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CHARACTERIZATION OF MULTIPLICATIVE METRIC COMPLETENESS

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Abstract. We established fixed point theorems in multiplicative metric spaces. The obtained results generalize Banach contraction principle in multiplicative metric spaces and also characterize completeness of the underlying multiplicative metric space.

1. Introduction and preliminaries

In 1970 Michael Grossman and Robert Katz [11] established a new calculus called *multiplicative calculus* also termed as *exponential calculus.* Florack and Van Assen [10] used the idea of multiplicative calculus in biomedical image analysis. Bashirov *et al.*[3] demonstrated the efficiency of multiplicative calculus over the Newtonian calculus. They elaborated that multiplicative calculus is more effective than Newtonian calculus for modeling various problems from different fields . Bashirov and Bashirova [4] used the concept of multiplicative calculus for deriving function that shows dynamics of literary text. Bashirov *et al.*[2] further demonstrated the usefulness of multiplicative calculus by proving the fundamental theorem of multiplicative calculus. By defining multiplicative distance they provided foundation for multiplicative metric spaces. Özavsar and Cevikel [13] presented the notion of multiplicative contraction mapping. Besides some other results, they proved the well known Banach contraction principle for such contraction in multiplicative metric spaces. HXiaoju *et al.*[12] established common fixed point theorems for weak commutative mappings in the setting of multiplicative metric space. Abbas *et al.*[1] established common fixed point results of quasi-weak commutative mappings on a closed ball in the framework of multiplicative metric spaces. Banach contraction principle has been a very advantageous and effectual means in nonlinear analysis. Generalization of the Banach contraction principle has been one of the most enquired branch of research. Banach theorem has many generalizations; (see [5, 6, 7, 8, 17]). Sarwar and Rome [16] established several generalizations of Banach contraction principle and proved Cantor intersection theorem in the framework of Multiplicative metric spaces. Tomonari Suzuki [18] proved a fixed point result which generalizes Banach theorem and characterizes metric completeness.

In the current article we prove fixed point results in the set up of multiplicative metric spaces. The derived results results generalized Banach contraction principle in multiplicative metric spaces and characterize completeness of the underlying multiplicative metric space. For various definitions and elements of multiplicative calculus we refer the reader to [1, 2, 3, 9, 11, 12, 13, 14, 15].

Definition 1.1. [2] *Let M be a nonempty set. A mapping $d : M \times \ M \to [1, \infty)$ is said to be multiplicative metric on M if the following condition are satisfied:*

(1) $d(x, y) \geq 1$ *for all* $x, y \in M$;
(2) $d(x, y) = 1$ *if and only if* $x = y$;
(3) $d(x, y) = d(y, x)$ *for all* $x, y \in M$;
(4) $d(x, z) \leq d(x, y).d(y, z)$ *for all* $x, y, z \in M$. *And the pair* (M, d) *is called multiplicative metric space.*

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Example 1.1. [2] *The mapping* $d^* : (0,\infty) \times (0,\infty) \to [1,\infty)$ *defined as* $d^*(x,y) = |\frac{x}{y}|^*$, *where* $|a|^* = \begin{cases} a & \text{if } a \geq 1 \\ \frac{1}{a} & \text{if } a < 1 \end{cases}$. *is a multiplicative metric.*

Definition 1.2. [1, 12, 13] *A sequence* $\{x_n\}$ *in a multiplicative metric space* (X,d) *is said to be multiplicative Cauchy sequence if for all* $\epsilon > 1$ *there exits a positive integer* n_0 *such that* $d(x_n, x_m) < \epsilon \ \ \forall n, m \geq n_0$.

Definition 1.3. [1, 12, 13] *A multiplicative metric space* (X,d) *is said to be complete if every multiplicative Cauchy sequence in* X *converges in* X.

Definition 1.4. [1, 12, 13] *Let* (X,d) *be a multiplicative metric space. A mapping* $f : X \to X$ *is called multiplicative contraction if there exists a real constant* $\lambda \in [0,1)$ *such that* $d(f(x_1), f(x_2)) \leq d(x_1, x_2)^\lambda \ \ \forall x, y \in X$.

2. Main results

This section studied two fixed point theorems in the setting of multiplicative metric spaces. The first result generalized the Banach contraction principal while the second one characterizes multiplicative metric completeness.

Theorem 2.1. *Let* (M,d) *be a complete multiplicative metric space. Let* f *be a mapping on* M *and* $\varphi : [0,1) \to (1/2, 1]$ *a non increasing function defined as follows*

$$\varphi(\gamma) = \begin{cases} 1 & \text{if } 0 \leq \gamma \leq (\sqrt{5}-1)/2, \\ (1-\gamma)\gamma^{-2} & \text{if } (\sqrt{5}-1)/2 < \gamma < 1/\sqrt{2}, \\ (1+\gamma)^{-1} & \text{if } 1/\sqrt{2} \leq \gamma < 1. \end{cases}$$

Let there exists $\gamma \in [0,1)$ *such that*

$$d(x, fx)^{\varphi(\gamma)} \leq d(x,y) \quad \Rightarrow \quad d(fx, fy) \leq d(x,y)^\gamma \ \ \forall \ \ x, y \in M. \tag{1}$$

Then f *has a unique fixed point* z. *Furthermore* $lim_n f^n x = z$ *for all* $x \in M$.

Proof. As $\varphi(\gamma) \leq 1$ therefor $d(x, fx)^{\varphi(\gamma)} \leq d(x, fx) \ \ \forall \ \ x \in M$. condition (1) implies

$$d(fx, f^2x) \leq d(x, fx)^\gamma \ \ \forall \ \ x \in M. \tag{2}$$

Fix $v \in M$ and define a sequence $\{v_n\}$ in M by $v_n = f^n v$. Relation (2) implies that $d(v_n, v_{n+1}) \leq d(v, fv)^{\gamma^n}$. Therefor $\prod_{n=1}^{\infty} d(v_n, v_{n+1}) \leq d(v, fv)^{\frac{\gamma}{1-\gamma}} < \infty$. It means $\{v_n\}$ is a Cauchy sequence. As M is complete so $\{v_n\}$ converges to some point $z \in M$. We show that

$$d(fx, z) \leq d(x, z)^\gamma \ \ \forall \ \ x \in M \backslash \{z\}. \tag{3}$$

For $x \in M \backslash \{z\}$ there will be some positive integer m such that $d(v_n, z) \leq d(x,z)^{1/3}$ $\forall \ \ n \geq m$. We have

$$\begin{aligned} d(v_n, fv_n)^{\varphi(\gamma)} &\leq d(v_n, fv_n) = d(v_n, v_{n+1}) \leq d(v_n, z) d(z, v_{n+1}) \\ &\leq d(x,z)^{2/3} = \frac{d(x,z)}{d(x,z)^{1/3}} \leq \frac{d(x,z)}{d(v_n,z)} \leq d(v_n, x). \end{aligned}$$

Using hypothesis of theorem, we get $d(v_{n+1}, fx) \leq d(v_n, x)^\gamma$ for $n \geq m$. Letting $n \to \infty$, we get $d(z, fx) \leq d(z, x)^\gamma$. Hence (3) is proved. Now let us suppose by the way of contradiction that $f^i z \neq z$ for all $i \in N$. Then (3) gives

$$d(f^{i+1}z, z) \leq d(fz, z)^{\gamma^i} \ \ \forall \ \ i \in N. \tag{4}$$

Now consider the following cases.

- $0 \leq \gamma \leq (\sqrt{5}-1)/2$,
- $(\sqrt{5}-1)/2 < \gamma < 1/\sqrt{2}$,
- $1/\sqrt{2} \leq \gamma < 1$.

When $0 \leq \gamma \leq (\sqrt{5}-1)/2$, then $\gamma^2 + \gamma - 1 \leq 0$, also $2\gamma^2 \leq 3 - \sqrt{5} < 1$.
If $d(f^2z, z) < d(f^2z, f^3z)$, then

$$d(z, fz) \leq d(z, f^2z) d(f^2z, fz) < d(f^2z, f^3z) d(f^2z, fz) \leq d(z, fz)^{\gamma^2+\gamma} \leq d(z, fz).$$

Which is contradiction. Therefor $d(f^2z,z) \geq d(f^2z, f^3z) = d(f^2z, f \circ f^2z)^{\varphi(\gamma)}$.
Using hypothesis of the theorem and (4),we have

$$\begin{aligned} d(z,fz) &\leq d(z,f^3z)d(f^3z,fz) \leq d(z,fz)^{\gamma^2}d(f^2z,z)^{\gamma} \leq d(z,fz)^{\gamma^2}d(fz,z)^{\gamma^2} \\ &= d(fz,z)^{2\gamma^2} < d(fz,z). \end{aligned}$$

Which is contraction. And when $(\sqrt{5}-1)/2 < \gamma < 1/\sqrt{2}$ then $2\gamma^2 < 1$. If we suppose $d(f^2z,z) < d(f^2z,f^3z)^{\varphi(\gamma)}$, then using (2) we have

$$\begin{aligned} d(z,fz) &\leq d(z,f^2z)d(f^2z,fz) < d(f^2z,f^3z)^{\varphi(\gamma)}d(f^2z,fz) \leq d(z,fz)^{\varphi(\gamma)\gamma^2}d(z,fz)^{\gamma} \\ &= d(z,fz)^{\varphi(\gamma)\gamma^2+\gamma} = d(z,fz)^{(1-\gamma)\gamma^{-2}\gamma^2+\gamma} = d(z,fz), \end{aligned}$$

giving a contradiction. Hence $d(f^2z,z) \geq d(f^2z,f^3z)^{\varphi(\gamma)} = d(f^2z, f\circ f^2z)^{\varphi(\gamma)}$. And this, like the preceding case, produces the following contradiction.

$$d(z,fz) \leq d(z,fz)^{2\gamma^2} < d(z,fz).$$

Finally when $1/\sqrt{2} \leq \gamma < 1$. Then for $x,y \in M$, either $d(x,fx)^{\varphi(\gamma)} \leq d(x,y)$ or $d(fx,f^2x)^{\varphi(\gamma)} \leq d(fx,y)$. In case $d(x,fx)^{\varphi(\gamma)} > d(x,y)$ and $d(fx,f^2x)^{\varphi(\gamma)} > d(fx,y)$, then using multiplicative triangular inequality and (2), we have

$$\begin{aligned} d(x,fx) &\leq d(x,y)d(y,fx) < d(x,fx)^{\varphi(\gamma)}d(fx,f^2x)^{\varphi(\gamma)} = (d(x,fx)d(fx,f^2x))^{\varphi(\gamma)} \\ &= d(x,fx)^{(1+\gamma)\varphi(\gamma)} = d(x,fx)^{(1+\gamma)(1+\gamma)^{-1}} = d(x,fx). \end{aligned}$$

Which is again contradiction. Now since $d(v_{2n},v_{2n+1})^{\varphi(\gamma)} \leq d(v_{2n},z)$ or $d(v_{2n+1},v_{2n+2})^{\varphi(\gamma)} \leq d(v_{2n+1},z)$ $\forall n \in N$. Therefore using hypothesis of the theorem, either $d(v_{2n+1},fz) \leq d(v_{2n},z)^{\gamma} \leq d(v_{2n},z)$ or $d(v_{2n+2},fz) \leq d(v_{2n+1},z)^{\gamma} \leq d(v_{2n+1},z)$ $\forall$ $n \in N$. Now $\{v_n\}$ converges to z, but the above inequalities indicate that there is a subsequence of $\{v_n\}$ which converges to fz. Therefore $fz = z$. This contradicts the supposition. Hence in all the above cases, there will be some $i \in N$ such that $f^iz = z$. As $\{f^nz\}$ is a Cauchy sequence, therefore $fz = z$. In order to show uniqueness of the fixed point of f, let $w \in M\backslash\{z\}$ be another fixed point of f. Then using (3), we have the contradiction, $d(w,z) = d(fw,z) \leq d(w,z)^{\gamma} < d(w,z)$. Hence z is the only fixed point of f in M. □

Theorem 2.2. *Let (M,d) be a multiplicative metric space and φ be a mapping as defined in Theorem 2.1. For $\gamma \in [0,1)$ and $\beta \in (0,\varphi(\gamma)]$, let $S_{\gamma,\beta}$ be the family of mappings f on M satisfying the following:*
(1) For $x,y \in M$, $d(x,fx)^{\beta} \leq d(x,y) \quad \Rightarrow \quad d(fx,fy) \leq d(x,y)^{\gamma}$.
Let $T_{\gamma,\beta}$ be the family of mappings f on M satisfying (1) and the following:
(2) $f(M)$ is countably infinite.
(3) Every subset of $f(M)$ is closed.
Then the following are equivalent:
(a) M is complete.
(b) Every mapping $f \in S_{\gamma,\beta}$ has a fixed point for all $\gamma \in [0,1)$.
(c) There exist $\gamma \in (0,1)$ and $\beta \in (0,\varphi(\gamma)]$ such that every mapping $f \in T_{\gamma,\beta}$ has a fixed point.

Proof. As $\beta \leq \varphi(\gamma)$, therefore using Theorem 2.1, (a) implies (b). And as $T_{\gamma,\beta} \subset S_{\gamma,\beta}$, therefore (b) implies (c). Next we prove that (c) implies (a). Let (c) holds but M is not complete. It means there exists a Cauchy sequence $\{v_n\}$ which doesn't converge in M. Define a mapping $g : M \to [1,\infty)$ by $g(x) = lim_n d(x,v_n)$ for $x \in M$. With the properties of multiplicative metric, the following are obvious:
(i) $g(x)/g(y) \leq d(x,y) \leq g(x)g(y)$ for all $x,y \in M$,
(ii) $g(x) > 1$ for all $x \in M$ and
(iii) $\lim_n g(v_n) = 1$.
Define a mapping f on M as follows: As for each $x \in M$, $g(x) > 1$ and $\lim_n g(v_n) = 1$, therefore there exists $\eta \in N$ such that $g(v_\eta) \leq g(x)^{\frac{\gamma\beta}{3+\gamma\beta}}$. For $f(x) = v_\eta$,

$$\text{(5)} \qquad \text{obviously} \quad g(fx) \leq g(x)^{\frac{\gamma\beta}{3+\gamma\beta}} \quad \text{and } fx \in \{v_n : n \in N\} \quad \text{for all } x \in M.$$

This implies that $g(fx) < g(x)$ for all $x \in M$ therefore $fx \neq x$ for all $x \in M$. That is f has no fixed point. Now since $f(M) \subset \{v_n : n \in N\}$, therefore condition (2) is satisfied. Obviously every subset of

$f(M)$ is closed, that is (3) holds. In order to prove (1), fix $x, y \in M$ such that $d(x, fx)^\beta \leq d(x, y)$. In case where $g(y) > (g(x))^2$,

$$\begin{aligned}(i) \text{ and } (5) \text{ imply that } d(fx, fy) &\leq g(fx)g(fy) \leq (g(x)g(y))^{\frac{\gamma\beta}{3+\gamma\beta}} \leq (g(x)g(y))^{\frac{\gamma}{3}} \\ &< (g(x)g(y))^{\frac{\gamma}{3}} (\frac{g(y)}{(g(x))^2})^{\frac{2\gamma}{3}} \leq (\frac{g(y)}{g(x)})^\gamma \leq d(x, y)^\gamma.\end{aligned}$$

And when $g(y) \leq (g(x))^2$, then again using (i) and (5), we have

$$d(x, y) \geq d(x, fx)^\beta \geq (\frac{g(x)}{g(fx)})^\beta \geq (\frac{g(x)}{g(x)^{\frac{\gamma\beta}{3+\gamma\beta}}})^\beta = g(x)^{\frac{3\beta}{3+\gamma\beta}}.$$

$$\begin{aligned}\text{And therefore } d(fx, fy) &\leq g(fx)g(fy) \leq (g(x)g(y))^{\frac{\gamma\beta}{3+\gamma\beta}} \leq (g(x)(g(x))^2)^{\frac{\gamma\beta}{3+\gamma\beta}} \\ &\leq g(x)^{\frac{3\gamma\beta}{3+\gamma\beta}} = \big(g(x)^{\frac{3\beta}{3+\gamma\beta}}\big)^\gamma \leq d(x, y)^\gamma.\end{aligned}$$

Therefore (1) is proved. Hence $f \in T_{\gamma,\beta}$. And by (c), f has a fixed point. which is contradiction. Consequently M is complete. This completes the proof. □

We conclude with the following example which supports Theorem 2.1.

Example 2.1. *Let $M = R^+$,set of positive real numbers. Consider the multiplicative metric $d : M \times M \to [1, \infty)$ defined by $d(x, y) = e^{|x-y|}$. Then (M, d) is complete multiplicative metric space. Let φ be a mapping as defined in Theorem 2.1. $T : M \to M$ be mapping defined by $T(x) = \frac{1}{5+x}$, such that $d(x, fx)^{\varphi(\gamma)} = d(x, fx)^{\varphi(\frac{1}{2})} = d(x, fx) = e^{|x-\frac{1}{5+x}|} \leq e^{|x-y|} = d(x, y)$ then $d(fx, fy) = e^{|x-y||\frac{1}{(5+x)(5+y)}|} \leq e^{\frac{1}{2}|x-y|} = d(x, y)^\gamma \quad \forall x, y \in M$.*
Obviously T has unique fixed point $0.1925824036 \in M$.

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