_n} x_0$, we get $\langle \xi - x_n, x_0 - x_n\rangle \le 0, \forall\ \xi \in C_n$. Since $x^* \in \Omega \subset C_n$, we have

$$\langle \xi - x^*, x_0 - x^*\rangle \le 0, \forall\ \xi \in \Omega.$$

Thus, $x^* = P_\Omega x_0$, completing the proof. □

If for each $i = 1, 2, \cdots, m$, $T_i : K \to CB(K)$ is multivalued nonexpansive mappings, then we have the following result.

Corollary 3.2. *Let K be a nonempty closed and convex subset of a real Hilbert space H, F be a bi-function from $K \times K$ to $\mathbb{R}$ satisfying $(A1) - (A4)$, and let φ be a proper lower semicontinuous and convex function from K to $\mathbb{R} \cup \{+\infty\}$ such that $K \cap dom\varphi \ne \emptyset$. Let $T_i : K \to CB(K)$ be multivalued nonexpansive mappings with $\Omega := \cap_{i=1}^m F(T_i) \cap GMEP(F, \varphi, B) \ne \emptyset$. Assume that for $p \in \bigcap_{i=1}^m F(T_i)$, $T_i p = \{p\}$ and that either $(B1)$ or $(B2)$ holds with $\{\alpha_{n,i}\} \subset [0, 1)$ satisfying the condition $\lim\inf_{n\to\infty} \alpha_{n,i}\alpha_{n,0} > 0\ \forall\ i = 1, 2, \cdots, m$. Define the sequence $\{x_n\}$ as follows: $x_1 \in K = C_1$,*

$$\begin{cases} \{F(u_n, y) + \varphi(y) - \varphi(u_n) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \forall\ y \in K, \\ y_n = \alpha_{n,0}u_n + \sum_{i=1}^m \alpha_{n,i}x_n^i, \quad x_n^i \in T_i u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0,\ n \ge 0, \end{cases} \tag{3.7}$$

where the sequence $r_n \in (0, \infty)$ with $\liminf_{n\to\infty} r_n > 0$ and $\sum_{i=0}^m \alpha_{n,i} = 1$. Then the sequence $\{x_n\}$ converges strongly to $P_\Omega x_0$.

Setting $\varphi \equiv 0$ in Theorem 3.1, we have the following result.

Corollary 3.3. *Let K be a nonempty closed and convex subset of a real Hilbert space H. Let F be a bifunction from $K \times K \to \mathbb{R}$ satisfying $(A1) - (A4)$. Let $T_i : K \to CB(K)$ be multivalued k_i-strictly pseudo-contractive mappings, $k_i \in (0, 1)$, $i = 1, \ldots, m$ with $\Omega := \cap_{i=1}^m F(T_i) \cap EP(F, \varphi) \ne \emptyset$. Assume* *that for* $p \in \bigcap_{i=1}^m F(T_i),\ T_i p = \{p\}$ *and that either* $(B1)$ *or* $(B2)$ *holds with* $\{\alpha_{n,i}\} \subset [0,1)$ *satisfying the condition* $\lim\inf_{n\to\infty} \alpha_{n,i}\alpha_{n,0} > 0\ \forall\ i = 1, 2, \cdots, m$. *Define the sequence* $\{x_n\}$ *as follows:* $x_1 \in K = C_1$,

$$\left\{\begin{array}{l} \{F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall\ y \in K, \\ y_n = \alpha_{n,0}u_n + \sum\limits_{i=1}^{m}\alpha_{n,i}x_n^i, \quad x_n^i \in T_i u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0,\ n \geq 0, \end{array}\right. \tag{3.8}$$

where the sequence $r_n \in (0, \infty)$ *with* $\liminf_{n\to\infty} r_n > 0$ *and* $\sum_{i=0}^m \alpha_{n,i} = 1$. *Then the sequence* $\{x_n\}$ *converges strongly to* $P_\Omega x_0$.

Setting $F \equiv 0$ in Theorem 3.1, we have the following result.

Corollary 3.4. *Let* K *be a nonempty closed and convex subset of a real Hilbert space* H. *Let* F *be a bifunction from* $K \times K \to \mathbb{R}$ *satisfying* $(A1) - (A4)$. *Let* $T_i : K \to CB(K)$ *be multivalued* k_i*-strictly pseudo-contractive mappings,* $k_i \in (0,1),\ i = 1, \ldots, m$ *with* $\Omega := \cap_{i=1}^m F(T_i) \cap CMP(\varphi) \neq \emptyset$. *Assume that for* $p \in \bigcap_{i=1}^m F(T_i),\ T_i p = \{p\}$ *and that either* $(B1)$ *or* $(B2)$ *holds with* $\{\alpha_{n,i}\} \subset (k, 1)$. *Define the sequence* $\{x_n\}$ *as follows:* $x_1 \in K = C_1$,

$$\left\{\begin{array}{l} \{\varphi(y) - \varphi(u_n) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall\ y \in K, \\ y_n = \alpha_{n,0}u_n + \sum\limits_{i=1}^{m}\alpha_{n,i}x_n^i, \quad x_n^i \in T_i u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0,\ n \geq 0, \end{array}\right. \tag{3.9}$$

where the sequence $r_n \in (0, \infty)$ *with* $\liminf_{n\to\infty} r_n > 0$ *and* $\sum_{i=0}^m \alpha_{n,i} = 1$. *Then the sequence* $\{x_n\}$ *converges strongly to* $P_\Omega x_0$.

Setting $F \equiv 0, \varphi \equiv 0$ in Theorem 3.1, we have the following result.

Corollary 3.5. *Let* K *be a nonempty closed and convex subset of a real Hilbert space* H. *Let* F *be a bifunction from* $K \times K \to \mathbb{R}$ *satisfying* $(A1) - (A4)$. *Let* $T_i : K \to CB(K)$ *be multi valued* k_i*-strictly pseudo-contractive mappings,* $k_i \in (0,1),\ i = 1, \cdots, m$ *with* $\Omega := \cap_{i=1}^m F(T_i) \neq \emptyset$. *Assume that for* $p \in \bigcap_{i=1}^m F(T_i),\ T_i p = \{p\}$ *and that* $\{\alpha_{n,i}\} \subset (k, 1)$. *Define the sequence* $\{x_n\}$ *as follows:* $x_1 \in K = C_1$,

$$\left\{\begin{array}{l} \{y_n = \alpha_{n,0}u_n + \sum\limits_{i=1}^{m}\alpha_{n,i}x_n^i, \quad x_n^i \in T_i u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0,\ n \geq 0, \end{array}\right. \tag{3.10}$$

where the sequence $\sum_{i=0}^m \alpha_{n,i} = 1$. *Then the sequence* $\{x_n\}$ *converges strongly to* $P_\Omega x_0$.

Remark 3.6. 1. Theorem 3.1 is a significant improvement on Theorem 3.1 of [11] for the following reasons;
(a) it solves two major problems, generalized mixed equilibrium problem and common fixed point problem.
(b) To prove Theorem 3.1 of [11], compactness assumptions were placed on the operators $T_i,\ i = 1, 2, \cdots, N$, this is dispensed with in the proof of Theorem 3.1of this paper.

2. Theorem 3.1 of this paper generalizes Theorem 3.1 of [6] from multivalued nonexapnsive mappings to multivalued strictly pseudo contractive mapping and from mixed equilibrium problem to generalized mixed equilibrium problem.

3. Corollary 3.4 solves the convex minimization problem (1.7).

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