International Journal of Analysis and Applications
ISSN 2291-8639
Volume 10, Number 2 (2016), 77-84
http://www.etamaths.com

CHARACTERIZATIONS OF p-WAVELETS ON POSITIVE HALF LINE USING THE WALSH-FOURIER TRANSFORM

ABDULLAH*

Abstract. In this paper, we study the characterization of wavelets on positive half line by means of two basic equations in the Fourier domain. We also give another characterization of wavelets.

1. Introduction

The characterization of wavelets of $L^2(\mathbb{R})$ was obtained by Gripenberg [7] in terms of two basic equations involving the Fourier transform of the wavelets (see also [8]). This result was generalized by Calogero [3] for wavelets associated with a general dilation matrix. Bownik [2] provided a new approach to characterizing multiwavelets in $L^2(\mathbb{R}^n)$. This characterization was obtained by using the result about shift invariant systems and quasi-affine systems in [4] and [9].

Farkov [5] has given general construction of compactly supported orthogonal p-wavelets in $L^2(\mathbb{R}^+)$. Farkov et al. [6] gave an algorithm for biorthogonal wavelets related to Walsh functions on positive half line. On the other hand, Shah and Debnath [10] have constructed dyadic wavelet frames on the positive half-line $\mathbb{R}^+$ using the Walsh-Fourier transform and have established a necessary condition and a sufficient condition for the system $\{\psi_{j,k}(x) = 2^{j/2}\psi(2^j x \ominus k) : j \in \mathbb{Z}, k \in \mathbb{Z}^+\}$ to be a frame for $L^2(\mathbb{R}^+)$. Further, A constructive procedure for constructing tight wavelet frames on positive half-line using extension principles was recently considered by Shah in [11], in which he has pointed out a method for constructing affine frames in $L^2(\mathbb{R}^+)$. Moreover, the author has established sufficient conditions for a finite number of functions to form a tight affine frames for $L^2(\mathbb{R}^+)$.

In the present paper, we study characterization of wavelet on positive half line by using the results on affine and quasi-affine frames on positive half-line. The paper is structured as follows. In Section 2, we introduce some notations and preliminaries related to the operations on positive half-line $\mathbb{R}^+$ including the definition of the Walsh-Fourier transform. In section 3, some results on the affine and quasi-affine systems on positive half-line is given and use them to provide a characterization of wavelets.

2. Notations and preliminaries on Walsh-Fourier Analysis

Let p be a fixed natural number greater than 1. As usual, let $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{Z}^+ = \{0, 1, ...\}$. Denote by $[x]$ the integer part of x. For $x \in \mathbb{R}^+$ and for any positive integer j, we set

$$x_j = [p^j x](\mathrm{mod}\, p),\ x_{-j} = [p^{1-j} x](\mathrm{mod}\, p), \tag{2.1}$$

where $x_j,\ x_{-j} \in \{0, 1, ..., p-1\}$.

Consider the addition defined on $\mathbb{R}^+$ as follows:

$$x \oplus y = \sum_{j<0} \xi_j p^{-j-1} + \sum_{j>0} \xi_j p^{-j} \tag{2.2}$$

2010 *Mathematics Subject Classification.* 42C15, 40A30.
Key words and phrases. wavelets; affine frame; quasi-affine frame; Wash-Fourier transform.

with

$$\xi_j = x_j + y_j (\bmod p), \ j \in \mathbb{Z}\backslash\{0\}, \tag{2.3}$$

where $\xi_j \in \{0, 1, 2, ..., p-1\}$ and $x_j, \ y_j$ are calculated by (2.1). Moreover, we write $z = x \ominus y$ if $z \oplus y = x$.

For $x \in [0, 1)$, let $r_0(x)$ be given by

$$r_0(x) = \begin{cases} 1, \ x \in \left[0, \dfrac{1}{p}\right), \\ \\ \varepsilon_p^j, \ x \in \left[jp^{-1}, (j+1)p^{-1}\right), \ j = 1, 2, ..., p-1, \end{cases} \tag{2.4}$$

where $\varepsilon_p = \exp\left(\frac{2\pi i}{p}\right)$. The extension of the function r_0 to $\mathbb{R}^+$ is defined by the equality $r_0(x+1) = r_0(x), \ x \in \mathbb{R}^+$. Then the generalized Walsh functions $\{\omega_m(x)\}_{m \in \mathbb{Z}^+}$ are defined by

$$\omega_0(x) = 1, \ \omega_m(x) = \prod_{j=0}^{k} \left(r_0\left(p^j x\right)\right)^{\mu_j},$$

where $m = \sum_{j=0}^{k} \mu_j p^j, \ \mu_j \in \{0, 1, 2, ..., p-1\}, \ \mu_k \neq 0$.

For $x, \ \omega \in \mathbb{R}^+$, let

$$\chi(x, \omega) = \exp\left(\frac{2\pi i}{p} \sum_{j=1}^{\infty} (x_j \omega_{-j} + x_{-j} \omega_j)\right), \tag{2.5}$$

where x_j and ω_j are calculated by (2.1).

We observe that

$$\chi\left(x, \ \frac{m}{p^{n-1}}\right) = \chi\left(\frac{x}{p^{n-1}}, \ m\right) = \omega_m\left(\frac{x}{p^{n-1}}\right) \ \forall\, x \in [0, \ p^{n-1}), \ m \in \mathbb{Z}^+.$$

The Walsh-Fourier transform of a function $f \in L^1(\mathbb{R}^+)$ is defined by

$$\tilde{f}(\omega) = \int_{\mathbb{R}^+} f(x) \overline{\chi(x, \omega)} dx, \tag{2.6}$$

where $\chi(x, \omega)$ is given by (2.5).

If $f \in L^2(\mathbb{R}^+)$ and

$$J_a f(\omega) = \int_0^a f(x) \overline{\chi(x, \ \omega)} dx \ (a < 0), \tag{2.7}$$

then $\tilde{f}$ is defined as limit of $J_a f$ in $L^2(\mathbb{R}^+)$ as $a \to \infty$.

The properties of Walsh-Fourier transform are quite similar to the classical Fourier transform. It is known that systems $\{\chi(\alpha, .)\}_{\alpha=0}^{\infty}$ and $\{\chi(., \alpha)\}_{\alpha=0}^{\infty}$ are orthonormal bases in $L^2(0, 1)$. Let us denote by $\{\omega\}$ the fractional part of ω. For $l \in \mathbb{Z}^+$, we have $\chi(l, \omega) = \chi(l, \{\omega\})$.

If $x, y, \omega \in \mathbb{R}^+$ and $x \oplus y$ is p-adic irrational, then

$$\chi(x \oplus y, \omega) = \chi(x, \omega)\chi(y, \omega), \ \chi(x \ominus y, \omega) = \chi(x, \omega)\overline{\chi(y, \omega)}, \tag{2.8}$$

3. Characterization of p-Wavelets

Definition 3.1. Let $\Psi = \{\psi^1, \psi^2, ..., \psi^L\}$ be a finite family of functions in $L^2(\mathbb{R}^+)$. The affine system generated by Ψ is the collection

$$X(\Psi) = \{\psi^l_{j,k} : 1 \le l \le L, j \in \mathbb{Z}, k \in \mathbb{Z}^+\}$$

where $\psi^l_{j,k}(x) = p^{j/2}\psi^l(p^j x \ominus k)$. The quasi-affine system generated by Ψ is

$$\tilde{X}(\Psi) = \{\tilde{\psi}^l_{j,k} : 1 \le l \le L, j \in \mathbb{Z}, k \in \mathbb{Z}^+\},$$

where

$$\tilde{\psi}^l_{j,k}(x) = \begin{cases} p^{j/2}\psi^l(p^j x \ominus k), \ j \ge 0, \ k \in \mathbb{Z}^+ \\ p^j \psi^l(p^j(x \ominus k)), \ j < 0, \ k \in \mathbb{Z}^+. \end{cases} \tag{3.1}$$

We say that Ψ is a set of basic wavelets of $L^2(\mathbb{R}^+)$ if the affine system $X(\Psi)$ forms an orthonormal basis for $L^2(\mathbb{R}^+)$.

Definition 3.2. $X \subset L^2(\mathbb{R}^+)$ is a Bessel family if there exists $b > 0$ so that

$$\sum_{\eta \in X} | < f, \eta > |^2 \le b\|f\|^2 \text{ for } f \in L^2(\mathbb{R}^+). \tag{3.2}$$

If, in addition, there exists a constant $a > 0$, $a \le b$ such that

$$a\|f\|^2 \le \sum_{\eta \in X} | < f, \eta > |^2 \le b\|f\|^2 \text{ for all } f \in L^2(\mathbb{R}^+), \tag{3.3}$$

then X is called a frame. The frame is tight if we can choose a and b such that $a = b$. The (quasi) affine system $X(\Psi)$ (resp. $X^q(\Psi)$) is a (quasi) affine frame if (3.3) holds for $X = X(\Psi)$ ($X = X^q(\Psi)$).

In [9], Chui, Shi and Stöckler have obsereved the relationship between affine and quasi-affine frame in $\mathbb{R}^n$. In [1], we have extended their result to positive half line.

Theorem 3.3. Let Ψ be a finite subset of $L^2(\mathbb{R}^+)$. Then

(a) $X(\Psi)$ is a Bessel family if and only if $\tilde{X}(\Psi)$ is a Bessel family. Furthermore, their exact upper bounds are equal.

(b) $X(\Psi)$ is an affine frame if and only if $\tilde{X}(\Psi)$ is a quasi-affine frame. Furthermore, their lower and upper exact bounds are equal.

Definition 3.4. Given $\{t_i : i \in \mathbb{N}\} \subset l^2(\mathbb{Z}^+)$, define the operator $H : l^2(\mathbb{Z}^+) \to l^2(\mathbb{N})$ by

$$H(v) = (< v, t_i >)_{i \in \mathbb{N}}.$$

If H is bounded then $\tilde{G} = H * H : l^2(\mathbb{Z}^+) \to l^2(\mathbb{N})$ is called the dual Gramian of $\{t_i : i \in \mathbb{N}\}$.

Observe that $\tilde{G}$ is a non negative definite operator on $l^2(\mathbb{Z}^+)$. Also, note that for $r, s \in \mathbb{Z}^+$, we have

$$< \tilde{G}e_r, e_s > = < He_r, He_s > = \sum_{i \in \mathbb{N}} \overline{t_i(r)} t_i(s),$$

where $\{e_i : i \in \mathbb{Z}^+\}$ is the standard basis of $l^2(\mathbb{Z}^+)$.

The following result characterizes when the system of translates of a given family of functions is a frame in terms of the dual Gramian.

Theorem 3.5. Let $\{\varphi_i : i \in \mathbb{N}\} \subset L^2(\mathbb{R}^+)$. Then for a.e. $\xi \in [0, 1)$, let $\tilde{G}(\xi)$ denote the dual Gramian of $\{t_i = (\hat{\varphi}_i(\xi \oplus k))_{k \in \mathbb{Z}^+} : i \in \mathbb{N}\} \subset l^2(\mathbb{Z}^+)$. The system of translates $\{T_k\varphi_i : k \in \mathbb{Z}^+, i \in \mathbb{N}\}$ is a frame for $L^2(\mathbb{R}^+)$ with constants a, b if and only if $\tilde{G}(\xi)$ is bounded for a.e. $\xi \in [0, 1/2)$ and

$$A\|v\|^2 \le \langle \tilde{G}(\xi)v, v\rangle \le B\|v\|^2 \text{ for } v \in l^2(\mathbb{Z}^+),\ \text{a.e. } \xi \in [0, 1/2),$$

that is, the spectrum of $\tilde{G}(\xi)$ is contained in [a,b] for a.e. $\xi \in [0, 1/2)$

We first prove a lemma which gives necessary and sufficient conditions for the orthonormality of an affine system.

Lemma 3.6. Suppose $\Psi = \{\psi^1, \psi^2, ..., \psi^L\} \subset L^2(\mathbb{R}^+)$. The affine system $X(\Psi)$ is orthonormal in $L^2(\mathbb{R}^+)$ if and only if

$$\sum_{k\in\mathbb{Z}^+} \hat{\psi}^l(\xi \oplus k)\overline{\hat{\psi}^m(p^j(\xi \oplus k))} = \delta_{j,0}\delta_{l,m} \text{ for a.e. } \xi \in \mathbb{R}^+,\ 1 \le l, m \le L,\ j \ge 0. \tag{3.4}$$

Proof. By a simple change of variables

$$\langle \psi^l_{j,k}, \psi^{l'}_{j',k'}\rangle = \delta_{l,l'}\delta_{j,j'}\delta_{k,k'},\ j, j' \in \mathbb{Z},\ k, k' \in \mathbb{Z}^+,\ 1 \le l, l' \le L$$

is equivalent to

$$\langle \psi^l_{j,k}, \psi^{l'}_{0,0}\rangle = \delta_{l,l'}\delta_{j,0}\delta_{k,0},\ j \ge 0,\ k \in \mathbb{Z}^+,\ 1 \le l, l' \le L.$$

Now, let $1 \le l, l' \le L,\ j \ge 0,\ k \in \mathbb{Z}^+$. Then

$$\begin{aligned}
&\delta_{l,l'}\delta_{j,0}\delta_{k,0} = \langle \hat{\psi}^l_{j,k}, \hat{\psi}^{l'}_{0,0}\rangle \\
&= \int_{\mathbb{R}^+} p^{-j/2}\hat{\psi}^l(p^{-j}\xi)\overline{\chi(k, p^{-j}\xi)}\overline{\hat{\psi}^{l'}(\xi)}d\xi \\
&= \int_{\mathbb{R}^+} p^{j/2}\hat{\psi}^l(\xi)\overline{\chi(k, \xi)}\overline{\hat{\psi}^{l'}(p^j\xi)}d\xi \\
&= \sum_{n\in\mathbb{Z}^+} p^{j/2}\int_{n+[0,1/2)} \hat{\psi}^l(\xi)\overline{\hat{\psi}^{l'}(p^j\xi)}\overline{\chi(k, \xi)}d\xi \\
&= p^{j/2}\int_{[0,1/2)} \left[\sum_{n\in\mathbb{Z}^+} \hat{\psi}^l(\xi \oplus n)\overline{\hat{\psi}^{l'}(p^j(\xi \oplus n))}\right]\overline{\chi(k, \xi)}d\xi \\
&= p^{j/2}\int_{[0,1/2)} K(\xi)\overline{\chi(k, \xi)}d\xi,
\end{aligned}$$

where $K(\xi) = \left[\sum_{n\in\mathbb{Z}^+} \hat{\psi}^l(\xi \oplus n)\overline{\hat{\psi}^{l'}(p^j(\xi \oplus n))}\right]$. The interchange of summation and integration is justified by

$$\begin{aligned}
&\int_{[0,1/2)} \sum_{n\in\mathbb{Z}^+} \left|\hat{\psi}^l(\xi \oplus n)\overline{\hat{\psi}^{l'}(p^j(\xi \oplus n))}\right| d\xi = \int_{\mathbb{R}^+} \left|\hat{\psi}^l(\xi)\right|\left|\hat{\psi}^{l'}(p^j\xi)\right| d\xi \\
&\le p^{-j/2}\|\psi^l\|^2\|\psi^{l'}\|^2 < \infty.
\end{aligned}$$

The above computation shows that all Fourier coefficients of $K(\xi) \in L^1([0, 1/2))$ are zero except for the coefficient corresponding to $k = 0$ which is 1 if $j = 0$ and $l = l'$. Therefore, $K(\xi) = \delta_{j,0}\delta_{l,l'}$ for a.e. $\xi \in [0, 1/2)$.

Suppose we have $\Psi = \{\psi^1, ..., \psi^L\} \subset L^2(\mathbb{R}^+)$. Define $\mathcal{D}_j$ as follows:

$$\mathcal{D}_j = \begin{cases} \{0,1,...,p^j-1\}, \ j \geq 0, \\ 0, \ j < 0. \end{cases}$$

Since the quasi affine system $X^q(\Psi)$ is invariant under shifts by $k \in \mathbb{Z}^+$, we have

$$X^q(\Psi) = \{T_k\varphi : k \in \mathbb{Z}^+, \ \varphi \in \mathcal{A}\}, \ \mathcal{A} = \{\tilde{\psi}^l_{j,d} : j \in \mathbb{Z}, \ d \in \mathcal{D}_j, l = 1, ..., L\}.$$

The dual Gramian $\tilde{G}(\xi)$ of the quasi affine system $X^q(\Psi)$ at $\xi \in [0,1/2)$ is defined as the dual Gramian of $\{(\hat{\varphi}(\xi \oplus k))_{k\in\mathbb{Z}^+} : \varphi \in \mathcal{A}\} \subset l^2(\mathbb{Z}^+)$.

For $s \in \mathbb{Z}^+ \backslash p\mathbb{Z}^+$, define the function

$$t_s(\xi) = \sum_{l=1}^{L}\sum_{j=0}^{\infty} \hat{\psi}^l(p^j\xi)\overline{\hat{\psi}^l(p^j(\xi \oplus s))}.$$

In the following lemma we compute the dual Gramian $\hat{G}(\xi)$ of the quasi-affine system $X^q(\Psi)$ at $\xi \in [0,1/2)$ in terms of the Fourier transforms of the functions in Ψ.

Lemma 3.7. Let $\Psi = \{\psi^1, \psi^2, ..., \psi^L\} \subseteq L^2(\mathbb{R}^+)$ and $\tilde{G}$ be the dual Gramian of $X^q(\Psi)$ at $\xi \in [0,1/2)$. Then

$$< \tilde{G}(\xi)e_k, e_k >= \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}} |\hat{\psi}^l(p^j(\xi \oplus k))|^2 \text{ for } k \in \mathbb{Z}^+ \tag{3.7}$$

and

$$< \tilde{G}(\xi)e_k, e_{k'} >= t_{p^{-m}(k'\ominus k)}(p^{-m}\xi \oplus p^{-m}\xi) \text{ for } k \neq k' \in \mathbb{Z}^+, \tag{3.8}$$

where $m = \max\{j \geq 0 : p^{-j}(k' \ominus k) \in \mathbb{Z}^+\}$, and functions $t_s, s \in \mathbb{Z}^+ \backslash p\mathbb{Z}^+$ are given by (3.6).

Proof. For $k, k' \in \mathbb{Z}^+$, we have

$$< \tilde{G}(\xi)e_k, e_{k'} >= \sum_{\varphi\in\mathcal{A}} \hat{\varphi}(\xi \oplus k)\overline{\hat{\varphi}(\xi \oplus k')}$$

$$= \sum_{l=1}^{L}\sum_{j<0} \hat{\psi}^l(p^{-j}(\xi \oplus k))\overline{\hat{\psi}^l(p^{-j}(\xi \oplus k'))}$$

$$+ \sum_{l=1}^{L}\sum_{j\geq 0} \hat{\psi}^l(p^{-j}(\xi \oplus k))\overline{\hat{\psi}^l(p^{-j}(\xi \oplus k'))} \left[\sum_{d\in D_j} p^{-j}\chi(k, p^j d)\overline{\chi(k', p^j d)}\right].$$

The expression in the bracket is equal to

$$\sum_{d\in D_j} p^{-j}\chi(k, p^j d)\overline{\chi(k', p^j d)} = \sum_{d\in D_j} p^{-j}\chi((k \ominus k'), p^j d)$$

$$= \begin{cases} 1 \text{ if } k \ominus k' \in p^j\mathbb{Z}^+ \\ 0 \text{ otherwise.} \end{cases}$$

Therefore, if $k = k'$, then

$$< \tilde{G}(\xi)e_k, e_k >= \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}} |\hat{\psi}^l(p^j(\xi \oplus k))|^2$$

If $k \neq k'$, let $m = \max\{j \geq 0 : k \ominus k' \in p^j\mathbb{Z}^+\}$. Then

$$< \tilde{G}(\xi)e_k, e_{k'} >= \sum_{l=1}^{L}\sum_{j=-\infty}^{m} \hat{\psi}^l(p^{-j}(\xi \oplus k))\overline{\hat{\psi}^l(p^{-j}(\xi \oplus k'))}$$

$$= \sum_{l=1}^{L}\sum_{j=-m}^{\infty} \hat{\psi}^l(p^{j}(\xi \oplus k))\overline{\hat{\psi}^l(p^{j}(\xi \oplus k'))}$$

$$= \sum_{l=1}^{L}\sum_{j\geq 0} \hat{\psi}^l(p^{j-m}(\xi \oplus k))\overline{\hat{\psi}^l(p^{j-m}(\xi \oplus k'))}$$

$$= \sum_{l=1}^{L}\sum_{j\geq 0} \hat{\psi}^l(p^{j}(p^{-m}\xi \oplus p^{-m}k))\overline{\hat{\psi}^l(p^{j}(p^{-m}\xi \oplus p^{-m}k \oplus p^{-m}(k' \ominus k)))}$$

$$= t_{p^{-m}(k'\ominus k)}(p^{-m}\xi \oplus p^{-m}k)).$$

In the following theorem, we provide a characterization of wavelets in terms of two basic equations.

Theorem 3.8. Suppose $\Psi = \{\psi^1, \psi^2, ..., \psi^L\} \subset L^2(\mathbb{R}^+)$. The affine system $X(\Psi)$ is a tight frame with constant 1 for $L^2(\mathbb{R}^+)$, i.e.,

$$\|f\|_2^2 = \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^+} | < f, \psi_{l,j,k} > |^2 \text{ for all } f \in L^2(\mathbb{R}^+)$$

if and only if

$$\sum_{l=1}^{L}\sum_{j\in\mathbb{Z}} |\hat{\psi}^l(p^j\xi)|^2 = 1 \text{ for a. e. } \xi \in \mathbb{R}^+, \tag{3.9}$$

and

$$t_s(\xi) = 0 \text{ for a. e. } \xi \in \mathbb{R}^+ \text{ and for all } s \in \mathbb{Z}^+ \backslash p\mathbb{Z}^+. \tag{3.10}$$

In particular, Ψ is a set of basic wavelets of $L^2(\mathbb{R}^+)$ if and only if $\|\psi^l\|_2 = 1$ for $l = 1, 2, ..., L$ and (3.9) and (3.10)hold.

Proof. By Theorem 3.3, $X(\Psi)$ is a tight frame with constant 1 if and only if $X^q(\Psi)$ is a tight frame with constant 1. By Theorem 3.5, this is equivalent to the spectrum $\tilde{G}(\xi)$ consisting of single point 1 i.e. $\tilde{G}(\xi)$ is identity on $l^2(\mathbb{Z}^+)$ for a.e. $\xi \in [0, 1/2)$. By Lemma 3.7, this is equivalent to (3.9) and (3.10). By Theorem 1.8, section 7.1 in [8], a tight frame $X(\Psi)$ is an orthonormal basis if and only if $\|\psi^l\|_2 = 1$ for $l = 1, 2, ..., L$.

Theorem 3.9. Suppose $\Psi = \{\psi^1, \psi^2, ..., \psi^L\} \subseteq L^2(\mathbb{R}^+)$. Then the following are equivalent:
(i) $X(\Psi)$ is a tight frame with constant 1.
(ii) Ψ satisfies (3.9)
(iii) Ψ satisfies

$$\sum_{l=1}^{L}\int_{\mathbb{R}^+} |\hat{\psi}^l(\xi)|^2 \frac{d\xi}{|\xi|} = \int_D \frac{d\xi}{|\xi|} \tag{3.11}$$

where $D \subset \mathbb{R}^+$ is such that $\{p^jD : j \in \mathbb{Z}\}$ is a partition of $\mathbb{R}^+$.

Proof. It is obvious from Theorem 3.8 that (i) $\Rightarrow$ (ii). To show (ii) implies (iii), assume that (3.9) holds, then

$$\sum_{l=1}^{L}\int_{\mathbb{R}^+}|\hat{\psi}^l(\xi)|^2\frac{d\xi}{|\xi|}=\sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}\int_{p^jD}|\hat{\psi}^l(\xi)|^2\frac{d\xi}{|\xi|}$$

$$=\sum_{l=1}^{L}\int_{D}\sum_{j\in\mathbb{Z}}|\hat{\psi}^l(p^j\xi)|^2\frac{d\xi}{|\xi|}$$

$$=\int_{D}\frac{d\xi}{|\xi|}$$

To prove (iii) $\Rightarrow$ (i), we assume that (3.11) holds. Since $X(\Psi)$ is a Bessel family with costant 1, then $X^q(\Psi)$ is also a Bessel family with constant 1 by Theorem 3.3 (a). Let $\tilde{G}(\xi)$ be the dual Gramian of $X^q(\Psi)$ at $\xi\in\left[0,\frac{1}{2}\right)$. By Theorem 3.5, we have $\|\tilde{G}(\xi)\|\leq 1$ for a.e. $\xi\in\left[0,\frac{1}{2}\right)$. In particular, $\|\tilde{G}(\xi)e_k\|\leq 1$. Hence,

$$1\geq\|\tilde{G}(\xi)e_k\|^2=\sum_{k'\in\mathbb{Z}^+}|<\tilde{G}(\xi)e_k,e_{k'}>|^2$$

$$=|<\tilde{G}(\xi)e_k,e_k>|^2+\sum_{k'\in\mathbb{Z}^+,k\neq k'}|<\tilde{G}(\xi)e_k,e_{k'}>|^2. \quad (3.12)$$

By lemma 3.7, we have

$$\sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}\left|\hat{\psi}^l(p^j(\xi\oplus k))\right|^2\leq 1 \text{ for } k\in\mathbb{Z}^+, a.e. \xi\in[0,1/2).$$

Hence,

$$\int_D\frac{d\xi}{|\xi|}=\sum_{l=1}^{L}\int_{\mathbb{R}^+}|\hat{\psi}^l(\xi)|^2\frac{d\xi}{|\xi|}=\int_D\left(\sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}\left|\hat{\psi}^l(p^j\xi)\right|^2\right)\frac{d\xi}{|\xi|}\leq\int_D\frac{d\xi}{|\xi|},$$

we have $\sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}\left|\hat{\psi}^l(p^j\xi)\right|^2=1$ for a.e. $\xi\in D$ and hence for a.e. $\xi\in\mathbb{R}^+$, i.e., equation (3.9) holds. By Lemma 3.7 and (3.9), $|<\tilde{G}(\xi)e_k,e_k>|^2=1$ for all $k\in\mathbb{Z}^+$. Hence by (3.12), it follows that $<\tilde{G}(\xi)e_k,e_{k'}>=0$ for $k\neq k'$ so that $\tilde{G}(\xi)$ is the identity operator on $l^2(\mathbb{Z}^+)$. Hence, by Theorem 3.5, $X^q(\Psi)$ is a tight frame with constant 1. So is $X(\Psi)$ by Theorem 3.3.

Theorem 3.10. Suppose $\Psi=\{\psi^1,\psi^2,...,\psi^L\}\subseteq L^2(\mathbb{R}^+)$. Then the following are equivalent:
(a) Ψ is a set of basic wavelets of $L^2(\mathbb{R}^+)$.
(b) Ψ satisfies (3.4) and (3.9).
(c) Ψ satisfies (3.4) and (3.11).

Proof. It follows from Theorem 3.9 and Lemma 3.7 that $(a)\Rightarrow(b)\Rightarrow(c)$. We now prove that (c) implies (a). Assume that Ψ satisfies (3.4) and (3.11). The equation (3.4) implies that $X(\Psi)$ is an orthonormal system, hence it is a Bessel family with constant 1. By Theorem 3.9 and (3.11), $X(\Psi)$ is a tight frame with constant 1. Since each ψ^l has L^2 norm 1, it follows that $X(\Psi)$ is an orthonormal basis for $L^2(K)$. That is, Ψ is a set of basic wavelets of $L^2(K)$.

References

[1] Abdullah, Affine and quasi-affine frames on positive half line, J. Math. Ext., in press.
[2] M. Bownik, On characterizations of multiwavelets in $L^2(\mathbb{R}^n)$, Proc. Amer. Math. Soc., 129 (2001), 3265-3274.
[3] A. Calogero, A characterization of wavelets on general lattices, J. Geom. Anal. 10 (2000), 597-622.
[4] C. K. Chui, X. Shi and J. Stöckler, Affine frames, quasi-affine frames, and their duals, Adv. Comput. Math., 8 (1998), 1-17.

[5] Y. A. Farkov, Orthogonal p-wavelets on R+, in Proceedings of International Conference Wavelets and Splines, St. Petersberg State University, St. Petersberg (2005), 4-6.
[6] Y. A. Farkov, A. Y. Maksimov and S. A. Stroganov, On biorthogonal wavelets related to the Walsh functions, Int. J. Wavelets, Multiresolut. Inf. Process. 9(3) (2011), 485-499.
[7] G. Gripenberg, A necessary and sufficient condition for the existence of a father wavelet, Stud. Math. 114 (1995), 207-226.
[8] E. Hernández and G. Weiss, A First Course on Wavelets, CRC Press, New York, 1996.
[9] A. Ron and Z. Shen, Frames and stable bases for shift invariant subspaces of $L^2(\mathbb{R}^d)$, Canad. J. Math., 47 (1995), 1051-1094.
[10] F A Shah and L. Debnath, Dyadic wavelet frames on a half-line using the Walsh-Fourier transform, Integ. Trans. Special Funct., 22(2011), 477-486.
[11] F. A. Shah, Tight wavelet frames generated by the Walsh polynomials, Int. J. Wavelets, Multiresolut. Inf. Process., 11 (2013), Article ID 1350042.

Department of Mathematics, ZHDC, University of Delhi, JLN Marg, New Delhi-110 002, India

*Corresponding author: abd.zhc.du@gmail.com